



Porous media - Poromechanics Hydro-mechanical couplings

Stéphane Bonelli

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Milieux poreux Poro-Mécanique Couplages hydro-mécanique

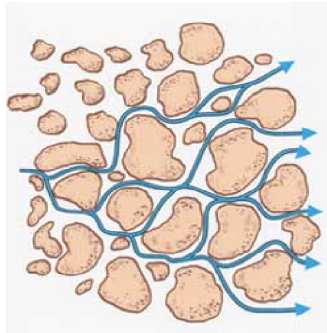
Parcours d'approfondissement – Mécanique – M3S

Stéphane Bonelli

Milieux poreux

Poro-Mécanique

Couplages hydro-mécanique



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Archimède (287 av. J.-C., 212 av. J.-C.)

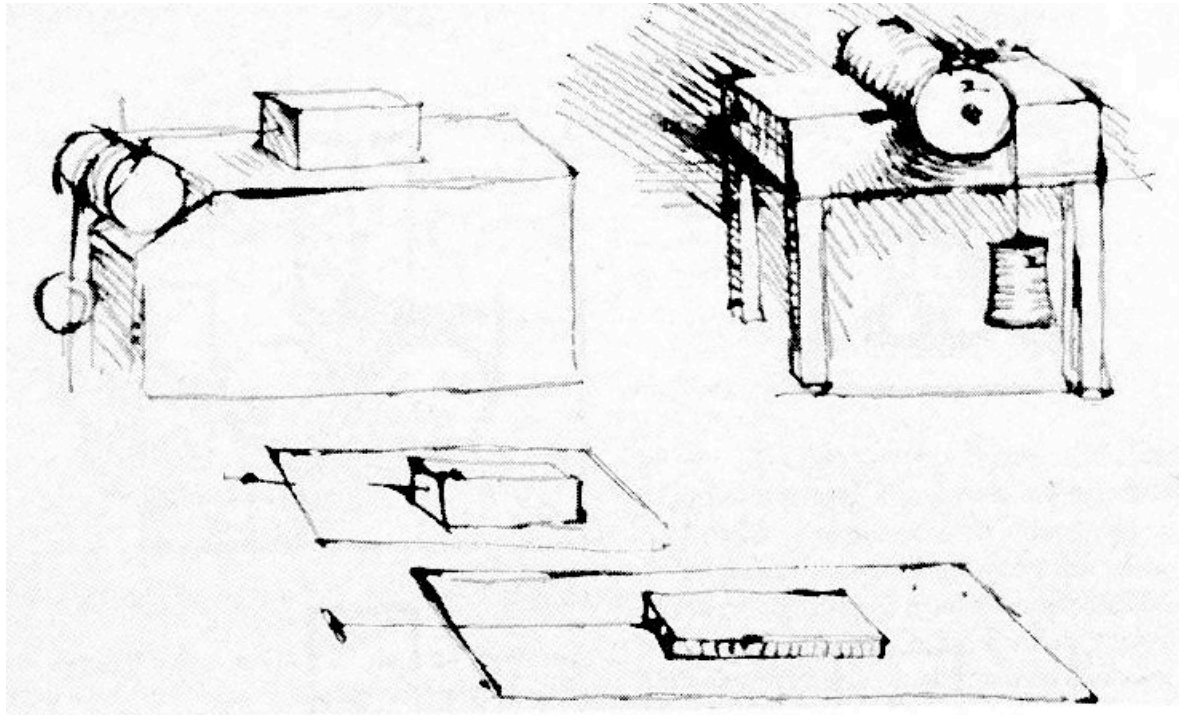
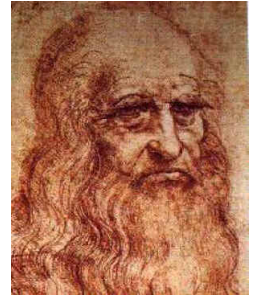


Expérience physique sur deux boules de métal,
l'une en or, l'autre en argent,
toutes deux plongées dans l'eau.

Théorème d'Archimède

Tout corps plongé dans un fluide, entièrement mouillé par celui-ci ou traversant sa surface libre, subit une force verticale, dirigée de bas en haut et égale au poids du volume de fluide déplacé ; cette force est appelée « poussée d'Archimède ».
(ce théorème fut ensuite démontré au XVI^e siècle).

De Vinci (1452, 1519)



Machine à étudier les frottements de De Vinci (1508)

Galilée (1564, 1642)

DISCORSI E DIMOSTRAZIONI MATEMATICHE, intorno à due nuove scienze

Attenenti alla
MECANICA & i MOVIMENTI LOCALI;

del Signor
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Serenissimo
Grand Duca di Toscana.

Con una Appendice del centro di gravità d'alcuni Solidi.



IN LEIDA.
Appresso gli Elsevirii. M. D. C. XXXVIII.

Title of the famous book of Galileo (1638) which
founded mechanics of materials

114 DIALOGO SECONDO
*fin qui dichiarate, non sarà difficile l'intender la ragione, onde au-
nenga, che un Prisma, ò Cilindro solido di vetro, acciaio, legno, ò
altra materia frangibile, che sospeso per lungo sotterrà gravissimo
peso, che gli sia attaccato, mà in trauerfo (come poco fa dicuamo) da
minor peso assai potrà tal volta essere spezzato, secondo che la sua
lunghezza cederà la sua grossezza. Imperò che figuriamoci il Pris-
ma solido A B, C D fitto in un muro dalla parte A B, e nell'altra
estremità s'intenda la forza del Peso E. (intendendo sempre il mu-
ro esser eretto all'Orizzonte, & il Prisma, ò Cilindro fitto nel muro
ad angoli retti) è manifesto che douendosi spezzare si romperà nel*

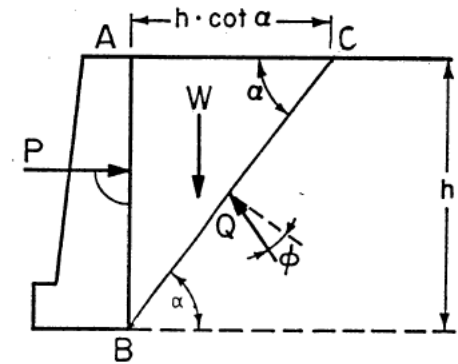
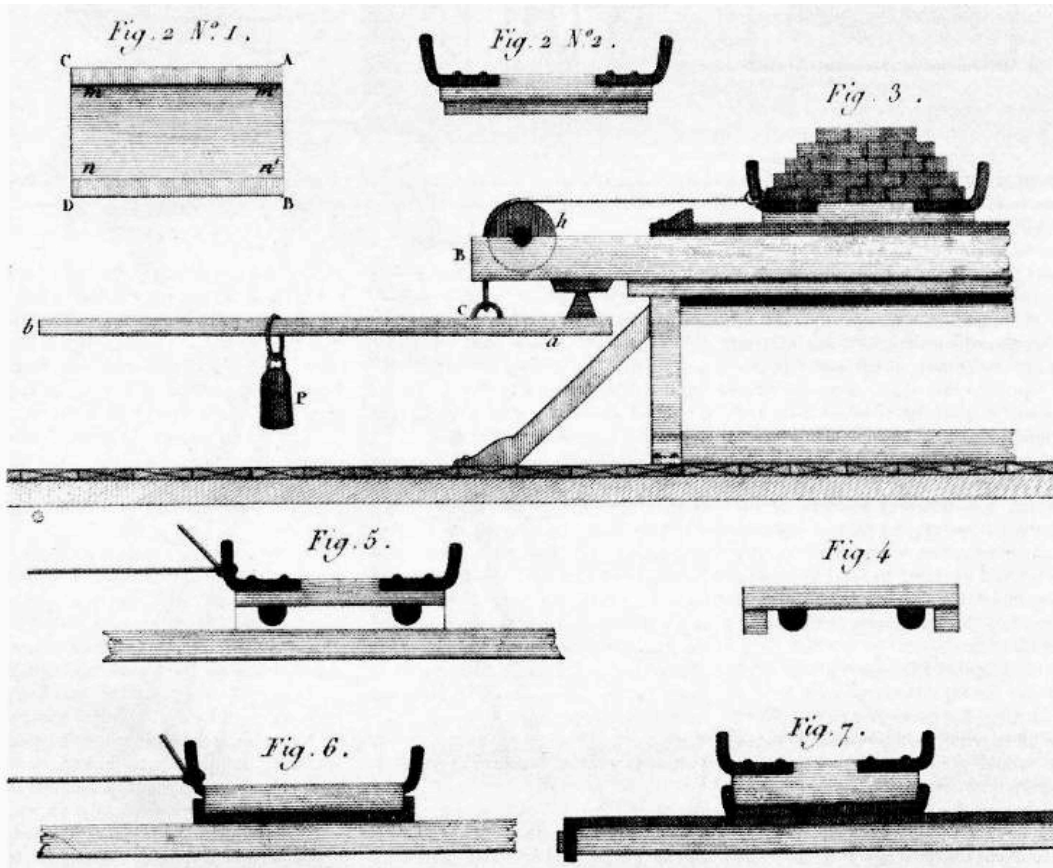
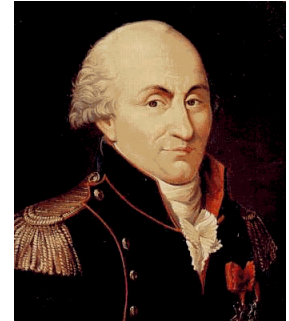


*luogo B, doue
il taglio del
muro serue
per sostegno, e
la B C per la
parte della
Leua, doue si
pone la forza,
e la grossezza
del solido B A
è l'altra parte
della Leua,
nella quale è
posta la resi-
stenza, che
consiste nel-
lo staccamen-
to, che s'hà
da fare della
parte del soli-
do B D, che è
fuor del muro, da quella che è dentro; e per le cose dichiarate il mo-
mento della forza posta in C al momento della resistenza che stà
nella*



GALILEI (1638)

Coulomb (1736, 1806)



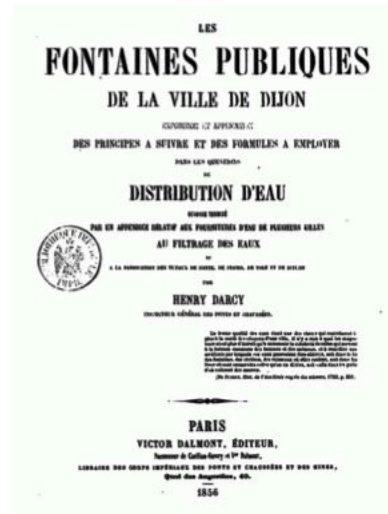
C.A.COULOMB, 1773

$$\alpha = \frac{\pi}{4} + \frac{\phi}{2}:$$

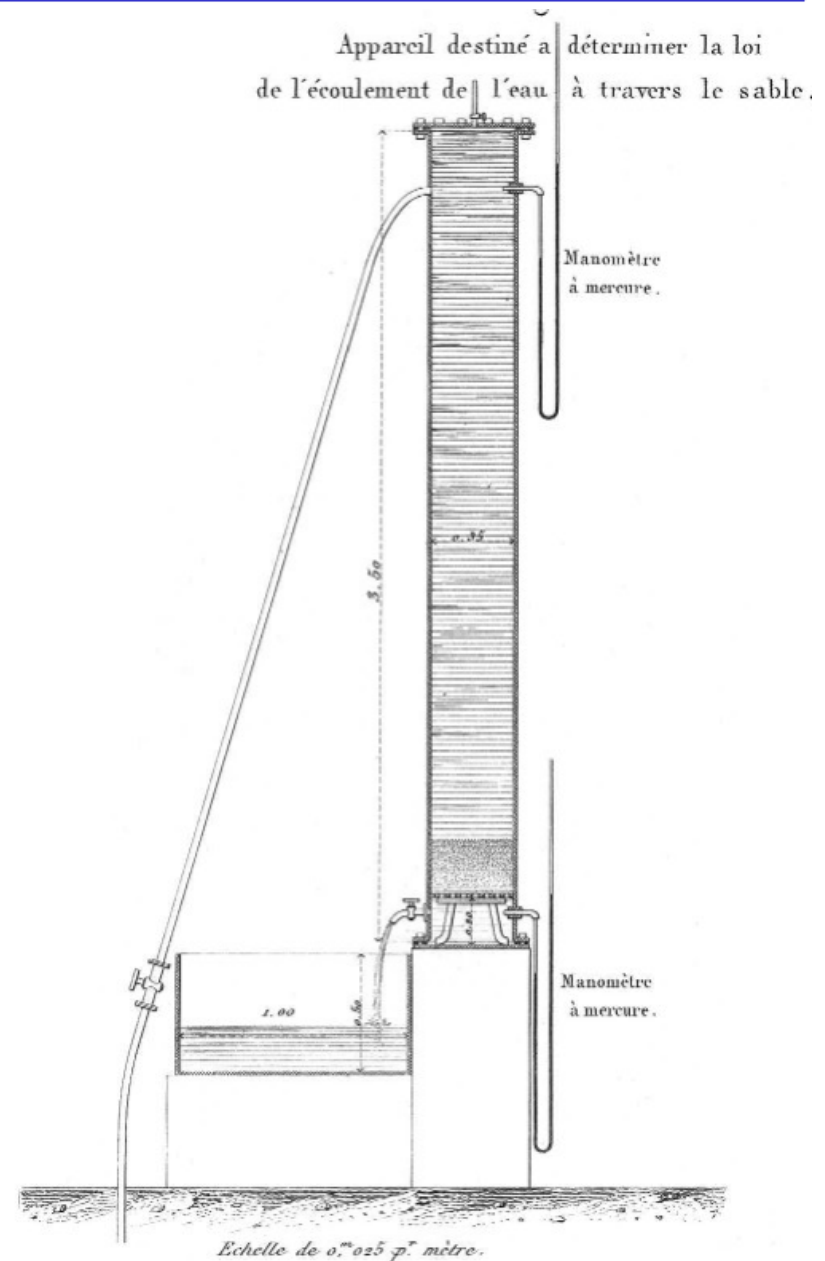
$$P = \frac{1}{2} \rho g h^2 (1 - \sin \phi) / (1 + \sin \phi)$$

Machine à étudier les frottements de Coulomb

Darcy (1803, 1858)



Appareil destiné à déterminer
la loi de l'écoulement de l'eau à travers le sable
de Darcy.



Terzaghi (1883, 1963)

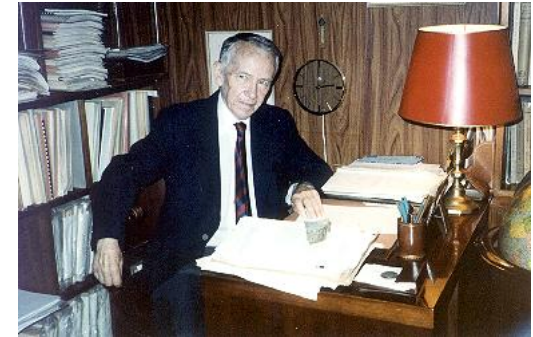


Karl von Terzaghi first proposed the relationship for effective stress in 1936.

$$\sigma' = \sigma - p$$

For him, the term 'effective' meant the calculated stress that was effective in moving soil, or causing displacements. It represents the average stress carried by the soil skeleton.

Biot (1905, 1985)



A theory for acoustic propagation in a porous and elastic medium developed by M.A. Biot. Compressional and shear velocities can be calculated by standard elastic theory from the composite density, shear and bulk modulus of the total rock.

The problem is how to determine these from the properties of the constituent parts.

Biot showed that the composite properties could be determined from the porosity and the elastic properties (density and moduli) of the fluid, the solid material, and the empty rock skeleton, or framework.

To account for different frequencies of propagation, it is also necessary to know the frequency, the permeability of the rock, the viscosity of the fluid and a coefficient for the inertial drag between skeleton and fluid.

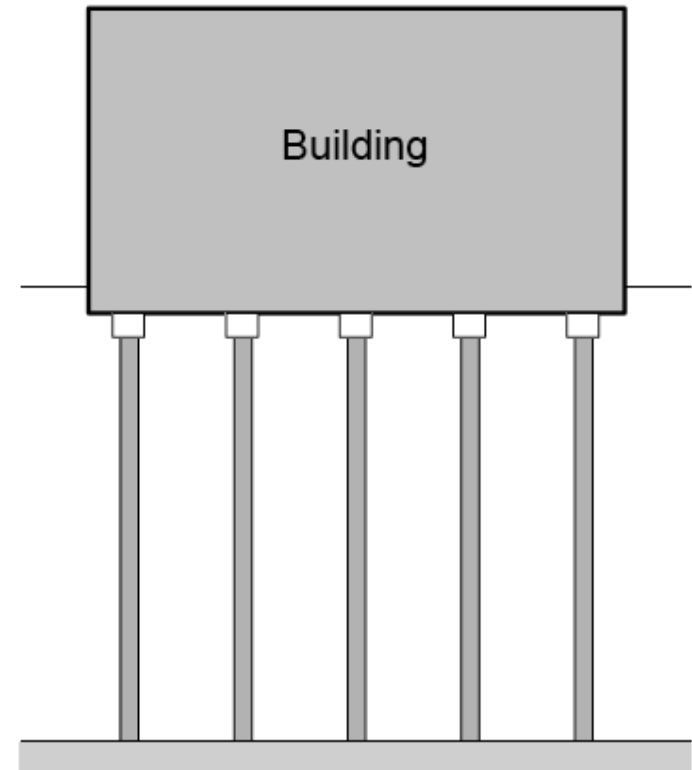
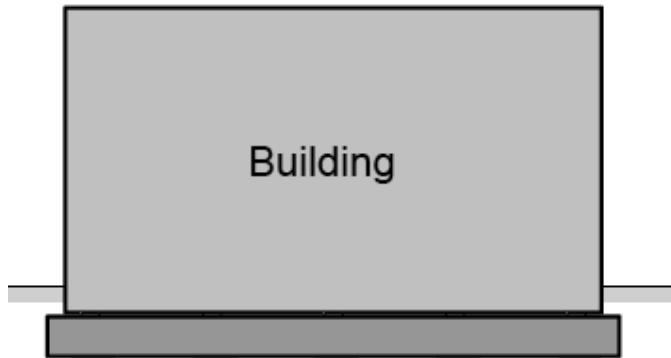
❑ WHAT PROBLEMS DOES IT ADDRESS?

FOUNDATIONS

SHALLOW

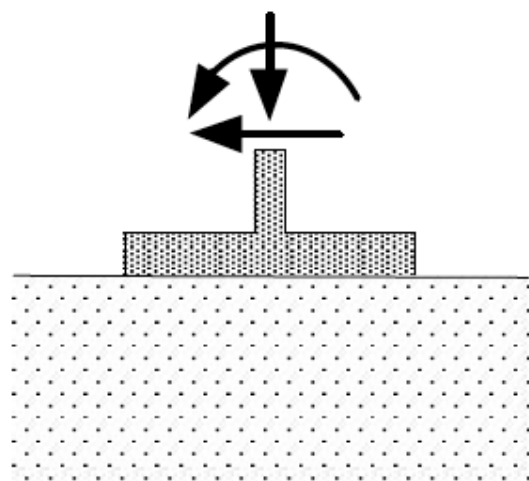
vs.

DEEP

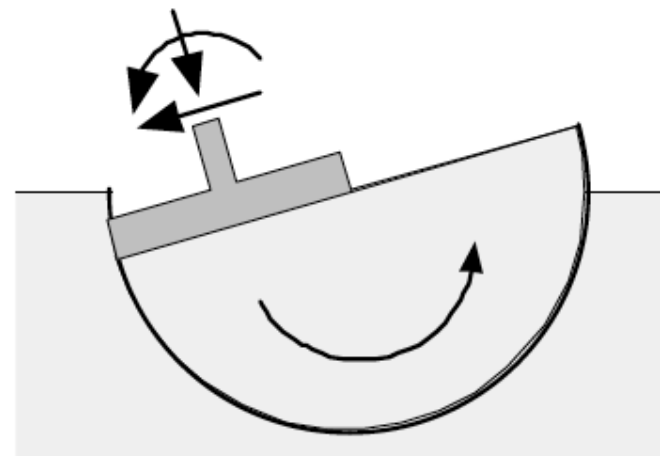


Rock

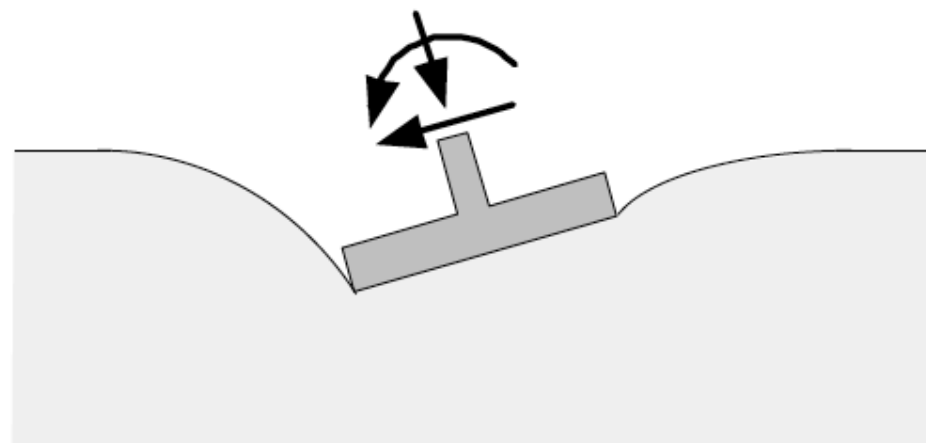
❑ WHAT ARE THE ISSUES?



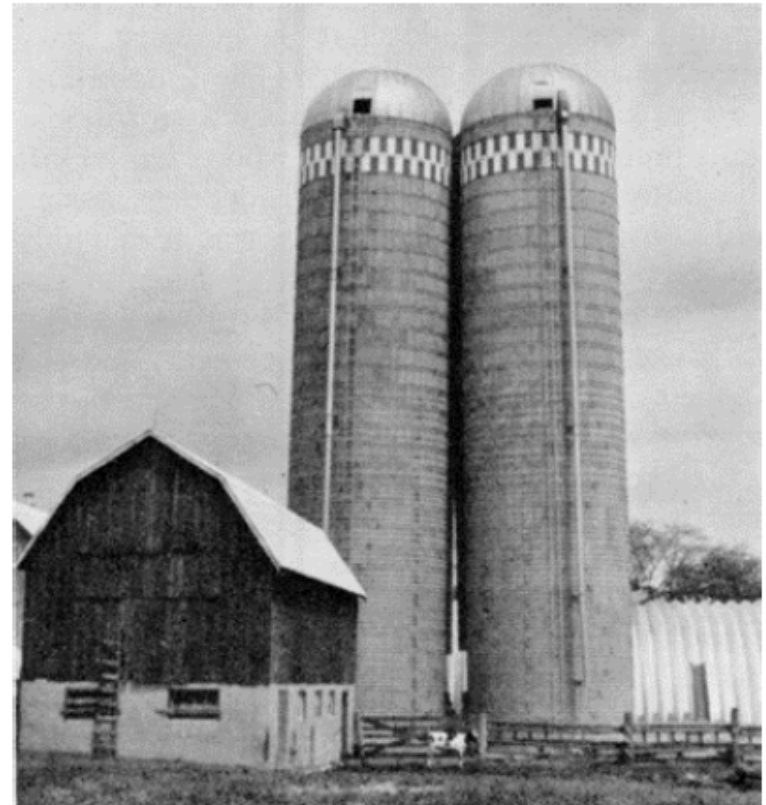
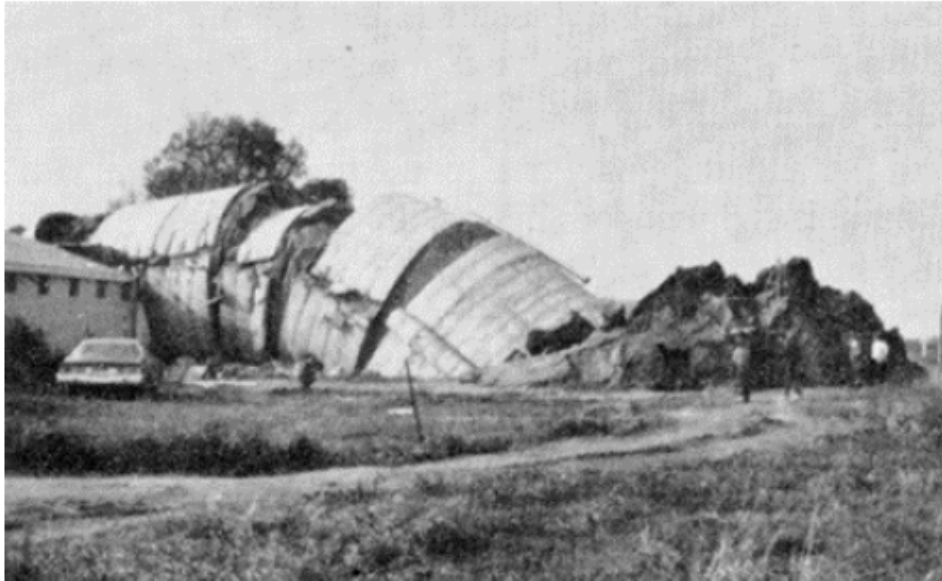
STRENGTH



STIFFNESS



STRENGTH



STIFFNESS

Including ones on the sea bottom



FOUNDATIONS

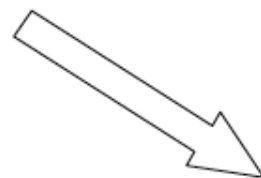
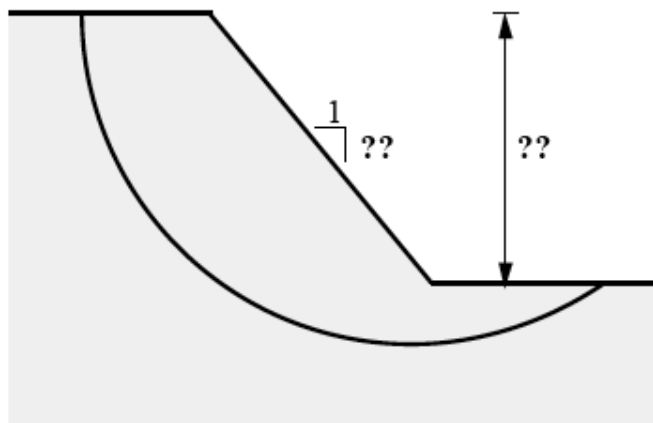


Metro Link Light Rail Stations,
St. Louis, MO

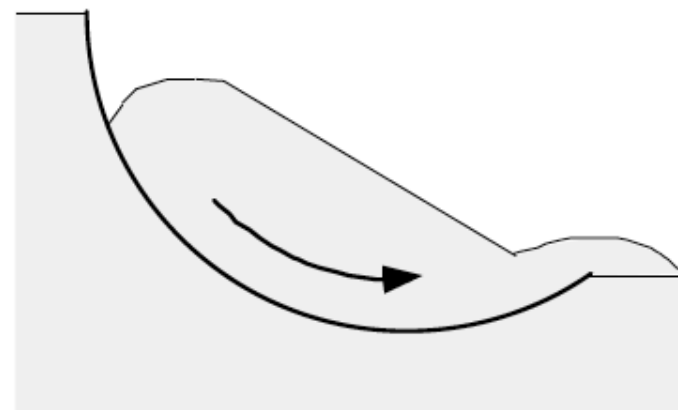


California Palace of the
Legion of Honor,
San Francisco, CA

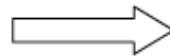
❑ WHAT ARE THE ISSUES?



STABILITY



SLOPES/LANDSLIDES

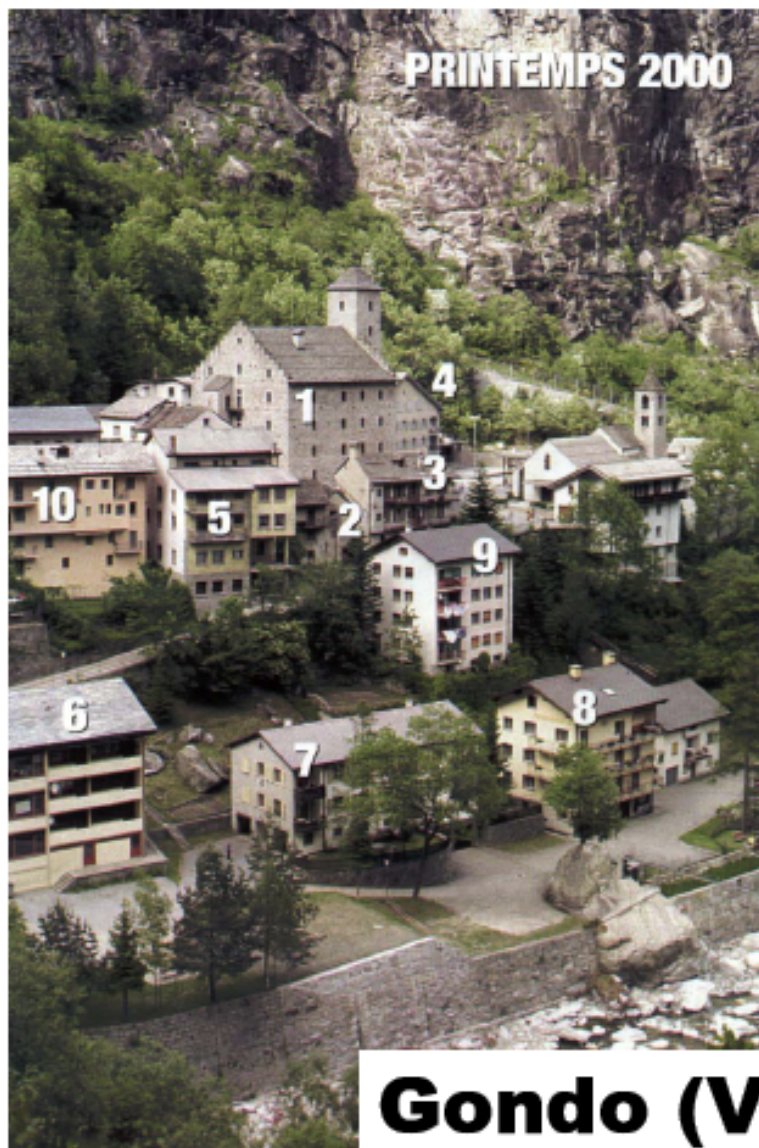


Jizukiyama Landslide, Japan, 1985

Vagnhärad Landslide, Sweden, 1997



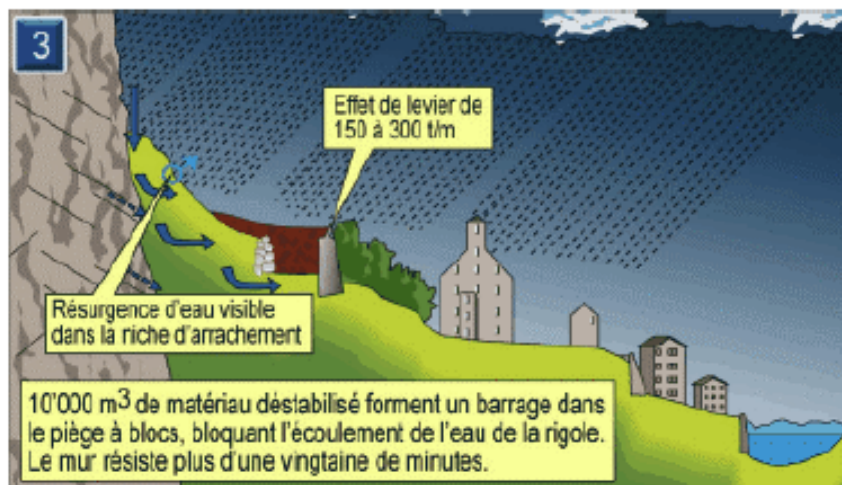
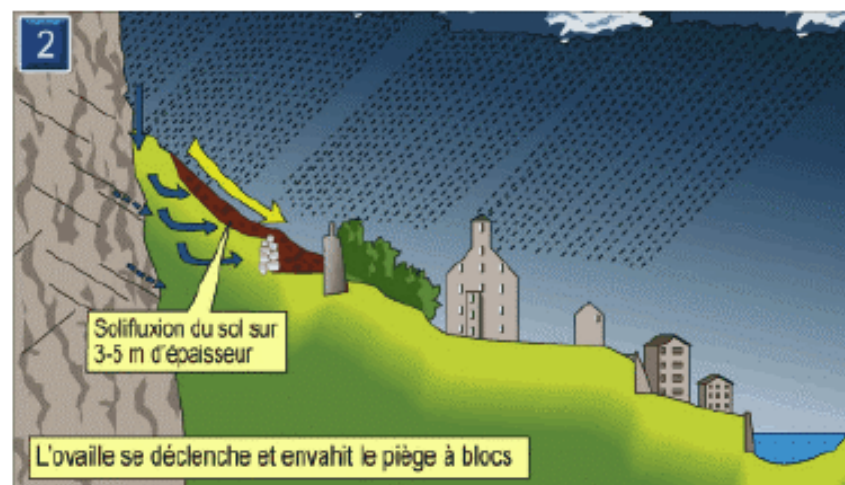
SLOPES/LANDSLIDES



Gondo (VS): avant, après

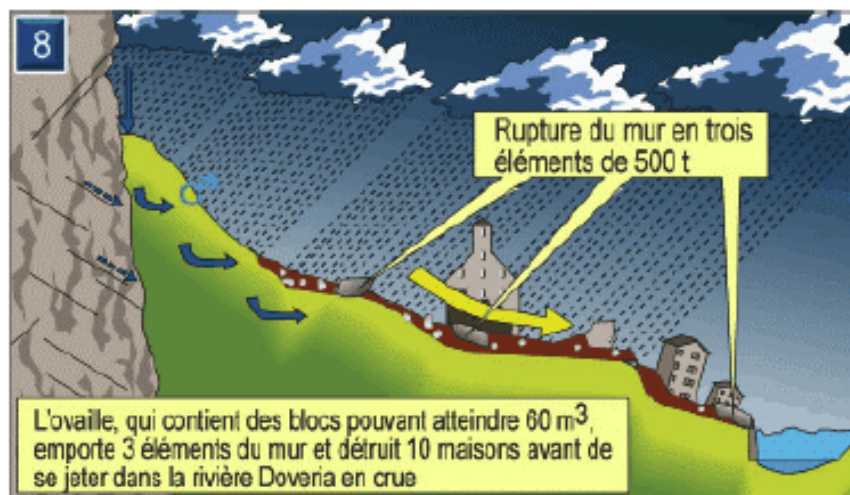
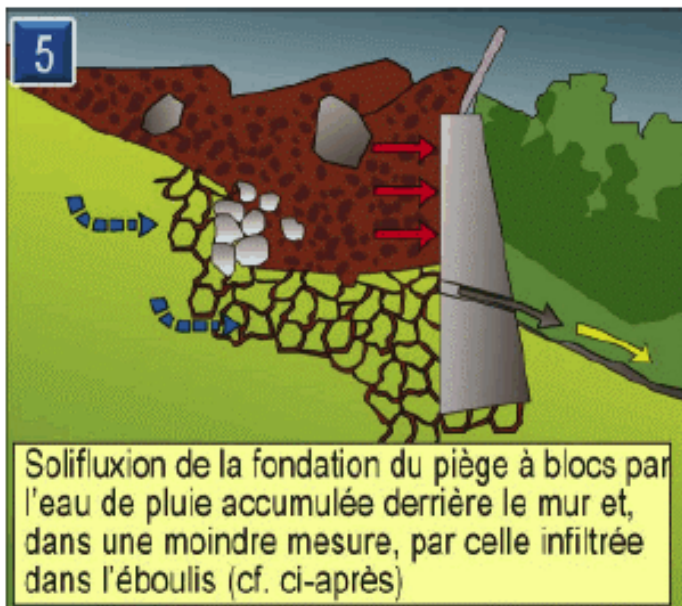
Photo L'illustré

Gondo: mécanisme possible (source: CREALP, SION)



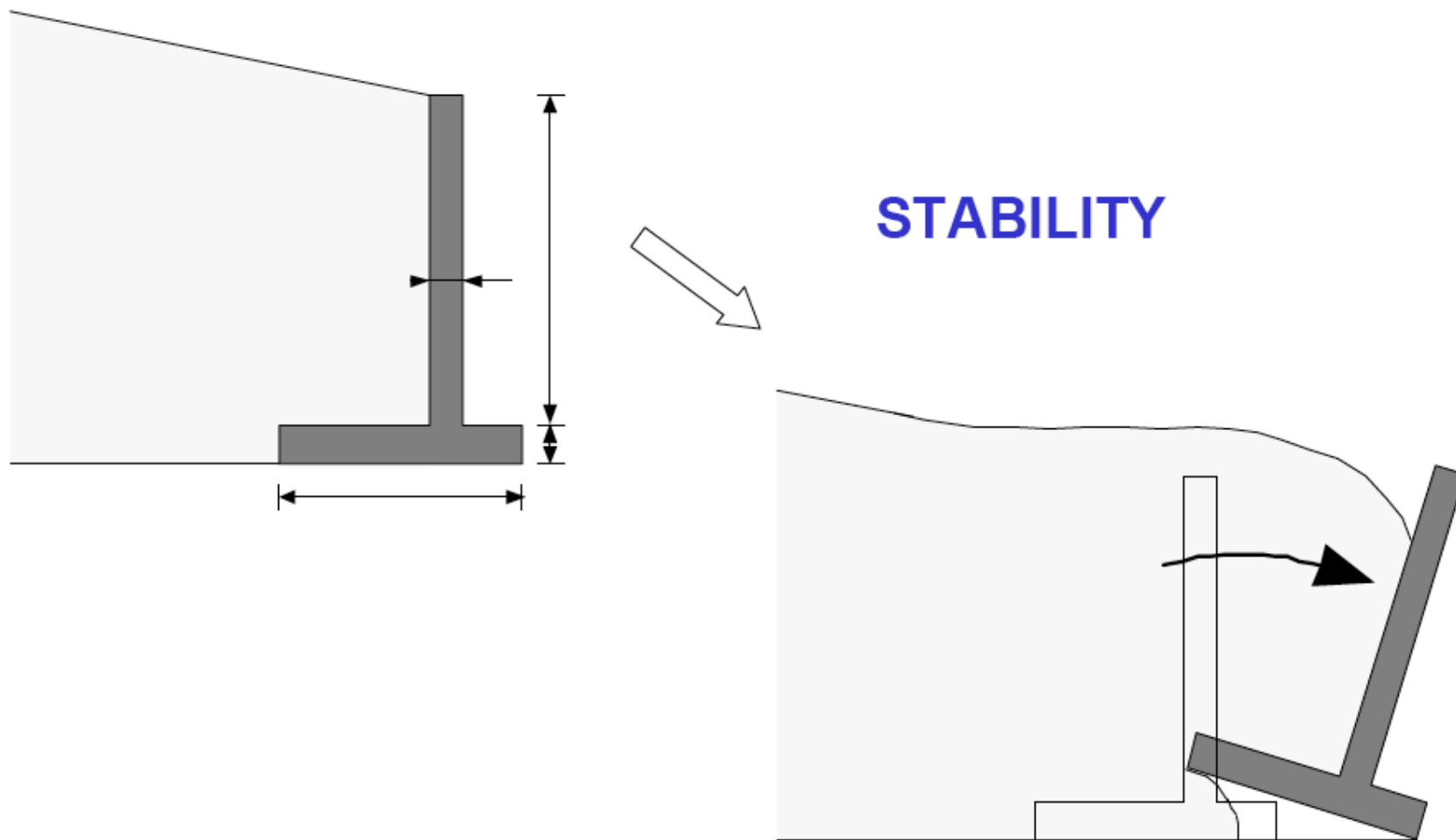
http://www.crealp.ch/f_principal.html

Gondo: mécanisme possible (source: CREALP, SION)



http://www.crealp.ch/f_principal.html

□ WHAT ARE THE ISSUES?



