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Bounded Confidence Model with Fixed Uncertainties and Extremists: The Opinions Can Keep Fluctuating Indefinitely

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Abstract

The bounded confidence model and its variants applied to moderate and extremist agents exhibit three types of attractors: central clusters, double extreme and single extreme clusters. These attractors are observed when the models include a dynamics on the uncertainties tending to decrease the moderate uncertainties when interacting with extremists. We show here that a new stationary state appears when the uncertainties are fixed, for large uncertainties of the moderates. In this stationary state, the opinions of moderate agents keep fluctuating without clustering, altogether forming a stable density which shape changes significantly when the parameters vary.

Keywords:

Bounded Confidence Model, Opinion Dynamics, Noise, Fluctuations, Extremists

Introduction

- 1.1 Understanding how extremist minorities can influence moderate majorities is a major issue in social sciences. The spreading of minority opinions (Galam 2002) is of growing concern in order to evaluate the impact of new emerging opinions: spreading of a reform proposal, debates on fashion products etc. ... In social psychology for instance, Moscovici brought substantial advances in this understanding with famous experiments about the influence of minorities in small groups (Moscovici et al. 1969). It is more generally an issue in the history of the last century with the rise of extreme ideologies leading to totalitarian states or in the more recent period with terrorist attacks perpetrated by religious extremists. In these cases, the propagation of extremist views is considered as a cause of murder and destruction. However, opinions that were considered extremist two centuries ago, for instance defending equality between men and women, have become the majority view in western culture and hence the propagation of extremism can also be considered as a major progress. Therefore this issue of extremism propagation should be considered without any value judgment.
- 1.2 Computer simulations can provide some insights on this issue, particularly when considering large populations on which it is impossible to perform experiments. This approach requires expressing clearly some hypotheses on the rules of interactions modifying agent opinions and then to explore the consequences of these hypotheses on the collective behaviour of interacting agents. Several models are available and already addressed the issue of extreme opinion propagation or minority influence.
- 1.3 For some of them (Axelrod 1997; Galam & Moscovici 1991; Sznadj-Weron 2005), the opinion can only take two values (e.g. A or B) (see Castellano et al. 2009 for a review of opinion dynamics models). The general mechanism is based on a function of the opinion in the local neighbourhood that determines if agents switch their opinion or not. Binary opinions are simple and fit to decision-making issues, but it is impossible to state if opinion A is more extreme than opinion B.
- 1.4 Other models include more possible values for the opinion. We consider here one of the most popular continuous opinion models, the bounded confidence (BC) model (Deffuant et al. 2001; Hegselmann & Krause 2002; Krause 2000; Weisbuch et al. 2002) from which several analytical models were developed (Canuto et al. 2012; Tabatabaei et al. 2014). Its principles are very close to the theory of social comparison (Festinger 1954) and to social judgment theory (Sherif & Hovland 1961). The agents are

characterized by an initial opinion (generally supposed to be between -1 and 1 as in the present study, or between 0 and 1) and a threshold that can be interpreted as an uncertainty around this opinion. The dynamics in the version of (Deffuant et al. 2001) picks random pairs of agents that influence each other by slightly moving their opinions towards each other, but only if the distance between their opinions is lower than the uncertainty. When all the agents share the same uncertainty, the opinions progressively converge to a set of opinion clusters, with all agents of a cluster sharing the same opinion and with an increasing number of clusters when the uncertainty decreases.

- 1.5 The model has been criticized because when, for some randomly chosen interactions with a low probability, the confidence is not bounded (i.e. any opinion can attract any other), the opinion distribution always converges to a single cluster (possibly after a very long number of iterations) (Mäs et al. 2010; Mäs & Flache 2013; Takács et al. 2014). However for other types of noise: adding to the opinion of each individual, a random number uniformly distributed between -0.3 and +0.3, multiplied by the uncertainty of this individual, the qualitative behaviour of the model holds, keeping a diversity of clusters (Deffuant 2006; Pineda et al. 2011), which weakens the criticisms.
- 1.6 With variants of this model, Castellano et al. (2009) and Deffuant (2006) study the impact of extremists on moderate agents. The extremists are initialized with extreme opinions (+1 or -1) and all share the same small uncertainty, whereas the moderates are initialized with an opinion uniformly drawn between -1 and +1 and the same larger uncertainty. Moreover, in these models, both opinions and uncertainties are modified during interactions. The extremists tend to reduce the uncertainty of the moderates during interactions. By construction of the model, the extremists are more influential and less likely to change their opinions and uncertainties. More recently, there is a renewed interest in studying the effect of extremism propagation such as Hegselmann (2014) and Hegselmann and Krause (2015) who studied the influence of "radical" individuals on "normal" individuals, "radicals" being agents with low uncertainty and an opinion close to the upper bound.
- 1.7 Depending on the initial number of extremists and the initial uncertainty of the moderates, three convergence types appear: central clusters, double extreme clusters, single extreme cluster. Recently Meadows and Cliff (2012) claimed to have found a different attractor map in the parameter space, but Deffuant et al. (2013) show that the difference is due to a different criterion for stopping the simulations. The single extreme cluster was somehow unexpected and subject of specific studies (Deffuant & Weisbuch 2008). More recent models of continuous opinions, instead of adding extremists in their initial population, study dynamics including a negative influence which can also lead to the emergence of polarization and extremism (Flache & Mäs 2008; Huet et al. 2008; Huet & Deffuant 2010; Jager & Amblard 2004; Mäs & Flache 2013; Moussaïd 2013).
- 1.8 This paper comes back to the original BC model with fixed uncertainties and studies its behaviour when adding extremists. To the knowledge of the authors, such a study is not reported in the literature. The primary goal of this paper is to improve the knowledge of the BC model, broadly studied by scholars. Our main result is that the model with fixed uncertainties exhibits a stationary state that has not been observed in models with changing uncertainties. In this stationary state, the opinions of the moderate agents keep fluctuating with different amplitudes depending on the parameters (uncertainty, proportion of extremists) whereas the opinion density distribution remains stable.
- 1.9 We study this stationary state in more details with a large number of agents (10 millions) and a model of evolving density, corresponding to the limit case of an infinity of agents. The density distribution shows several patterns, depending on the uncertainty of moderates, the proportion of extremists and sometimes the intensity of the interactions.
- 1.10 In the next section, we describe the model and the rules determining the convergence types. Section 3 presents the map of steady states in the parameter space and gives more details about the new stationary state. The final section is devoted to a discussion of this result.