RETAINING STRUCTURES



Cantilever Wall



Anchored Wall

RETAINING STRUCTURES

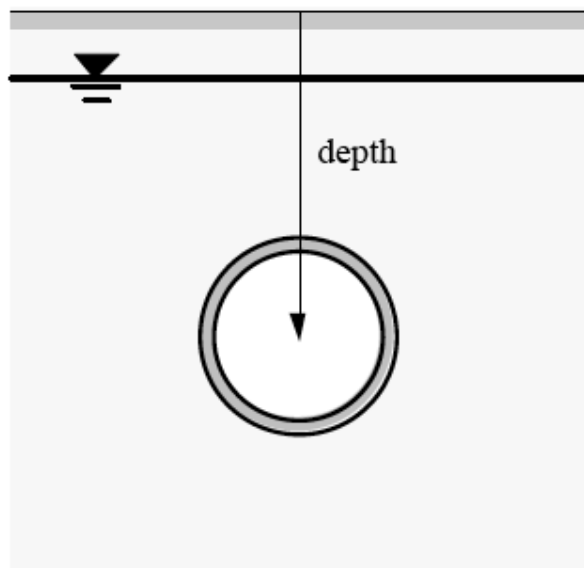


RETAINING STRUCTURES

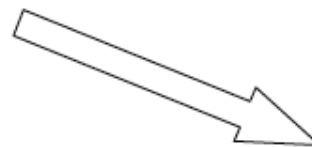
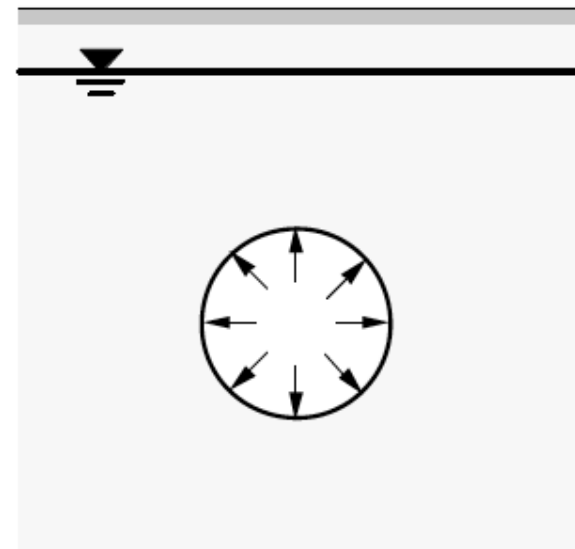


Boston Central Artery Project – *The Big Dig*

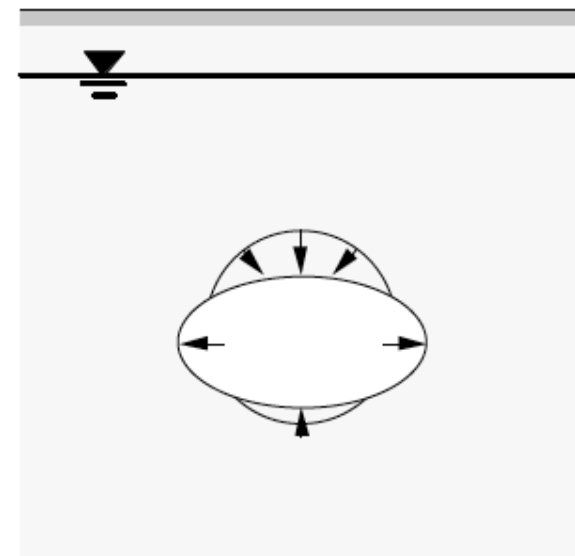
□ WHAT ARE THE ISSUES?



SUPPORT



CONVERGENCE

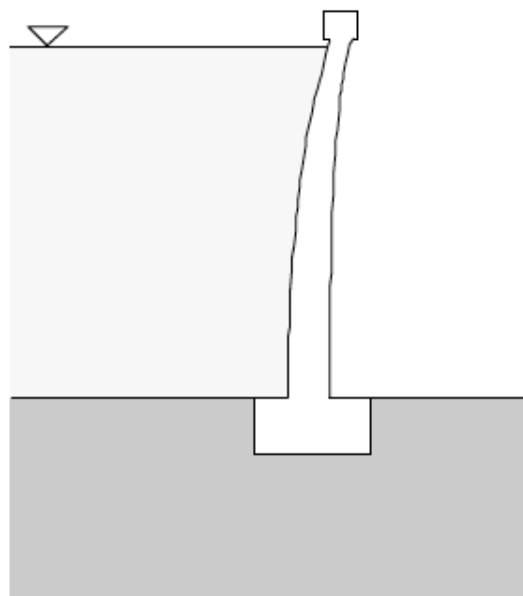




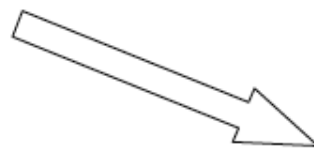
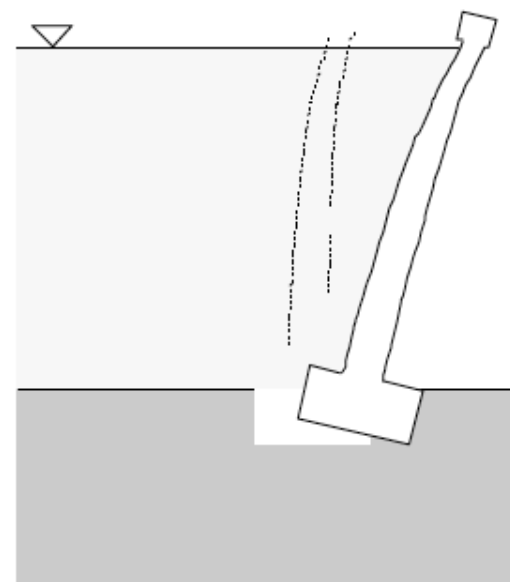
English Channel Tunnel - ***Chunnel***



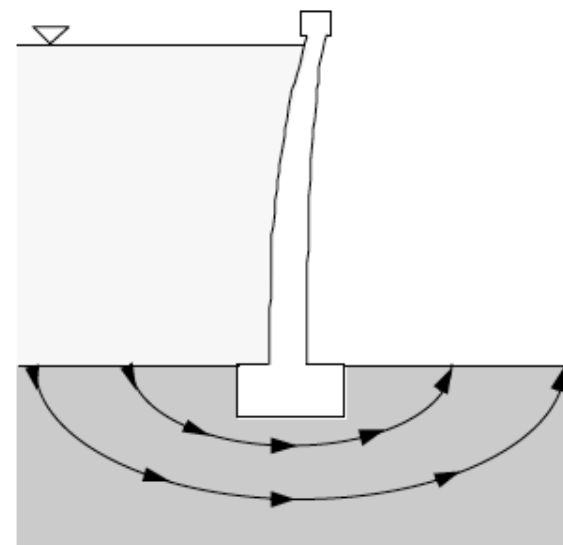
□ WHAT ARE THE ISSUES?



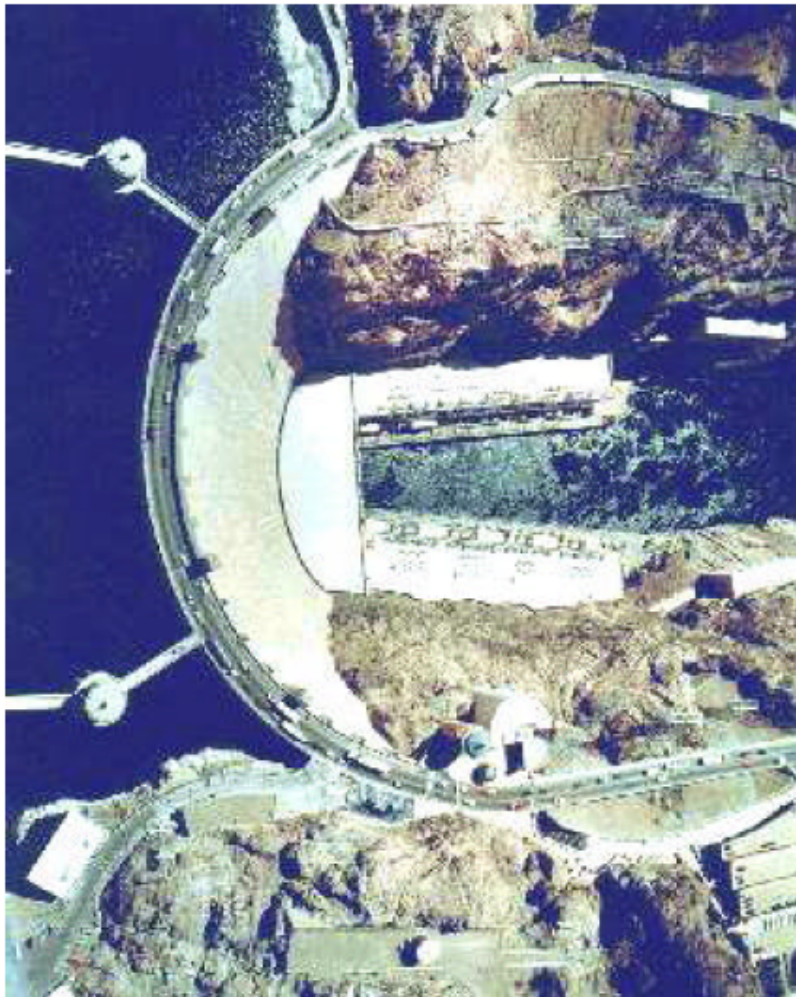
STABILITY



SEEPAGE



DAMS



Hoover Dam,
Colorado River, Nevada

East Side Reservoir Project
Riverside County, California



DAMS



Before



After



Malpasset Dam, France.
Failed December 2, 1959

DAMS



Before



...during

...and after failure

Teton Dam, Idaho.
Failed June 5, 1976



Niigata, Japan 1964



Adapazari, Turkey 1999



A partially sunken house illustrates the challenge of understanding how grains of soil interact with each other and under what conditions they will support structures. (National Geophysical Data Center)

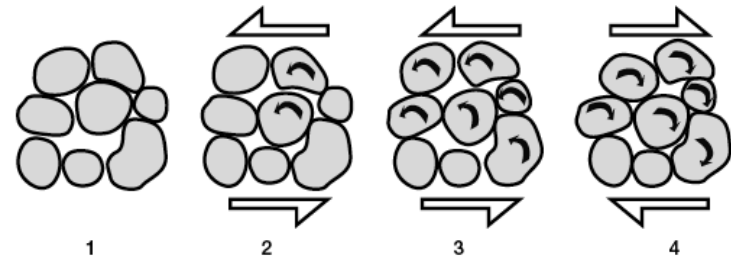


Figure 1. The packing of particles can change radically during cyclic shear; (1) a large hole is maintained by the particle interlocking; (2) a small counterclockwise strain causes the hole to collapse; (3) large shear strain causes more holes to form; (4) holes will collapse when the strain direction is reversed (Youd, 1977).



Figure 2. Partially sunken houses in San Francisco and the slumped sides of Copernicus crater on the Moon share one geologic fact: soil liquefaction. (USGS, NASA)

□ WHAT WE DESIGN FOR?

- AVOID FAILURE



- LIMIT DEFORMATIONS





Some of the problems are not new,
...but now we know (many of) the answers

Questions

1. A possible explanation of the leaning of the Pisa tower is that the subsoil contains a compressible Clay layer of variable thickness.

On what side of the tower would that clay layer be thickest ?

2. Another possible explanation for the leaning of the Pisa tower is that in earlier ages (before the start of the building of the tower, in 1173), a heavy structure stood near that location.

On what side of the tower would that building have been ?

Equations de bilan (HM linéaire)

2 phases (solide=matrice minérale, fluide=eau)

donc 2 équations de bilan de masse

et 2 équations d'équilibre

Masses

$$\dot{n} = (1 - n) \text{tr} \underline{\underline{\dot{\varepsilon}}}$$



variation de porosité

$$n \dot{\rho}_f + \rho_f \text{tr} \underline{\underline{\dot{\varepsilon}}} + \text{div}(\rho_f \vec{q}) = 0$$



*vitesse moyenne
d'écoulement
du fluide*

Equilibres

$$\text{div} \underline{\underline{\sigma}} + \rho \vec{g} = \vec{0}$$



contraintes du milieu diphasique

$$-\overrightarrow{\text{grad} p} + \rho_f \vec{g} = \vec{I}$$



*force de volume
d'interaction
fluide/solide*

Lois de comportement (HM linéaire)

2 phases (solide=matrice minérale, fluide=eau)

donc 3 modèles de comportement (au moins)

Fluide (compressibilité) :
$$p = \chi_f \log \frac{\rho_f}{\rho_f^0}$$

Solide (élasticité) :
$$\underline{\underline{\sigma}} + p \underline{\underline{1}} = 2G \underline{\underline{\varepsilon}} + \left(\chi - \frac{2}{3} G \right) (\text{tr} \underline{\underline{\varepsilon}}) \underline{\underline{1}}$$

Interaction (diffusion) :
$$\vec{q} = - \frac{k}{\mu_f} \vec{I}$$

Contenu du cours

Milieu poreux diphasique : lois de bilan, PPV à deux champs de vitesse

Diffusion linéaire (Darcy), Consolidation linéaire (Biot)

Elastoplasticité (Mohr-Coulomb, Cam-Clay)

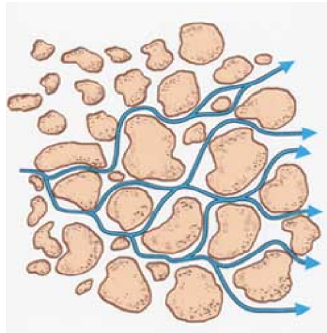
Aperçu de quelques enrichissements de modélisation sur études de cas :

- HM non saturé (solide/eau/air)
- THM (T=thermique) : stockage profond de déchets nucléaires (exothermiques) , injection d'eau froide dans un puit de forage pétrolier
- THMC (C=chimique) : transport de polluant (huile lourde, pesticide) dans les nappes phréatique

Milieux poreux

Poro-Mécanique

Couplages hydro-mécanique

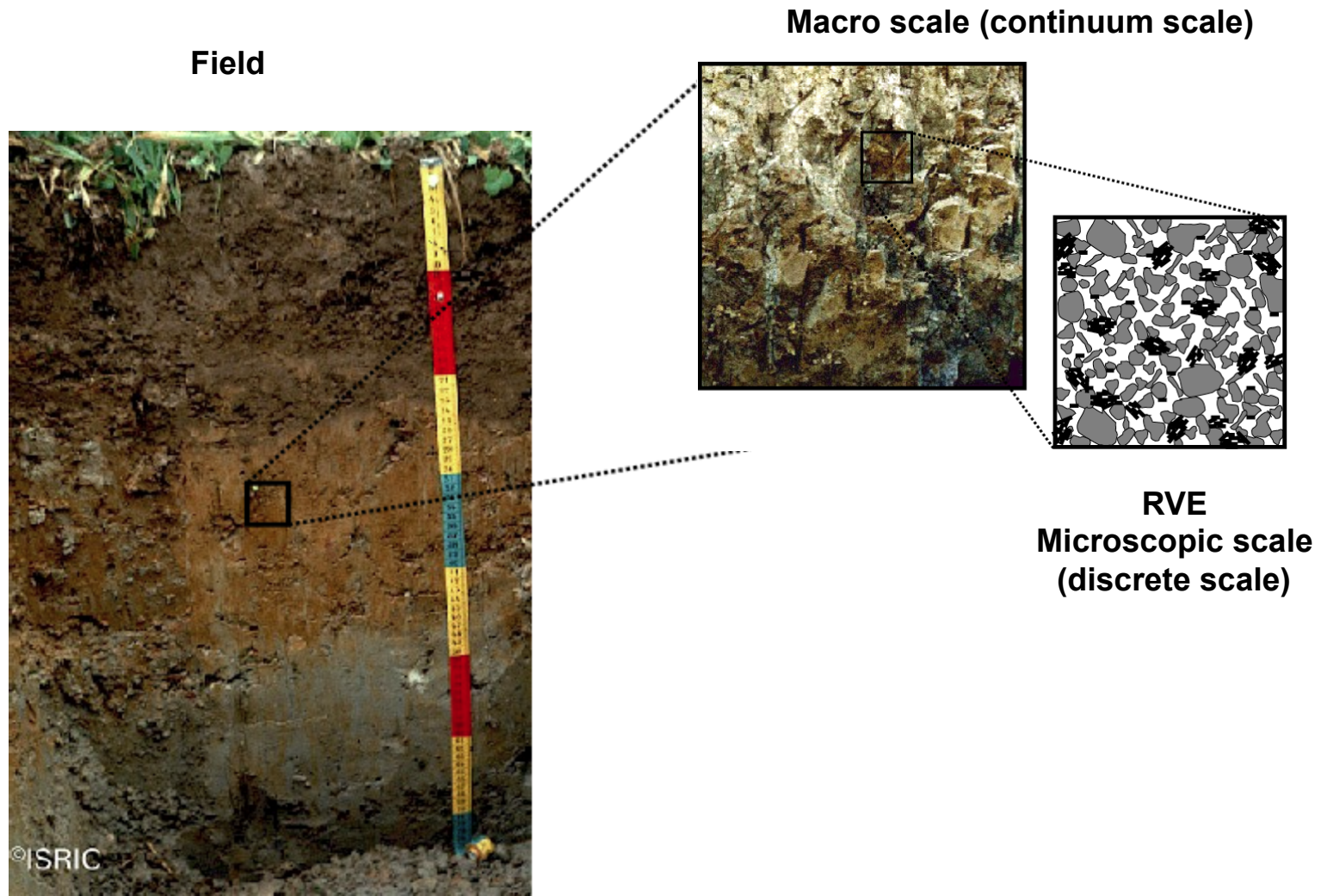


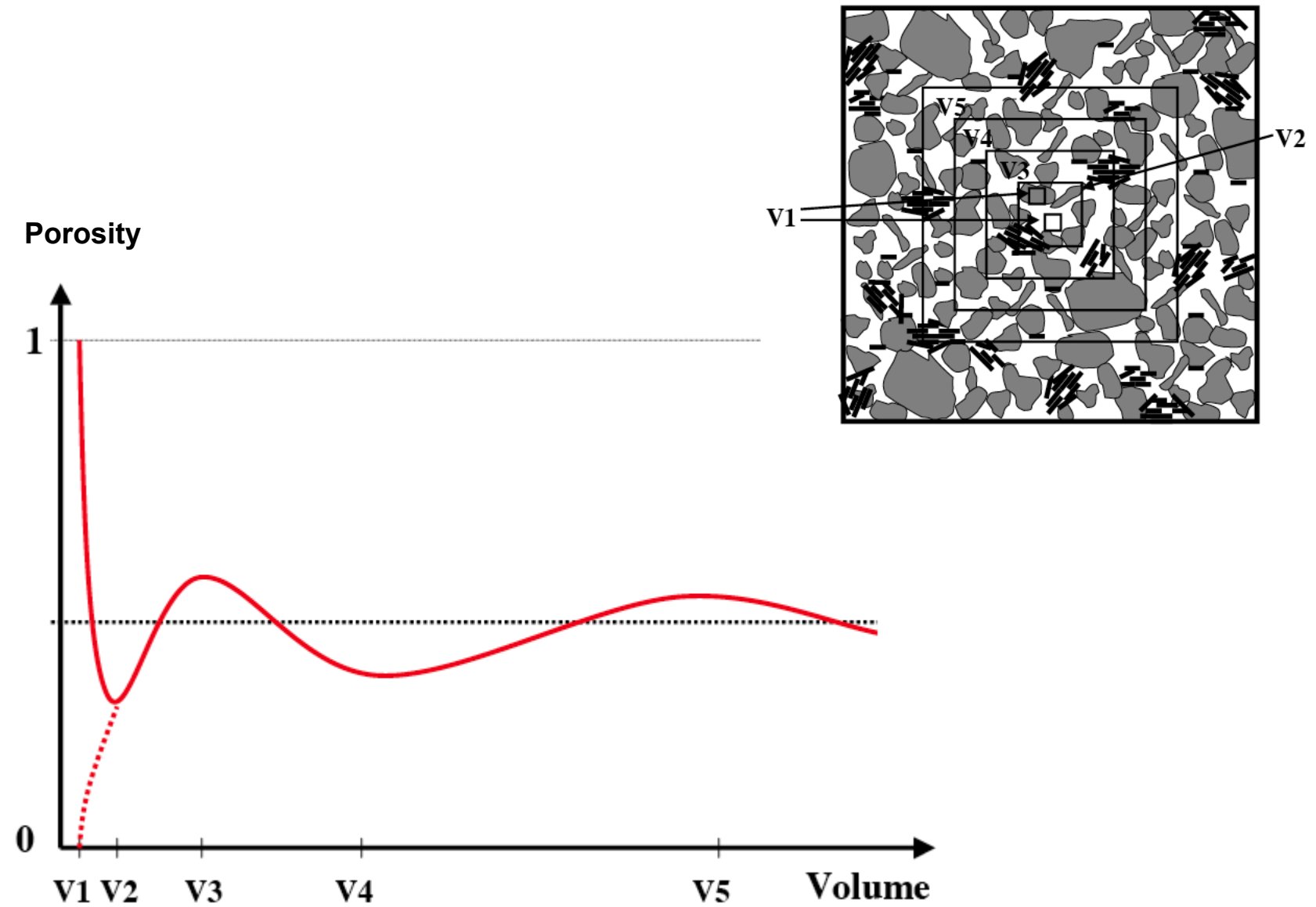
Micro-scale: grains, pores, examples
Macro-scale: porosity, specific surface

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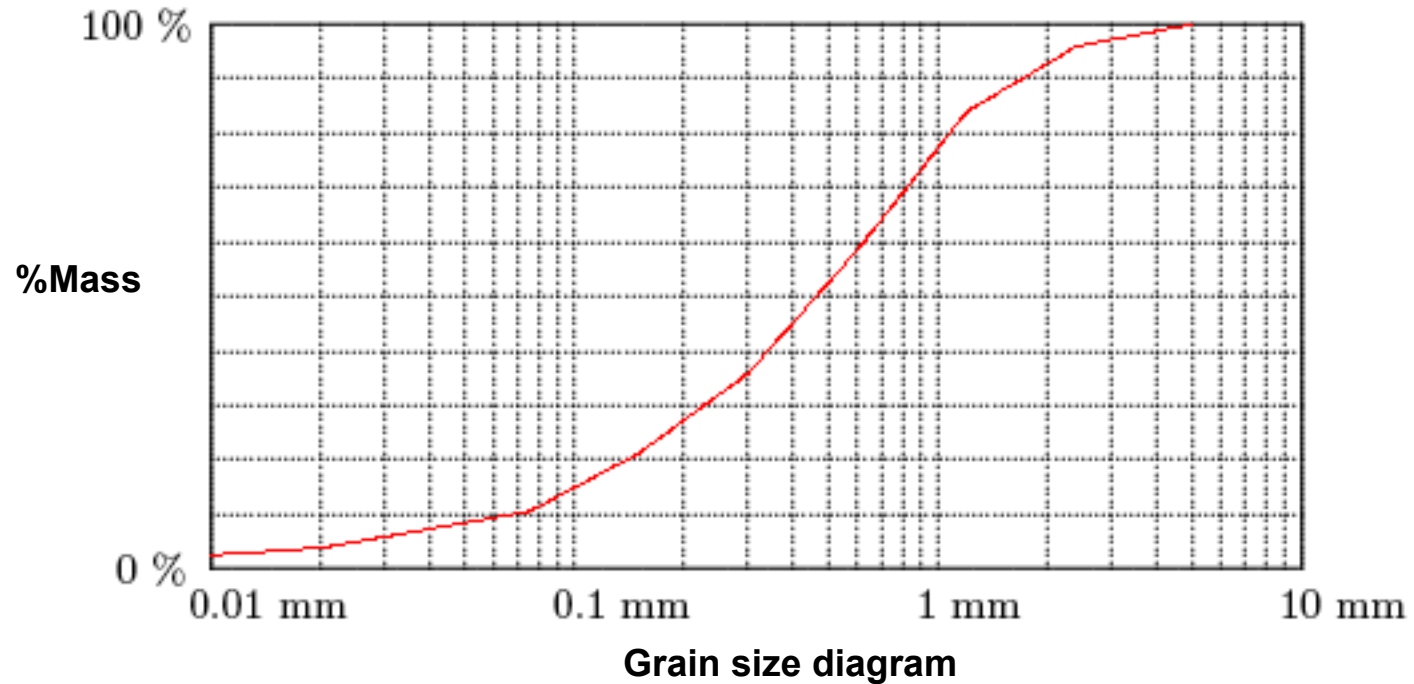


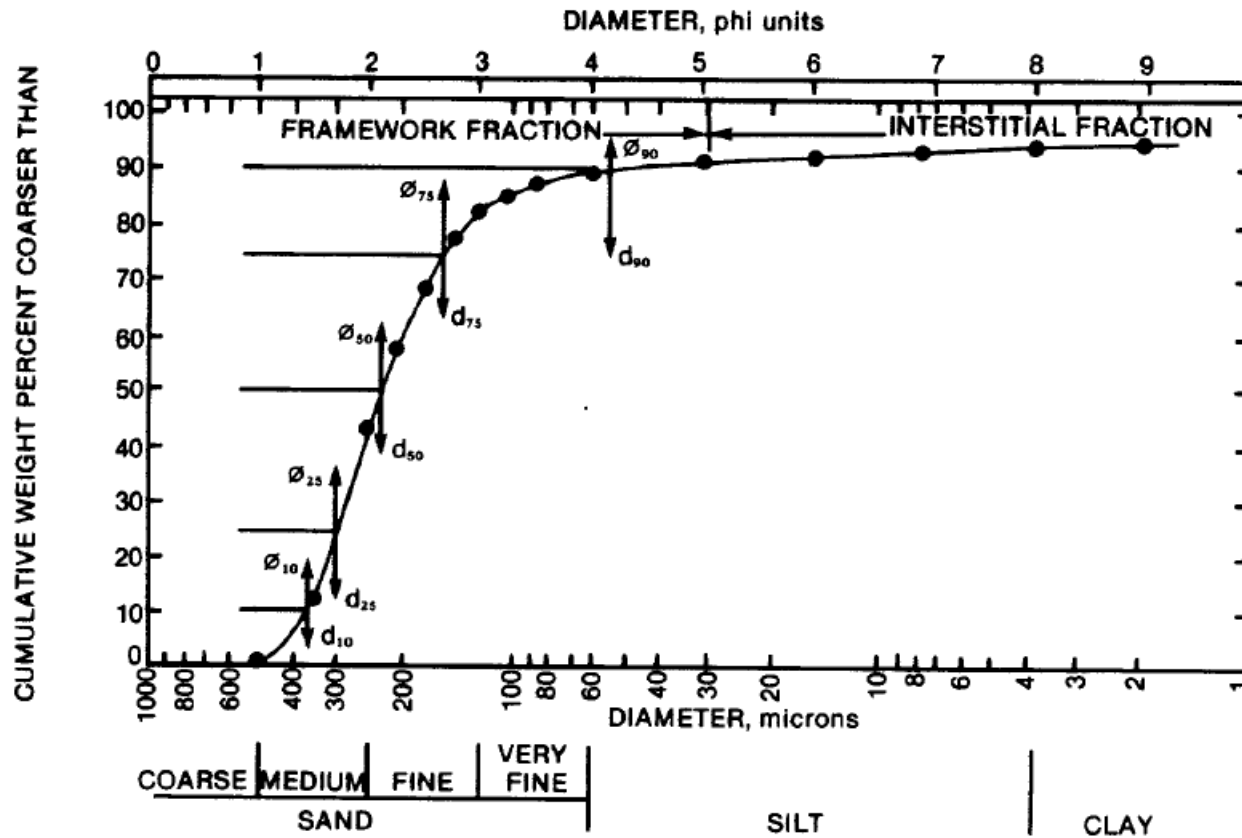
Grain size

Soil type	min.	max.
clay		0.002 mm
silt	0.002 mm	0.063 mm
sand	0.063 mm	2 mm
gravel	2 mm	63 mm

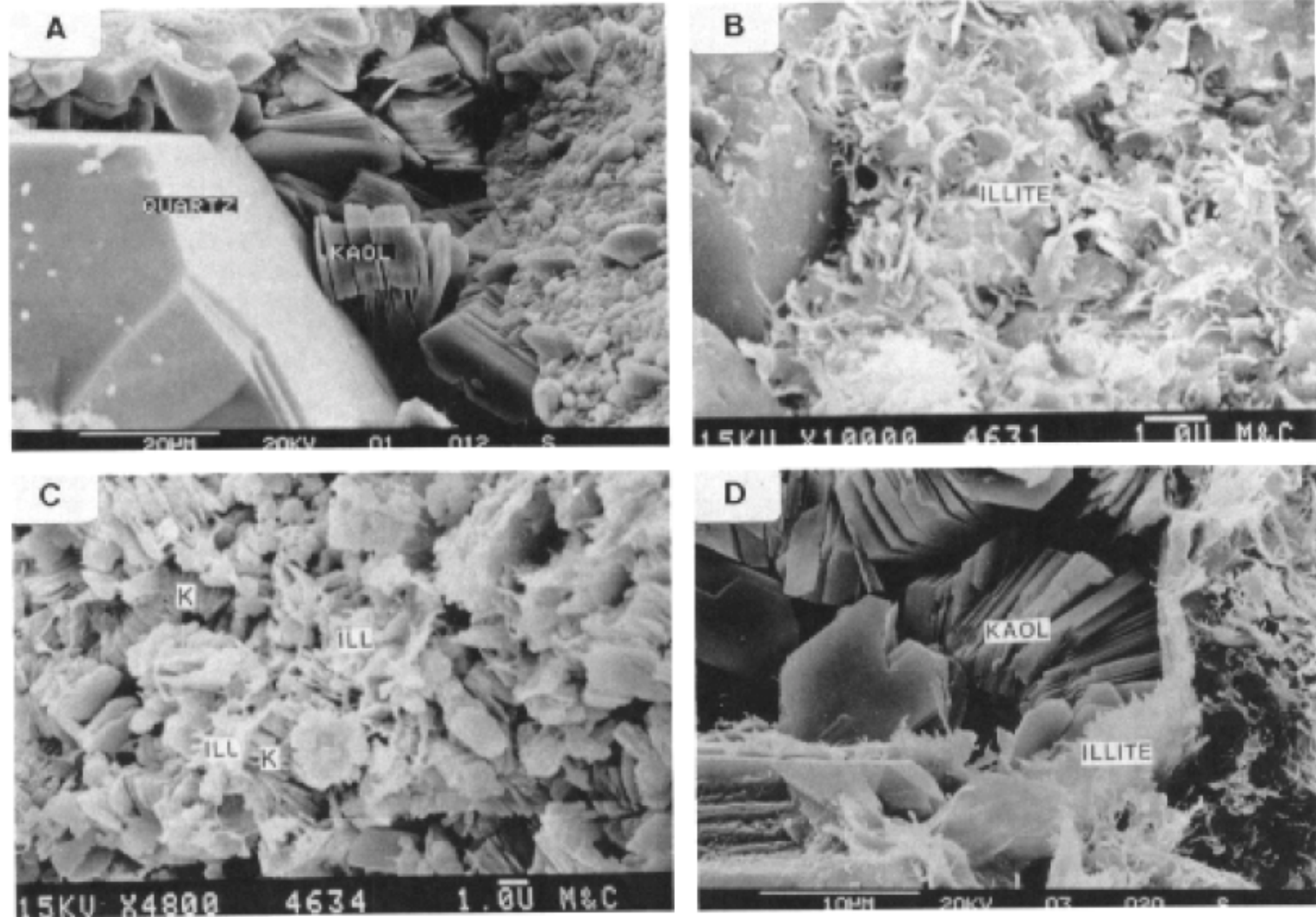
Density

$$\rho_{solids} \approx 2700 \pm 50 \text{ kg/m}^3$$

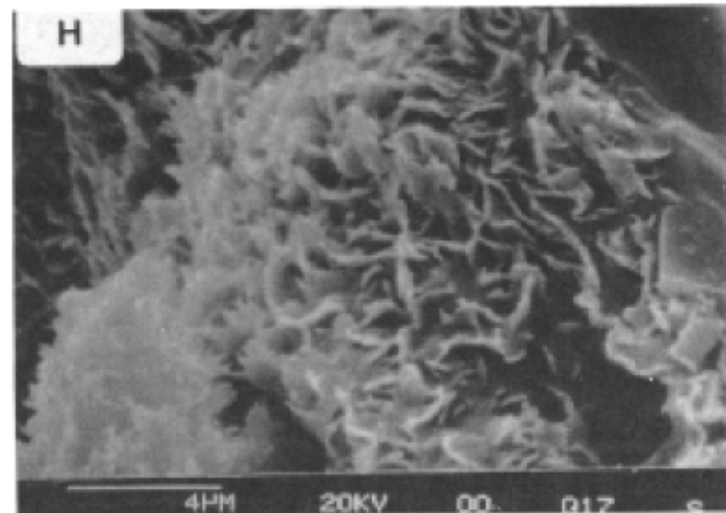
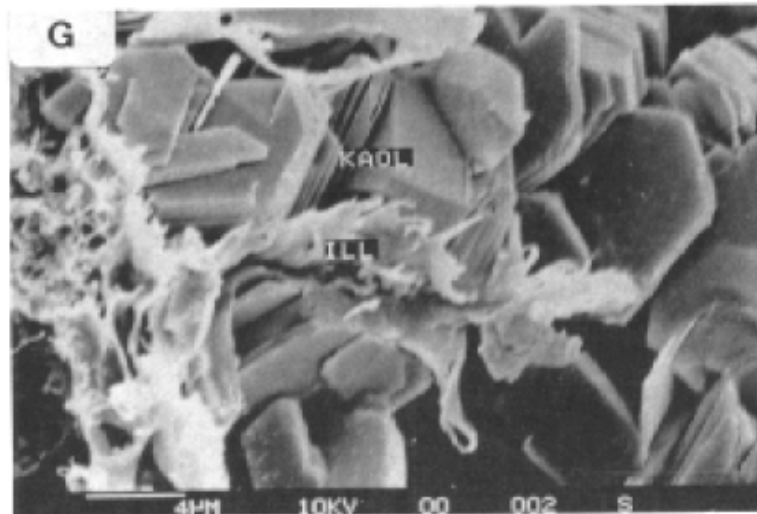
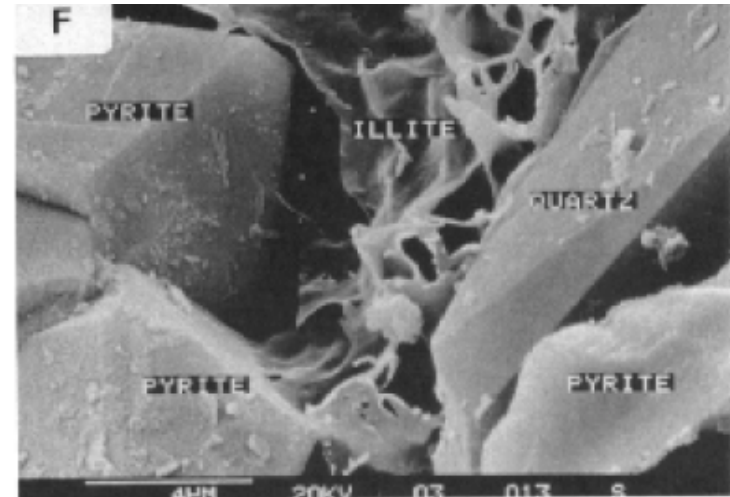
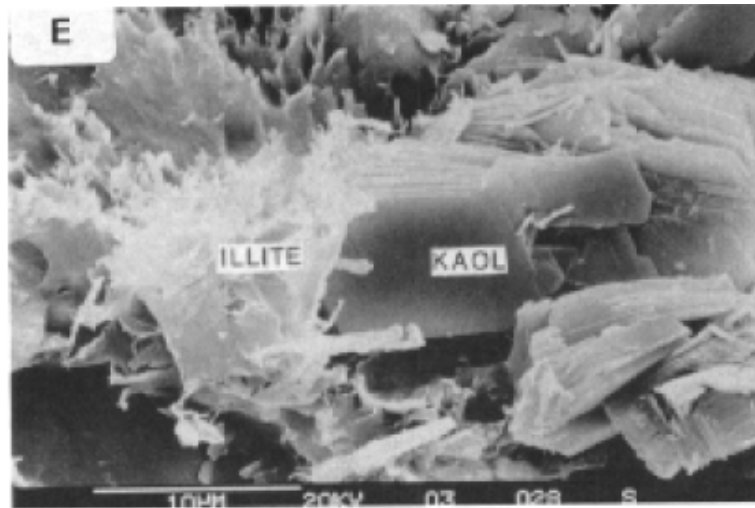




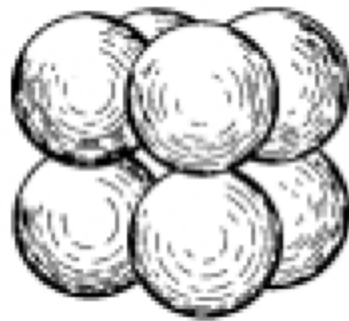
Cumulative grain size distribution; percent by weight greater than a given diameter on a logarithm scale (Jorden and Campbell, 1984)



**SEM photomicrographs of kaolinite and illite in sandstone
(Houseknecht and Pittman, 1992)**



**SEM photomicrographs of kaolinite and illite in sandstone
(Houseknecht and Pittman, 1992)**

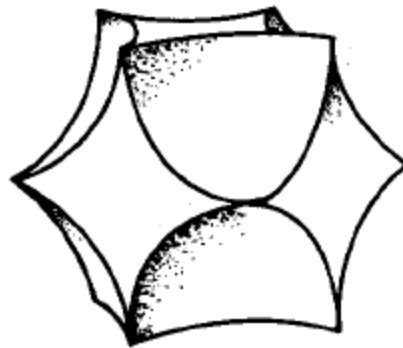


Cubic

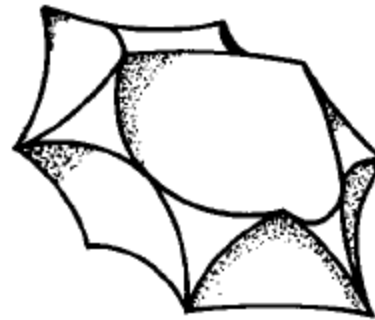


Rhombohedral

Packs of spherical beads
(Collins 1961)

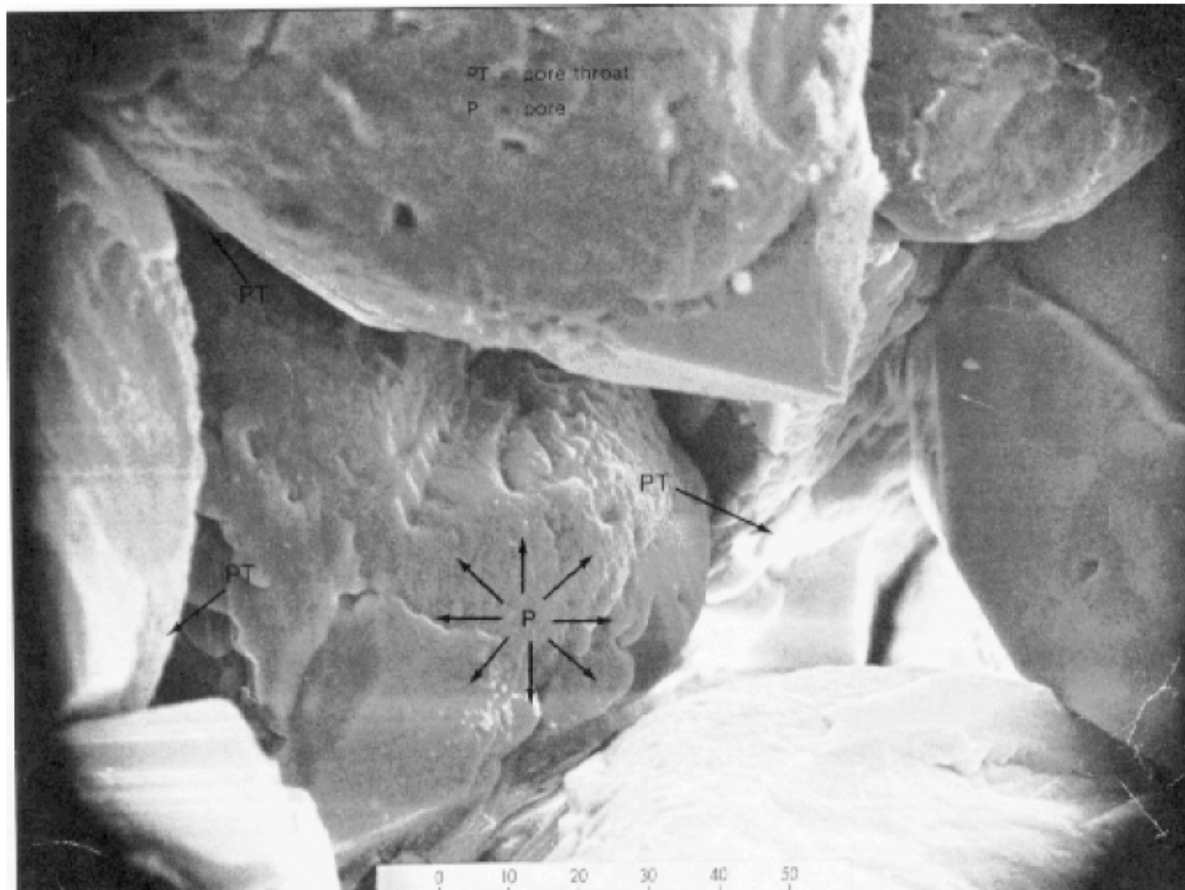


Cubic



Rhombohedral

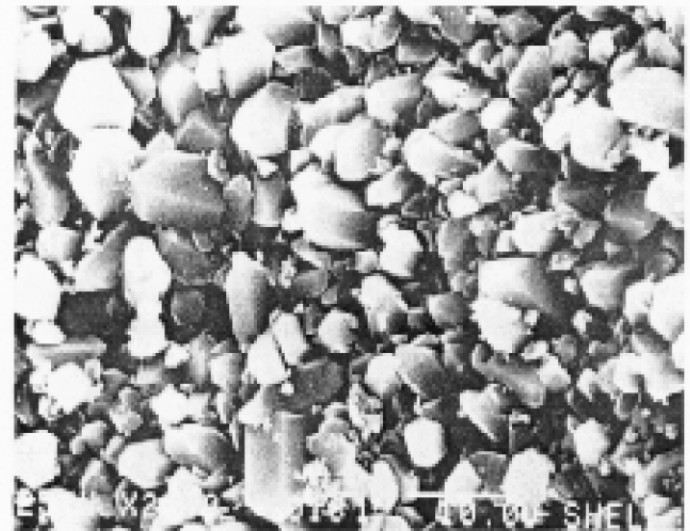
Pore space of spherical bead pack
(Collins 1961)



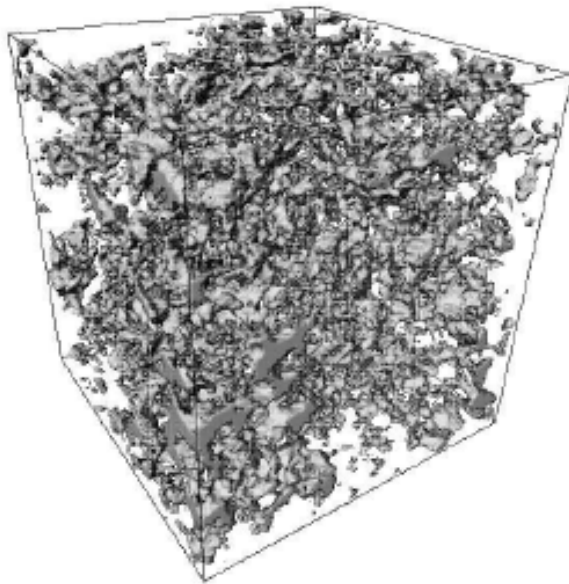
SEM microphotograph of bore body and pore throat (Jordon 1984)



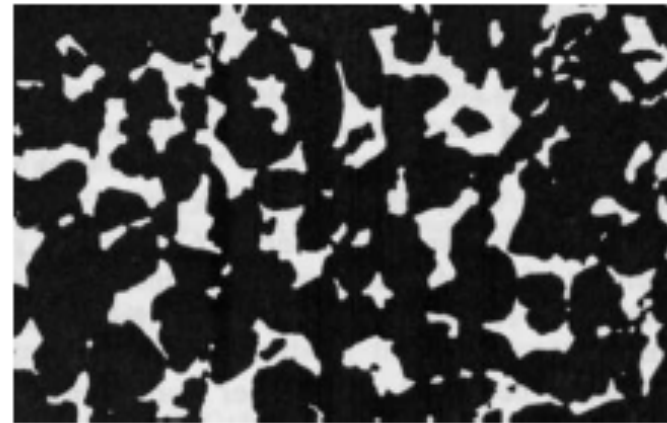
Kaolinite
SEM of micropores in
kaolinite. Note 10 μm scale.
(Swanson 1985)



Chert
SEM of micropores in
chert. Note 10 μm scale.
(Swanson 1985)

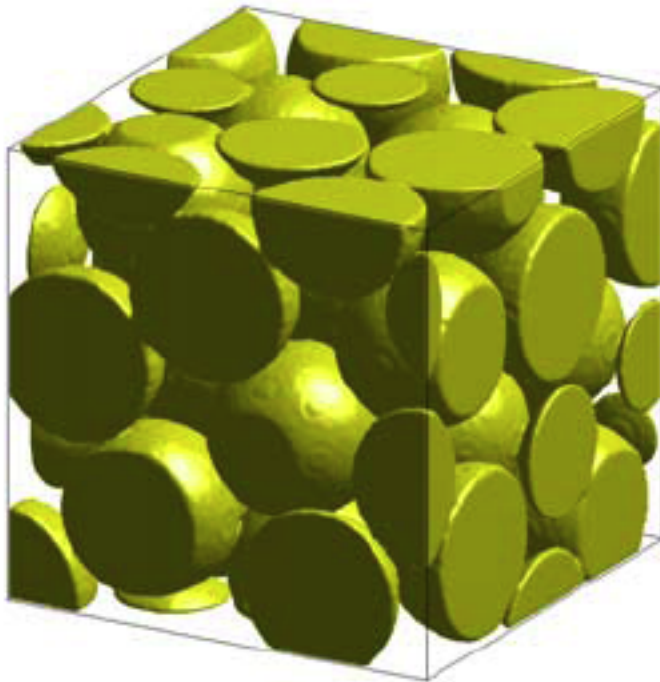


Espace des pores d'un grès de la mer du Nord (données Statoil).

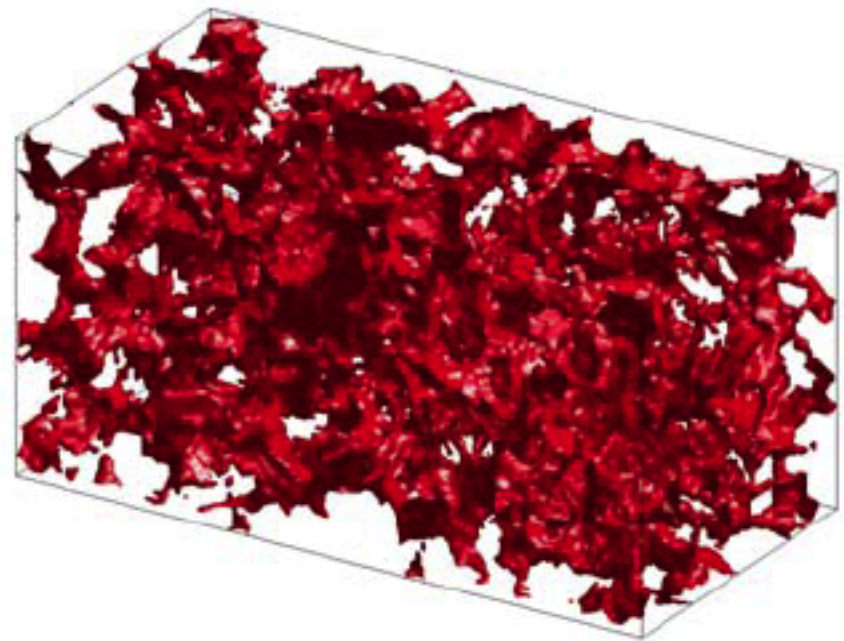


Exemple de coupe d'un poreux (grès).

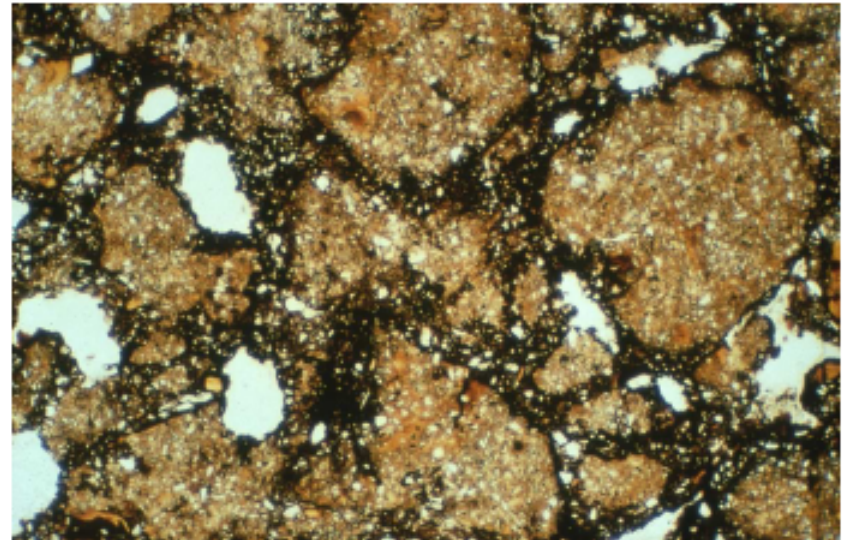
Finney-pack
(Random-dense pack of spheres)

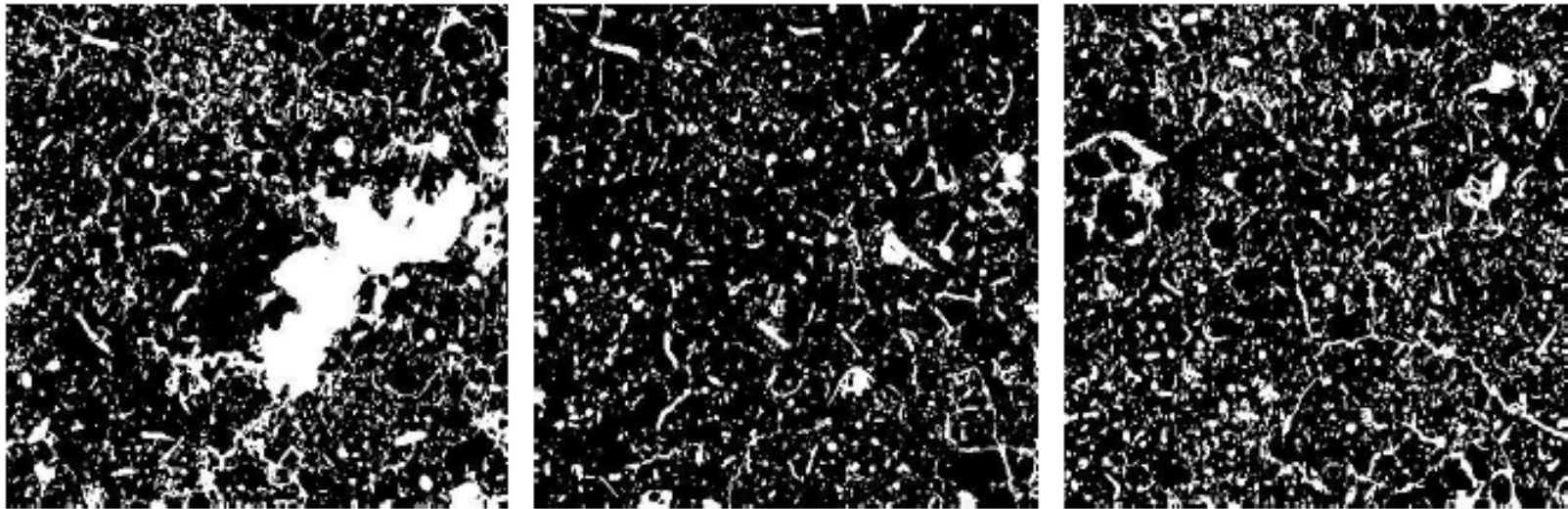


Fontainebleau Sandstone
(By X-ray microtomography)

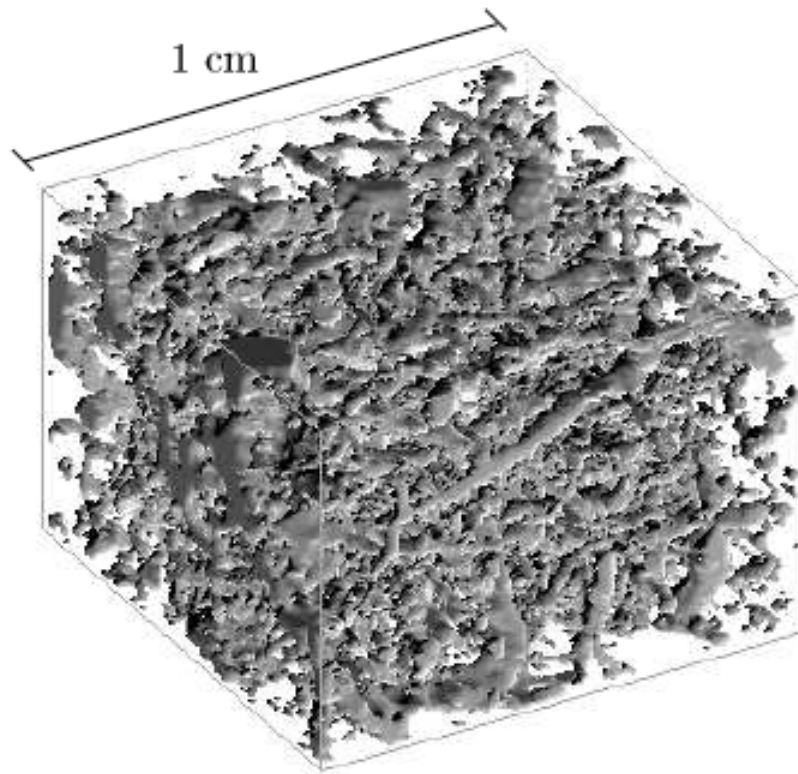


Thin section of about 1 mm length from a loamy-clay soil. Clearly distinguishable are the system of macropores with diameters of some 0.1 mm, small soil aggregates (lighter shades of brown) with sizes of about 0.3 mm, and the system of meso- and micropores. (Image courtesy of H.-J. Vogel)





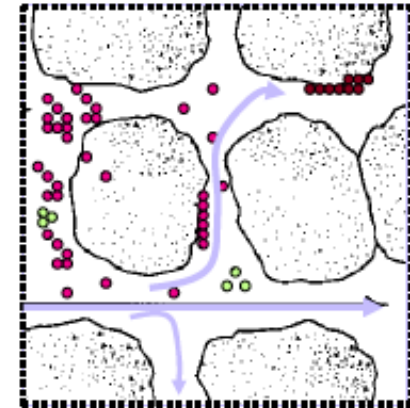
Horizontal cross-sections through a sample taken at 0.4 m depth from a loamy-clay soil near Beauce, France [*Cousin et al.*, 1996]. The side length of the square sections is 48 mm with a resolution of 0.12 mm. The smallest visible pores thus are comparable to the largest pores in Figure 3.1. The vertical distance between the sections shown here is 6 mm. White represents the pore space, black the soil matrix which itself is again porous at a smaller scale. (Data courtesy of I. Cousin)

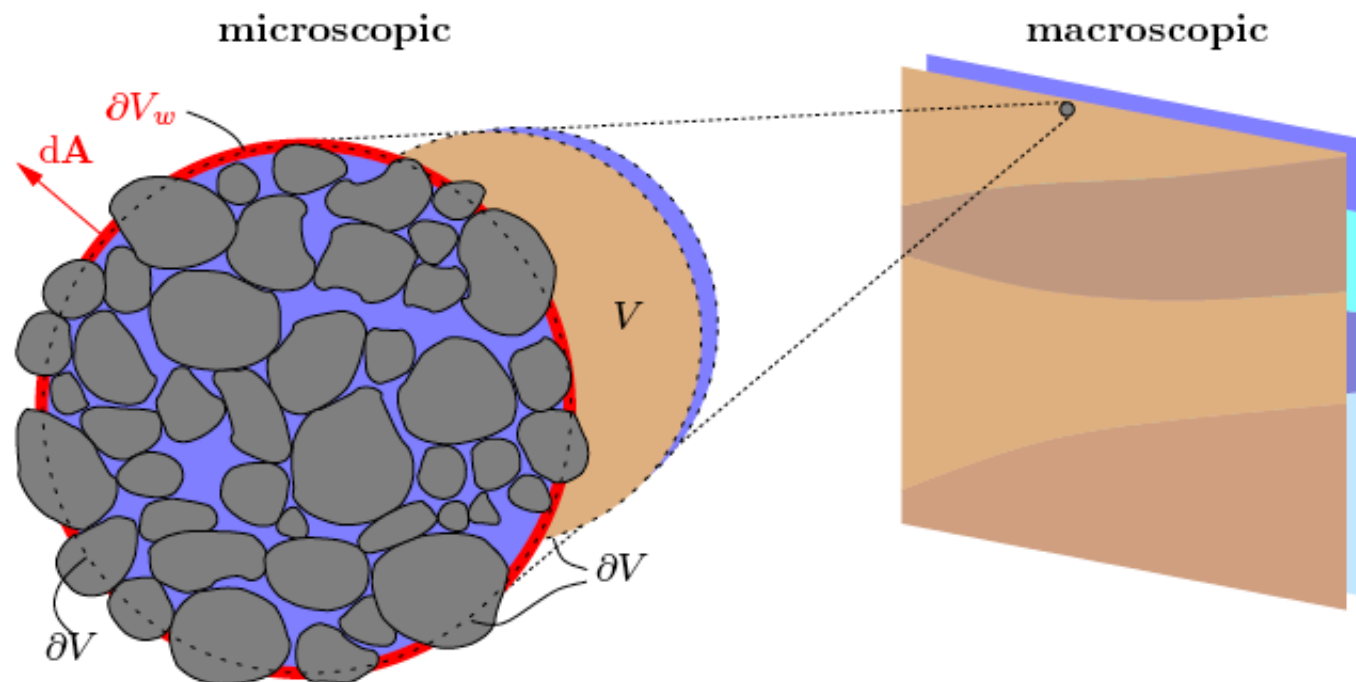


Three-dimensional reconstruction of the macropore system for a selection from the dataset shown in Figure 3.2. Resolution is 0.12 mm horizontally and 0.10 mm vertically. (Image courtesy of H.-J. Vogel)

- Approche microscopique
 - de mettre en évidence les phénomènes à prendre en compte par la loi macroscopique
 - de donner des moyens pour obtenir les paramètres des lois macroscopiques (e.g. perméabilité)
- Approche macroscopique
 - on moyenne le phénomène sur un volume plus grand que les pores (VER)
 - l'acte de moyennation implique une perte d'information, donc des informations supplémentaires éventuellement empiriques doivent être considérées (ex: loi de Darcy)

- Transport de masse
 - Écoulement
 - Diffusion
 - Dispersion
- Transfert de masse
 - Sorption (Adsorption, chimisorption, échange de ions)
 - Atténuation (biodégradation, décomposition radioactive)
 - Transformations (dissolution/précipitation)





Transition from pore-scale (microscopic) to continuum (macroscopic) representation. Consider a macroscopic volume V with boundary ∂V (dotted line). Microscopically, the detailed distribution of all the phases is available, e.g., of the water phase $V_w \subset V$ with external boundary $\partial V_w \subset \partial V$ (red line). Macroscopically, the phases and possibly other quantities are replaced by the superposition of continuous fields (uniformly colored regions). These fields may vary in space, but on a much larger scale than that of the averaging volume.

Transition to the continuum scale

Consider a representative elementary volume (REV) constituted of two-phases, Solids and pores (always filled with some fluid in the real world)

Phase indicator function

$$\chi_i(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \text{phase } i \\ 0 & \text{otherwise} \end{cases}$$

Pore volume

$$\chi_{pore}(\mathbf{x}) + \chi_{solids}(\mathbf{x}) = 1$$

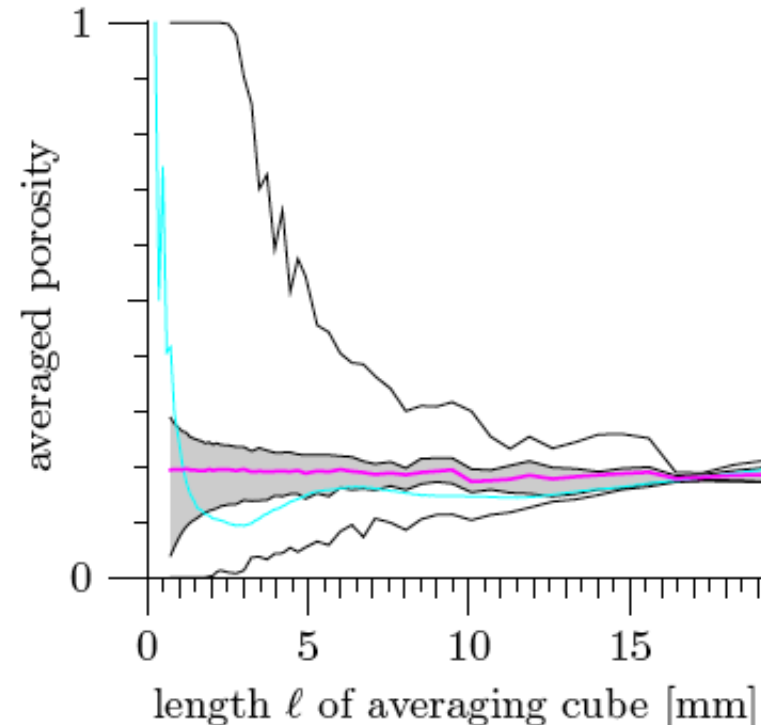
$$V_{pore} = \int_V \chi_{pore}(\mathbf{x}) dV$$

$$V_{pore} + V_{solids} = V$$

Solid volume

$$V_{solids} = \int_V \chi_{solids}(\mathbf{x}) dV$$

Estimated porosity of soil sample from Figure 3.2 as a function of averaging cube's length. The cyan curve represents a particular location. The other curves represent the ensemble of all cubes: average (magenta), minimum and maximum, and the two quartiles. Half of all values are within the gray band. The linear extent of a reasonable REV would be some 17 mm.

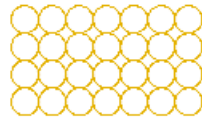


Porosity: $n = \frac{V_{pore}}{V_{total}}$

Compacity: $1 - n = \frac{V_{solids}}{V_{total}}$

N: nb spheres
 D_{Grain} : diameter

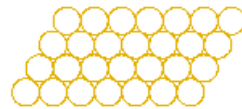
$$V_{solids} = N \frac{\pi D_{Grain}^3}{6}$$



Cubic array.

Loose arrangement

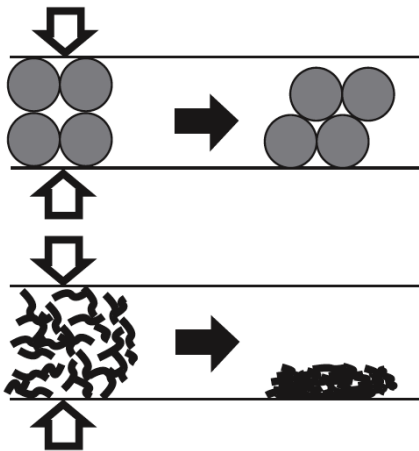
$$V_{total} = ND_{Grain}^3 \Rightarrow n = 1 - \frac{\pi}{6} \approx 0.476$$



Rhombic array.

Dense arrangement

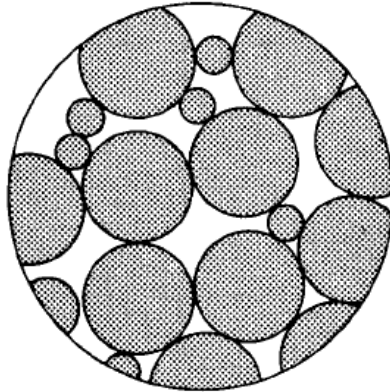
$$V_{total} = N \frac{D_{Grain}^3}{\sqrt{2}} \Rightarrow n = 1 - \frac{\pi}{\sqrt{18}} \approx 0.259$$



Practical situations for granular materials: $0.25 \leq n \leq 0.45$

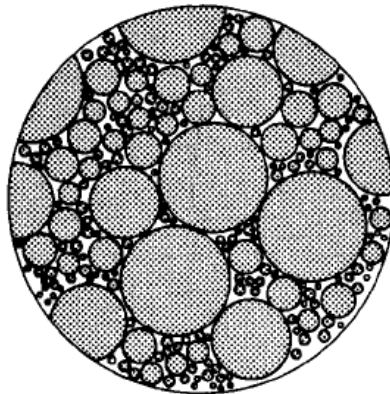
Practical situations for fine materials: $0.05 \leq n \leq 0.70$

Effect of sorting on porosity (Bear, 1972)



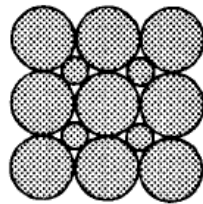
(a) WELL SORTED MATERIAL $n \sim 32\%$

$$n = 32\%$$



(b) POORLY SORTED MATERIAL $n \sim 17\%$

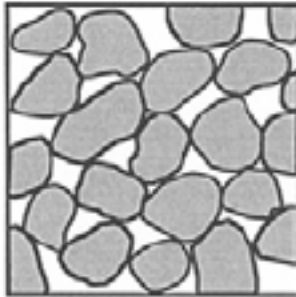
$$n = 17\%$$



(c) CUBIC ARRANGEMENT OF SPHERICAL
GRAINS OF TWO SIZES $n \sim 12.5\%$

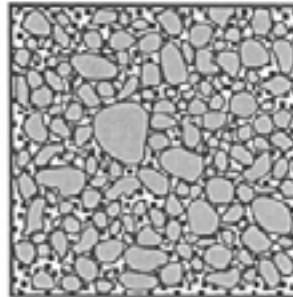
$$n = 12.5\%$$

well sorted gravel,
very high hydraulic
conductivity, K



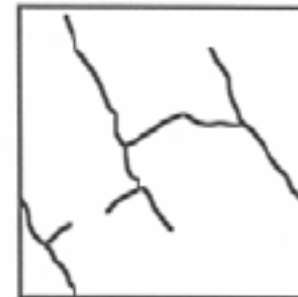
high porosity

poorly sorted sand
and gravel,
intermediate K



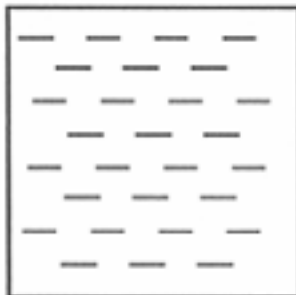
medium porosity

sparsely fractured
granite, low K



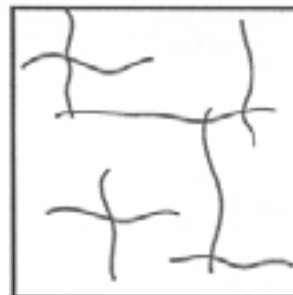
low porosity

unweathered
marine clay,
very low K



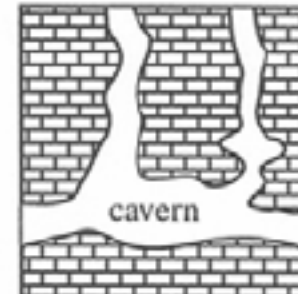
medium-high porosity

fractured glacial
clay till,
low-medium K

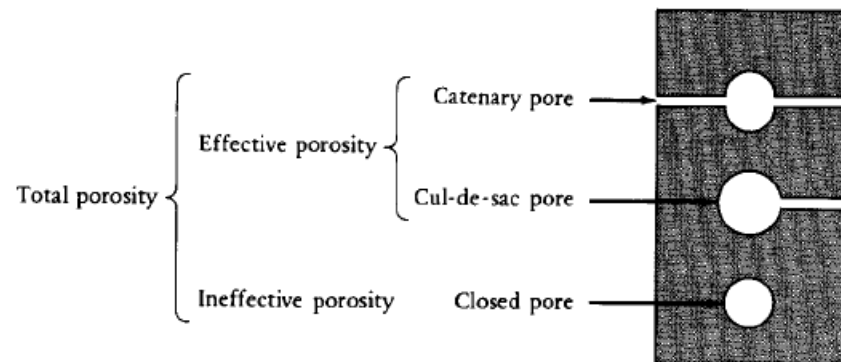


medium porosity

karstic limestone,
very high K



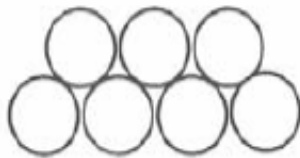
low-medium porosity



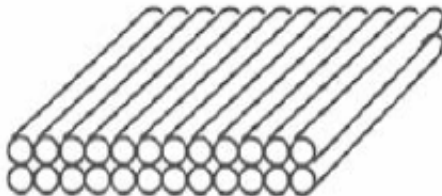
The three basic types of porosity. (Selley 1985)



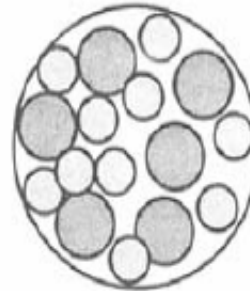
a. cubically packed uniform spheres, $\phi = 48\%$



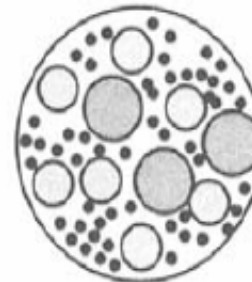
b. rhombohedral packing of uniform spheres, $\phi = 26\%$



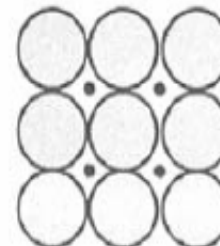
c. cubic packing of equally sized cylindrical rods, $\phi = \pi/4$



d. well sorted material, $\phi = 32\%$

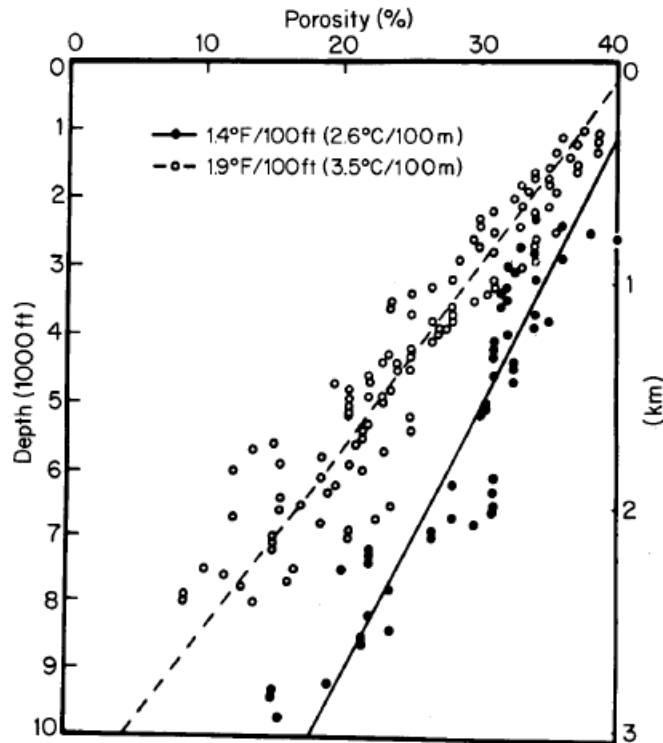


e. poorly sorted materials, $\phi = 17\%$

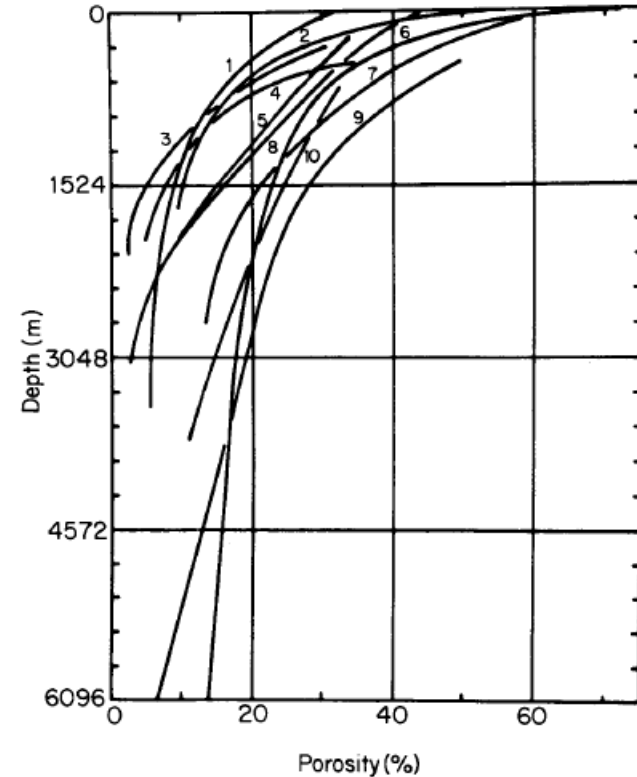


f. cubic arrangement of two sizes of spheres, $\phi = 12.5\%$

Porosity, examples



Porosity of sandstones with depth for two geothermal gradients [Jenyon 1990 (Magara 1980)]



Porosity of clay/shale as a function of depth [Jenyon 1990 (Magara 1980)]

Specific surface

Pore surface (m²)

$$A_{pore} = \partial V_{pore} = \int_V \|\nabla \chi_{pore}(\mathbf{x})\| dV$$

Pore specific surface (m⁻¹) (pore surface per unit REV volume)

$$a_{V_{pore}} \stackrel{\text{def}}{=} \frac{A_{pore}}{V_{total}}$$

$$\partial V_{solids} - \partial V_{pore} = \text{contact area}$$

Negligible in granular soils

$$a_{V_{pore}} \approx a_{V_{solids}}$$

but not in fine soils

Solid surface

$$A_{solids} = \partial V_{solids} = \int_V \|\nabla \chi_{solids}(\mathbf{x})\| dV$$

Solids specific surface (m⁻¹) (solids surface per unit REV volume)

$$a_{V_{solids}} \stackrel{\text{def}}{=} \frac{A_{solids}}{V_{total}}$$

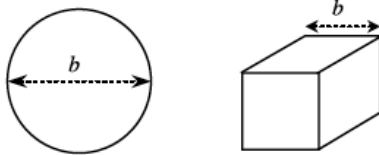
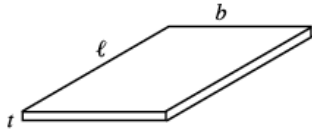
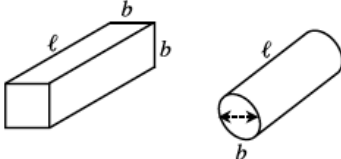
Solids mass specific surface (m²/kg) (solids surface per unit REV mass)

$$a_{W_{solids}} \stackrel{\text{def}}{=} \frac{A_{solids}}{M_{solids}}$$

$$a_{V_{solids}} = (1 - n) \rho_{solids} a_{W_{solids}}$$

Specific surface

Table 1. Specific surface and particle geometry: the smallest dimension controls the specific surface.