The model and convergence patterns

The bounded confidence model (BC) with extremist and fixed uncertainties

- 2.1 Following the model of Deffuant (2006), in a population of N agents, we consider two types of agents, the moderates and the extremists. The moderate agents share the same uncertainty u and the extremists the uncertainty u_e (a positive real number). We take here a very small value of u_e compared to u (typically $u_e \leq u/100$). It leads to very small opinion variations of extremists: extremist opinions can be considered as quasi static in view of these small variations. The opinions of the moderate agents are uniformly drawn between -1 and +1. We choose an opinion range equal to $[-1, +1]$, as in the original paper about the model with extremists (Deffuant et al. 2002), because it facilitates the interpretation of extreme opinions against (-1) or in favour (+1) a given issue; this choice has of course no impact on the patterns emerging from the dynamics. The proportion of extremists in the population is denoted p_e and half of them is initialised with opinion +1 and the other half with opinion -1.
- 2.2 The dynamics of the opinions is the BC model in its version of Deffuant et al. (2001) and the uncertainties remain constant, which is the major difference with Deffuant et al. (2002) and Deffuant (2006).
- 2.3 At each time step, $\frac{N}{2}$ random pairs interact: each agent is paired once in one period. A randomly drawn agent i with opinion x_i , an uncertainty u_i meets with a randomly drawn agent j with opinion x_j and uncertainty u_j . Their opinions are modified according

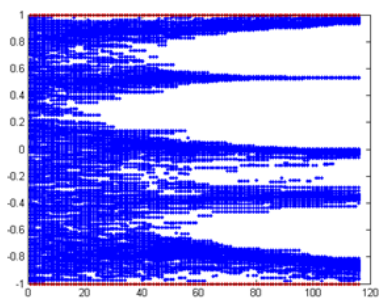
the following rule:

$$\begin{cases} \text{if } |x_i - x_j| < u_i \text{ then } x_i := x_i + \mu(x_j - x_i) \\ \text{if } |x_i - x_j| < u_j \text{ then } x_j := x_j + \mu(x_i - x_j) \end{cases} \quad (1)$$

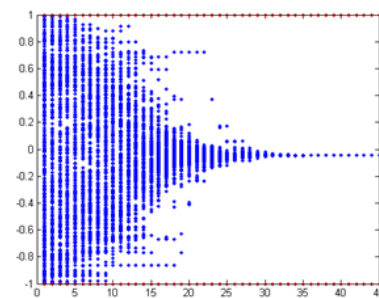
2.4 In this equation, μ is a parameter of the model that rules the intensity of the influence (generally $0 < \mu \leq 0.5$). The interaction updating mainly depends on two issues: 1) the number of individuals involved in the interactions; 2) the updating procedure (synchronous or asynchronous, see Alizadeh & Cioffi-Revilla 2015 for more details). In our model we chose pair interactions, as in Deffuant et al. (2002) and Deffuant (2006). We also chose a synchronal updating, i.e. all individuals' opinions are updated simultaneously at each time step. Therefore, we have a simultaneous updating of the opinions as in the probability distribution version of the model described in section 3.

Indicators and rules defining convergence patterns

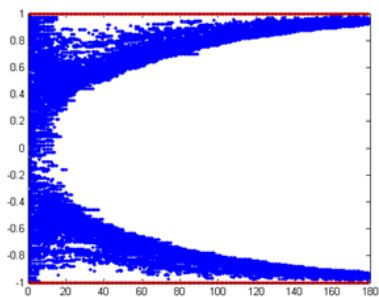
2.5 Deffuant et al. (2002) and Deffuant (2006) identify three main patterns of convergence: central, double extreme and single extreme convergence (see Figure 1).



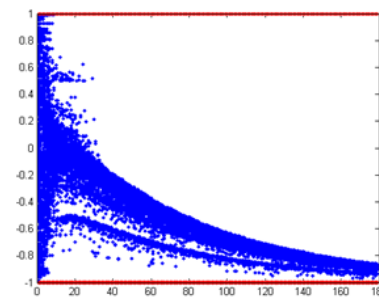
$p_e = 0.05, u = 0.25$, central clusters, red dots in figure 2



$p_e = 0.05, u = 0.75$, single central cluster, light blue in figure 2



$p_e = 0.05, u = 0.5$, double extreme, green in figure 2



$p_e = 0.05, u = 1$, single extreme, orange in figure 2

Figure 1. Examples of opinion trajectories for different usual convergence patterns. The vertical axis is the opinion value; the horizontal axis is the number of iterations; the opinions of the 400 agents are represented at each time step. p_e is the proportion of extremists, u is the uncertainty of the moderates.

2.6 We use the same rules as in Deffuant (2006) for classifying automatically the convergence patterns, that we are recalling in this section. We sum all opinion variations of all individuals at each time step. If this sum is lower than 0.001, we will consider the convergence of the model and the simulation will stop. The indicators are based on the generalized number of clusters and the average absolute value of moderate opinions. We define the generalized number of clusters in a population involving k opinion clusters x_i , each with a proportion r_i of agents as:

$$c = \frac{1}{\sum_{i=1}^k r_i^2} \quad (2)$$

2.7 It corresponds to a smooth number of clusters obtained following the method defined in Derrida and Flyvbjerg (1986). Note that the detailed algorithm can be found in Deffuant (2006) and in Appendix A. A cluster of opinions contains opinions which are

different from each other from a maximum value equal to ε .

2.8 We also define the average of the absolute value of the moderate opinions, noted X , which indicates how extreme the population is:

$$X = \frac{\sum_{i=1}^N |x_i|}{N} \quad (3)$$

2.9 Finally, we combine these two indicators in order to characterize the convergence type. We use the following rules, inspired from Deffuant (2006):

- $X < 0.75$ and $c < 1.33$: convergence to a single moderate cluster, represented in "light blue" on Figure 2;
- $X > 0.75$, $c > 1.66$ and $c < 2.33$: convergence to two extreme clusters, represented in "green" on Figure 2;
- $X > 0.75$ and $c < 1.66$: convergence to a single extreme cluster, represented in "orange" on Figure 2;
- $c > 2.33$: convergence to several clusters, represented in "red" on Figure 2;
- no convergence after 2000 time steps (the opinions keep fluctuating), represented in "dark blue" on Figure 2.