Geometry	Equations $S_s = \frac{A_s}{M}$	Examples
Sphere or cube: 	$S_s = \frac{6}{b\rho_w G_s}$	Amorphous clay minerals Allophane, hollow spherules $b = 50 \text{ \AA}$, $G_s = 2.65$ $S_s = 453 \text{ m}^2/\text{g}$ (Extreme case: $b = 9.6 \text{ \AA}$, then $S_s = 2358 \text{ m}^2/\text{g}$)
Thin plate:  $\ell \gg t$ and $b \gg t$	$S_s = \frac{2}{t\rho_w G_s}$	Sheet structure clay minerals Montmorillonite (extreme case: fully swollen) $t = 9.6 \text{ \AA}$, $G_s = 2.65$ $S_s = 786 \text{ m}^2/\text{g}$
Prism and rod:  $\ell \gg b$	$S_s = \frac{4}{b\rho_w G_s}$	Chain structure clay minerals Palygorskite, thread $b = 100 \text{ \AA}$, $G_s = 2.65$ $S_s = 151 \text{ m}^2/\text{g}$

Note: G_s , specific gravity of the particle mineral; (S_s platy particle dimensions $b \times b \times t$)/(S_s cube $b \times b \times b$) = $(\beta + 2)/3$, where $\beta = b/t$; ρ_w , mass density of water (1 g/cm^3).

Specific surface

Examples	Mass specific surface $a_{W_{solids}} \text{ (m}^2 / \text{g)}$	Volume specific surface $a_{V_{solids}} \text{ (m}^{-1}\text{)}$
Gravel $D_{grain} = 1 \text{ cm}$	10^{-4}	400
Fine silt $D_{grain} = 200 \text{ } \mu\text{m}$	10^{-2}	10^4 (1 ha/m ³)
Montmorillonite $e = 10 \text{ } \text{\AA} = 10^{-6} \text{ mm}$	750	8×10^8 (800 km ² /m ³ or 800 m ² /cm ³) <i>the surface of a football field in a thimble</i>

Granular material

$$\begin{cases} V_{solids} = N \frac{\pi D_{Grain}^3}{6} \\ A_{solids} = N \pi D_{Grain}^2 \end{cases}$$

N: nb spheres
 D_{Grain} : diameter

$\Rightarrow$

$$V_{solids} = (1 - n) V_{total}$$

$$\begin{cases} a_{V_{solids}} = \frac{6(1 - n)}{D_{Grain}} \\ a_{V_{solids}} = \frac{6}{\rho_{solids} D_{Grain}} \end{cases}$$

Hydraulic radius

$$\begin{aligned}
 \text{Pore hydraulic radius} &\stackrel{\text{def}}{=} \frac{\text{Cross section available for flow}}{\text{Wetted perimeter}} \\
 &= \frac{\text{Volume available for flow}}{\text{Wetted surface}} \\
 &= \frac{(\text{Volume of pores})/(\text{Total volume})}{(\text{Surface of pores})/(\text{Total Volume})} \\
 &= \frac{n}{a_{V_{\text{Pore}}}}
 \end{aligned}$$

Tube

$$\text{Tube hydraulic radius} = \frac{R_{\text{tube}}}{2}$$

Granular material

$$\text{Pore hydraulic radius} = \frac{nD_{\text{Grain}}}{6(1-n)}$$

$$\text{Mean pore radius} = \frac{2n}{a_{V_{\text{Pore}}}} \approx \frac{2n}{(1-n)\rho_{\text{solids}} a_{W_{\text{solids}}}}$$

$$\begin{aligned}
 \text{Mean pore radius} &= \frac{nD_{\text{Grain}}}{3(1-n)} \\
 &(\text{granular material})
 \end{aligned}$$

Questions

1. What is the density of a dry soil (i.e filled with air) ?
2. What is the density of a water-saturated soil ?
3. Assume $n=0.3$ for a sand. What is the weight of 1 m³ of this sand in dry conditions ?
4. Fill the pores of this sand with water. What is the volume of the water than the sand could contain ? Then, what is the density of the saturated sand ?
5. A building is constructed on a clay layer of 5 m thickness, with initial porosity of 50%, on top of a stiff sand. After the construction, the clay porosity is reduced to 40%.
What is the settlement of the soil ?
6. The void ratio is another engineering quantity widely used in porous mechanics.
Void ratio is defined as follows:

$$e \stackrel{def}{=} \frac{V_{pore}}{V_{solids}}$$

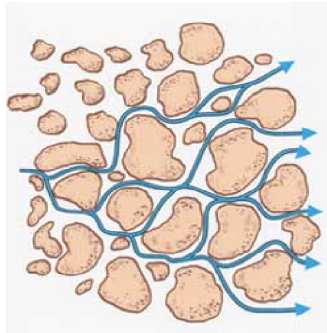
Express the void ratio as a function of the porosity.

7. Express the volume strain as a function of the porosity.
8. Express the volume strain rate as a function of the porosity.

Milieux poreux

Poro-Mécanique

Couplages hydro-mécanique



Balance equations:
mass, momentum, energy and entropy
State laws and dissipations

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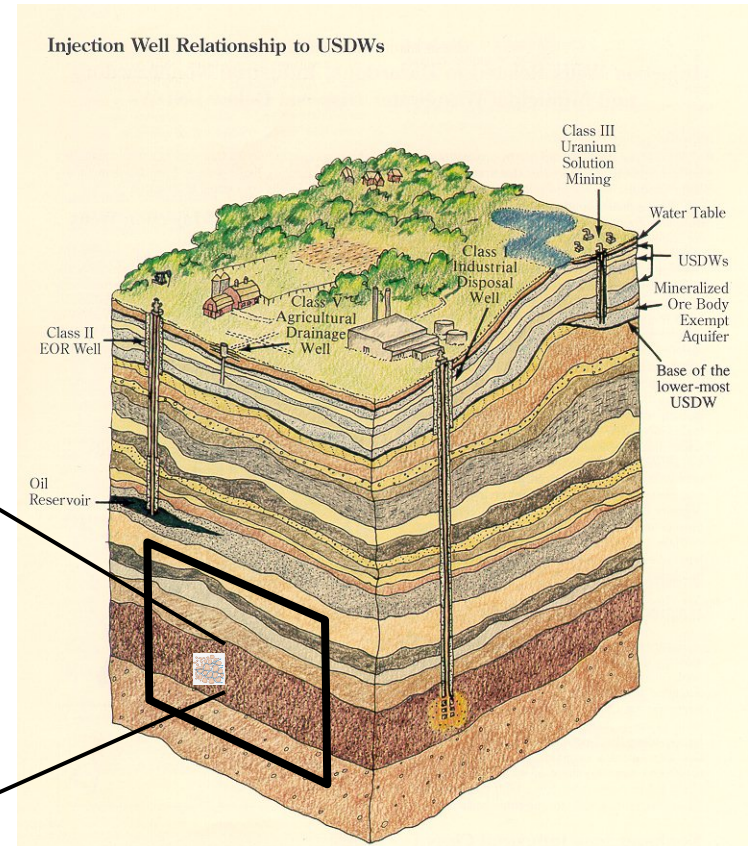
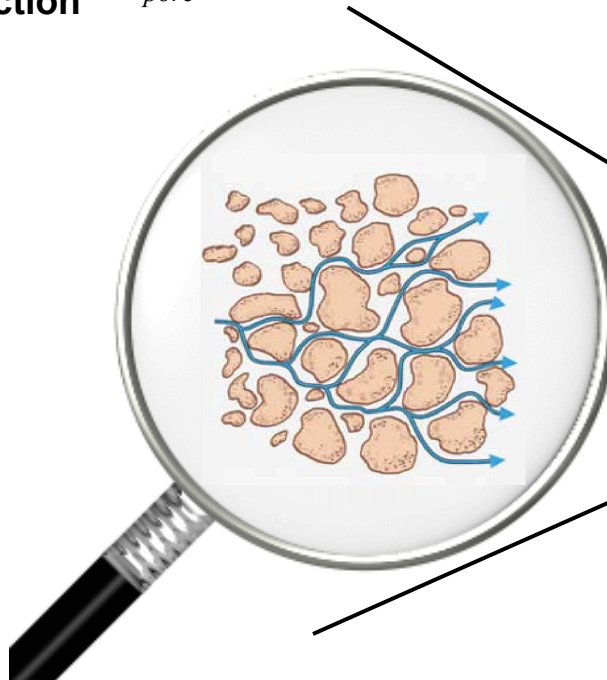
stephane.bonelli@irstea.fr

REV: $V = V_{solids} \cup V_{pore}$

Micro
solids velocity $\hat{v}_s$

Micro
flow velocity $\hat{v}_w$

Pore
Indicator function χ_{pore}



ω Macroscopic
control volume

Basic quantities

REV: $V = V_{solids} \cup V_{pore}$

Averaging operator: $\langle \bullet \rangle = \frac{1}{V} \int_V \bullet dV$

Porosity: $n = \langle \chi_{pore} \rangle_{def}$

Mean pore flow velocity: $\mathbf{v}_w \stackrel{def}{=} \frac{\langle \chi_{pore} \rho_w \tilde{\mathbf{v}}_w \rangle}{\langle \chi_{pore} \rho_w \rangle}$

Macro control volume: ω

Mean solid velocity: $\mathbf{v}_s \stackrel{def}{=} \frac{\langle \chi_{solids} \rho_s \tilde{\mathbf{v}}_s \rangle}{\langle \chi_{solids} \rho_s \rangle}$

Control volume velocity: $\mathbf{v}$
(to be defined)

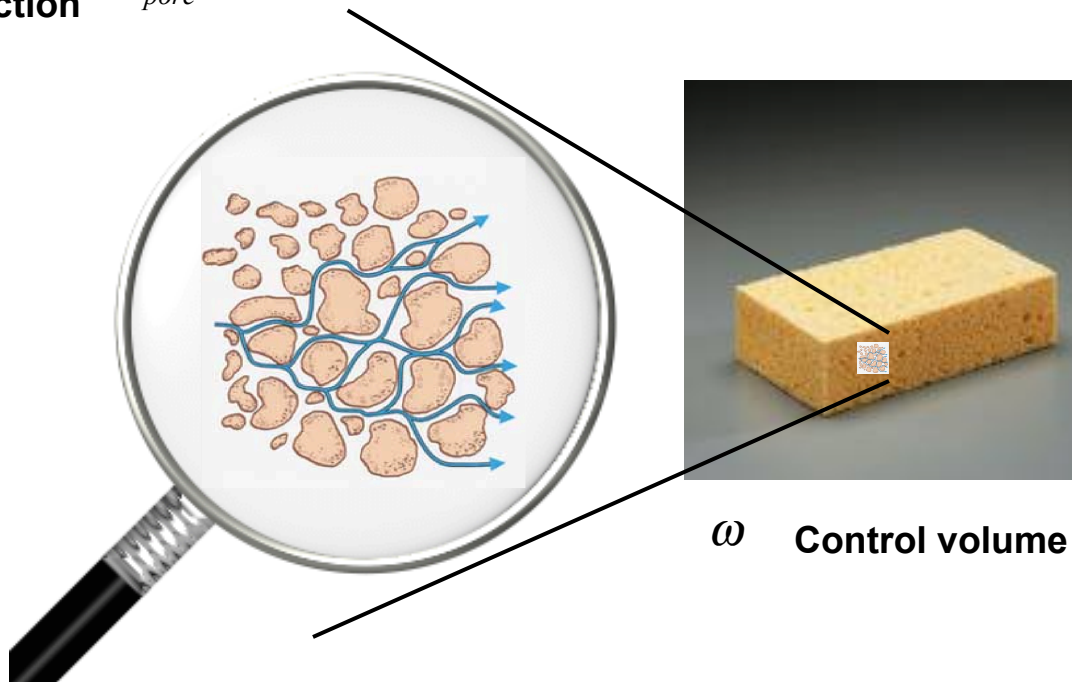
**The control volume can *not* be a material volume:
It can not be defined with the same material particles
at two different moments
 $\mathbf{v}$ can not be equal at the same time to $\mathbf{v}_s$ and to $\mathbf{v}_w$**

REV: $V = V_{\text{solids}} \cup V_{\text{pore}}$

Micro
solids velocity $\hat{\mathbf{v}}_s$

Micro
flow velocity $\hat{\mathbf{v}}_w$

Pore
Indicator function χ_{pore}



ω Control volume

Total time derivatives

Derivative of quantities (scalars, vectors, tensors):

Derivative with respect to the solid: $\frac{d^s}{dt}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \mathbf{v}_s \cdot \nabla(\bullet)$

Derivative with respect to the fluid: $\frac{d^w}{dt}(\bullet) = \frac{\partial}{\partial t}(\bullet) + \mathbf{v}_w \cdot \nabla(\bullet)$

Reynolds transport theorem (scalars, vectors, tensors):

$$\frac{d}{dt} \int_{\omega} (\bullet) d\omega = \int_{\omega} \frac{\partial}{\partial t} (\bullet) dx + \int_{\partial\omega} (\bullet) \mathbf{v} \cdot \mathbf{n} da$$

$\mathbf{v}$: control volume velocity
(to be defined)

Solid total time derivatives of volume integrals

$\mathbf{v} = \mathbf{v}_s$: the derivative is taken by following the solid in its movement

Derivative with respect to the solid:

$$\begin{aligned}
 \frac{d^s}{dt} \int_{\omega} (\bullet) dx & \stackrel{def}{=} \left. \frac{d}{dt} \right|_{\mathbf{v}=\mathbf{v}_s} \int_{\omega} (\bullet) dx \\
 &= \int_{\omega} \frac{\partial}{\partial t} (\bullet) dx + \int_{\partial\omega} (\bullet) \mathbf{v}_s \cdot \mathbf{n} da \\
 &= \int_{\omega} \frac{\partial}{\partial t} (\bullet) dx + \int_{\omega} \nabla \cdot [(\bullet) \mathbf{v}_s] dx \\
 &= \int_{\omega} \frac{\partial}{\partial t} (\bullet) + \mathbf{v}_s \cdot \nabla (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_s dx
 \end{aligned}$$

$$\frac{d^s}{dt} \int_{\omega} (\bullet) dx = \int_{\omega} \left[\frac{d^s}{dt} (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_s \right] dx$$

Fluid total time derivatives of volume integrals

$\mathbf{v} = \mathbf{v}_w$: the derivative is taken by following the fluid in its movement

Derivative with respect to the fluid:

$$\frac{d^w}{dt} \int_{\omega} (\bullet) dx = \frac{d}{dt} \bigg|_{\mathbf{v}=\mathbf{v}_w} \int_{\omega} (\bullet) dx$$

$$= \int_{\omega} \left[\frac{d^w}{dt} (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_w \right] dx$$

And also

$$= \int_{\omega} \left[\frac{\partial}{\partial t} (\bullet) + \mathbf{v}_w \cdot \nabla (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_w \right] dx$$

$$= \int_{\omega} \left[\frac{\partial}{\partial t} (\bullet) + \mathbf{v}_s \cdot \nabla (\bullet) + (\mathbf{v}_w - \mathbf{v}_s) \cdot \nabla (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_s + (\bullet) \nabla \cdot (\mathbf{v}_w - \mathbf{v}_s) \right] dx$$

$$= \int_{\omega} \left[\frac{d^s}{dt} (\bullet) + (\bullet) \nabla \cdot \mathbf{v}_s + \nabla [(\bullet)(\mathbf{v}_w - \mathbf{v}_s)] \right] dx$$

$$\frac{d^w}{dt} \int_{\omega} (\bullet) dx = \frac{d^s}{dt} \int_{\omega} (\bullet) dx + \int_{\partial \omega} (\bullet)(\mathbf{v}_w - \mathbf{v}_s) \cdot \mathbf{n} da$$

Solid mass balance

Solid mass of any domain $M_s(\omega^t, t) = \int_{\omega} (1-n) \rho_s dx$

Assumption:

- No mass exchange between phases

Mass balance $\forall \omega, \frac{d^s}{dt} M_s(\omega, t) = 0$

$\Leftrightarrow$
localization
theorem

$$\underbrace{\nabla \cdot \mathbf{v}_s}_{\text{variation of total volume}} = \underbrace{\frac{1}{1-n} \frac{d^s n}{dt}}_{\text{variation of porosity}} - \underbrace{\frac{1}{\rho_s} \frac{d \rho_s}{dt}}_{\text{variation of solid volume}}$$

Assumption:

- Homogeneous and rigid solids

(relevant for many - but not all - porous media, irrelevant for rocks, for example)

$\forall \omega,$
 $\frac{d^s}{dt} M_s(\omega, t) = 0$

$\Leftrightarrow$
localization
theorem

$$\underbrace{\nabla \cdot \mathbf{v}_s}_{\text{variation of total volume}} = \underbrace{\frac{1}{1-n} \frac{d^s n}{dt}}_{\text{variation of porosity}}$$

Bulk equation to be used in the following (VPP, energy balance, ...)

$$\text{tr } \boldsymbol{\varepsilon} = \ln \left[\frac{1-n^0}{1-n} \right]$$

Bulk equation to be used to evaluate the porosity which appears to be a secondary unknown

Fluid mass balance

Fluid mass of any domain:
$$M_w(\omega^t, t) = \int_{\omega} n \rho_w dx$$

Assumption:

- No mass exchange between phases

Mass balance $\forall \omega,$

$$\frac{d^w}{dt} M_w(\omega^t, t) = 0 \quad \Leftrightarrow \quad \underbrace{\frac{d}{dt} \int_{\omega} n \rho_w dx}_{\text{variation of fluid mass in } \omega} + \underbrace{\int_{\partial \omega} \rho_w \mathbf{q} \cdot \mathbf{n} da}_{\text{mass flux of fluid crossing } \partial \omega} = 0$$

localization theorem

Average relative pore-fluid velocity:
$$\mathbf{q} = n(\mathbf{v}_w - \mathbf{v}_s)$$

Fluid mass balance

Assumption:

- Homogeneous and rigid solids

$$\forall \omega,$$

$$\frac{d^w}{dt} M_w(\omega^t, t) = 0$$

$\Leftrightarrow$
localization
theorem

**Bulk equation to be discretized
(FDM, FEM, FVM, ...)**

$$\underbrace{n \frac{d^s \rho_w}{dt}}_{\text{fluid density influence}} + \underbrace{\rho_w \nabla \cdot \mathbf{v}_s}_{\text{solid matrix porosity influence}} + \underbrace{\nabla \cdot (\rho_w \mathbf{q})}_{\text{fluid mass diffusion}} = 0$$

$\Leftrightarrow$

**Bulk equation to be used in the following
(VPP, energy balance, ...)**

$$\underbrace{\frac{n}{\rho_w} \frac{d^w \rho_w}{dt}}_{\text{fluid volume strain rate}} + \underbrace{\nabla \cdot \mathbf{v}_s}_{\text{total volume strain rate}} + \underbrace{\nabla \cdot \mathbf{q}}_{\text{fluid diffusion}} = 0$$

Total time derivatives of mass integrals

Accounting for the solid mass balance equations,
the derivative of mass integrals with respect to the solid reads

$$\frac{d^s}{dt} \int_{\omega} (1-n) \rho_s(\bullet) dx = \int_{\omega} (1-n) \rho_s \frac{d^s}{dt}(\bullet) dx$$

Accounting for the fluid mass balance equations,
the derivative of mass integrals with respect to the fluid reads

$$\frac{d^w}{dt} \int_{\omega} n \rho_w(\bullet) dx = \int_{\omega} n \rho_w \frac{d^w}{dt}(\bullet) dx$$

The total time derivative of mass integrals of mixture quantities are therefore

$$\frac{D}{dt} \int_{\omega} n \rho_w(\bullet) dx = \int_{\omega} (1-n) \rho_s \frac{d^s}{dt}(\bullet) dx + \int_{\omega} n \rho_w \frac{d^w}{dt}(\bullet) dx$$

where $\rho(\bullet) = (1-n) \rho_s(\bullet) + n \rho_w(\bullet)$

Mixture kinetic energy

$$K(\omega, \mathbf{v}_s, \mathbf{v}_w) = \int_{\omega} \underbrace{\frac{1}{2}(1-n)\rho_s (\mathbf{v}_s)^2}_{\text{solid kinetic energy}} + \underbrace{\frac{1}{2}n\rho_w (\mathbf{v}_w)^2}_{\text{fluid kinetic energy}} dx$$

Total time derivative of the mixture kinetic energy

$$\frac{D}{dt} K(\omega, \mathbf{v}_s, \mathbf{v}_w) = \int_{\omega} (1-n)\rho_s \mathbf{v}_s \cdot \boldsymbol{\gamma}_s dx + \int_{\omega} n\rho_w \mathbf{v}_w \cdot \boldsymbol{\gamma}_w dx$$

$$\boldsymbol{\gamma}_s = \frac{d^s}{dt} \mathbf{v}_s \quad : \text{solid acceleration}$$

$$\boldsymbol{\gamma}_w = \frac{d^w}{dt} \mathbf{v}_w \quad : \text{fluid acceleration}$$

Virtual inertia

In terms of $(\hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w)$

$$A(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) = \int_{\omega} \underbrace{(1-n)\rho_s \hat{\mathbf{v}}_s \cdot \boldsymbol{\gamma}_s}_{\text{Solids}} dx + \int_{\omega} \underbrace{n\rho_w \hat{\mathbf{v}}_w \cdot \boldsymbol{\gamma}_w}_{\text{Fluid}} dx$$

In terms of $(\hat{\mathbf{v}}_s, \hat{\mathbf{q}})$

$$A(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = \int_{\omega} \underbrace{\left[\underbrace{(1-n)\rho_s \cdot \boldsymbol{\gamma}_s + n\rho_w \boldsymbol{\gamma}_w}_{\text{Barycentric acceleration}} \right] \cdot \hat{\mathbf{v}}_s}_{\text{Mixture}} dx + \int_{\omega} \underbrace{\rho_w \boldsymbol{\gamma}_w \cdot \hat{\mathbf{q}}}_{\text{Pore fluid}} dx$$

However, the current modelling often use a simplified description (more for numerical reasons than for physical evidences)

$$\boldsymbol{\gamma}_w \approx \boldsymbol{\gamma}_s$$

Therefore

$$A(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = \int_{\omega} \underbrace{\rho \boldsymbol{\gamma}_s \cdot \hat{\mathbf{v}}_s}_{\text{Mixture}} dx + \int_{\omega} \underbrace{\rho_w \boldsymbol{\gamma}_s \cdot \hat{\mathbf{q}}}_{\text{Pore fluid}} dx$$

Internal virtual power

Given: the set of virtual velocities $H(\omega) = \{(\hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) \text{ k.a.}\}$

Assumption:

- **First gradient theory**

$$\forall \omega, \forall (\hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) \in H(\omega)$$

$$P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) = - \int_{\omega} \underbrace{\mathbf{r}_s \cdot \hat{\mathbf{v}}_s + \mathbf{r}_w \cdot \hat{\mathbf{v}}_w}_{\text{order zero}} + \underbrace{\mathbf{T}_s : \nabla \hat{\mathbf{v}}_s + \mathbf{T}_w : \nabla \hat{\mathbf{v}}_w}_{\text{order one}} dx$$

Assumption:

- **Material indifference**

$$P^{\text{int}}(\omega^t, t) = 0 \text{ for any rigid translation} \quad \Leftrightarrow \quad \mathbf{r}_s + \mathbf{r}_w = 0$$

$$P^{\text{int}}(\omega^t, t) = 0 \text{ for any rigid rotation} \quad \Leftrightarrow \quad (\mathbf{T}_s + \mathbf{T}_w)_{\text{skew}} = 0$$

$$P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) = - \int_{\omega} \underbrace{\mathbf{r}_w \cdot (\hat{\mathbf{v}}_w - \hat{\mathbf{v}}_s)}_{\text{order zero}} + \underbrace{\boldsymbol{\sigma} : \mathbf{D}(\hat{\mathbf{v}}_s) + \mathbf{T}_w : \nabla (\hat{\mathbf{v}}_w - \hat{\mathbf{v}}_s)}_{\text{order one}} dx$$

$$\mathbf{D}(\hat{\mathbf{v}}_s) = (\nabla \hat{\mathbf{v}}_s)_{\text{sym}} \quad \text{Virtual strain rate}$$

$$\boldsymbol{\sigma} = \mathbf{T}_s + \mathbf{T}_w \quad \text{Total stress}$$

Internal virtual power

New choice for the set of virtual velocities

$$H(\omega) = \{(\hat{\mathbf{v}}_s, \hat{\mathbf{q}}) \text{ k.a.} \}$$

Assumption:

- Inviscid fluid (at the macro-scale)

$$\mathbf{T}_w = -np\mathbf{I}$$

$$P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = - \int_{\omega} \mathbf{f}_w \cdot \mathbf{q} + \boldsymbol{\sigma} : \mathbf{D}(\hat{\mathbf{v}}_s) - p \nabla \cdot \hat{\mathbf{q}} dx \quad \forall \omega, \forall (\hat{\mathbf{v}}_s, \hat{\mathbf{q}}) \in H(\omega)$$

$\mathbf{f}_w$

Vector of solid/fluid interaction

Significance can best be assessed by inserting the fluid mass balance equation

$$P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = - \int_{\omega} \underbrace{\mathbf{f}_w \cdot \mathbf{q}}_{\substack{\text{solid/fluid} \\ \text{interaction} \\ \text{term}}} + \underbrace{(\boldsymbol{\sigma} + p\mathbf{I}) : \mathbf{D}(\hat{\mathbf{v}}_s)}_{\substack{\text{effective} \\ \text{stress} \\ \text{solid matrix} \\ \text{term}}} + \underbrace{\frac{np}{\rho_w} \frac{d^w \rho_w}{dt}}_{\substack{\text{pore-fluid} \\ \text{term}}} dx$$

External loading virtual power

In term of $(\hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w)$

$$P^{\text{ext}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{v}}_w) = \underbrace{\int_{\omega} (1-n)\rho_s \mathbf{g} \cdot \hat{\mathbf{v}}_s + n\rho_w \mathbf{g} \cdot \hat{\mathbf{v}}_w dx}_{\text{bulk loading}} + \underbrace{\int_{\partial\omega} \underbrace{\mathbf{t}_s \cdot \hat{\mathbf{v}}_s}_{\text{solids}} + \underbrace{\mathbf{t}_w \cdot \hat{\mathbf{v}}_w}_{\text{fluid}} da}_{\text{boundary loading}}$$

With the new choice of virtual velocities $(\hat{\mathbf{v}}_s, \hat{\mathbf{q}})$

Noticing that, for an inviscid fluid in a porous medium $\mathbf{t}_w = -np_{\text{ext}} \mathbf{n}$

$$P^{\text{ext}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = \underbrace{\int_{\omega} \underbrace{\rho \mathbf{g} \cdot \hat{\mathbf{v}}_s}_{\text{mixture}} + \underbrace{\rho_w \mathbf{g} \cdot \hat{\mathbf{q}}}_{\text{pore-fluid}} dx}_{\text{bulk loading}} + \underbrace{\int_{\partial\omega} \underbrace{\mathbf{t} \cdot \hat{\mathbf{v}}_s}_{\text{mixture}} - \underbrace{p_{\text{ext}} \hat{\mathbf{q}} \cdot \mathbf{n}}_{\text{pore-fluid}} da}_{\text{boundary loading}}$$

$$\rho = (1-n)\rho_s + n\rho_w \quad \text{Mixture density}$$

$\mathbf{t}$ Total traction vector on boundary

$$\forall \omega, \forall (\hat{\mathbf{v}}_s, \hat{\mathbf{q}}) \in H(\omega) \quad P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) + P^{\text{ext}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) - A(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = 0$$

Movement equations

$$\begin{array}{l} \text{Dynamics VPP} \\ \Leftrightarrow \end{array} \left\{ \begin{array}{l} \text{Mixture} \\ \left\{ \begin{array}{l} \nabla \cdot \boldsymbol{\sigma} + \rho(\mathbf{g} - \boldsymbol{\gamma}_s) = 0 \text{ in } \omega \quad (\text{movement eq.}) \\ \boldsymbol{\sigma}_{skew} = 0 \text{ in } \omega \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \text{ on } \partial\omega \quad (\text{boundary condition}) \end{array} \right. \\ \text{Pore-fluid} \\ \left\{ \begin{array}{l} -\nabla p + \rho_w(\mathbf{g} - \boldsymbol{\gamma}_s) = \mathbf{f}_w \text{ in } \omega \quad (\text{movement eq.}) \\ p = p_{ext} \text{ on } \partial\omega \quad (\text{boundary condition}) \end{array} \right. \end{array} \right.$$

$\forall \omega, (\mathbf{v}_s, \mathbf{q})$: actual velocities

$$\frac{D}{dt} K(\omega, \mathbf{v}_s, \mathbf{q}) = P^{\text{int}}(\omega, \mathbf{v}_s, \mathbf{q}) + P^{\text{ext}}(\omega, \mathbf{v}_s, \mathbf{q})$$

$$\forall \omega, \forall (\hat{\mathbf{v}}_s, \hat{\mathbf{q}}) \in H(\omega) \quad P^{\text{int}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) + P^{\text{ext}}(\omega, \hat{\mathbf{v}}_s, \hat{\mathbf{q}}) = 0$$

Equilibrium equations

$$\begin{array}{l} \text{Quasi-static VPP} \Leftrightarrow \left\{ \begin{array}{l} \text{Mixture} \\ \left\{ \begin{array}{ll} \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = 0 & \text{in } \omega \quad (\text{equilibrium eq.}) \\ \boldsymbol{\sigma}_{skew} = 0 & \text{in } \omega \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} & \text{on } \partial\omega \quad (\text{boundary condition}) \end{array} \right. \\ \text{Pore-fluid} \\ \left\{ \begin{array}{ll} -\nabla p + \rho_w \mathbf{g} = \mathbf{f}_w & \text{in } \omega \quad (\text{equilibrium eq.}) \\ p = p_{ext} & \text{on } \partial\omega \quad (\text{boundary condition}) \end{array} \right. \end{array} \right. \end{array}$$

Archimède' s theorem vs. Terzaghi' s principle

Assume hydrostatic conditions: $\mathbf{f}_w = 0$

Inserting the fluid equilibrium eq. into the mixture equilibrium eq. yields:

Solid matrix equilibrium equation

$$\left\{ \begin{array}{ll} \nabla \cdot \boldsymbol{\sigma}' + \rho' \mathbf{g} = 0 & \text{in } \omega \quad (\text{equilibrium eq.}) \\ \boldsymbol{\sigma}'_{skew} = 0 & \text{in } \omega \\ \boldsymbol{\sigma}' \cdot \mathbf{n} = \mathbf{t} + p_{ext} \mathbf{n} & \text{on } \partial\omega \quad (\text{boundary condition}) \end{array} \right.$$

$$\begin{aligned} \rho' &= \rho - \rho_w \\ &= (1 - n)(\rho_s - \rho_w) \end{aligned}$$

Buyoant mixture density (Archimede, 250 av. J.-C.)

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\mathbf{I}$$

Effective stress (Terzaghi, 1925)

Balance of equations and unknowns

Unknowns		Equations	Dimension	
n	Porosity	$\frac{d^s n}{dt} = (1 - n) \nabla \cdot \mathbf{v}_s$	Solid mass balance eq.	1
p	Pore pressure	$n \frac{d^s \rho_w}{dt} + \rho_w \nabla \cdot \mathbf{v}_s + \nabla \cdot (\rho_w \mathbf{q}) = 0$	Fluid mass balance eq.	1
ρ_w	Fluid density	?	Fluid behaviour	1
$\mathbf{v}_s$	Solid matrix velocity	$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = 0$	Mixture equilibrium eq.	3
$\mathbf{q}$	Pore-fluid velocity	$-\nabla p + \rho_w \mathbf{g} = \mathbf{f}_w$	Pore-fluid equilibrium eq.	3
$\mathbf{f}_w$	Solid/fluid interaction	?	Solid/fluid behaviour	3
$\mathbf{D}$	Strain rate	$\mathbf{D} = \left(\nabla \mathbf{v}_s \right)_{sym}$	Geometric relationship	6
$\boldsymbol{\sigma}'$	Effective stress	?	Solid matrix behaviour	6
$\boldsymbol{\sigma}$	Total stress	$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p \mathbf{I}$	Terzaghi' s principle	6

Balance of energy

Global internal energy: $E^{\text{int}}(\omega) = \int_{\omega} \rho e dx$

Global heat power: $P^{\text{heat}}(\omega) = - \int_{\partial\omega} \mathbf{q}_{\theta} \cdot \mathbf{n} da$

$\mathbf{q}_{\theta}$: Heat flux vector

$$\rho e = \underbrace{(1-n)\rho_s e_s}_{\text{solids}} + \underbrace{n\rho_w e_w}_{\text{fluid}} + \underbrace{\rho e^{\text{mix}}}_{\text{coupling term}} \quad : \text{Mixture internal energy}$$

Balance of energy

$$\frac{D}{dt} E^{\text{int}}(\omega) + \frac{D}{dt} K(\omega) = P^{\text{heat}}(\omega) + P^{\text{ext}}(\omega)$$

$\Leftrightarrow$
kinetic energy theorem

$$\frac{D}{dt} E^{\text{int}}(\omega) = P^{\text{heat}}(\omega) - P^{\text{int}}(\omega)$$

$$\frac{D}{dt} E^{\text{int}}(\omega) = P^{\text{heat}}(\omega) - P^{\text{int}}(\omega)$$

$\Leftrightarrow$
localization
theorem

Assumption: $e^{\text{mix}} = 0$
(no internal energy coupling term)

Mixture energy equation

$$(1-n)\rho_s \frac{d^s}{dt} e_s + n\rho_w \frac{d^w}{dt} e_w + \nabla \cdot \mathbf{q}_\theta = \mathbf{f}_w \cdot \mathbf{q} + \boldsymbol{\sigma} : \mathbf{D} - p \nabla \cdot \mathbf{q}$$

$\Leftrightarrow$
Inserting the fluid mass balance equation

$$\underbrace{\left[(1-n)\rho_s \frac{d^s}{dt} e_s - \boldsymbol{\sigma}' : \mathbf{D} \right]}_{\text{solids}} + n \underbrace{\left[\rho_w \frac{d^w}{dt} e_w - \frac{p}{\rho_w} \frac{d^w \rho_w}{dt} \right]}_{\text{pore fluid}} + \underbrace{\nabla \cdot \mathbf{q}_\theta}_{\text{heat}} = \underbrace{\mathbf{f}_w \cdot \mathbf{q}}_{\text{solid/fluid interaction}}$$

Imbalance of entropy

Global entropy:
$$S(\omega) = \int_{\omega} \rho s dx$$

$$\rho s = \underbrace{(1-n)\rho_s s_s}_{\text{solids}} + \underbrace{n\rho_w s_w}_{\text{fluid}} + \underbrace{\rho s^{mix}}_{\text{coupling term}} : \text{Mixture entropy}$$

Assumption: thermal equilibrium of each phase, having therefore the same absolute temperature

Imbalance of entropy

$$\frac{D}{dt} S(\omega) \geq - \int_{\partial\omega} \frac{1}{T} \mathbf{q}_{\theta} \mathbf{n} da$$

T **Absolute temperature**

Assumption: $s^{mix} = 0$
(no internal entropy coupling term)

Volume intrinsic dissipation

$$\Phi_m \stackrel{def}{=} T \left[(1-n) \rho_s \frac{d^s}{dt} s_s + n \rho_w \frac{d^w}{dt} s_w \right] + \nabla \cdot \mathbf{q}_\theta$$

Volume heat dissipation

$$\Phi_\theta \stackrel{def}{=} -\frac{1}{T} \mathbf{q}_\theta \cdot \nabla T$$

$$\frac{D}{dt} S(\omega) \geq - \int_{\partial\omega} \frac{1}{T} \mathbf{q}_\theta \mathbf{n} da$$

$\Leftrightarrow$
localization
theorem

$$\Phi_m + \Phi_\theta \geq 0$$

Dissipations

$$\Psi_s = e - s_s T \quad : \text{Solid matrix free energy}$$

$$\Psi_w = e_w - s_w T \quad : \text{Fluid matrix free energy}$$

Inserting Φ_m i, as well as Ψ_s and Ψ_w in the energy equation yields

$$\Phi_m = \underbrace{\mathbf{f}_w \cdot \mathbf{q}}_{\text{solid/fluid interaction}} + \underbrace{\left[\boldsymbol{\sigma}' : \mathbf{D} - (1-n)\rho_s s_s \frac{d^s}{dt} T - (1-n)\rho_s \frac{d^s}{dt} \Psi_s \right]}_{\text{solid matrix}} + n \underbrace{\left[\frac{p}{\rho_w} \frac{d^w \rho_w}{dt} - \rho_w s_w \frac{d^w}{dt} T - \rho_w \frac{d^w}{dt} \Psi_w \right]}_{\text{pore fluid}}$$

State variables

State variables

T Temperature

ρ_w Fluid density

$\boldsymbol{\varepsilon} = \left(\nabla \mathbf{u}_s \right)_{sym}$ Elastic small strain

Assumptions

$$\Psi_w \equiv \Psi_w(T, \rho_w) \quad \Psi_s \equiv \Psi_s(T, \boldsymbol{\varepsilon})$$

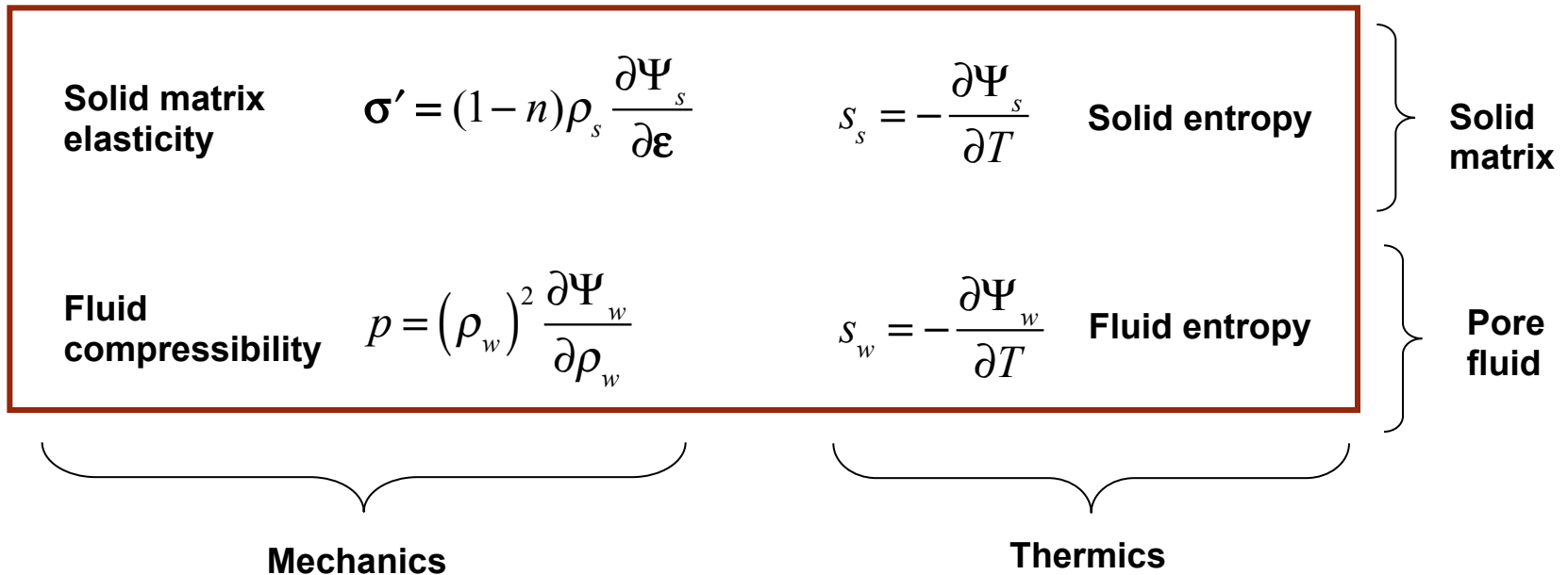
(Matrix elasticity, fluid compressibility and thermal effects)

$$\mathbf{D}(\mathbf{v}_s) = \left(\nabla \mathbf{v}_s \right)_{sym} = \left(\nabla \left[\frac{d^s}{dt} \mathbf{u}_s \right] \right)_{sym} = \frac{d^s}{dt} \left(\nabla \mathbf{u}_s \right)_{sym} = \frac{d^s}{dt} \boldsymbol{\varepsilon} \quad (\text{Small strains})$$

Therefore

$$\begin{aligned} \Phi_m = \underbrace{\mathbf{f}_w \cdot \mathbf{q}}_{\text{solid/fluid interaction}} &+ \underbrace{\left[\left(\boldsymbol{\sigma}' - (1-n)\rho_s \frac{\partial \Psi_s}{\partial \boldsymbol{\varepsilon}} \right) : \mathbf{D} + (1-n)\rho_s \left[-s_s - \frac{\partial \Psi_s}{\partial T} \right] \frac{d^s}{dt} T \right]}_{\text{solid matrix}} \\ &+ n \underbrace{\left[\left(\frac{p}{\rho_w} - \rho_w \frac{\partial \Psi_w}{\partial \rho_w} \right) \frac{d^w \rho_w}{dt} + \rho_w \left(-s_w - \frac{\partial \Psi_w}{\partial T} \right) \frac{d^w}{dt} T \right]}_{\text{pore fluid}} \end{aligned}$$

By usual reasoning, the state laws are as follows



The intrinsic dissipation reduces to the solid/fluid interaction

$$\Phi_m = \mathbf{f}_w \cdot \mathbf{q}$$

Energy equation and dissipations

Finally, the energy equation - which is not yet the heat equation - reads

$$T \left[(1-n) \rho_s \frac{d^s}{dt} s_s + n \rho_w \frac{d^w}{dt} s_w \right] + \nabla \cdot \mathbf{q}_\theta = \mathbf{f}_w \cdot \mathbf{q}$$

The dissipations are

$$\Phi_m = \mathbf{f}_w \cdot \mathbf{q} \qquad \Phi_\theta \stackrel{def}{=} - \frac{1}{T} \mathbf{q}_\theta \cdot \nabla T$$

A sufficient - but not necessary condition to fulfill the imbalance entropy is

$$\Phi_m \geq 0 \qquad \Phi_\theta \geq 0$$

Balance of equations and unknowns: poro-mechanics

Unknowns		Equations	Dimension	
n	Porosity	$\frac{d^s n}{dt} = (1 - n) \nabla \cdot \mathbf{v}_s$	Solid mass balance eq.	1
p	Pore pressure	$n \frac{d^s \rho_w}{dt} + \rho_w \nabla \cdot \mathbf{v}_s + \nabla \cdot (\rho_w \mathbf{q}) = 0$	Fluid mass balance eq.	1
ρ_w	Fluid density	$p = (\rho_w)^2 \frac{\partial \Psi_w}{\partial \rho_w}$	Fluid behaviour	1
$\mathbf{u}_s$	Solid matrix displacement	$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = 0$	Mixture equilibrium eq.	3
$\mathbf{q}$	Pore-fluid velocity	$-\nabla p + \rho_w \mathbf{g} = \mathbf{f}_w$	Pore-fluid equilibrium eq.	3
$\mathbf{f}_w$	Solid/fluid interaction	$\mathbf{f}_w \cdot \mathbf{q} \geq 0$	Solid/fluid dissipation	3
$\boldsymbol{\varepsilon}$	Small strain	$\boldsymbol{\varepsilon} = (\nabla \mathbf{u}_s)_{sym}$	Geometric relationship	6
$\boldsymbol{\sigma}'$	Effective stress	$\boldsymbol{\sigma}' = (1 - n) \rho_s \frac{\partial \Psi_s}{\partial \boldsymbol{\varepsilon}}$	Solid matrix behaviour	6
$\boldsymbol{\sigma}$	Total stress	$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p \mathbf{I}$	Terzaghi' s principle	6

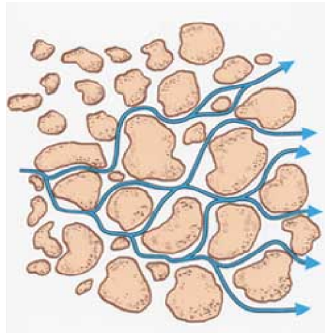
Balance of equations and unknowns: thermics

Unknowns		Equations	Dimension	
s_s	Solid entropy	$s_s = -\frac{\partial \Psi_s}{\partial T}$	Solid matrix behaviour.	1
s_w	Fluid entropy	$s_w = -\frac{\partial \Psi_w}{\partial T}$	Fluid matrix behaviour.	1
T	Absolute temperature	$T \left[(1-n)\rho_s \frac{d^s}{dt} s_s + n\rho_w \frac{d^w}{dt} s_w \right] + \nabla \cdot \mathbf{q}_\theta = \mathbf{f}_w \cdot \mathbf{q}$	Energy equation	1
$\mathbf{q}_\theta$	Heat vector		Thermal dissipation	1
		$-\frac{1}{T} \mathbf{q}_\theta \cdot \nabla T \geq 0$		

Milieux poreux

Poro-Mécanique

Couplages hydro-mécanique



Thermal diffusion, mass diffusion
Fourier's and Darcy's laws
Heat and seepage equations

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Balance of equations and unknowns: deformations

Unknowns		Equations	Dimension	
n	Porosity	$\frac{d^s n}{dt} = (1 - n) \nabla \cdot \mathbf{v}_s$	Solid mass balance eq.	1
$\mathbf{u}_s$	Solid matrix displacement	$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = 0$	Mixture equilibrium eq.	3
$\boldsymbol{\varepsilon}$	Small strain	$\boldsymbol{\varepsilon} = \left(\nabla \mathbf{u}_s \right)_{sym}$	Geometric relationship	6
$\boldsymbol{\sigma}'$	Effective stress	$\boldsymbol{\sigma}' = (1 - n) \rho_s \frac{\partial \Psi_s}{\partial \boldsymbol{\varepsilon}}$	Solid matrix behaviour	6
$\boldsymbol{\sigma}$	Total stress	$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p \mathbf{I}$	Terzaghi' s principle	6

The total stress is defined by the static equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad \left(\text{tr } \boldsymbol{\sigma} < 0 \Leftrightarrow \text{compression} \right)$$

The effective stress is defined as follows (Terzaghi, 1925)

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\mathbf{I}$$

The effective stress differs than the total stress only on the isotropic part

$$\frac{1}{3} \text{tr } \boldsymbol{\sigma}' = \frac{1}{3} \text{tr } \boldsymbol{\sigma} + p$$

$$\boldsymbol{\sigma}'^d = \boldsymbol{\sigma}^d$$

The behaviour law of the solid matrix involves the effective stress, for example, isotropic linear elasticity in small strains

$$\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon} + \text{tr } \boldsymbol{\varepsilon} \left(\chi - \frac{2G}{3} \right) \mathbf{I}$$

$$G = \frac{E}{2(1+\nu)}$$

(shear modulus)

$$\chi = \frac{E}{3(1-2\nu)}$$

(bulk modulus)

Balance of equations and unknowns: thermics

Unknowns		Equations	Dimension	
s_s	Solid entropy	$s_s = -\frac{\partial \Psi_s}{\partial T}$	Solid matrix behaviour.	1
s_w	Fluid entropy	$s_w = -\frac{\partial \Psi_w}{\partial T}$	Fluid matrix behaviour.	1
T	Absolute temperature	$T \left[(1-n)\rho_s \frac{d^s}{dt} s_s + n\rho_w \frac{d^w}{dt} s_w \right] + \nabla \cdot \mathbf{q}_\theta = \mathbf{f}_w \cdot \mathbf{q}$	Energy equation	1
$\mathbf{q}_\theta$	Heat vector		Thermal dissipation	1
		$-\frac{1}{T} \mathbf{q}_\theta \cdot \nabla T \geq 0$		

Specific heat of solids and fluid

Solid matrix

$$\left\{ \begin{array}{l} s_s = -\frac{\partial \Psi_s}{\partial T} \\ \Psi_s \equiv \Psi_s(T, \epsilon) \end{array} \right. \Rightarrow T \frac{d^s}{dt} s_s = c_s \frac{d^s}{dt} T - T \left(\frac{\partial \Psi_s}{\partial T \partial \epsilon} \right) \frac{d^s}{dt} \epsilon$$

$c_s = -T \frac{\partial^2 \Psi_s}{\partial T^2}$ **Specific heat of the solid matrix**

Pore fluid

$$\left\{ \begin{array}{l} s_w = -\frac{\partial \Psi_w}{\partial T} \\ \Psi_w \equiv \Psi_w(T, \rho_w) \end{array} \right. \Rightarrow T \frac{d^w}{dt} s_w = c_w \frac{d^w}{dt} T - T \left(\frac{\partial \Psi_w}{\partial T \partial \rho_w} \right) \frac{d^w}{dt} \rho_w$$

$c_w = -T \frac{\partial^2 \Psi_w}{\partial T^2}$ **Specific heat of the pore fluid**

Specific heat of the porous medium

Solid matrix

Pore fluid

$$\left\{ \begin{array}{ll} s_s = -\frac{\partial \Psi_s}{\partial T} & s_w = -\frac{\partial \Psi_w}{\partial T} \\ \Psi_s \equiv \Psi_s(T, \boldsymbol{\varepsilon}) & \Psi_s \equiv \Psi_s(T, \rho_w) \end{array} \right. \Rightarrow$$

$$T \left[(1-n) \rho_s \frac{d^s}{dt} s_s + n \rho_w \frac{d^w}{dt} s_w \right] = \rho c \frac{d^s}{dt} T + \rho_w c_w \nabla T \cdot \mathbf{q} - T r_\Psi$$

$$\rho c = \underbrace{(1-n) \rho_s c_s}_{\text{solids}} + \underbrace{n \rho_w c_w}_{\text{fluid}} \quad \text{Specific heat of the mixture}$$

$$r_\Psi = (1-n) \rho_s \underbrace{\left(\frac{\partial \Psi_s}{\partial T \partial \boldsymbol{\varepsilon}} \right)}_{\text{solid dilatation}} \frac{d^s}{dt} \boldsymbol{\varepsilon} + n \rho_w \underbrace{\left(\frac{\partial \Psi_s}{\partial T \partial \rho_w} \right)}_{\text{fluid dilatation}} \frac{d^w}{dt} \rho_w \quad \text{Volume power due to dilatation}$$

Heat equation in the porous medium

$$\underbrace{\rho c \frac{d^s T}{dt} + \nabla \cdot \mathbf{q}_\theta}_{\text{Heat diffusion in the porous medium}} = \underbrace{\mathbf{f}_w \cdot \mathbf{q}}_{\substack{\text{Intrinsic dissipation} \\ \text{(elastic solid matrix)}}} + \underbrace{Tr_\Psi}_{\substack{\text{Power due to} \\ \text{dilatation}}} - \underbrace{\rho_w c_w \nabla T \cdot \mathbf{q}}_{\substack{\text{Heat transport by} \\ \text{seepage} \\ \text{(advection)}}}$$

Often neglected

Fourier's law

$$\Phi_\theta \geq 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \mathbf{q}_\theta = -\frac{\partial}{\partial \nabla T} \Omega_\theta(\nabla T, \text{state variables}) \\ \Omega_\theta = \frac{1}{2} (\nabla T) \cdot \boldsymbol{\kappa} \cdot (\nabla T) \\ \boldsymbol{\kappa} \quad \text{Symmetric definite positive order two tensor} \end{array} \right.$$

This is the Fourier's law

$$\mathbf{q}_\theta = -\boldsymbol{\kappa} \cdot \nabla T$$

Here, we have

state variables = (T, ρ_w, ϵ)

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}(\text{state variables})$$

Thermal conductivity of the porous medium

Unknowns	Equations	Dimension
p Pore pressure	$n \frac{d^s \rho_w}{dt} + \rho_w \nabla \cdot \mathbf{v}_s + \nabla \cdot (\rho_w \mathbf{q}) = 0$	Fluid mass balance eq. 1
ρ_w Fluid density	$p = (\rho_w)^2 \frac{\partial \Psi_w}{\partial \rho_w}$	Fluid behaviour 1
$\mathbf{q}$ Pore-fluid velocity	$-\nabla p + \rho_w \mathbf{g} = \mathbf{f}_w$	Pore-fluid equilibrium eq. 3
$\mathbf{f}_w$ Solid/fluid interaction	$\mathbf{f}_w \cdot \mathbf{q} \geq 0$	Solid/fluid dissipation 3

Water state law

The water state law can be found in many books

For current applications with FEM codes, the following state law is often used

$$\rho_w = \rho_w^{ref} \exp\left(\frac{p}{\chi_w}\right)$$

$$\chi_w \approx 2 \text{ GPa} \quad \text{Bulk water modulus}$$

$$\mathbf{f}_w \cdot \mathbf{q} \geq 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \mathbf{q} = \frac{\partial}{\partial \mathbf{f}_w} \Omega_{fs}(\mathbf{f}_w, \text{state variables}) \\ \Omega_{fs} = \frac{1}{2} \mathbf{f}_w \cdot \boldsymbol{\kappa}_{fs} \cdot \mathbf{f}_w \\ \boldsymbol{\kappa}_{fs} \quad \text{Symmetric definite positive order two tensor} \end{array} \right.$$

This is a diffusion law

$$\mathbf{q} = \boldsymbol{\kappa}_{fs} \cdot \mathbf{f}_w$$

Here, we have

state variables = (T, ρ_w, ϵ)

$\boldsymbol{\kappa}_{fs} = \boldsymbol{\kappa}_{fs}(\text{state variables})$ Hydraulic conductivity of the porous medium

Darcy' s law (mechanics-like form)

Actually, the hydraulic conductivity may be written as

$$\kappa_{fs} = \frac{1}{\rho_w \eta_w(T)} \Lambda_s(T, \epsilon)$$

Λ_s (m²) Geometric permeability of the solid matrix

η_s (m²/s) Kinematic fluid viscosity

Inserting the equilibrium eq. of the pore fluid yields

Darcy' s law $\mathbf{q} = \frac{1}{\rho_w \eta_w} \Lambda_s \cdot [-\nabla p + \rho_w \mathbf{g}]$

Darcy' s law (engineering-like form)

The hydraulic conductivity may also be written as

$$\kappa_{fs} = \frac{1}{\gamma_w} \mathbf{K} \quad \text{where} \quad \gamma_w = \rho_w g$$

$\mathbf{K}$ (m/s) Hydraulic permeability of the porous medium

g (m/s²) Gravitational constant

The hydraulic head is defined as follows

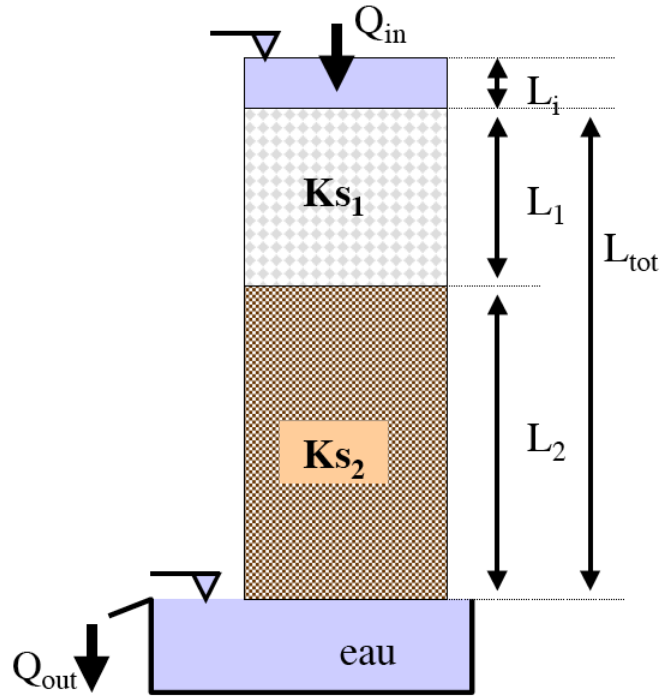
$$H = \frac{p - p_{atm}}{\gamma_w} + z \text{ (m)}$$

Inserting the equilibrium eq. of the pore fluid yields

Darcy' s law $\mathbf{q} = -\mathbf{K} \cdot \nabla H$

Type of soil	k (m/s)
gravel	$10^{-3} - 10^{-1}$
sand	$10^{-6} - 10^{-3}$
silt	$10^{-8} - 10^{-6}$
clay	$10^{-10} - 10^{-8}$

Where the (hidden) assumptions are: 1) incompressible fluid, 2) $p_{atm} = 0$ (reference pressure)



$$K_{s\,eff} = \frac{L_1 + L_2}{\frac{L_1}{K_{s1}} + \frac{L_2}{K_{s2}}}$$

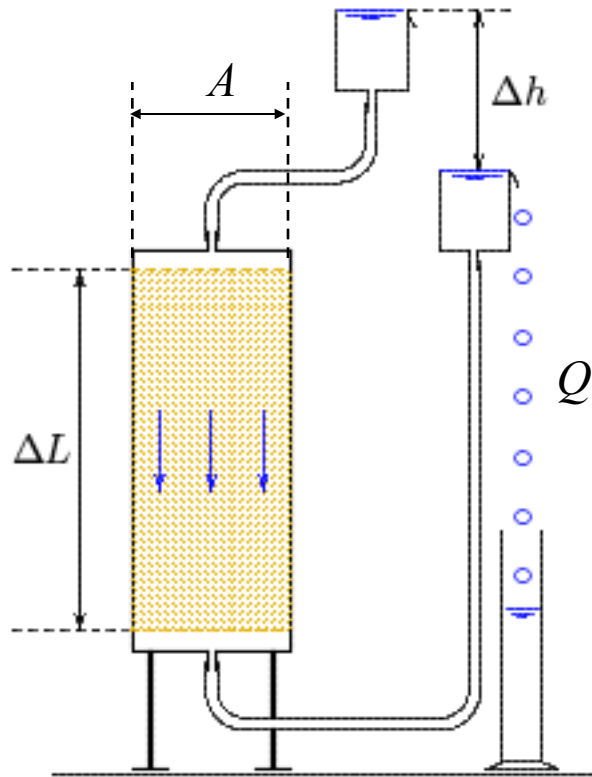
$$K_{eff,z} = \frac{\sum_j L_j}{\sum_j \frac{L_j}{K_j}}$$

**perpendiculairement aux strates:
moyenne harmonique**

$$K_{eff,x} = \frac{\sum_j K_j L_j}{\sum_j L_j}$$

**parallèlement aux strates:
moyenne géométrique**

Testing: constant head permeability test



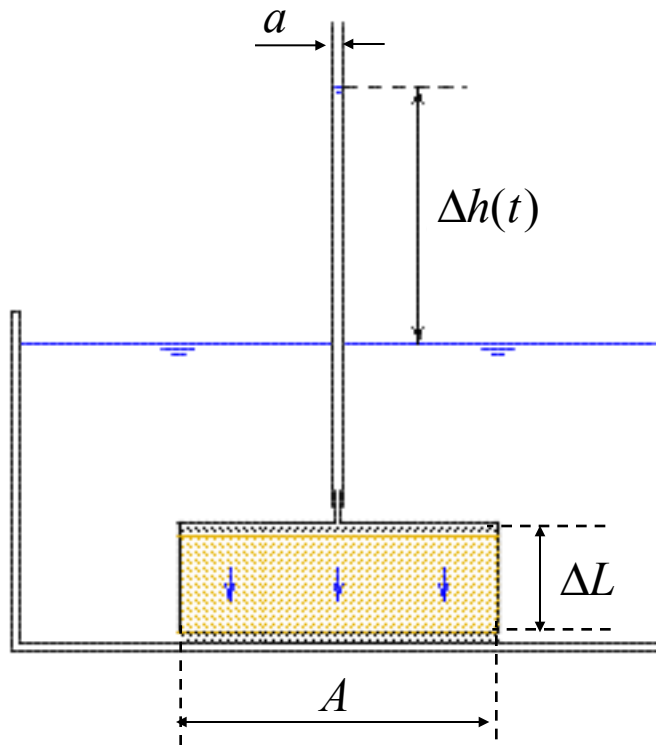
Δh : constant head drop (m)

Q : total discharge (m^3/s)

A : area of soil sample (m^2)

ΔL : length of soil sample (m)

Testing: falling head test



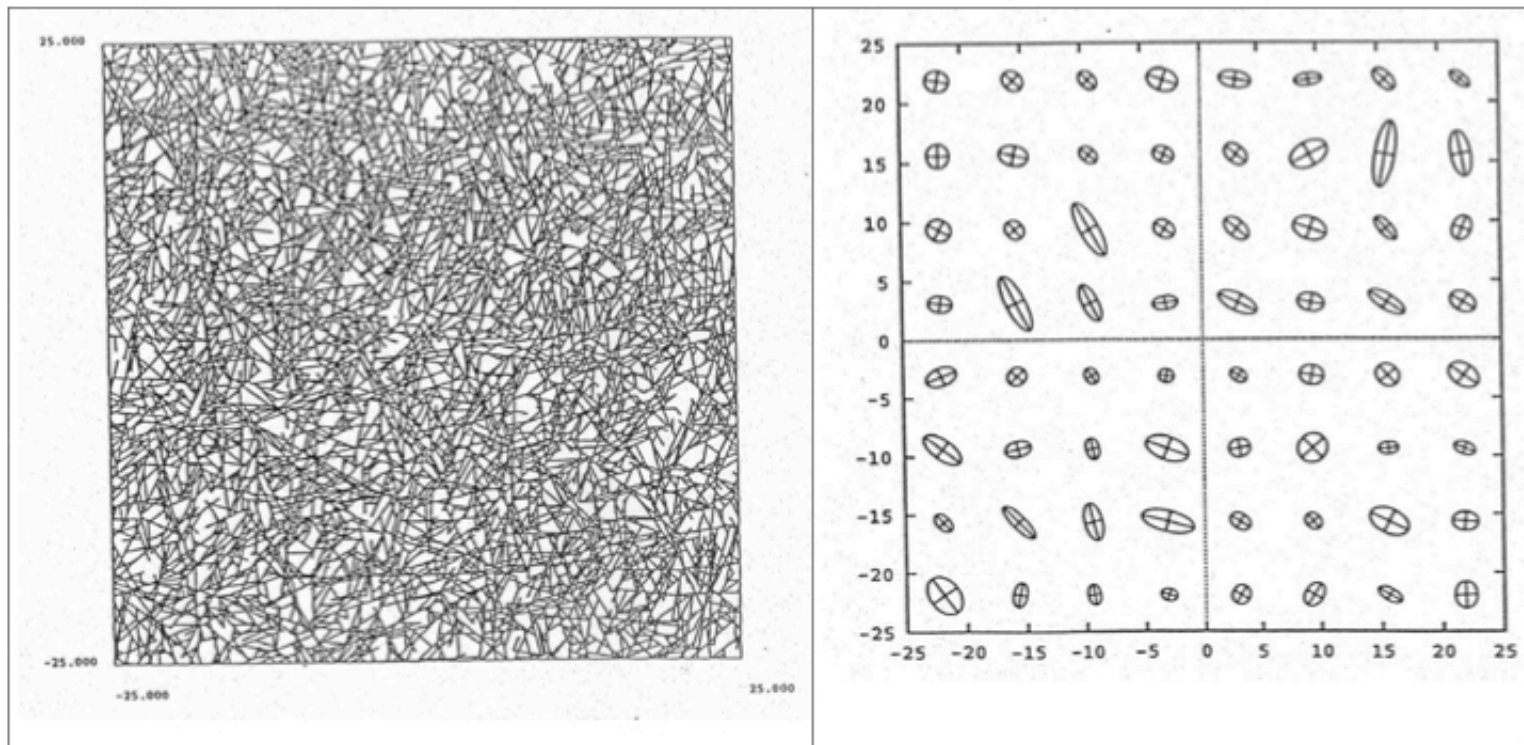
$\Delta h(t)$: variable head drop (m)

$Q(t)$: total discharge (m^3/s)

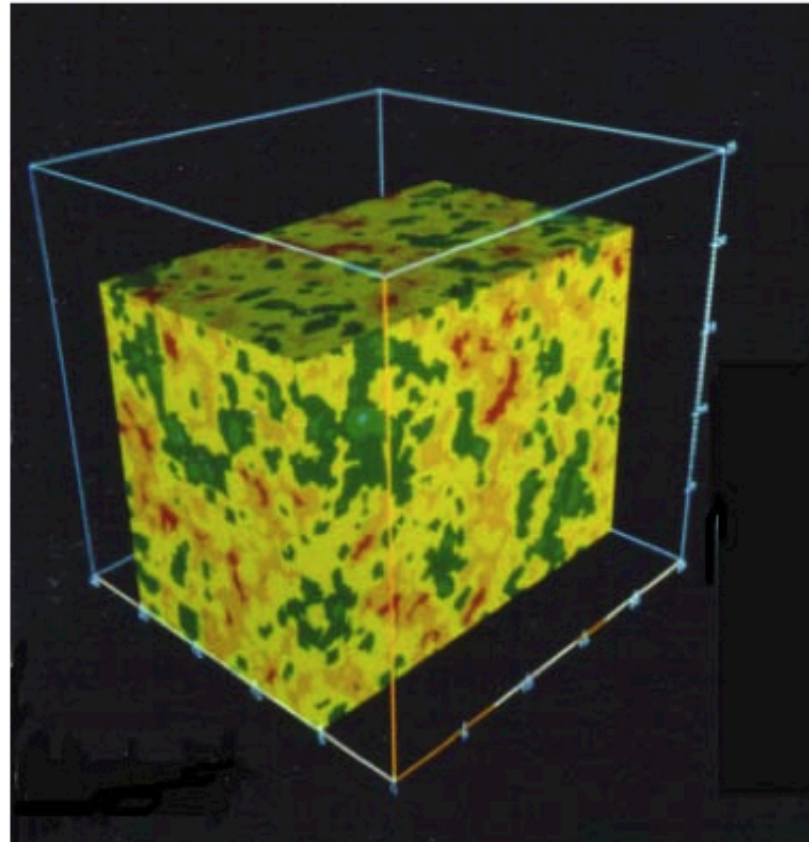
a : area of tube above soil (m^2)

A : area of soil sample (m^2)

ΔL : length of soil sample (m)



Réseau plan de fractures (environ 7000 fractures) et les perméabilités tensorielles équivalentes du domaine divisé en 64 sous-blocs (les K_{ij} sont représentées par les ellipses d'anisotropie). [source : R.A. et al. 1994]



PERMEABILITE D'UN MASSIF POREUX ALEATOIRE GENERE NUMERIQUEMENT EN 3D
(CHAMP ALEATOIRE AUTOCORRELE A STRUCTURE ISOTROPE)

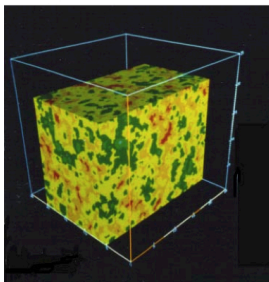
Résultats théorique et numérique connus

**Solide
rigide et imperméable
+
Ecoulement de Navier-Stokes
dans les pores**

$$\tilde{\nabla} \cdot \tilde{\mathbf{v}}_w = 0$$

$$\rho_w \left(\frac{\partial \tilde{\mathbf{v}}_w}{\partial t} + \tilde{\nabla} \tilde{\mathbf{v}}_w \cdot \tilde{\mathbf{v}}_w \right) = \tilde{\nabla} \tilde{\boldsymbol{\sigma}}$$

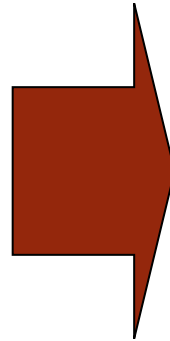
$$\tilde{\boldsymbol{\sigma}} = -\tilde{p}\mathbf{I} + \rho_w \eta_w \left(\tilde{\nabla} \tilde{\mathbf{v}}_w \right)_{sym}$$



PERMEABILITE D'UN MASSIF POREUX ALEATOIRE GENERE NUMERIQUEMENT EN 3D
(CHAMP ALEATOIRE AUTOCORRELLE A STRUCTURE ISOTROPE)

$$\underbrace{\tilde{l}}_{\text{pore scale}} \ll \underbrace{L}_{\text{REV scale}}$$

**Homogeneisation
(changement d'échelle
Micro $\rightarrow$ macro)**



$$\varepsilon = \frac{\tilde{l}}{L} \ll 1$$

**Matrice solide rigide
+
Ecoulement de Darcy**

$$\nabla \cdot \mathbf{q} = 0$$

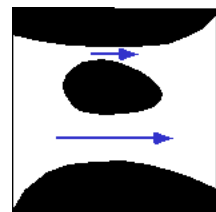
$$\mathbf{q} = -\frac{1}{\rho_w \eta_w} \Lambda_s \cdot \nabla p$$

avec

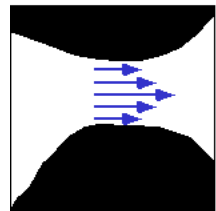
$$\mathbf{q} = \left\langle \chi_{pore} \tilde{\mathbf{v}}_w \right\rangle_{def}$$

p : terme du 1er ordre en ε

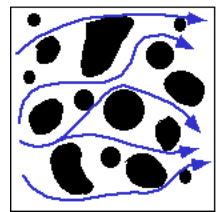
Origines microscopique de la dissipation



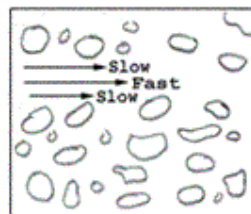
different
pore sizes



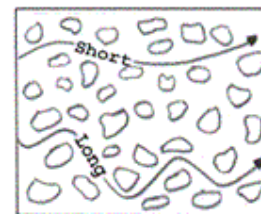
different
velocities
in a pore



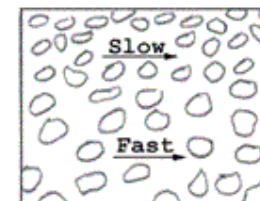
different
pathways
around the
grains



Friction in Pore

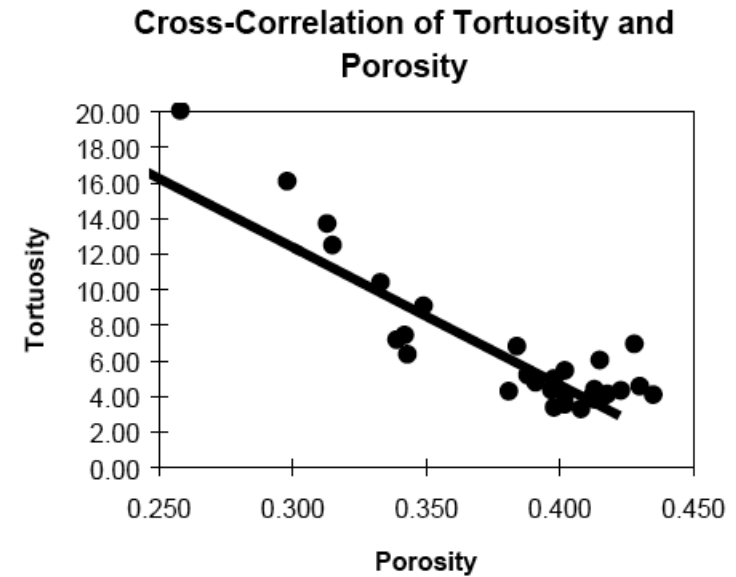
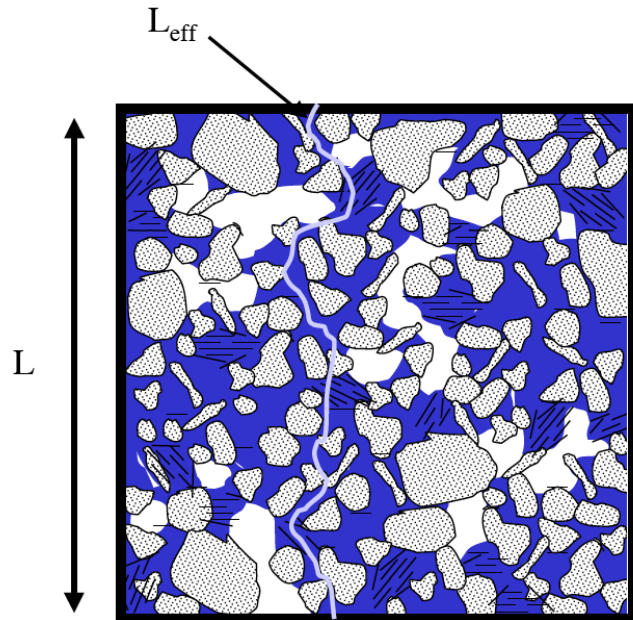


Path Length



Pore Size

Fine
Coarse



$$\tau \approx 35.6 - 77.3n$$

Tortuosity

$$\tau = \left(\frac{L_{eff}}{L} \right)^2$$

Rough approximation

$$\tau \approx \frac{25}{12} \approx 2$$

The porous medium viewed as a bundle of tubes

Hagen-Poiseuille's law

$$\langle \tilde{v}_w \rangle_{tubes} = \frac{R^2}{8\rho_w \eta_w} \frac{\Delta p}{L_{eff}}$$

Assumption

$$\langle \tilde{v}_w \rangle_{pore} = \frac{L}{L_{eff}} \langle \tilde{v}_w \rangle_{tubes}$$

Darcy's law

$$q = \frac{\lambda}{\rho_w \eta_w} \frac{\Delta p}{L}$$

$$q = n \langle \tilde{v}_w \rangle_{pore}$$

$$= n \frac{L}{L_{eff}} \langle \tilde{v}_w \rangle_{tubes}$$

$$= \frac{nR^2}{8\rho_w \eta_w \tau} \left(\frac{\Delta p}{L} \right)$$

$$\lambda = \frac{nR^2}{8\tau}$$

$$\lambda = \frac{n^3}{2\tau(a_v)^2}$$

Hydraulic radius

Tube

$$R_h = \frac{R}{2}$$

Porous medium

$$R_h = \frac{n}{a_v}$$

$$R = \frac{2n}{a_v}$$

The Kozeny-Karman relationship for granular materials

General relationship

$$\lambda = \frac{n^3}{2\tau(a_v)^2}$$

n : porosity

a_v : volume specific pore surface

τ : tortuosity

The Kozeny-Karman relationship for granular materials

$$\tau \approx \frac{25}{12}$$

$$a_v = (1-n) \frac{6}{D_{grain}}$$

$$\lambda = \frac{n^3 D_{grain}^2}{150(1-n)^2}$$



$$\lambda = \lambda_{ref} \left(\frac{n}{n_{ref}} \right)^3 \left(\frac{1-n_{ref}}{1-n} \right)^2$$

General relationship (assuming pore surface $\approx$ solid surface)

$$\lambda = \frac{n^3 d_{eq}^2}{2\tau(1-n)^2}$$

n : porosity

$a_{w_{solids}}$: mass specific solid surface

τ : tortuosity

$d_{eq} = (\rho_s a_{w_{solids}})^{-1}$: equivalent grain size



$$\lambda = \lambda_{ref} \left(\frac{n}{n_{ref}} \right)^3 \left(\frac{1-n_{ref}}{1-n} \right)^2$$

More accurate descriptions can be found in petrophysics, relating permeabilities and porosities to others physical quantities (like the cation exchange capacity, or the electrical conductivities)

In brief: Deformations

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = 0 \quad (\text{Total equilibrium eq.})$$

$$\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon} + \text{tr} \boldsymbol{\varepsilon} \left(\chi - \frac{2G}{3} \right) \mathbf{I} \quad (\text{Pore fluid state law})$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I} \quad (\text{Terzaghi's principle})$$

Constitutive laws

Material informations:

χ	Bulk solid matrix modulus
G	Shear solid matrix modulus

Coupling:

p Pore-pressure

T Temperature (For conciseness, strain due to dilatation is not explicated)

In brief: Seepage equation

$$n \frac{d^s \rho_w}{dt} + \rho_w \nabla \cdot \mathbf{v}_s + \nabla \cdot (\rho_w \mathbf{q}) = 0 \quad (\text{Pore fluid mass balance})$$

$$\rho_w = \rho_w^{ref} \exp\left(\frac{p}{\chi_w}\right) \quad (\text{Pore fluid state law})$$

$$\mathbf{q} = \frac{1}{\rho_w \eta_w} \Lambda_s \cdot [-\nabla p + \rho_w \mathbf{g}] \quad (\text{Darcy's law})$$

Constitutive laws

Material informations:

χ_w	Bulk water modulus
$\rho_w \eta_w$	Water viscosity
Λ	Geometric permeability

Coupling: $(n, \mathbf{v}_s)$ Solid matrix deformation on the mass balance $(\nabla \cdot \mathbf{v}_s)$ and the permeability Λ

T Temperature on ρ_w, η_w, χ_w

In brief: Heat equation

$$\rho c \frac{d^s T}{dt} + \nabla \cdot \mathbf{q}_\theta + \rho_w c_w \nabla T \cdot \mathbf{q} = 0 \quad (\text{Energy balance})$$