2.10 Using these rules, it is possible to classify the different convergence patterns obtained when the parameters vary.

Experiments and map of stationary states

3.1 We perform an experimental design with a population of 400 individuals and $\mu = 0.5$ for the BC model. The parameters varying in the experiments are: $p_e \in [0; 1]$ (the proportion of extremists) and $u \in [0.02; 2]$ (the uncertainty of moderates). We kept the ratio between moderates' and extremists' uncertainties at the value of 100. The map of convergence types is reported in Figure 2 (see Appendix B for more details). Figure 1 shows examples of the usual patterns already identified in the literature.

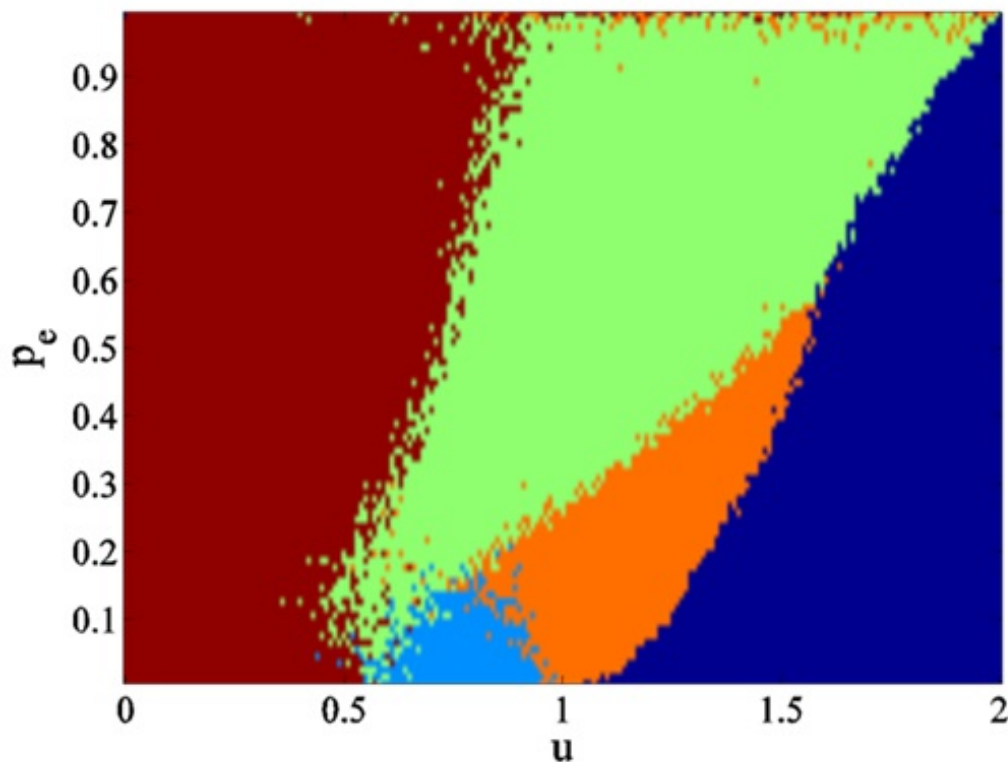


Figure 2. Map of convergence patterns for 400 agents. Each dot of the map represents a simulation and the dot's colour codes for the type of convergence: light blue: a single moderate cluster; green: double extreme clusters; orange: a single extreme cluster; red: several moderate clusters; dark blue: no clustering (opinions keep fluctuating).

3.2 The regions of central and double extreme patterns are similar to the ones found with the BC model with changing uncertainties. However, two main differences appear:

- the single extreme pattern appears in a different parameter zone (in orange on Figure 2) and in this zone the single extreme convergence is systematic, whereas with the model with changing uncertainties, there is a probability to get also a central convergence;
- in a large parameter zone (high values of u) the opinions do not stabilise. Even if we increase the number of time steps,

the moderate opinions do not cluster and keep fluctuating. With the model with changing uncertainties, the model converges to a single extreme or a moderate cluster for the same values of p_e and u . Simulations with 10 millions agents (see Figure 3) show that the distribution of the agents reaches a steady state. The stable distribution shows several maxima at opinions -0.5 ; -0.25 ; 0.25 ; 0.5 . The previous research did not report this pattern, as far as we know.

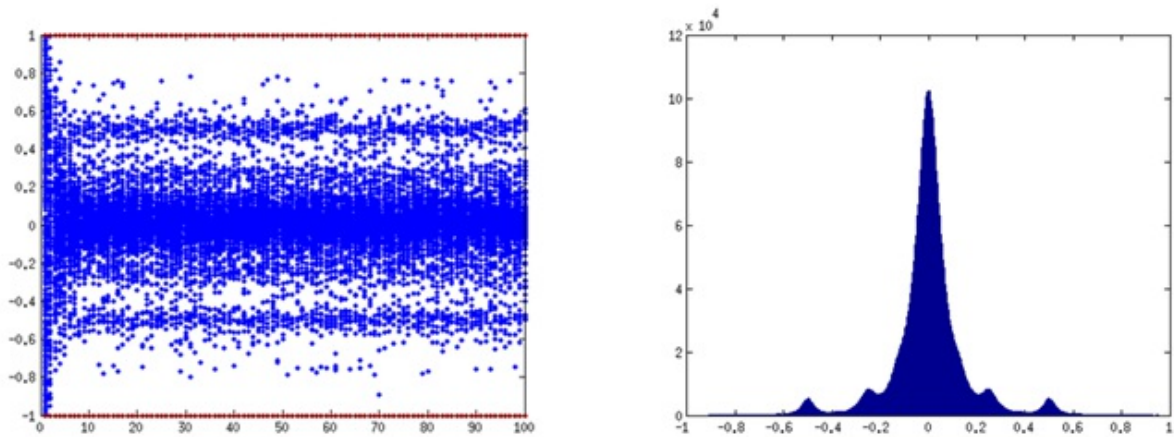


Figure 3. Stationary distribution of fluctuating moderates. $u=1.8$, $p_e = 0.05$ and $\mu = 0.5$. Left panel: time trajectories of the opinions with 400 agents. Right panel: stationary distribution with 10 Million agents, dark blue in Figure 2.

3.3 We now analyse these differences in more details.

Systematic single extreme in a restricted parameter zone

- 3.4 The single extreme convergence zone takes place for u around 1 and $p_e < 0.2$ (in orange on Figure 2). The process of single extreme convergence is similar to the one identified in previous research (Deffuant & Weisbuch 2008): in a first step a central cluster of opinions emerges. Then because the uncertainty is close to 1, when the distribution is narrow and not perfectly at the centre, there are significantly more agents that interact with only one extreme (positive or negative). Hence small fluctuations in this central cluster necessarily lead to the drift to this extreme (see Figure 1).
- 3.5 With the model with changing uncertainties, the single extreme convergence takes place for similar low proportion of extremists but also for initial uncertainties of the moderates much larger than 1. Indeed, the dynamics of uncertainties decreases these large uncertainties progressively when the moderates are in the central cluster because of the influence of extremists, until these uncertainties reach a value close to 1 that leads to the instability. However, the uncertainty may sometimes decrease below 1 before a significant drift towards one or the other extreme, leading to a central convergence (see Deffuant & Weisbuch 2008). This decrease below 1 does not take place when the uncertainties are fixed hence the instability of the central cluster systematically leads to a single extreme convergence when the uncertainty of the moderates is adequate.

Moderate opinions keep fluctuating for large values of uncertainty

- 3.6 When the opinions keep fluctuating, it seems particularly interesting to study a density distribution model derived from the agent model, and representing the case of an infinite number of agents. This type of model makes the different phenomena more apparent because the initial uniform distribution can be perfect and eliminate the random irregularities of the agent based model. The model includes an array of n values ρ_i representing the probability that agents have an opinion x located in an interval of size $1/n$ and centred on i/n :

$$\rho_i = P\left(\frac{2i-1}{2n} \leq x \leq \frac{2i+1}{2n}\right) \quad (4)$$

- 3.7 Initially, the density distribution of the moderate opinions is uniform, and for all i , $\rho_i = \frac{1-p_e}{n}$. The extreme values of the distribution are given a density $p_e/2$ of extremists. Then we classically compute the flows between the cells of the density because of the interactions and update iteratively (see for instance Deffuant & Weisbuch 2008 for details).
- 3.8 On Figure 4 we can check that the convergence density provides a good approximation of the one derived from the agent model with a large population. Therefore, we can use the density model to study the properties of this stationary state.

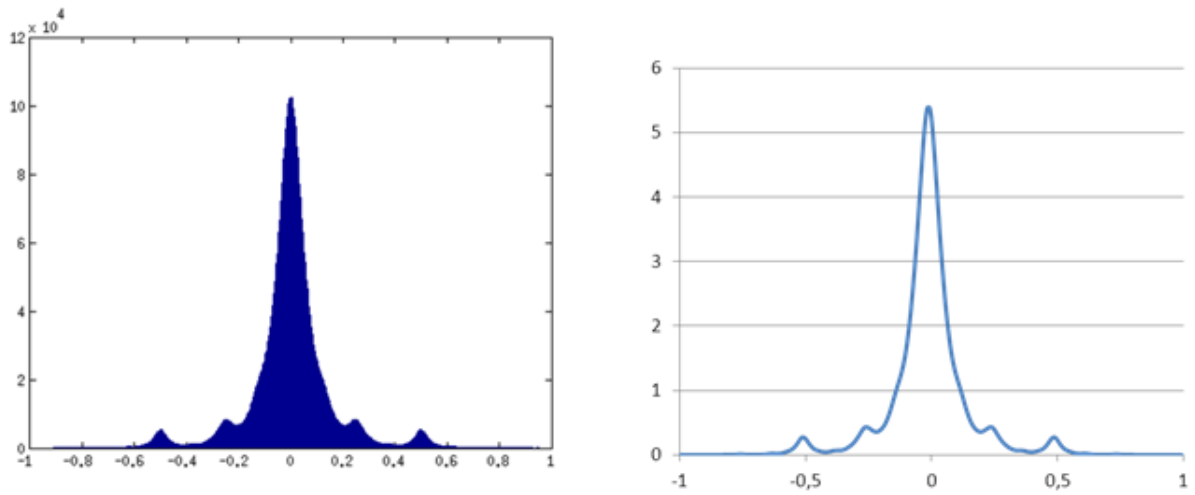


Figure 4. Stationary state with fluctuating moderates. Moderate uncertainty $u=1.8$ and proportion of extremists $p_e = 0.05$, intensity $\mu = 0.5$. Left panel: histogram of number of agents for the model with 10 Millions agents. Right panel: convergence density with the density distribution model.

3.9 The patterns of convergence distribution are the result of two opposite tendencies:

- while the opinions can be attracted by both extremes, the extremist influence makes them fluctuate strongly in both directions;
- the interactions between the moderates is averaging and tends to form a central peak of density, keeping the opinions in a zone where they can be attracted by both extremes.

3.10 These opposite tendencies reach an equilibrium leading to a stable density of opinions. However, the opinions keep fluctuating and the stable density is the result of averaging the presence of always changing agents.

3.11 Moreover, the model is ergodic: the positions of a single agent over a large number of time steps describe the same distribution as the positions of a large number of agents after convergence, as shown on Figure 5. This implies that the initial conditions are much less important: Many very different initial distributions of the moderates lead to the same stationary distribution.