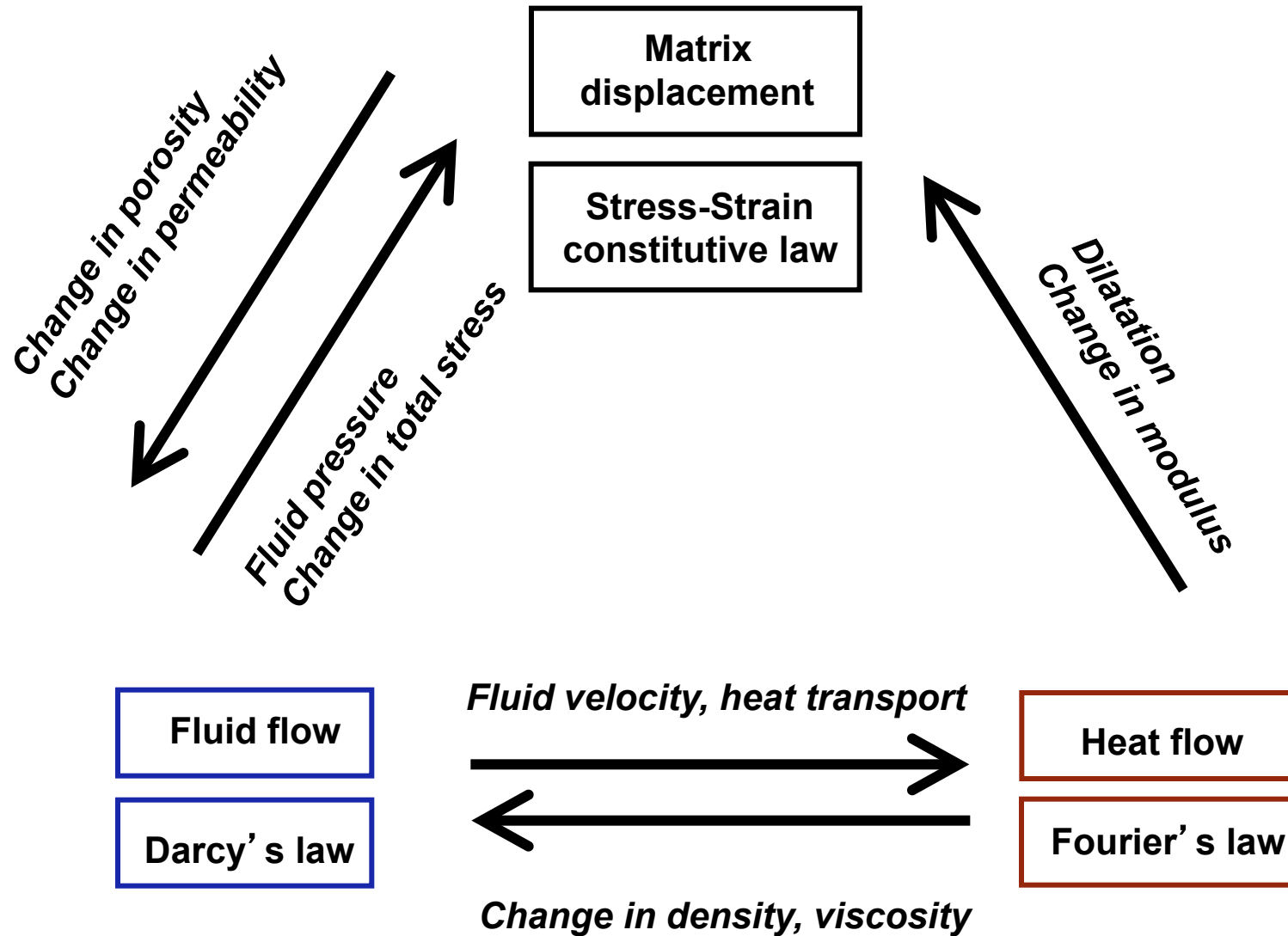
$$\mathbf{q}_\theta = -\kappa \cdot \nabla T \quad (\text{Fourier's law}) \quad \left. \vphantom{\mathbf{q}_\theta} \right\} \text{Constitutive law}$$

Material informations:

$$\rho c = \underbrace{(1-n)\rho_s c_s}_{\text{solids}} + \underbrace{n\rho_w c_w}_{\text{fluid}} \quad \text{Specific heat of the mixture}$$

$$\kappa = \kappa(\text{state variables}) \quad \text{Thermal conductivity of the porous medium}$$

Coupling: $\mathbf{q}$ heat transport by seepage flow (advection)



Couplages THMC: transport de particules

transport et transfert des polluants

$$\text{bilan de masse: } \text{div}(\underline{f} + \underline{f} + \underline{f}) + \partial_t m = 0$$

Diffusion
moléculaire

$$\underline{f} = -\underline{D}(c) \underline{\text{grad}} c$$

Convection

$$\underline{f} = \underline{V}_r c$$

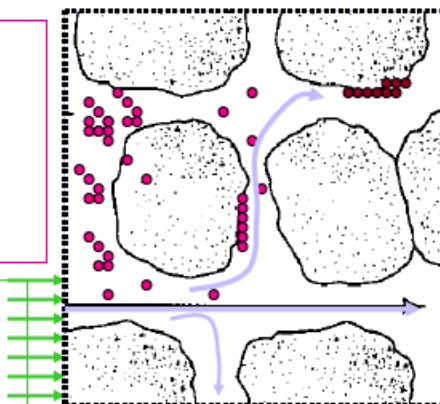
$\underline{V}_r$: loi de Darcy

Dispersion

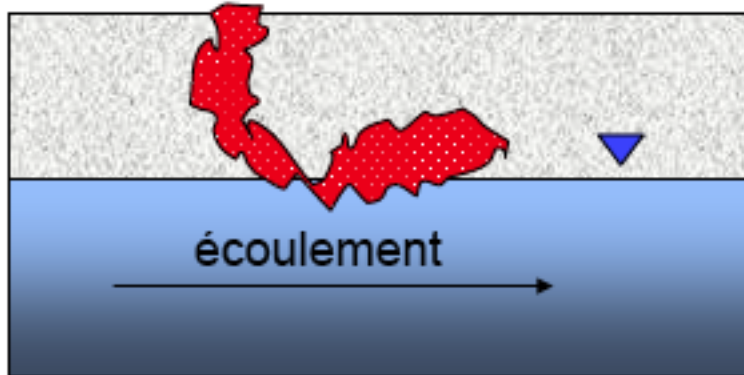
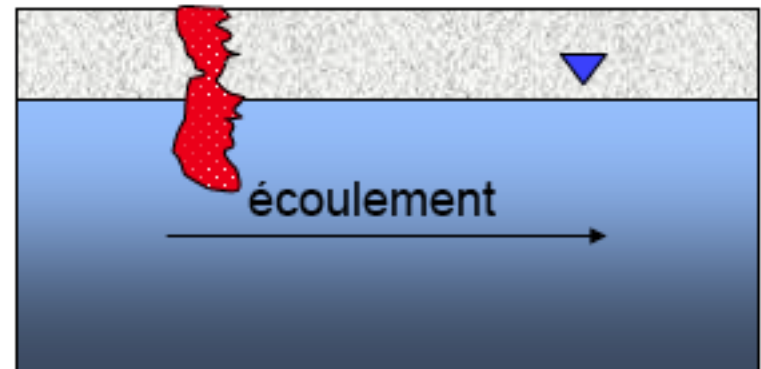
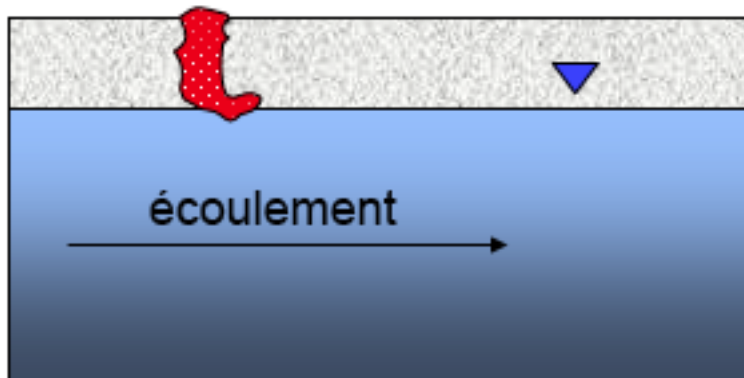
$$\underline{f} = \underline{D}_m(\underline{V}) \underline{\text{grad}} c$$

sorption,
atténuation,
transformation

$$\partial_t m$$



42



LNAPL
(hydrocarbures)

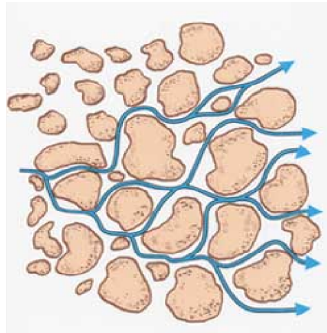
DNAPLS
(solvants chlorés)

38

Milieux poreux

Poro-Mécanique

Couplages hydro-mécanique



Elastoplasticité
Mohr-Coulomb
Cam-Clay

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Notations

Consider any order two-tensor $(\mathbf{a}, \mathbf{c}, \mathbf{b})$

Identity order-two tensor: $\mathbf{I} \underset{def}{/} \left\{ \forall \mathbf{a}, \mathbf{I} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{I} \right\}$

Trace operator: $\text{tr } \mathbf{a} \underset{def}{=} \mathbf{I} : \mathbf{a} = \mathbf{a} : \mathbf{I}$

Tensorial products

$\times \underset{def}{/} \left\{ \forall (\mathbf{a}, \mathbf{c}, \mathbf{b}) \text{ order two tensors, } [\mathbf{a} \times \mathbf{b}] : \mathbf{c} = (\mathbf{b} : \mathbf{c}) \mathbf{a} = \mathbf{c} : [\mathbf{b} \times \mathbf{a}] \right\}$

$\otimes \underset{def}{/} \left\{ \forall (\mathbf{a}, \mathbf{c}, \mathbf{b}) \text{ order two tensors, } [\mathbf{a} \otimes \mathbf{b}] : \mathbf{c} = \mathbf{a} \cdot \mathbf{c} \cdot \mathbf{b} = \mathbf{c} : [\mathbf{a}^T \otimes \mathbf{b}^T] \right\}$

Decomposition into spheric and deviatoric parts

$$\mathbf{a} = \underbrace{a_m \mathbf{I}}_{\text{spheric part}} + \underbrace{\mathbf{a}^d}_{\text{deviatoric part}}$$

$$a_m \mathbf{I} = \underbrace{\left[\frac{1}{3} \mathbf{I} \times \mathbf{I} \right]}_{\text{spheric projector}} : \mathbf{a} = \left(\frac{1}{3} \text{tr } \mathbf{a} \right) \mathbf{I}$$

$$\mathbf{a}^d = \underbrace{\left[\mathbf{I} \otimes \mathbf{I} - \frac{1}{3} \mathbf{I} \times \mathbf{I} \right]}_{\text{deviatoric projector}} : \mathbf{a} = \mathbf{a} - \left(\frac{1}{3} \text{tr } \mathbf{a} \right) \mathbf{I}$$

Invariants of order-two tensors

Eigenvalues of any symmetric order two tensor $\mathbf{a}$

$$\det(\mathbf{a} - \lambda \mathbf{I}) = 0 \quad \Leftrightarrow \quad \lambda = a_m + \lambda^d, \quad \det(\mathbf{a}^d - \lambda^d \mathbf{I}) = 0$$

$$\det(\mathbf{a}^d - \lambda^d \mathbf{I}) = 0 \quad \Leftrightarrow \quad (\lambda^d)^3 - \frac{1}{2}(\mathbf{a}^d : \mathbf{a}^d) \lambda^d - \det(\mathbf{a}^d) = 0$$

Trigo tools: $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ $\sin 3\omega = 3\sin \omega - 4\sin^3 \omega$

$$\lambda_i^d = \sqrt{\frac{2}{3} \mathbf{a}^d : \mathbf{a}^d} \cos \left[\theta + \frac{2\pi}{3} (1-i) \right]_{i=1,2,3} = \sqrt{\frac{2}{3} \mathbf{a}^d : \mathbf{a}^d} \sin \left[\omega + \frac{2\pi}{3} (2-i) \right]$$

$$\cos 3\theta \stackrel{\text{def } \theta}{=} \frac{27 \det \mathbf{a}^d}{2 \left(\frac{3}{2} \mathbf{a}^d : \mathbf{a}^d \right)^{3/2}}$$

$$\sin 3\omega \stackrel{\text{def } \omega}{=} - \frac{27 \det \mathbf{a}^d}{2 \left(\frac{3}{2} \mathbf{a}^d : \mathbf{a}^d \right)^{3/2}}$$

$$0 \leq \theta \leq \frac{\pi}{3}$$

$$\omega = \frac{\pi}{6} - \theta$$

$$-\frac{\pi}{6} \leq \omega \leq \frac{\pi}{6}$$

Principal stress invariants

Mean stress: $\sigma_m = \frac{1}{3} \text{tr } \boldsymbol{\sigma}$

Traction $\sigma_m > 0$  **Compression** $\sigma_m < 0$

Von-Mises equivalent stress (1913): $\sigma_{eq} = \sqrt{\frac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}$

Shear stress intensity

Lode's angle (1925): $\cos 3\theta_\sigma = \frac{27 \det \boldsymbol{\sigma}^d}{2 \sigma_{eq}^{3/2}}$ **or** $\sin 3\omega_\sigma = -\frac{27 \det \boldsymbol{\sigma}^d}{2 \sigma_{eq}^{3/2}}$

Shear type (pure shear, simple shear, extension, ...)

$$\sigma_I > \sigma_{II} > \sigma_{III}$$

$$\left\{ \begin{array}{l} \sigma_I = \sigma_m + \frac{2}{3} \sigma_{eq} \cos \theta_\sigma \\ \sigma_{II} = \sigma_m + \frac{2}{3} \sigma_{eq} \cos(\theta_\sigma - \frac{2\pi}{3}) \\ \sigma_{III} = \sigma_m + \frac{2}{3} \sigma_{eq} \cos(\theta_\sigma + \frac{2\pi}{3}) \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \sigma_I = \sigma_m + \frac{2}{3} \sigma_{eq} \sin(\omega_\sigma + \frac{2\pi}{3}) \\ \sigma_{II} = \sigma_m + \frac{2}{3} \sigma_{eq} \sin \omega_\sigma \\ \sigma_{III} = \sigma_m + \frac{2}{3} \sigma_{eq} \sin(\omega_\sigma - \frac{2\pi}{3}) \end{array} \right.$$

$$0 \leq \theta \leq \frac{\pi}{3}$$

$$\omega = \theta - \frac{\pi}{6}$$

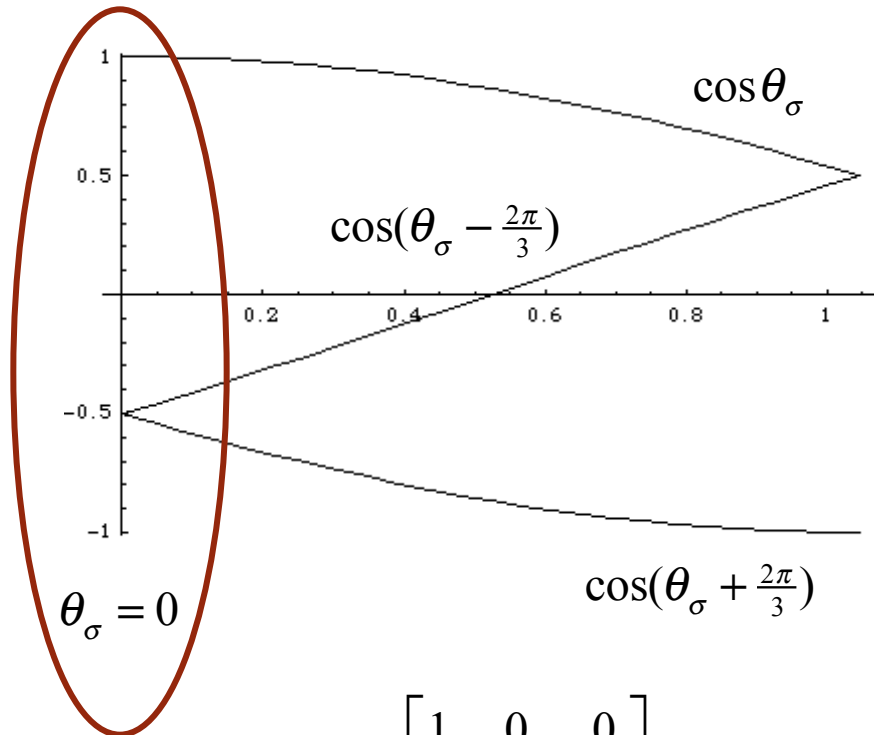
$$-\frac{\pi}{6} \leq \omega \leq \frac{\pi}{6}$$

Mean stress: $\sigma_m = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III})$

Von-Mises equivalent stress: $\sigma_{eq} = \sqrt{\frac{1}{2}[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2]}$

Lode's angle: $\cos 3\theta_\sigma = \frac{9(2\sigma_{III} - \sigma_I - \sigma_{II})(2\sigma_{II} - \sigma_I - \sigma_{III})(2\sigma_I - \sigma_{II} - \sigma_{III})}{2(\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 - \sigma_I\sigma_{II} - \sigma_{II}\sigma_{III} - \sigma_I\sigma_{III})^{3/2}}$

Shear type vs. Lode's angle: extension



$$\theta_\sigma = 0$$

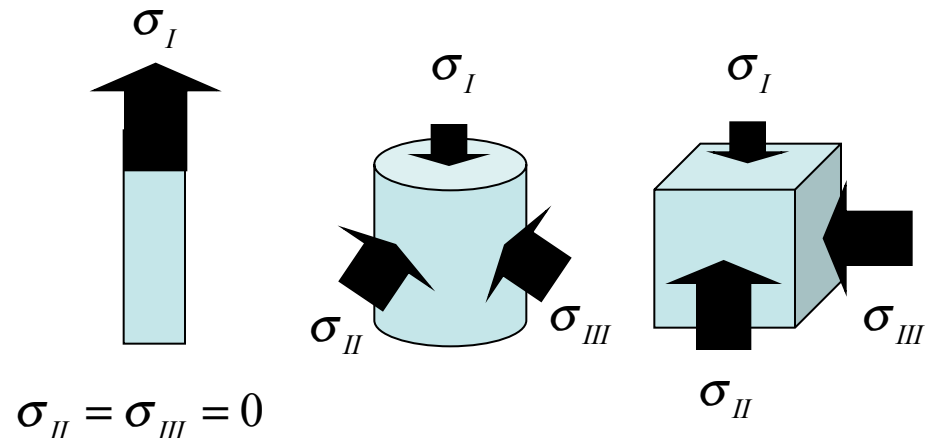
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_m + \sigma_I^d & 0 & 0 \\ 0 & \sigma_m - \frac{1}{2} \sigma_I^d & 0 \\ 0 & 0 & \sigma_m - \frac{1}{2} \sigma_I^d \end{bmatrix}$$

$$\sigma_I > \sigma_{II} = \sigma_{III}$$

Extension along σ_I

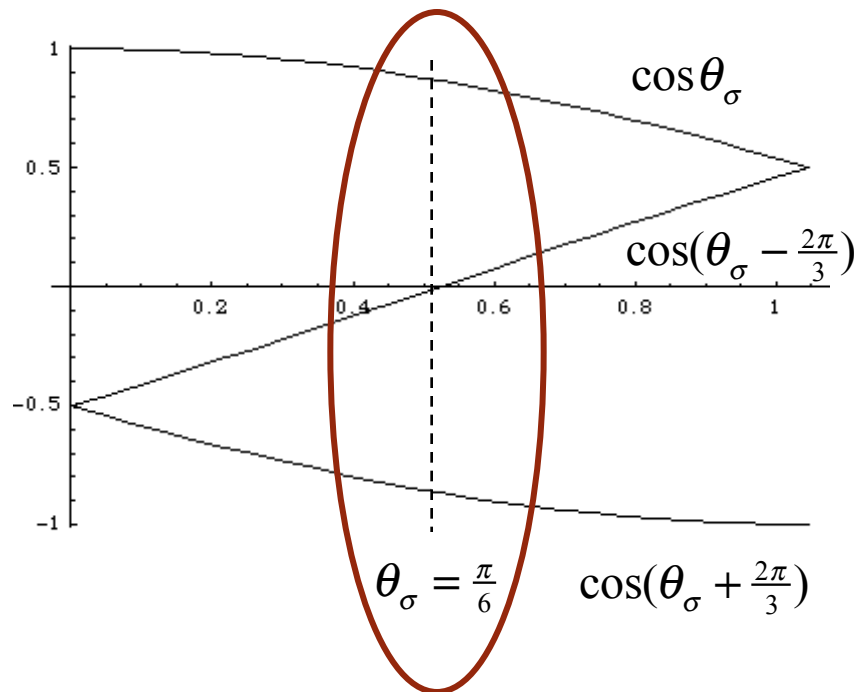
$$\boldsymbol{\sigma}^d = \frac{2}{3} \sigma_{eq} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{cases} \sigma_{II}^d = \sigma_{III}^d = -\frac{1}{2} \sigma_I^d \\ \sigma_I^d > 0 \end{cases} \Rightarrow \begin{cases} \sigma_{eq} = \frac{3}{2} \sigma_I^d \\ \cos 3\theta_\sigma = 1 \end{cases}$$



Examples

Shear type vs. Lode's angle: pure shear



$$\theta_\sigma = \frac{\pi}{6}$$

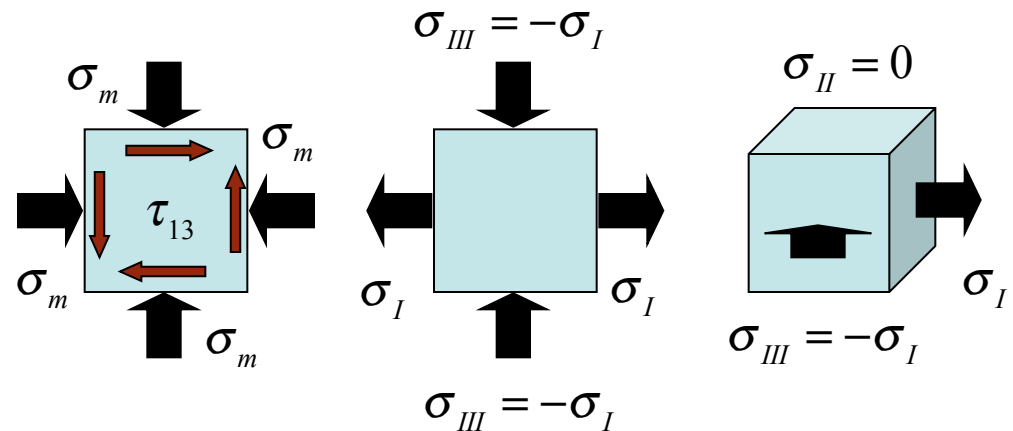
$$\sigma = \begin{bmatrix} \sigma_m + |\sigma_I^d| & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m - |\sigma_I^d| \end{bmatrix}$$

$$\sigma_I > \sigma_{II} = \sigma_m > \sigma_{III}$$

Shear orthogonal to σ_2

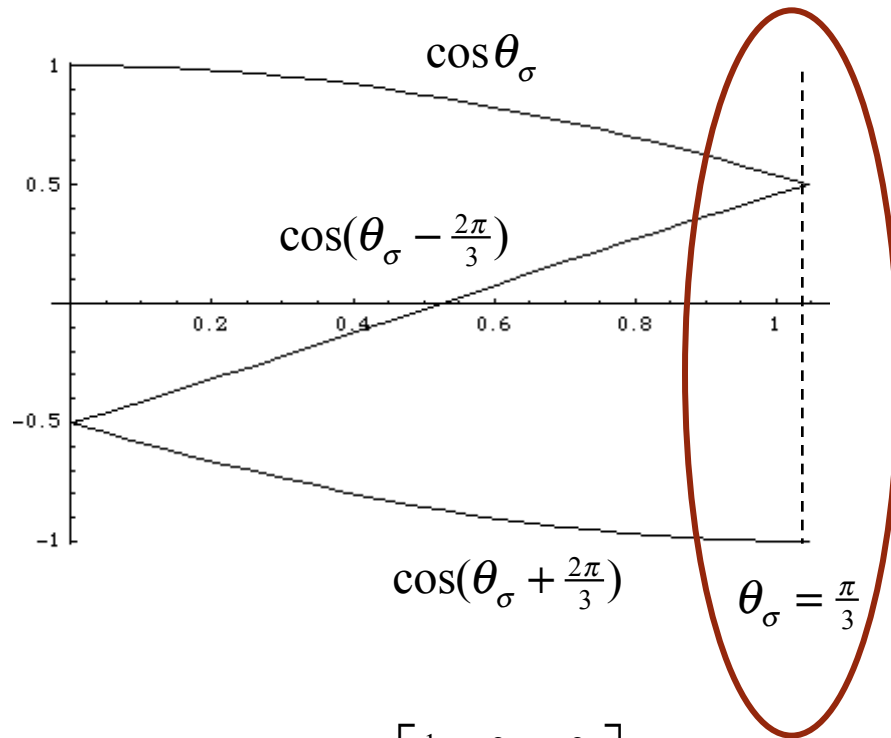
$$\sigma^d = \frac{2}{3} \sigma_{eq} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\begin{cases} \sigma_I^d = -\sigma_{III}^d \\ \sigma_I^d = 0 \end{cases} \Rightarrow \begin{cases} \sigma_{eq} = \sqrt{3} |\sigma_I^d| \\ \cos 3\theta_\sigma = 0 \end{cases}$$



Examples

Shear type vs. Lode's angle: compression



$$\theta_\sigma = \frac{\pi}{3}$$

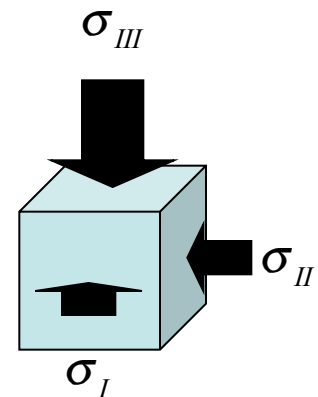
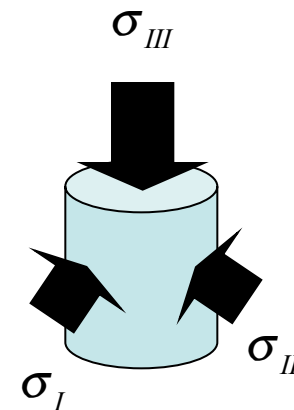
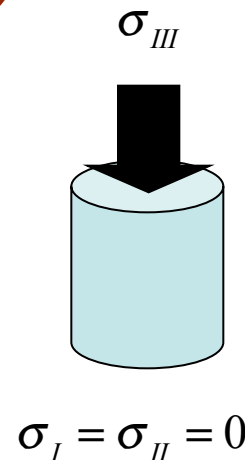
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_m + \frac{1}{2} \sigma_{III}^d & 0 & 0 \\ 0 & \sigma_m + \frac{1}{2} \sigma_{III}^d & 0 \\ 0 & 0 & \sigma_m - \sigma_{III}^d \end{bmatrix}$$

$$\sigma_I > \sigma_{II} = \sigma_{III}$$

Compression along σ_{III}

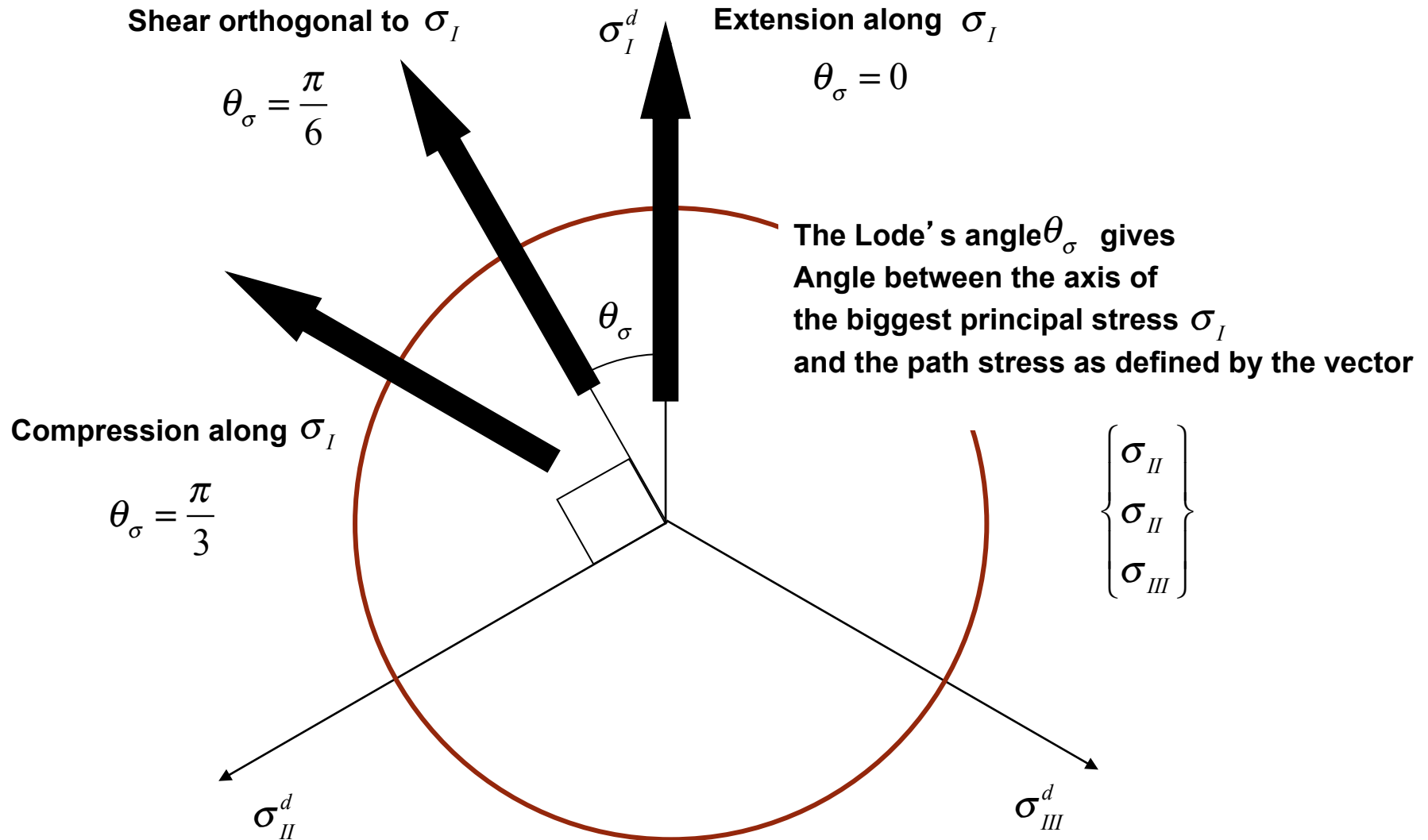
$$\boldsymbol{\sigma}^d = \frac{2}{3} \boldsymbol{\sigma}_{eq} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{cases} \sigma_I^d = \sigma_{II}^d = -\frac{1}{2} \sigma_{III}^d \\ \sigma_{III}^d < 0 \end{cases} \Rightarrow \begin{cases} \sigma_{eq} = -\frac{3}{2} \sigma_{III}^d \\ \cos 3\theta_\sigma = -1 \end{cases}$$



Examples

Shear type vs. Lode's angle



Volume strain rate: $\dot{\epsilon}_v = \text{tr } \dot{\epsilon}$

Dilatancy $\dot{\epsilon}_v > 0$ $\boxtimes$ **Contractancy** $\dot{\epsilon}_v < 0$

Equivalent strain rate $\dot{\epsilon}_{eq} = \sqrt{\frac{2}{3} \dot{\epsilon}^d : \dot{\epsilon}^d}$

Shear strain rate intensity

Power equivalence

$\sigma' : \dot{\epsilon} = \sigma_{eq} \dot{\epsilon}_{eq} + \sigma'_m \dot{\epsilon}_v$ if σ' and $\dot{\epsilon}$ have the same eigen vectors

Elastoplastic models for porous and for granular materials

Assumptions: small strains, isotropic material

(for conciseness only, things are much more complicated for large strains and/or anisotropy)

Strain decomposition

$$\boldsymbol{\varepsilon} = \underbrace{\boldsymbol{\varepsilon}^e}_{\text{reversible strain}} + \underbrace{\boldsymbol{\varepsilon}^p}_{\text{irreversible strain}}$$

Elasticity

$$\dot{\boldsymbol{\sigma}}' = \mathbf{D}^e(\boldsymbol{\sigma}') : \dot{\boldsymbol{\varepsilon}}^e$$

Plasticity

Yield locus

$$f(\boldsymbol{\sigma}', R) \leq 0$$

Flow rule (non associated, not standard, not generalized)

- Plastic strains

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \mathbf{Q}(\boldsymbol{\sigma}', R)$$

- Isotropic hardening

$$\dot{R} = -\dot{\lambda} H(R)$$

- Consistency condition

$$\dot{\lambda} \geq 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}', R) = 0$$

Elastic models for porous and for granular materials

Isotropic non-linear elasticity

$$\dot{\boldsymbol{\sigma}}' = \underset{\sim}{\mathbf{D}}^e(\boldsymbol{\sigma}') : \dot{\boldsymbol{\varepsilon}}^e$$

- usually written in incremental form
- not always thermodynamics consistent
- always non-linear
- usually depend on the mean effective stress $\sigma'_m = \frac{1}{3} \text{tr } \boldsymbol{\sigma}'$

$$\underset{\sim}{\mathbf{D}}^e(\boldsymbol{\sigma}') = 2 \underbrace{G(\sigma'_m)}_{\text{shear modulus}} \underbrace{\left[\mathbf{I} \otimes \mathbf{I} - \frac{1}{3} \mathbf{I} \times \mathbf{I} \right]}_{\text{deviatoric projector}} + 3 \underbrace{\chi(\sigma'_m)}_{\text{bulk modulus}} \underbrace{\left[\frac{1}{3} \mathbf{I} \times \mathbf{I} \right]}_{\text{spheric projector}}$$

$$G(\sigma'_m) = \chi(\sigma'_m) \frac{3(1-2\nu)}{2(1+\nu)} \quad : \text{shear modulus}$$

$$\chi(\sigma'_m) \quad : \text{bulk modulus}$$

$$\nu \quad : \text{constant Poisson's coefficient}$$

Granular materials

$$\chi(\sigma'_m) = \begin{cases} \chi^e \left(\frac{-\sigma'_m}{p_{ref}} \right)^n & \text{if } \sigma'_m < 0 \text{ (for compression only)} \\ \text{not defined} & \text{if } \sigma'_m > 0 \text{ (traction not allowed)} \end{cases}$$

where

χ^e : reference bulk modulus (kPa)

p_{ref} : reference stress (kPa)

n : exponent ($0 < n < 1$, usual value $n=0.6$)

Clays materials

$$\chi(\sigma'_m) = \begin{cases} \frac{1+e}{\kappa^e}(-\sigma'_m) & \text{if } \sigma'_m < 0 \text{ (for compression only)} \\ \text{not defined} & \text{if } \sigma'_m > 0 \text{ (traction not allowed)} \end{cases}$$

where

κ^e : elastic index (dimensionless)

e : initial void ratio, usually considered as constant
and equal the initial void ratio in small strains e^0

Questions

The void ratio is another engineering quantity widely used in porous mechanics.

Void ratio is defined as follows:

$$e = \frac{V_{pore}}{V_{solids}}$$

The solids material is rigid.

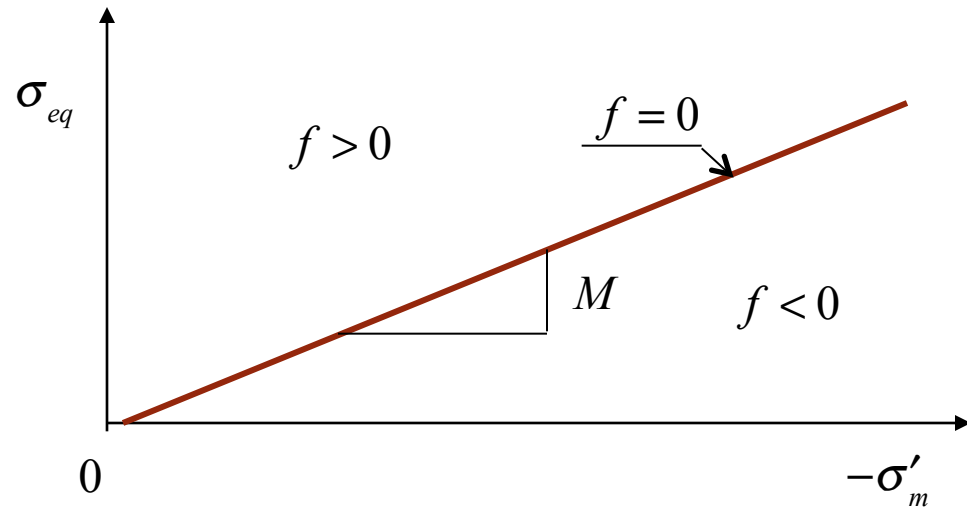
1. Express the void ratio as a function of the porosity.
2. Express the volume strain as a function of the porosity.
3. Express the volume strain as a function of the void ratio.
4. Express the volume strain rate as a function of the porosity.
5. Express the volume strain rate as a function of the void ratio.
6. Express the mean effective stress as a function of the void ratio for a clay matrix and a clay elastic model.