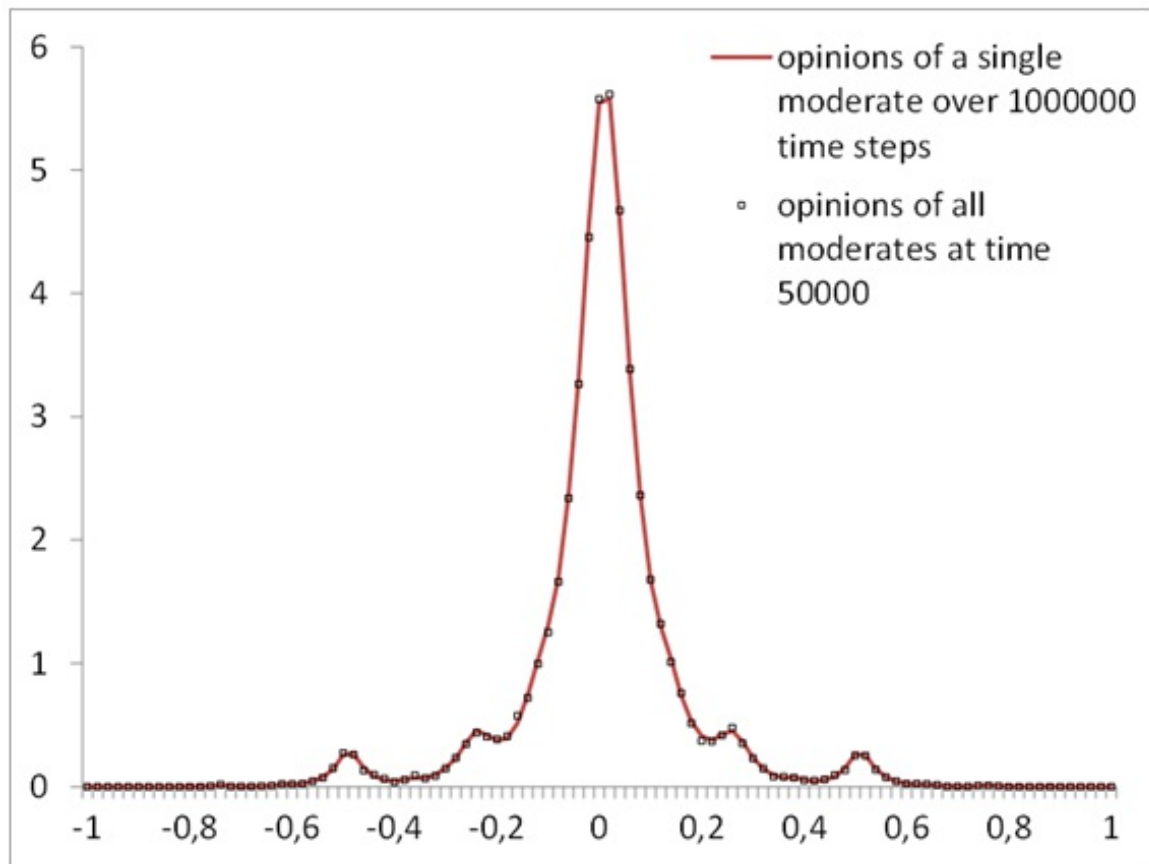
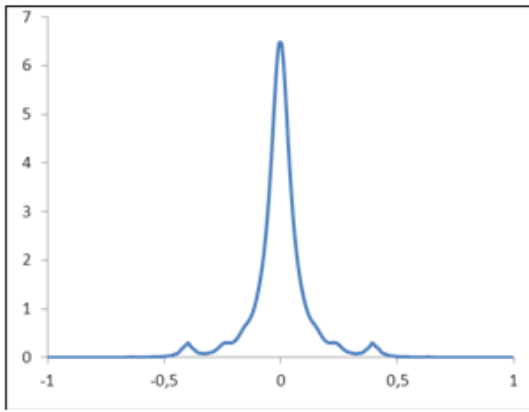


Figure 5. Ergodicity of the model. Comparing the density of opinions of a single agent over 1 Million time steps with the density of opinions of 1 Million agents at time step 50000 for $u = 1.8$, $p_e = 0.05$, $\mu = 0.5$.

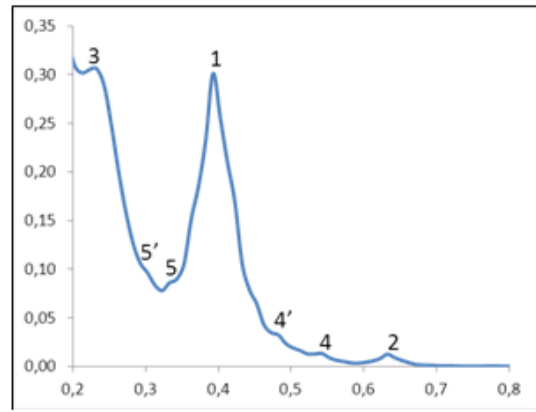
3.12 Depending on the values of the parameters we get very different patterns of distribution at convergence. For this purpose, we

study the density of opinions for different values of p_e , u and μ . Using the continuous model (see equation 4), these different densities of opinions are plotted in Figure 6. We note that:

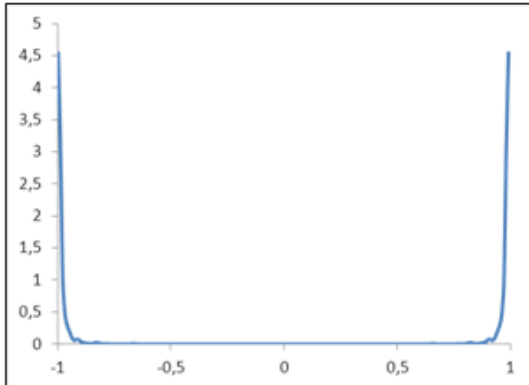
- When the proportion of extremists p_e is low, there is a central peak of density because the interactions between moderate agents tend to average their opinions. However, the uncertainty of moderates is large enough for the opinions located close to 0 to be attracted regularly by extremists of both sides. This creates secondary peaks at a distance μ of the central peak, and then the interaction of this secondary peak with the extremists creates a smaller peaks at distance $\mu + (1 - \mu) * \mu$ of the central peak, and so on. For instance on Figure 6 on the first line, for $\mu = 0.4$, we can observe several peaks of the distribution on the right panel. Peak 1 at $x_1 = \mu = 0.4$, is the result of interactions $i(C, E)$ between central peak C and extremists E . Peak 2, at $x_2 = x_1 + (1 - x_1) * \mu = 0.64$, is the result of interactions $i(1, E)$ between peak 1 and extremists. Peak 3, at $x_3 = x_1 - \mu x_1 = 0.24$ is the result of $i(1, C)$, peak 4, at $x_4 = x_2 - \mu(x_2 - x_1) = 0.544$, is the result of $i(2, 1)$, peak 4', at $x_{4'} = x_1 + \mu(x_2 - x_1) = 0.496$, is the result of $i(1, 2)$ Peak 5, at $x_5 = x_1 - \mu(x_1 - x_3) = 0.336$ is the result of $i(1, 3)$, peak 5', at $x_{5'} = x_3 + \mu(x_1 - x_3) = 0.3046$ is the result of $i(1, 3)$;
- When the proportion of extremists p_e is high (for instance 0.8), then the attraction of extremists prevents the central peak to appear, and the distribution tends to be higher in the vicinity of the extremists. When the uncertainty of moderates u is high, the probability of moderate opinions tends to be higher because the moderates attract each other for larger differences of opinions (see second row of Figure 6);
- Intermediate patterns take place for intermediate values of the parameters (see third line of Figure 6). Note the depression of the distribution at the centre and the complex geometry of local maxima and minima of variable sizes which are due to the local equilibrium between the attraction by extremists towards the extremes and towards the centre by moderates.



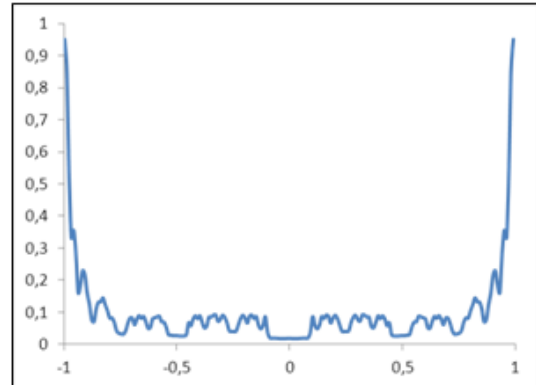
$$p_e = 0.05, u = 1.5, \mu = 0.4$$



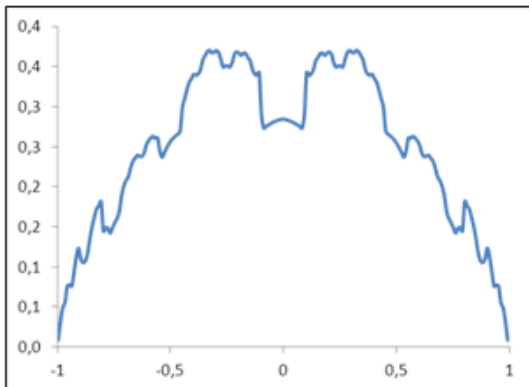
$$p_e = 0.05, u = 1.5, \mu = 0.4$$



$$p_e = 0.8, u = 1.5, \mu = 0.5$$



$$p_e = 0.8, u = 1.8, \mu = 0.5$$



$$p_e = 0.5, u = 1.8, \mu = 0.5$$

Figure 6. Patterns of moderates opinion density at convergence for high values of u and different values of p_e and μ . For small p_e , the distribution tends to be concentrated in the centre, while for large p_e , it is close to a double extreme clustering.

3.13 To summarize, the moderate opinion distribution tends to a central peak when the proportion of extremists p_e is low and tends to a double extreme peak when p_e is high (higher than 0.5). Moreover, a high uncertainty tends to increase the centrality of the distribution. These results are in line with the ones obtained previously with changing uncertainty models but the fixed uncertainties prevent the single extreme convergence for high uncertainties of the moderates. Moreover, the indefinite strong fluctuations of opinions is a really new feature. With the complex geometry of some densities, for instance for intermediate values of p_e , this new type of moderate opinions evolution shows a rich variety.

Discussion and conclusion

4.1 Theoretically, the convergence towards a stationary density distribution of opinions that keep fluctuating had already been identified (Blondel et al. 2013), especially when the interactions are not symmetric. However, in the currently published studies of standard variants of the BC model, only the models introducing directly some noise in the interactions exhibit such distributions (Pineda et al. 2009). In all the other studies, the model converges to a set of clusters in which all opinions are the same and stable. Therefore, the stationary state of the opinion density that we observe in the model with fixed uncertainties is new. Moreover, the variety of the distribution patterns is particularly striking compared with the always similar set of clusters obtained with the changing uncertainties models.

- 4.2 The qualitative explanation of this difference is rather straightforward. When moderates uncertainties are fixed and large, the moderates are submitted to the influence of both extremes and of the other moderates. They continuously fluctuate under these influences. This leads to ergodic dynamics (see Figure 4) which is a particularly new feature for the BC models. Indeed, the opinions of the agents take all the possible values of the distribution over time. The immediate consequence is that a large variety of initial distributions of the moderates opinions lead to the same stationary distribution (we tried with various non uniform distributions). This feature is interesting because in many papers about BC models and variants, the initial distribution of opinions is supposed uniform whereas when starting with a Gaussian distribution of opinions for instance, the number and location of convergence clusters is different. The new pattern is hence more robust.
- 4.3 Considering now the possible interpretation of this model in social processes, one can think of situations where hesitating agents are discussing with two very convinced groups with opposite opinions. This can reflect to some extent frequent political situations where two parties of convinced militants try to convince a majority of unconvinced moderates. The model assumes that the moderates easily change their opinion after discussing with an extremist (or a militant) of one side and easily change it again when discussing with an extremist of the other side or with an other uncertain agent. The model predicts a very high volatility of the moderate opinions (they change very frequently their opinion) when the proportion of extremists is high (say around 0.5), which could be related to "swing voters" (see empirical work of Battaglini et al. 2010). Moreover, the more uncertain, the more likely moderates are not to choose one side and remain neutral. According to the model, the global state of the opinion could appear stable from usual polls whereas many individual changes take place, and compensate each other.
- 4.4 Even if a whole population of very uncertain agents not choosing their own position, defining themselves as a member of a particular group is somewhat too much prototypical, it will be very interesting since it exhibits a different order at a different level. Indeed, if only a part of the population is in this state of a large uncertainty but not motivated to reduce it, this part will continue moving along the social order passing from one group of people with a lower uncertainty to another group. They represent what Granovetter (1973) has outlined as "weak" ties ensuring the exchange of information in the population. Moreover, considering real social order, this is strongly known in the classical problem of community detection that some people seem to be very mobile, members of several groups at the same time. Questioning about what these people bring to the social system is then a very hard and interesting question. That is the purpose of further investigation implying a more complex population and some analysis of the stable states.
- 4.5 Nevertheless, the results should be interpreted with caution as usual with such theoretical models (see Deffuant et al. 2006 for a general discussion on this issue), and especially because of the strong hypothesis that each agent interacts with any other. In particular this paper underlines the controversy about the interest of abstract models and the necessary caution in their interpretation. Further conclusions would require complementary investigations, considering for example various social networks constraining the interactions.

Appendix A: Computing the clusters, extracted from Deffuant (2006)

We only consider the clusters of initially moderate agents (i.e. initialised with uncertainty U), because our goal is to evaluate the effect on moderates of different configurations of extremists. To compute the clusters, we define a minimum distance ε between opinions, below which we consider that they belong to the same cluster. We compute the clusters as groups of agents such that between any couple of agents of opinions x and x' in the group, there is a list of agents in the group of opinions (x_1, x_2, \dots, x_k) making a chain of couples with a distance below ε . More precisely:

$$|x - x_1| < \varepsilon, |x' - x_k| < \varepsilon,$$

$$1 \leq i \leq k-1, |x_i - x_{i+1}| < \varepsilon.$$

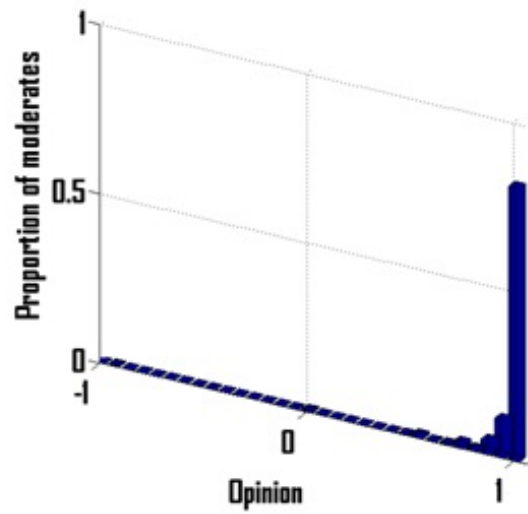
In practice, we chose $\varepsilon = 0.01$. The following pseudo-code can be used to compute the clusters. Initially moderate and extremist agents are tagged at the beginning of the simulation. Method `addAll()` adds all the elements of the argument (supposed to be a list) to the argument (supposed to be a list as well). Method `removeAll()` removes from the recipient all the elements of the argument (both are supposed to be lists).