Yield locus

$$f(\sigma') = \sigma_{eq} + M\sigma'_m$$

Evolution rule (not associated)

$$\dot{\epsilon}^p = \dot{\lambda} \left[\frac{M_\Psi}{3} \mathbf{I} + \frac{3}{2\sigma_{eq}} \sigma^d \right]$$

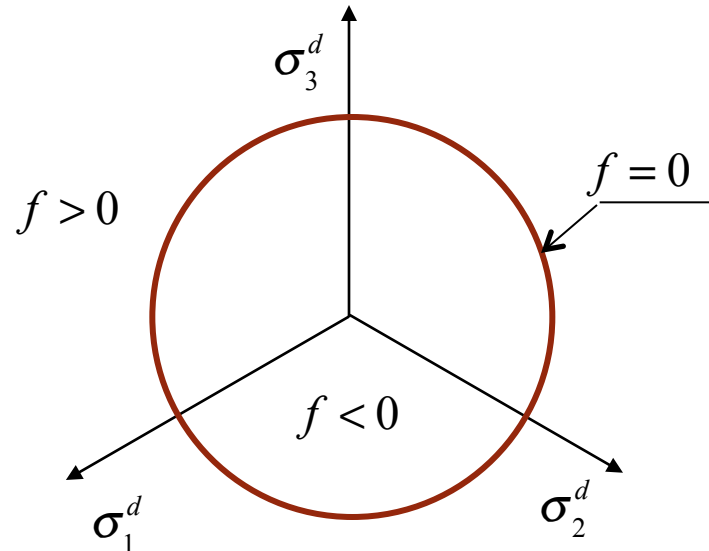


Two material constants

$$M = \left. \frac{\sigma_{eq}}{-\sigma'_m} \right|_{failure} \quad M_\Psi = \left. \frac{\dot{\epsilon}_v^p}{\dot{\epsilon}_{eq}^p} \right|_{failure}$$

$$\sigma : \dot{\epsilon}^p = \dot{\lambda} \sigma'_m (M_\Psi - M)$$

$$\begin{cases} \sigma : \dot{\epsilon}^p > 0 \\ \sigma' < 0 \end{cases} \Leftrightarrow 0 \leq M_\Psi \leq M$$



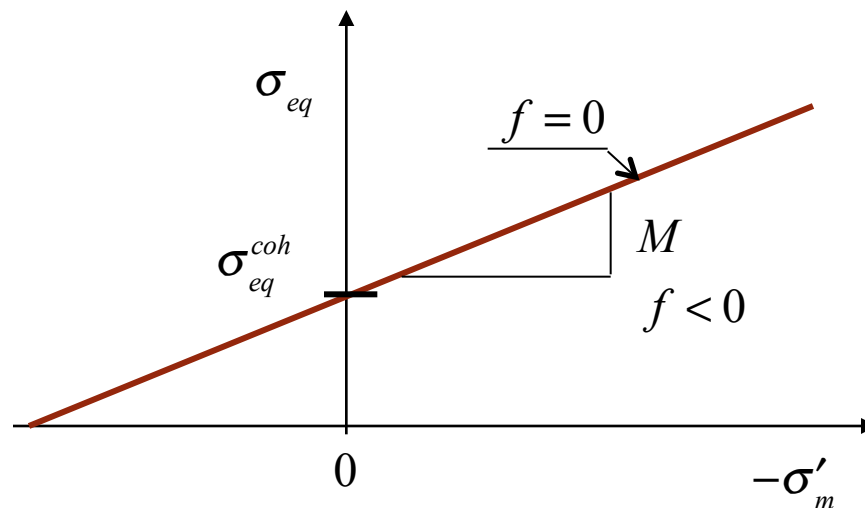
Drucker-Prager failure criterion (with cohesion)

Yield locus

$$f(\sigma') = \sigma_{eq} + M\sigma'_m - \sigma_{eq}^{coh}$$

Evolution rule (not associated)

$$\dot{\epsilon}^p = \dot{\lambda} \left[\frac{M_\Psi}{3} \mathbf{I} + \frac{3}{2\sigma_{eq}} \boldsymbol{\sigma}^d \right]$$

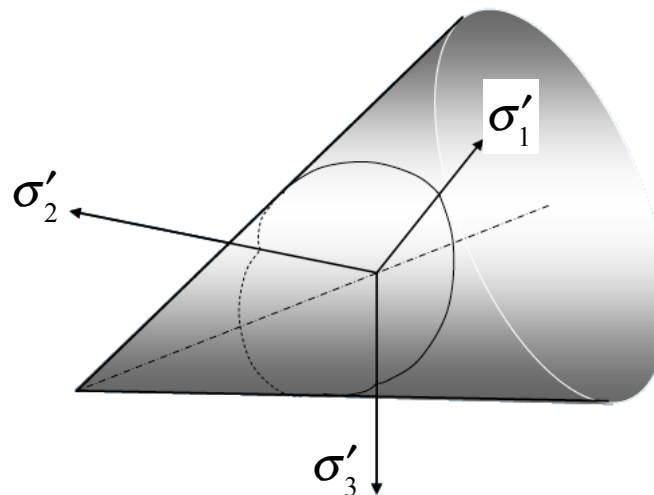


Three material constants

$$M = \left. \frac{\sigma_{eq}}{-\sigma'_m} \right|_{failure}$$

$$M_\Psi = \left. \frac{\dot{\epsilon}_v^p}{\dot{\epsilon}_{eq}^p} \right|_{failure}$$

$$\sigma_{eq}^{coh}$$



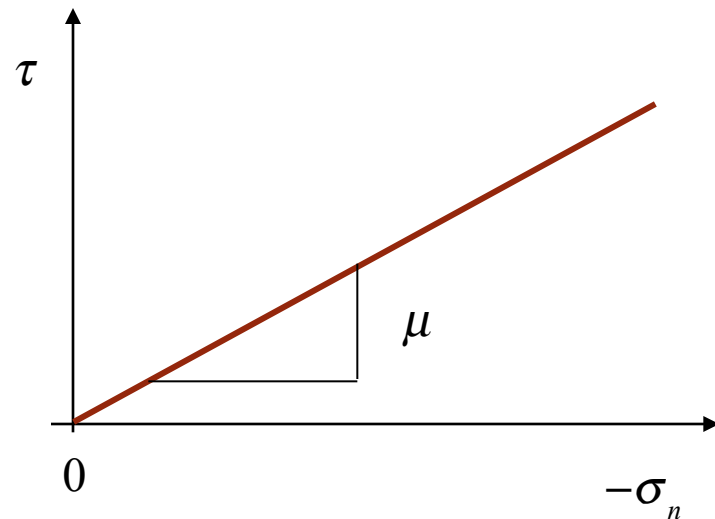
Coulomb friction criterion (1773)

Yield locus

$$\tau + \mu \sigma_n \leq 0$$

μ : friction coefficient

$$\tau = \frac{M_T g}{S} \quad \sigma_n = \frac{M_N g}{S}$$



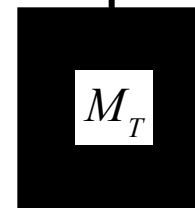
$$M_T < \mu M_N$$

No sliding



$$M_T > \mu M_N$$

Sliding

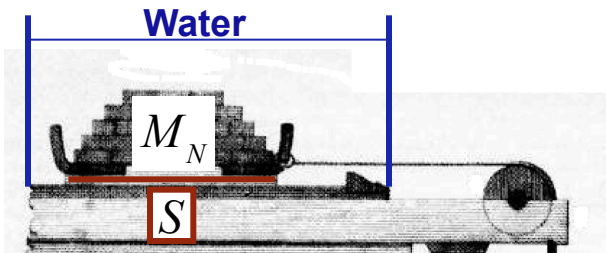
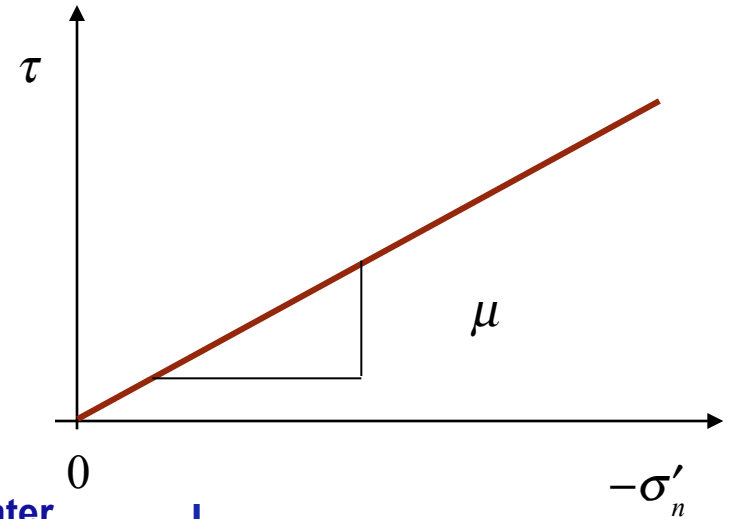


Coulomb friction criterion (1773) accounting for Archimèdes principle (-260)

Yield locus

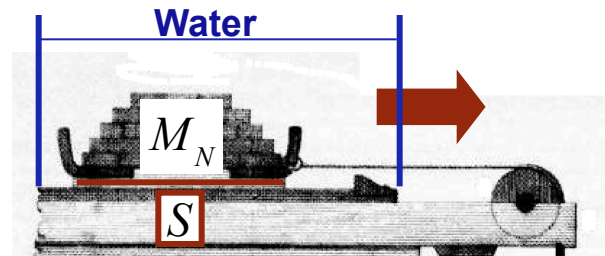
$$\tau + \mu \sigma'_n \leq 0$$

$$\tau = \frac{M_T g}{S} \quad \sigma'_n = \frac{(M_N - \rho_w V)g}{S}$$



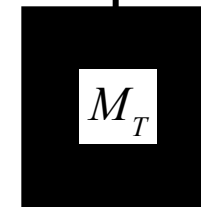
$$M_T < \mu(M_N - \rho_w V)$$

No sliding



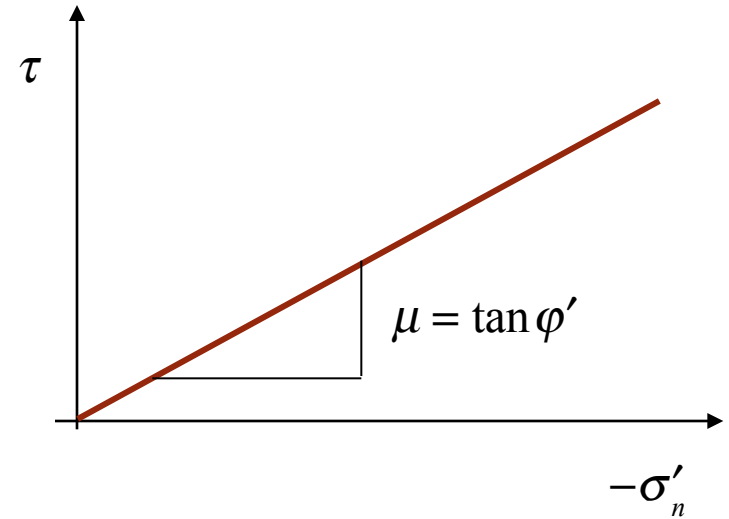
$$M_T > \mu(M_N - \rho_w V)$$

Sliding

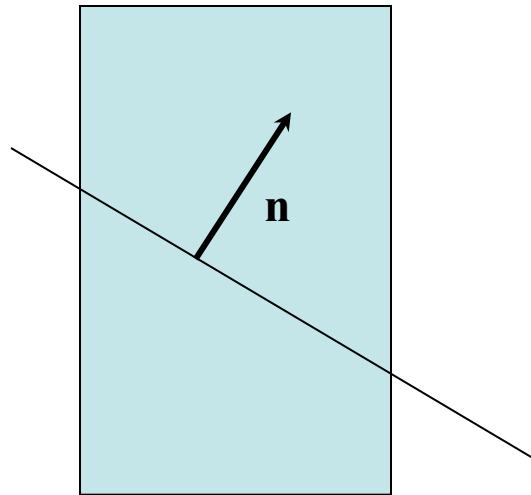


Yield locus

$$\forall \mathbf{n}, \begin{cases} \tau + \sigma' \tan \varphi'_n \leq 0 \\ \tau = [\mathbf{I} - \mathbf{n} \otimes \mathbf{n}] \cdot \boldsymbol{\sigma}' \\ \sigma' = [\mathbf{n} \otimes \mathbf{n}] \cdot \boldsymbol{\sigma}' \end{cases}$$



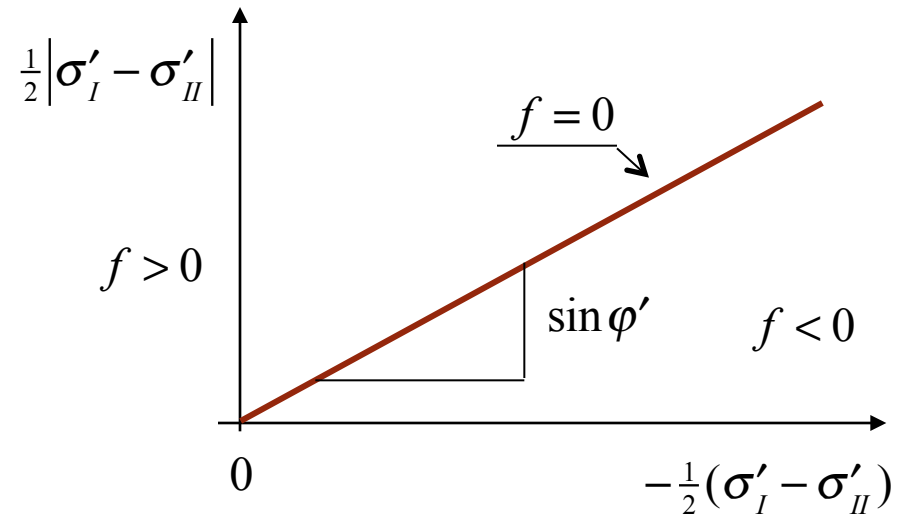
φ' : internal friction angle



Mohr-Coulomb failure criterion (cohesionless)

Yield locus

$$f(\boldsymbol{\sigma}') = \frac{1}{2}(\sigma'_I - \sigma'_{II}) + \frac{1}{2}(\sigma'_I + \sigma'_{II}) \sin \varphi'$$



$$f(\boldsymbol{\sigma}') \leq 0 \quad \Leftrightarrow \quad \frac{\sigma'_I}{\sigma'_{III}} \leq \frac{1 + \sin \varphi'}{1 - \sin \varphi'} = \tan^2 \left(\frac{\pi}{4} + \frac{\varphi'}{2} \right)$$

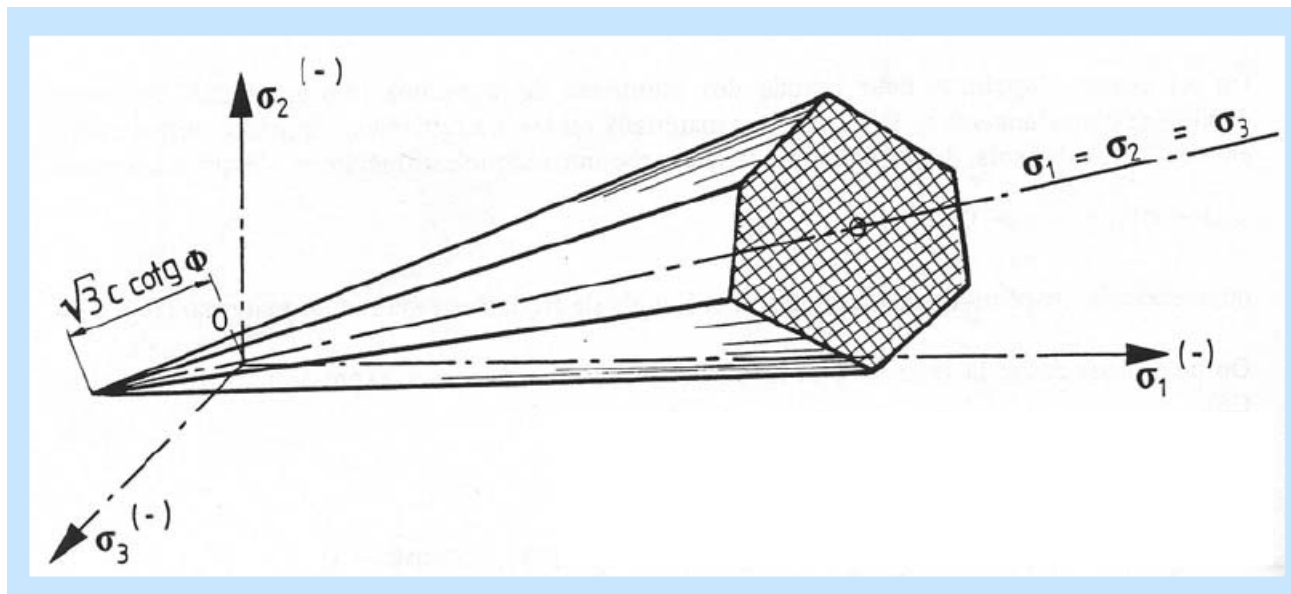
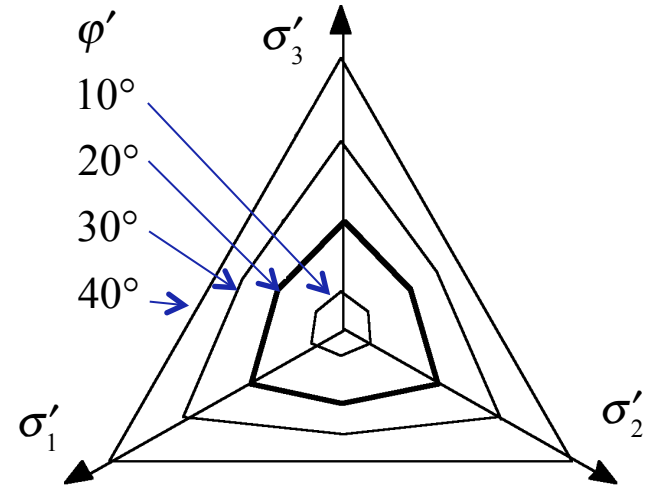
Mohr-Coulomb failure criterion (with cohesion)

Yield locus

$$f_1(\boldsymbol{\sigma}') = \frac{1}{2} |\sigma'_2 - \sigma'_3| + \frac{1}{2} (\sigma'_2 + \sigma'_3) \sin \varphi' - c' \cos \varphi'$$

$$f_2(\boldsymbol{\sigma}') = \frac{1}{2} |\sigma'_1 - \sigma'_3| + \frac{1}{2} (\sigma'_1 + \sigma'_3) \sin \varphi' - c' \cos \varphi'$$

$$f_3(\boldsymbol{\sigma}') = \frac{1}{2} |\sigma'_1 - \sigma'_2| + \frac{1}{2} (\sigma'_1 + \sigma'_2) \sin \varphi' - c' \cos \varphi'$$



Yield locus

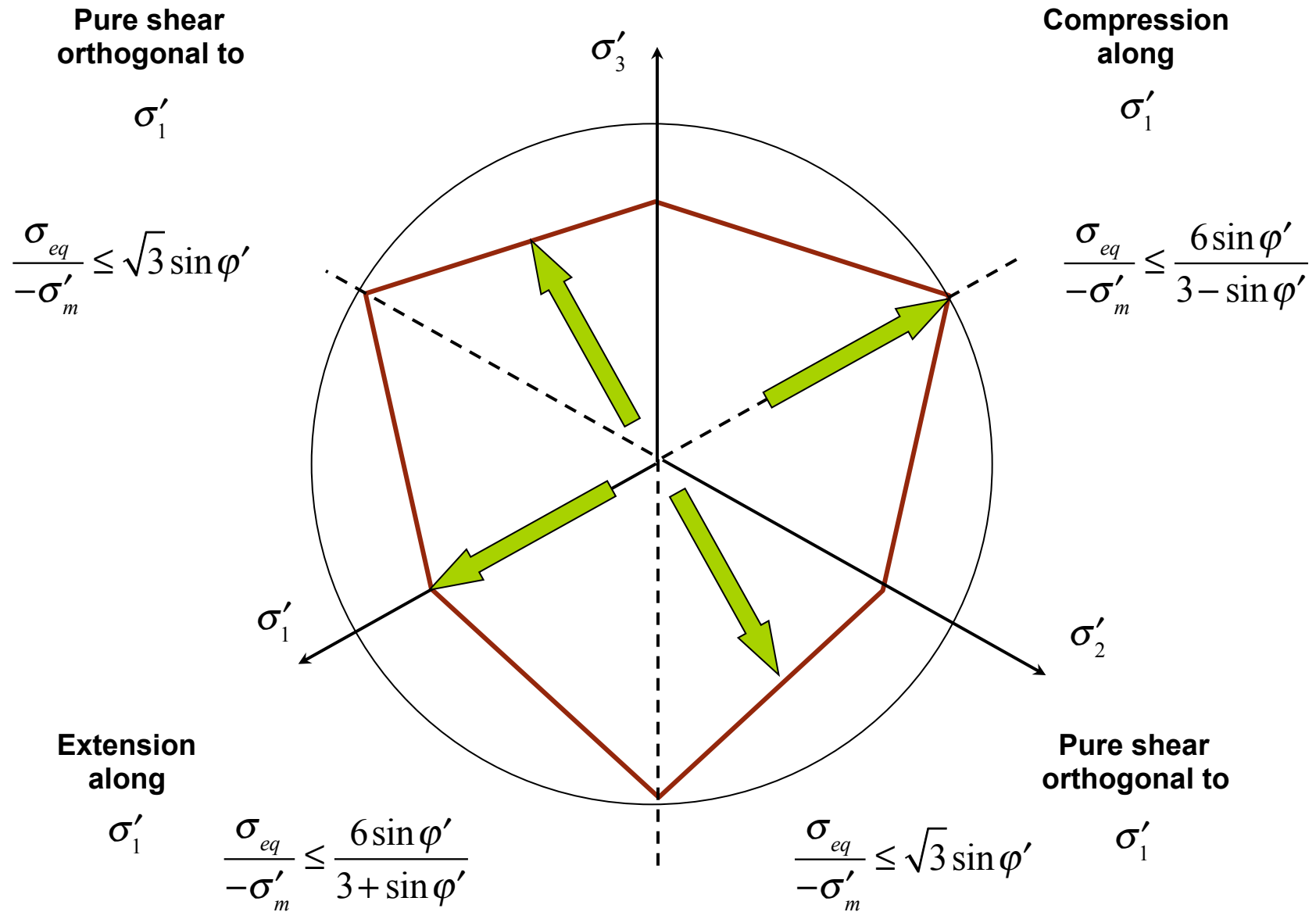
$$f(\boldsymbol{\sigma}') = \sigma_{eq} h(\omega_\sigma) + M \sigma'_m$$

$$M = \frac{6 \sin \varphi'}{3 - \sin \varphi'}$$

$$h(\omega_\sigma) = \frac{2}{3 - \sin \varphi'} (\sqrt{3} \cos \omega_\sigma + \sin \varphi' \sin \omega_\sigma)$$

Extension	Pure shear	Compression
$\theta_\sigma = 0$	$\theta_\sigma = \frac{\pi}{6}$	$\theta_\sigma = \frac{\pi}{3}$
$\frac{\sigma_{eq}}{-\sigma'_m} \leq \frac{6 \sin \varphi'}{3 + \sin \varphi'}$	$\frac{\sigma_{eq}}{-\sigma'_m} \leq \sqrt{3} \sin \varphi'$	$\frac{\sigma_{eq}}{-\sigma'_m} \leq \frac{6 \sin \varphi'}{3 - \sin \varphi'}$

Mohr-Coulomb failure criterion



Yield locus:

$$f(\sigma') = \sigma_{eq} h(\theta_\sigma) + M \sigma'_m$$

Matsuoka-Nakai (1974)

$$h(\theta_\sigma) = \left(\frac{1 + \gamma \cos 3\theta_\sigma}{1 - \gamma} \right)^{1/2} \quad \eta = \frac{\sigma_{eq}}{-M \sigma'_m} \quad \gamma = \frac{9 - \sin^2 \varphi'}{9(3 + \sin^2 \varphi')}$$

Lade-Duncan (1975)

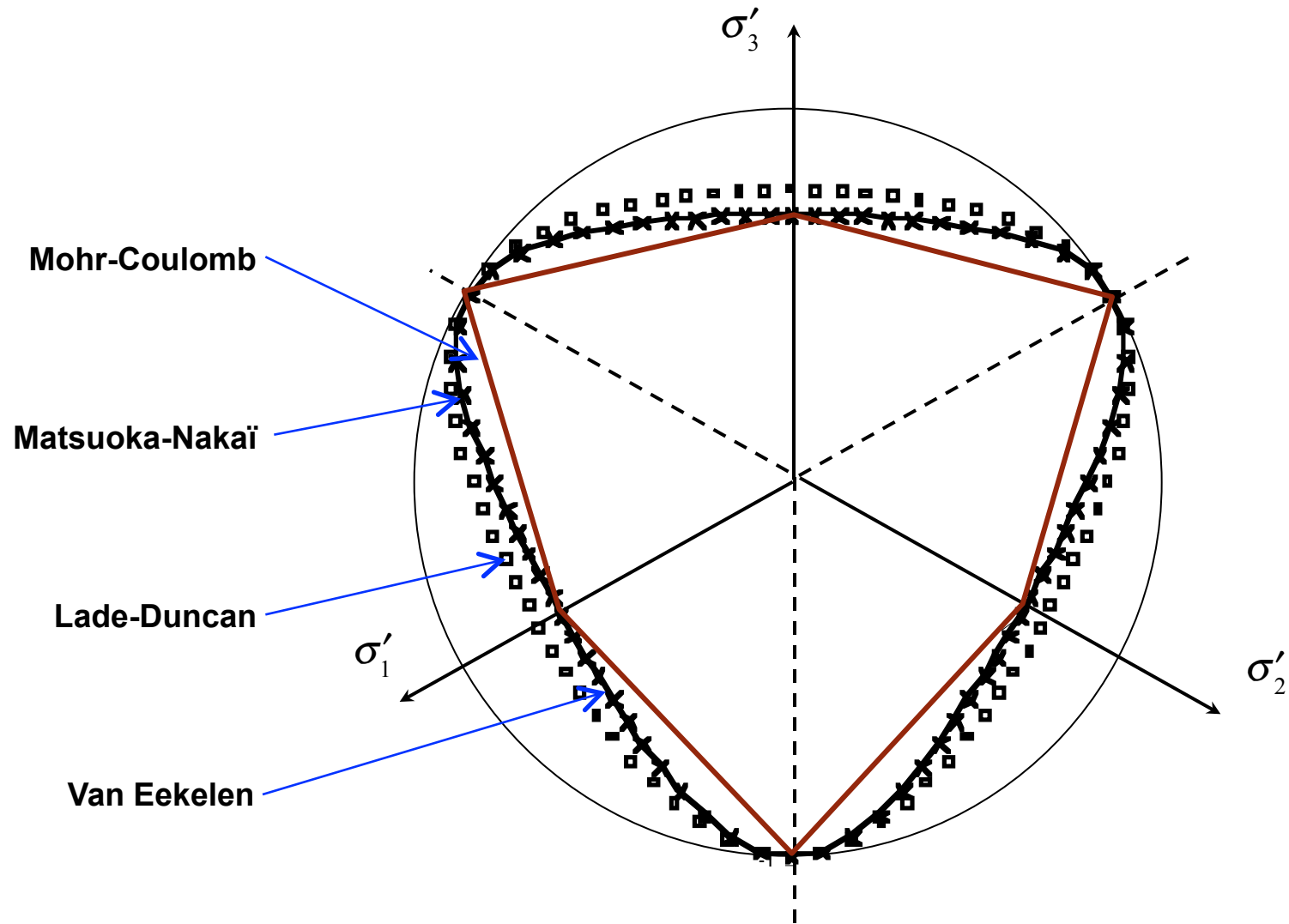
$$h(\theta_\sigma) = \left(\frac{1 + \gamma \cos 3\theta_\sigma}{1 - \gamma} \right)^{1/2} \quad \eta = \frac{\sigma_{eq}}{-M \sigma'_m} \quad \gamma = \frac{4}{9}$$

Van Eekelen (1980)

$$h(\theta_\sigma) = \left(\frac{1 + \gamma \cos 3\theta_\sigma}{1 - \gamma} \right)^k \quad k = 0.229 \quad \gamma = \frac{1 - r}{1 + r} \quad r = \left(\frac{3 - \sin \varphi'}{3 + \sin \varphi'} \right)^{1/k}$$

Seule valeur possible avec Abaqus: $k = 1$ (attention à la perte de convexité !!)

Valeur optimum assurant la convexité: $k = 0.229$



Cambridge elastoplastic models: Cam-Clay

The Cam-Clay model for isotropic media in small strains is usually described as follows

Strain decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$

Isotropic non-linear elasticity $\chi(\sigma'_m) = \frac{1+e}{\kappa^e}(-\sigma'_m) \quad G(\sigma'_m) = \chi(\sigma'_m) \frac{3(1-2\nu)}{2(1+\nu)}$

Plasticity

Yield locus $f(\boldsymbol{\sigma}', \sigma_c) = \sigma_{eq} + M\sigma'_m \ln\left(\frac{\sigma_c}{-\sigma'_m}\right)$

Flow rule

- Associated plastic potential $Q(\boldsymbol{\sigma}', \sigma_c) = \frac{\partial f}{\partial \boldsymbol{\sigma}'}(\boldsymbol{\sigma}', \sigma_c)$

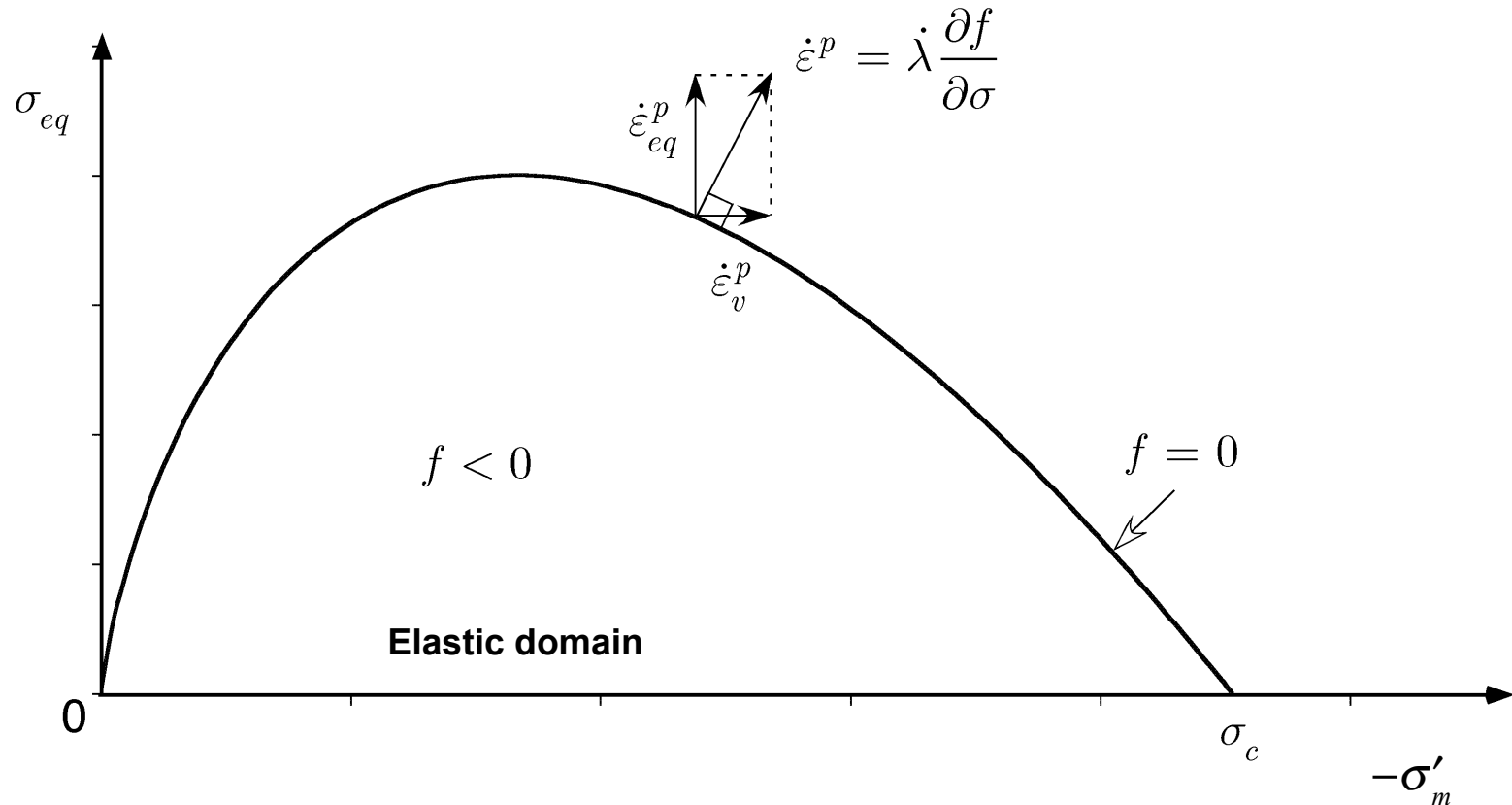
- Non associated isotropic hardening $\dot{\sigma}_c = \sigma_c^0 \exp(-\beta \varepsilon_v^p) \quad \varepsilon_v^p = \text{tr } \boldsymbol{\varepsilon}^p$

Material constants

$$\kappa^e, \nu, M, \beta$$

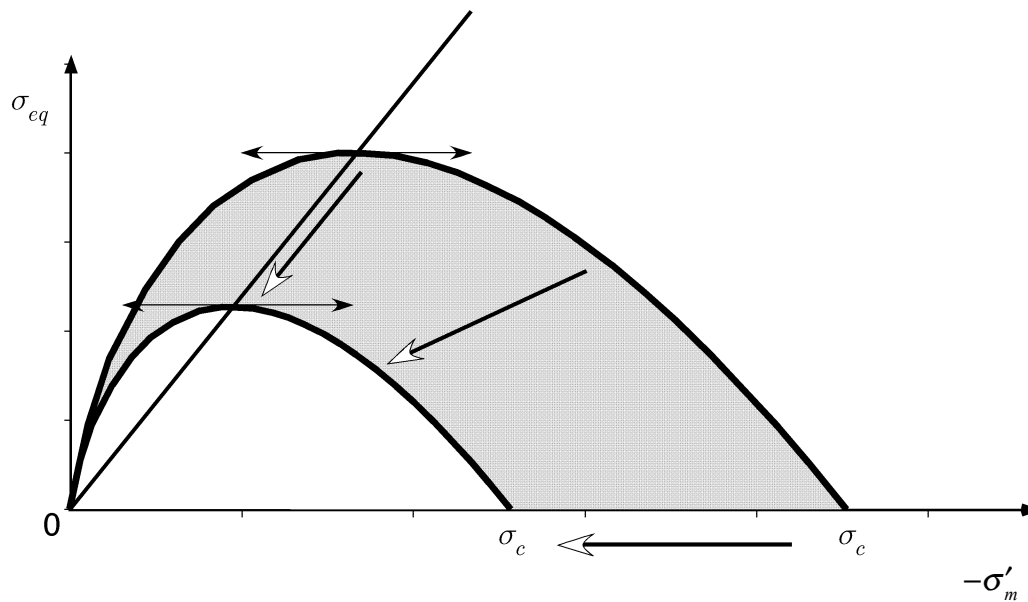
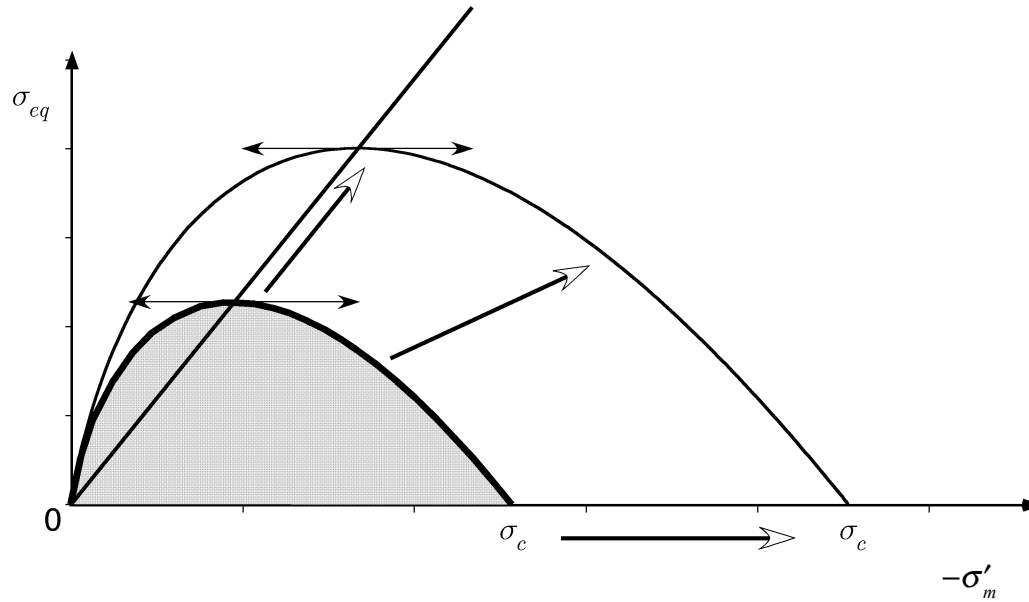
Initial state

$$\sigma_c^0, e^0$$



The isotropic hardening variable σ_c is the consolidation stress.

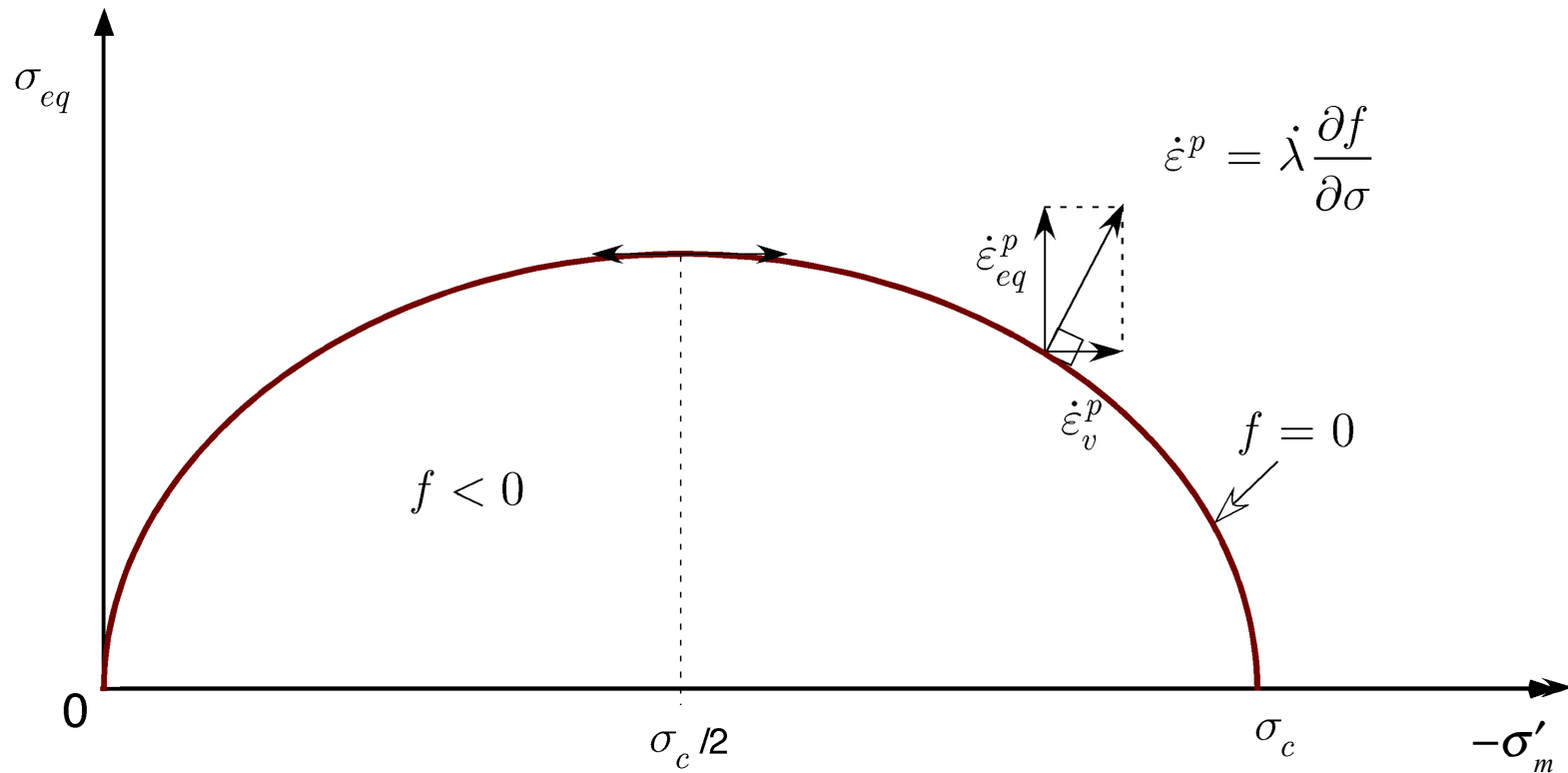
The material has some memory of the greatest consolidation stress undergone in the course of its history.