Appendix B: exploration of the stationary states according to the proportion of extremists p_e and the uncertainty u

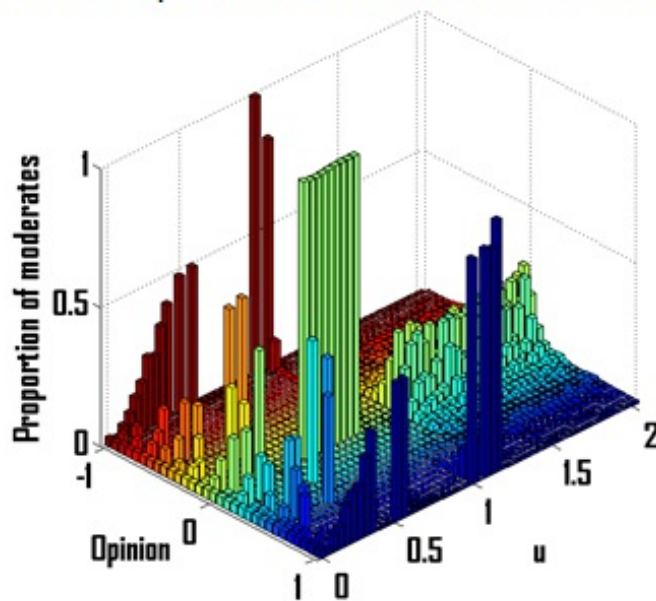
We propose here to give more details about the stationary states according to the proportion of extremists p_e and the uncertainty u .

B.1 Influence of p_e

Figures 7 and 8 show details about the convergence opinion distributions for different values of p_e . In the case of few extremists ($p_e=0.05$), the pattern obtained clearly depends on parameter u (see Figure 7). For low values of u , we have approximately $1/u$ clusters as in the model without extremists. For u around 0.5, we obtained double extreme clusters. For $0.5 < u < 0.9$, there is only one central cluster. A single extreme cluster comes after for $0.9 < u < 1.1$ before a state where opinion keep fluctuating ($u > 1.1$). Increasing the value of p_e decreases this zone of fluctuations and increases the polarization of the population (see Figure 8a) towards a double extreme highly polarized (see Figure 8b).

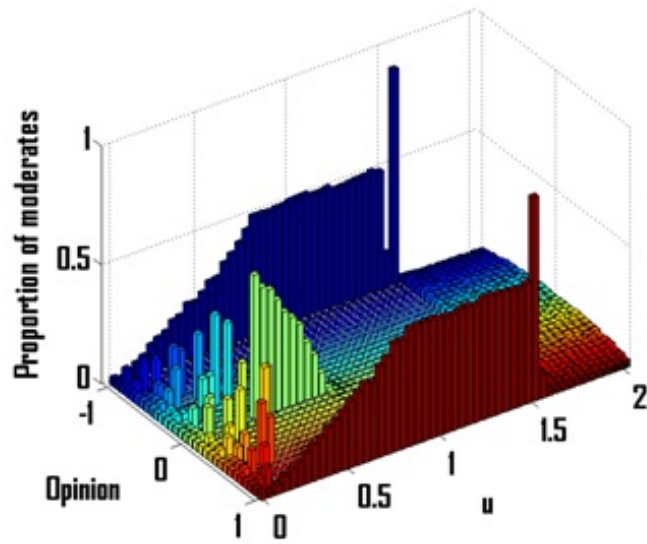


a - Final opinion distributions for $u=1$

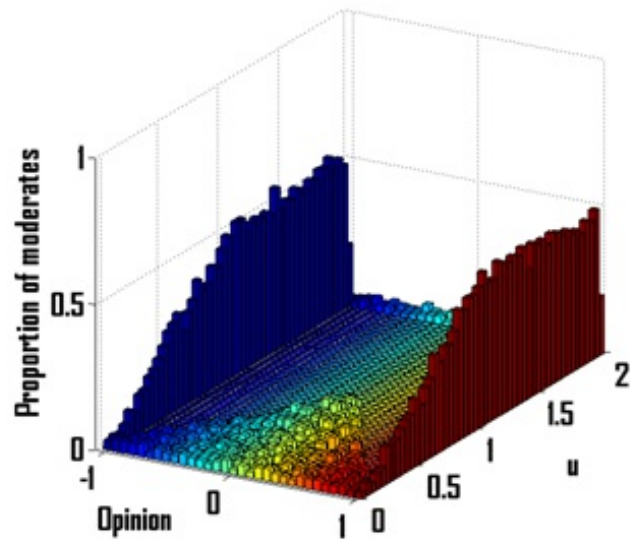


b - Final opinion distributions for different values of u

Figure 7. Opinion distributions after convergence for a proportion of extremists $p_e=0.05$ and for $u=1$ (convergence towards a single extreme, Figure a) and for various values of uncertainties u in ordinate (Figure b).



a : $p_e = 0.5$

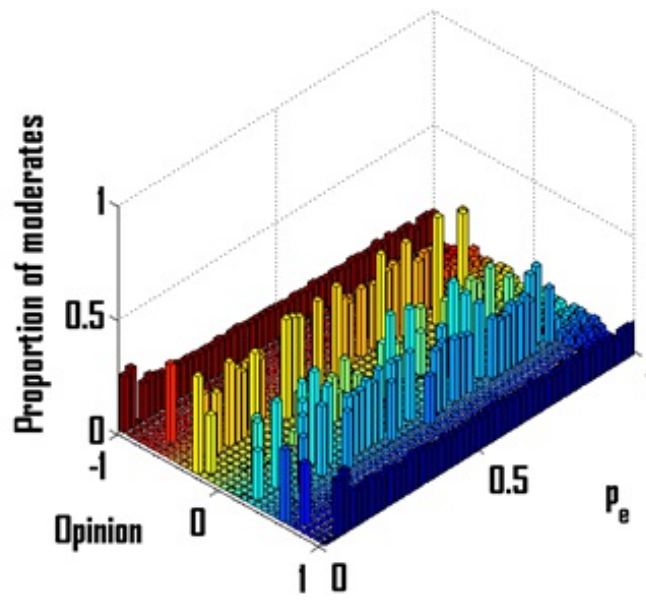


b : $p_e = 0.95$

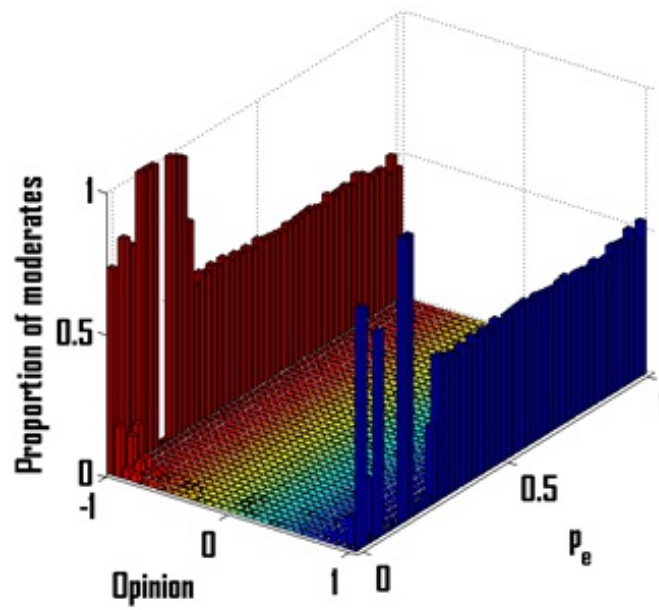
Figure 8. Opinion distributions after convergence, for different values of p_e , the proportion of extremists with (a) $p_e=0.5$ and (b) $p_e=0.95$ and various uncertainties u values in ordinate.

B.2 Influence of u

Figure 9 shows details about the convergence opinion distributions for different values of u . For low values of u , the proportion of extremists p_e does not affect the opinion densities except for very low or very high values of p_e for which opinions keep fluctuating (Figure 9a). For $u=1$ (Figure 9b), we can see that we have a threshold of p_e (around 0.25) for which we obtain a transition from a single extreme to a double extreme. For high values of u , opinions keep fluctuating except for high values of p_e where the population converges toward a double extreme (Figure 9c).



a : $u = 0.25$



b : $u = 1$

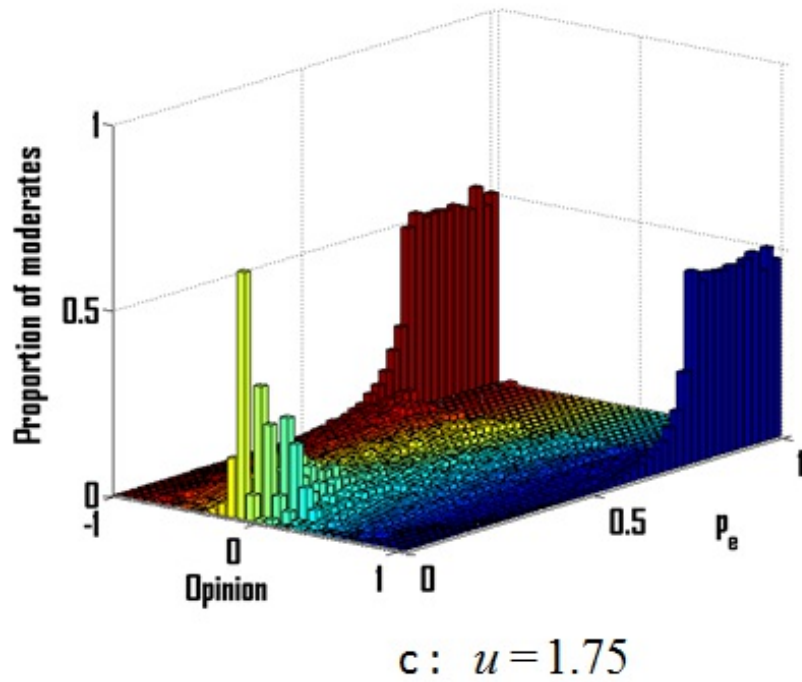


Figure 9. Opinion distributions after convergence for different values of u , the uncertainty of the moderates with (a) $u=0.25$; (b) $u=1.0$; (c) $u=1.75$ and various values of p_e , the proportion of extremists given in ordinate.

Appendix C: Source code of the model

```

clear all;close all
pe=0.5;
u=0.4;
mu=0.5;
N=400;
x(1,1:round((1-pe)*N))=rand(1,round((1-pe)*N))*2-1;
ux(1,1:round((1-pe)*N))=ones(1,round((1-pe)*N))*u;

x(1,round(((1-pe)*N+1)):round(((1-pe)*N+(1-(1-pe))/2*N)))=ones(1,round(((1-(1-pe))/2*N)));
x(1,round(((1-pe)*N+1+(1-(1-pe))/2*N)):N)=ones(1,round(((1-(1-pe))/2*N))*(-1);
ux(1,(find(abs(x(1,:))=1)))=u/100;
t=1;
criteria=0;
while criteria==0
    t=t+1;
    vect=randperm(N);
    distx=abs(x(t-1,vect(1:N/2))-x(t-1,vect((N/2+1):N)));
    r1=find(distx-ux(1,vect(1:N/2)) < 0);
    r2=find(distx-ux(1,vect((N/2+1):N)) < 0);
    dx(1,1:N)=0;
    dx(1,vect(r1))=(x(t-1,vect(r1+N/2))-x(t-1,vect(r1)))*mu;    dx(1,vect(r2+(N/2)))=(x(t-1,vect(r2))-x(t-1,vect(r2+(N/2))))*mu;
    x(t,:)=x(t-1,:)+dx(1,:);
    if (sum(abs(dx(1,:))) < 0.001
        criteria=1;
    end

plot(x(:,ux(1,:) > 0.1),'b')
hold on;
plot(x(:,ux(1,:) < 0.1),'r')
f(t-1) = getframe;
end

```

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