Cambridge elastoplastic models: modified Cam-Clay

The modified Cam-Clay model for isotropic media in small strains is the same as the Cam-Clay model, with the following yield locus

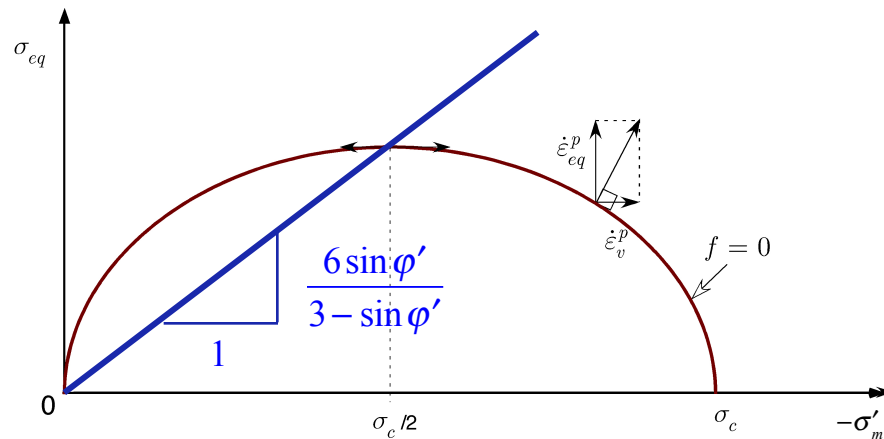
$$f(\sigma', \sigma_c) = \sigma_{eq}^2 + M\sigma'_m(\sigma'_m + \sigma_c)$$



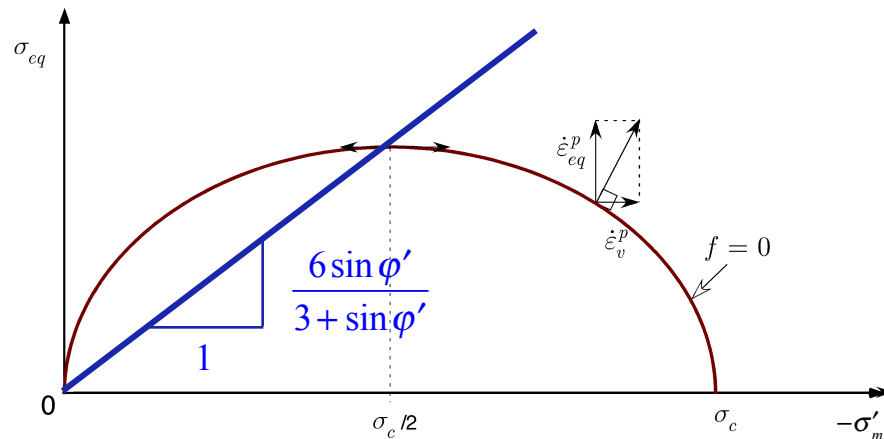
Equivalent expression:
$$f(\sigma', \sigma_c) = \sigma_{eq} + M\sigma'_m \sqrt{\frac{\sigma_c}{-\sigma'_m} - 1}$$

Yield locus

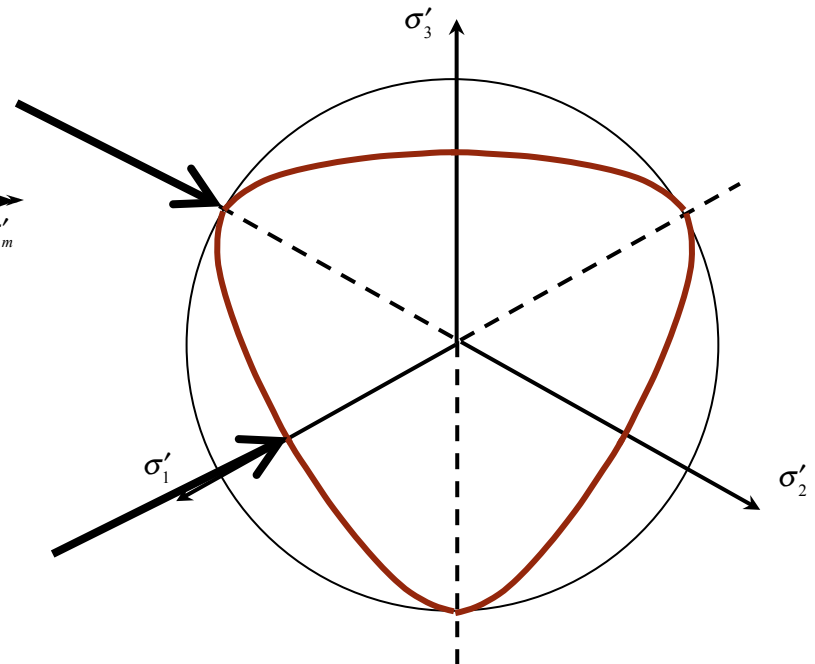
$$f(\sigma', \sigma_c) = \left[\sigma_{eq} h(\theta_\sigma) \right]^2 + M \sigma'_m (\sigma'_m + \sigma_c)$$



Compression



Extension



Questions

Modèle de Cam-Clay

1. Le point d'intersection de la surface de charge du modèle de Cam-Clay et de la droite définie par

$$\sigma_{eq} + M\sigma'_m = 0$$

est un point particulier. Le positionner sur le graphique.

2. Expliciter la vitesse de déformation plastique en fonction de σ_{eq} , σ'_m et $\dot{\lambda}$

(σ_c ne doit pas apparaître.)

3. Expliciter la condition cinématique du modèle de Cam-Clay, en fonction de σ_{eq} et σ'_m

Cette condition relie $\dot{\epsilon}_v^p$ et $\dot{\epsilon}_{eq}^p$

4. Quelle inéquation doit vérifier (σ'_m, σ_{eq}) pour que l'on ait une évolution avec dilatance plastique ?

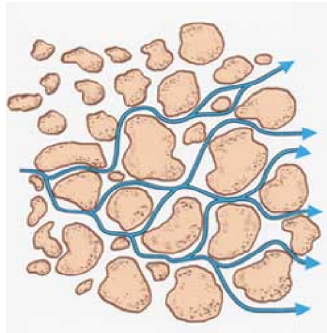
5. Quelle inéquation doit vérifier (σ'_m, σ_{eq}) pour que l'on ait une évolution avec contractance plastique ?

6. Comment évolue la variable d'écrouissage isotrope en dilatance plastique ?
Et en contractance plastique ?

Milieux poreux

Poro-Mécanique

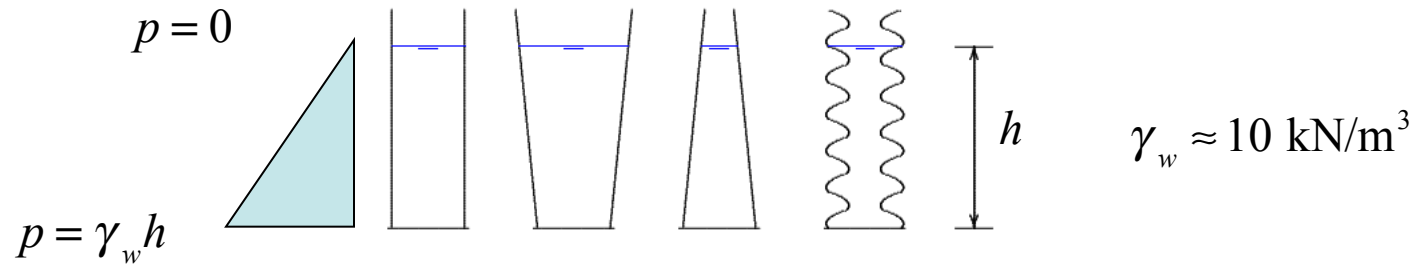
Couplages hydro-mécanique



Stresses in soils
Cases study

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stephane.bonelli@irstea.fr

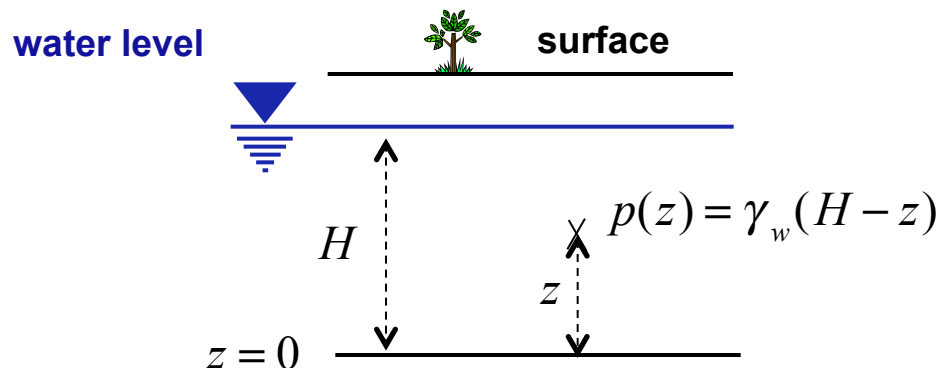
Pore pressures and hydraulic head



Hydrostatic water pressure depends upon depth only ($p > 0$ in the water)

The hydraulic head is defined as follows

$$H = \frac{p - p_{atm}}{\gamma_w} + z \quad \gamma_w = \rho_w g$$



Effective stress

The total stress is defined by the static equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad \left(\text{tr } \boldsymbol{\sigma} < 0 \Leftrightarrow \text{compression} \right)$$

The effective stress is defined as follows (Terzaghi, 1925)

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\mathbf{I}$$

The effective stress differs than the total stress only on the isotropic part

$$\frac{1}{3} \text{tr } \boldsymbol{\sigma}' = \frac{1}{3} \text{tr } \boldsymbol{\sigma} + p$$

$$\boldsymbol{\sigma}'^d = \boldsymbol{\sigma}^d$$

The behaviour law of the solid matrix involves the effective stress, for example, isotropic linear elasticity in small strains

$$\boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon} + \text{tr } \boldsymbol{\varepsilon} \left(\chi - \frac{2G}{3} \right) \mathbf{I}$$

$$G = \frac{E}{2(1+\nu)}$$

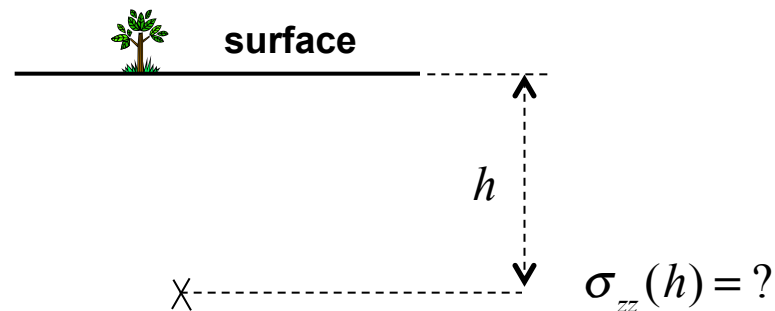
(shear modulus)

$$\chi = \frac{E}{3(1-2\nu)}$$

(bulk modulus)

Question (1/4): total vertical stress in a dry soil

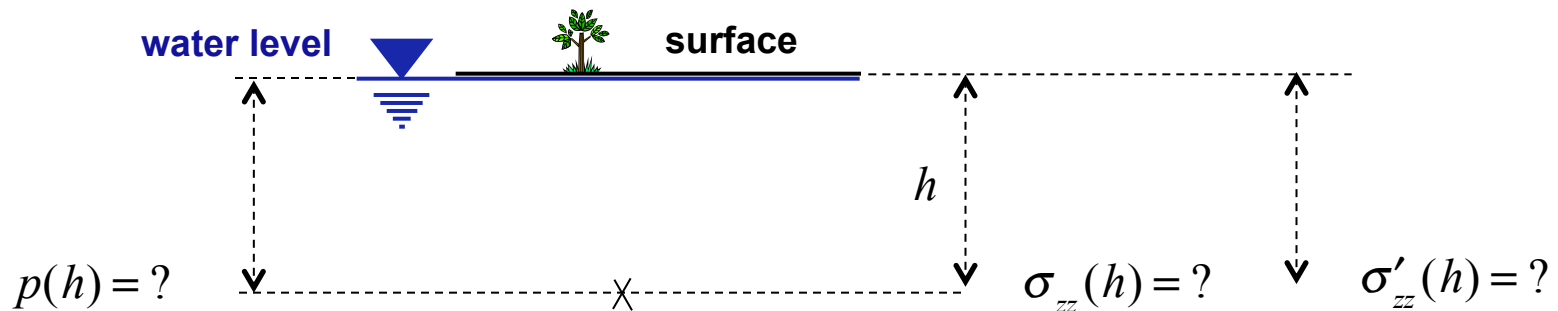
1. Assume a dry and homogeneous semi-infinite soil.
Express the total vertical stress.



Assume a porosity $n=0.3$, what is the total vertical stress for a 10 m depth (in kPa) ?

Question (2/4): total vertical stress in a water-saturated soil

2. Assume a water-saturated and homogeneous semi-infinite soil.
Express the total vertical stress and the pore-pressure.
Infer the vertical effective stress.

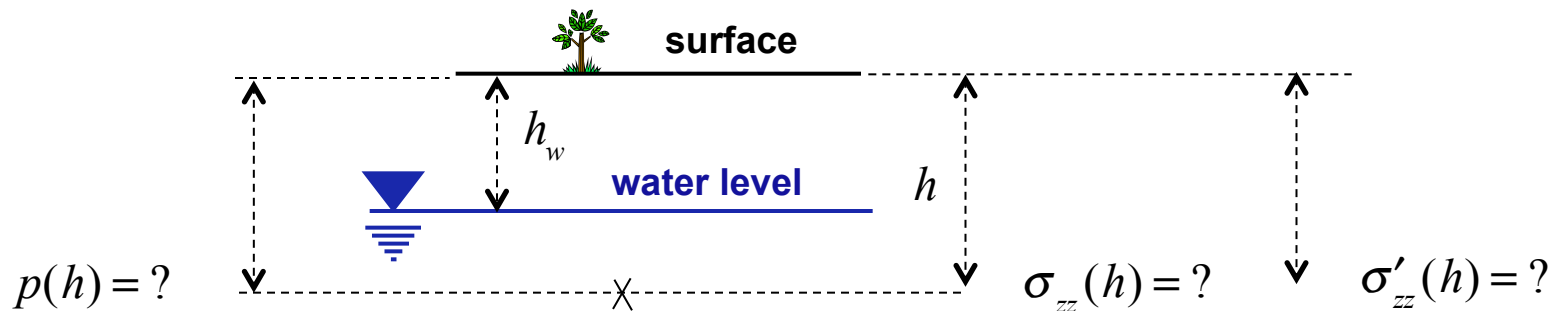


Assume a porosity $n=0.3$, for a 10 m depth what are (in kPa):

- the total vertical stress?
- the pore pressure ?
- the effective stress ?

Question (3/4): total vertical stress in a soil

3. Assume a semi-infinite soil with a homogeneous solid matrix.
Assume a hydrostatic water level.
Express the total vertical stress and the pore-pressure.
Infer the vertical effective stress.



Assume a porosity $n=0.3$, and a water level depth $h_w=5$ m. For a 10 m depth what are (in kPa):

- the total vertical stress?
- the pore pressure ?
- the effective stress ?

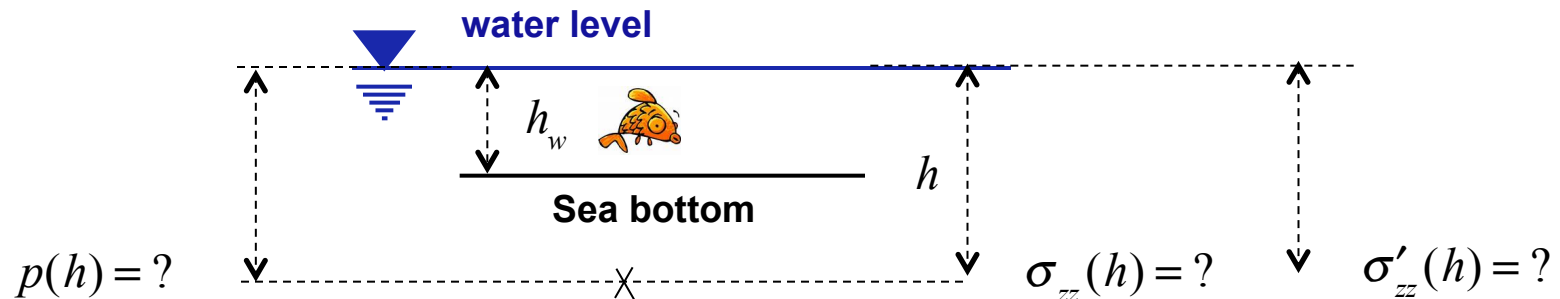
Question (4/4): stresses again

4. Assume a semi-infinite soil with a homogeneous solid matrix.

Assume a hydrostatic water level.

Express the total vertical stress and the pore-pressure.

Infer the vertical effective stress.



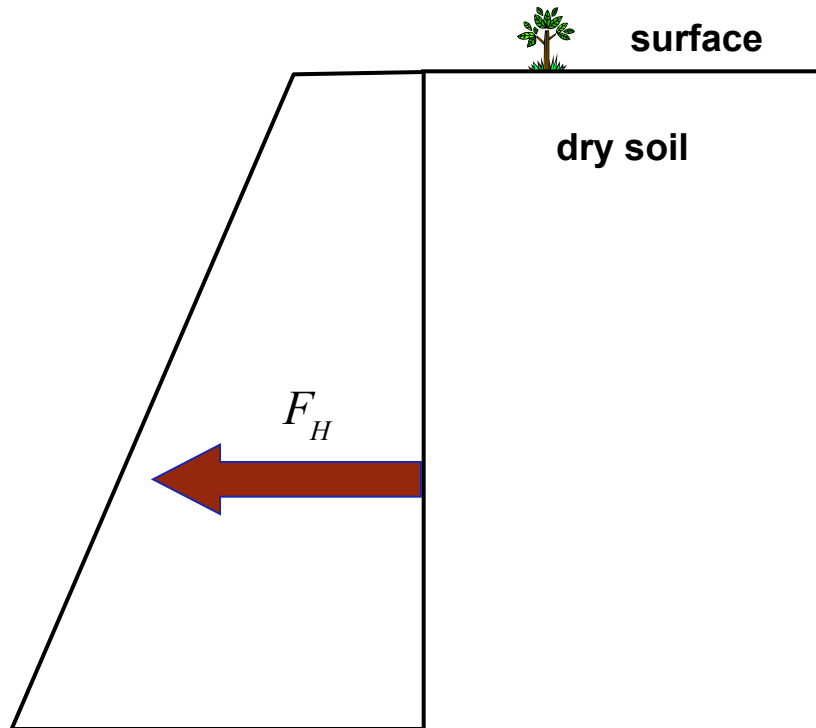
Assume a porosity $n=0.3$, and a water level depth $h_w=5$ m. For a 10 m depth what are (in kPa):

- the total vertical stress?
- the pore pressure ?
- the effective stress ?

Question (1/4): the retaining wall

1. Assume a rigid and impervious retaining wall below a semi-infinite dry soil.
Assume that the soil behaves elastically, with an homogeneous isotropic and linear elastic behaviour law relating the effective stresses σ' to the strain.
Assume a perfect wall/soil contact with no interface displacement.
Express the mean horizontal force exerted by the soil on the wall.

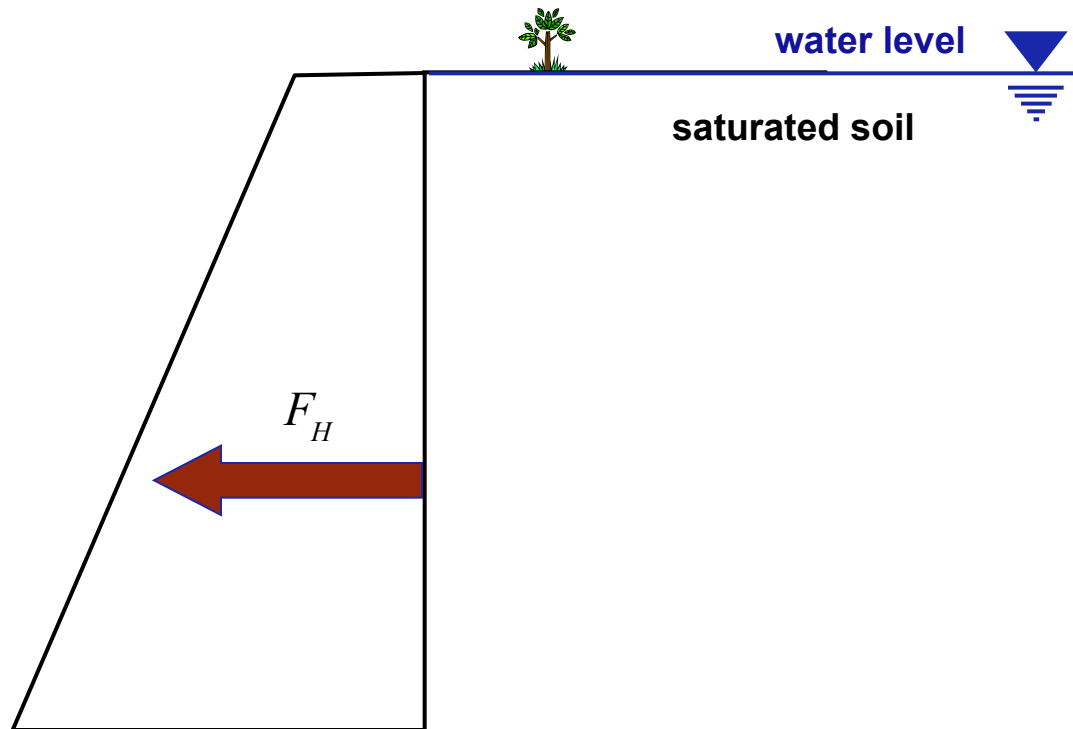
$$F_H = \int_0^H \sigma_{xx}(h) dh$$



Question (2/4): the retaining wall

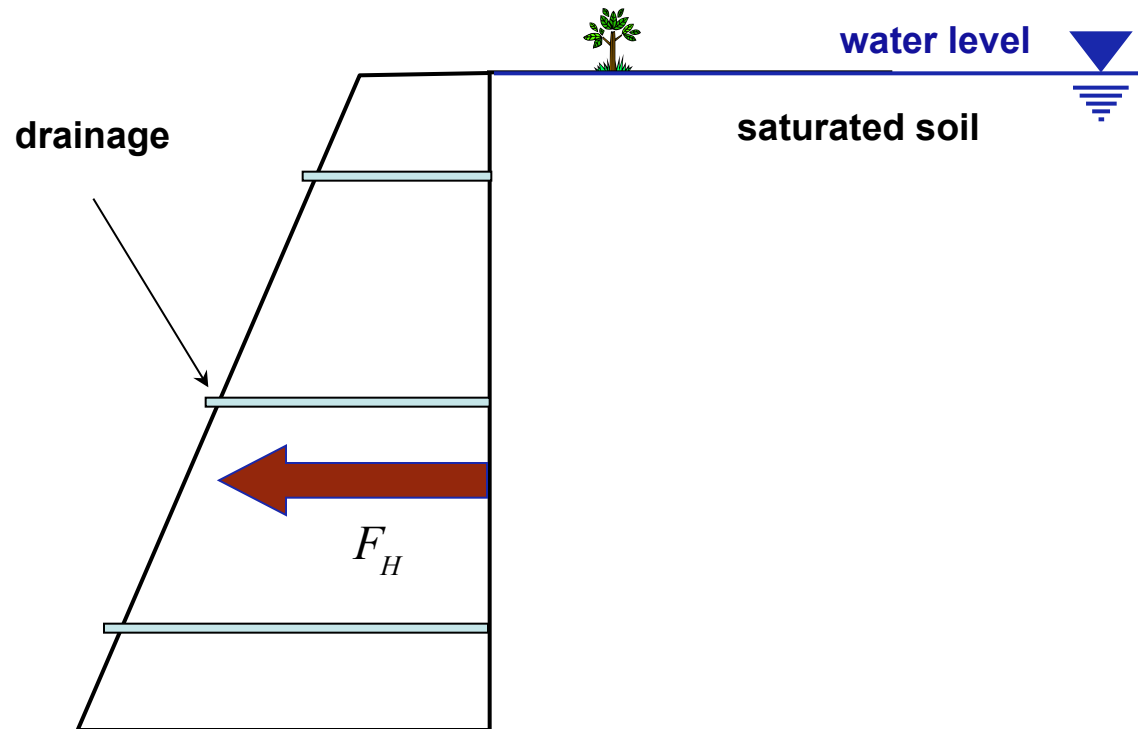
2. Now the soil is water-saturated.

Express the mean horizontal force exerted by this soil on the wall.



Question (3/4): the retaining wall

3. Now the soil is water-saturated but the wall is drained.
Express the mean horizontal force exerted by this soil on the wall.
Conclusion ?



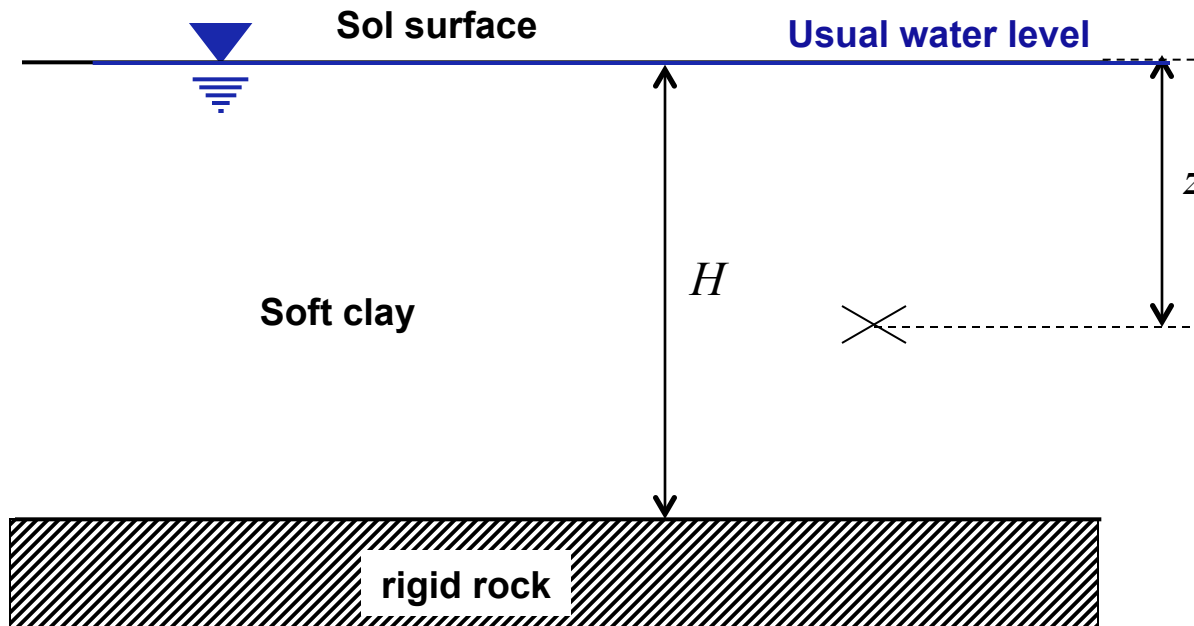
Question (4/4): the retaining wall



Question (1/3): long subsidence after a construction

1. We consider a soil constituted by a layer of soft clay, lying on a rigid rock. The clay is assumed saturated, and have a constant density.

Express the total vertical stress $\sigma_{zz}^0(z)$, the pore pressure $p^0(z)$ and the effective vertical stress $\sigma'_{zz}(z)$ as a function of depth z corresponding to this situation.



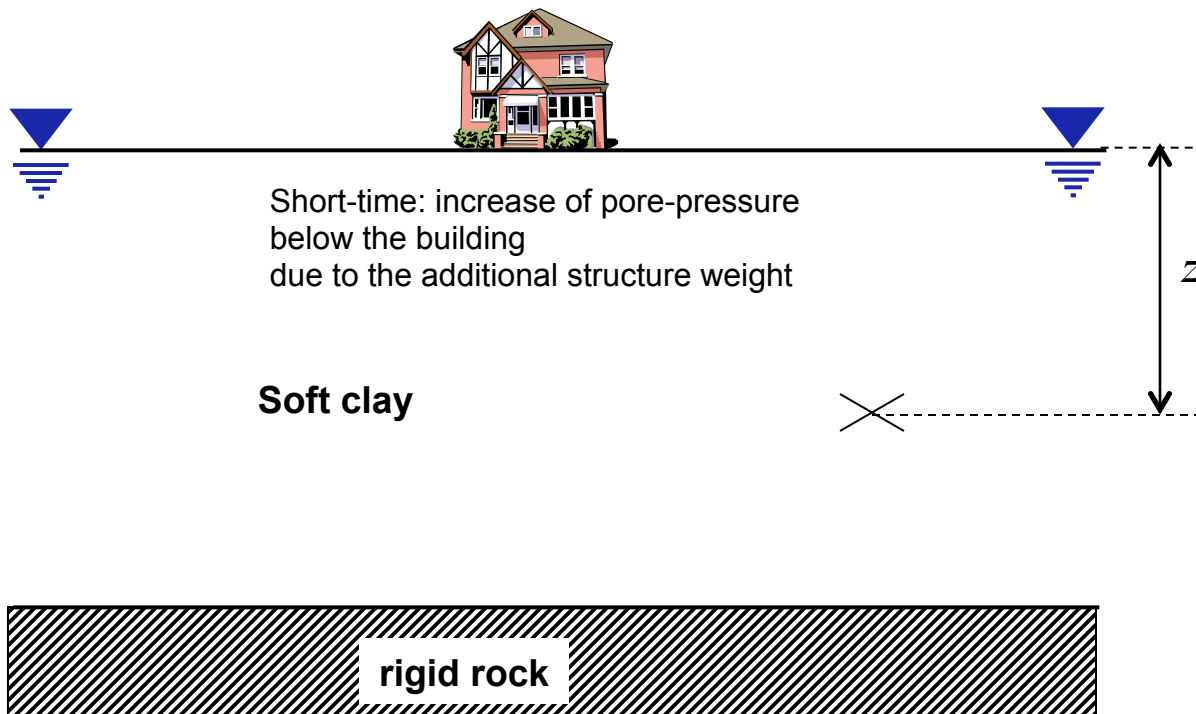
Question (2/3): long subsidence after a construction

2. A building is constructed very quickly. The pore-water is assumed to compensate integrally this excess loading. The clay is assumed saturated, and have a constant density.

Express the total vertical stress $\sigma_{zz}^1(z)$, the pore pressure $p^1(z)$

and the effective vertical stress $\sigma'_{zz}(z)$ as a function of depth z

corresponding to this situation, which is the « short-term » situation, just after the construction.



Question (3/3): long subsidence after a construction

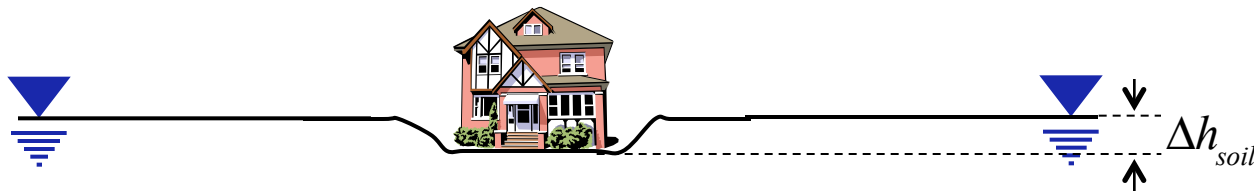
3. Now we skip to many years after the construction. The pore-pressures have dissipated. The clay is assumed saturated, and have a constant density. In addition, this clay behaves elastically as follows

$$\sigma'_{zz} = E_{\alpha d} \varepsilon_{zz} \quad E_{\alpha d} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

where ε_{zz} is the vertical strain, and $E_{\alpha d}$ is the oedometric modulus.

Give an estimation of the subsidence Δh_{soil} , considering that

$$E = 1\,000 \text{ kPa}, \nu = \frac{1}{3}, H = 12 \text{ m}$$



Long-term subsidence

Soft clay

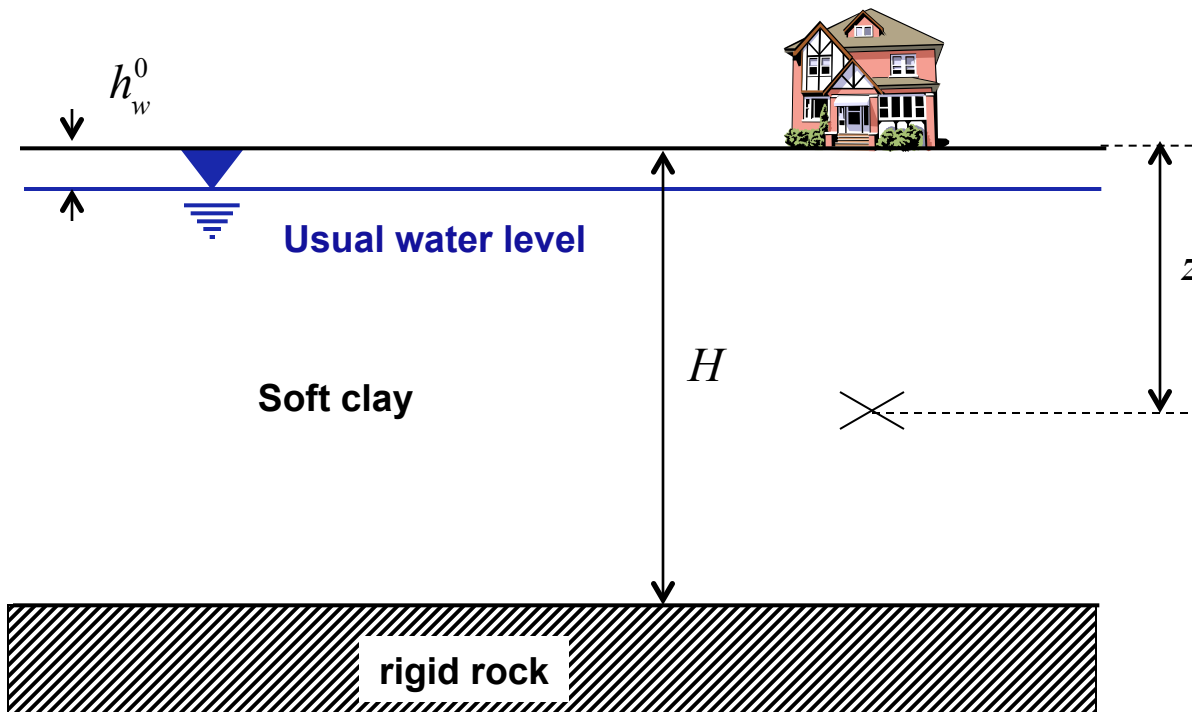


Question (1/4): subsidence by pumping

1. A building lies on a soil.

This soil is constituted by a layer of soft clay, lying on a rigid rock.
The clay is assumed saturated, and have a constant density.

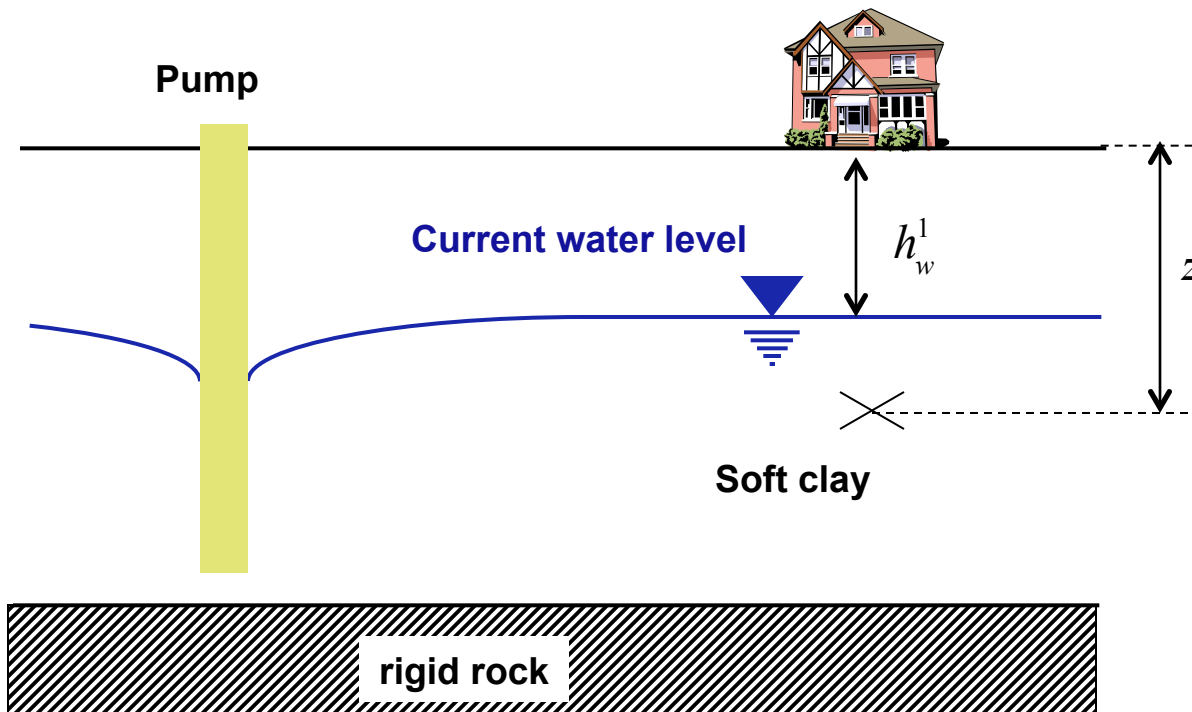
Express the total vertical stress $\sigma_{zz}^0(z)$, the pore pressure $p^0(z)$
and the effective vertical stress $\sigma'_{zz}{}^0(z)$ as a function of depth z
corresponding to this situation.



Question (2/4): subsidence by pumping

2. Somebody install a huge pump not too far from the building.
This water level falls down.
The clay is assumed saturated, and have a constant density.

Express the total vertical stress $\sigma_{zz}^1(z)$, the pore pressure $p^1(z)$ and the effective vertical stress $\sigma'_{zz}^1(z)$ as a function of depth z corresponding to this situation.



Question (3/4): subsidence by pumping

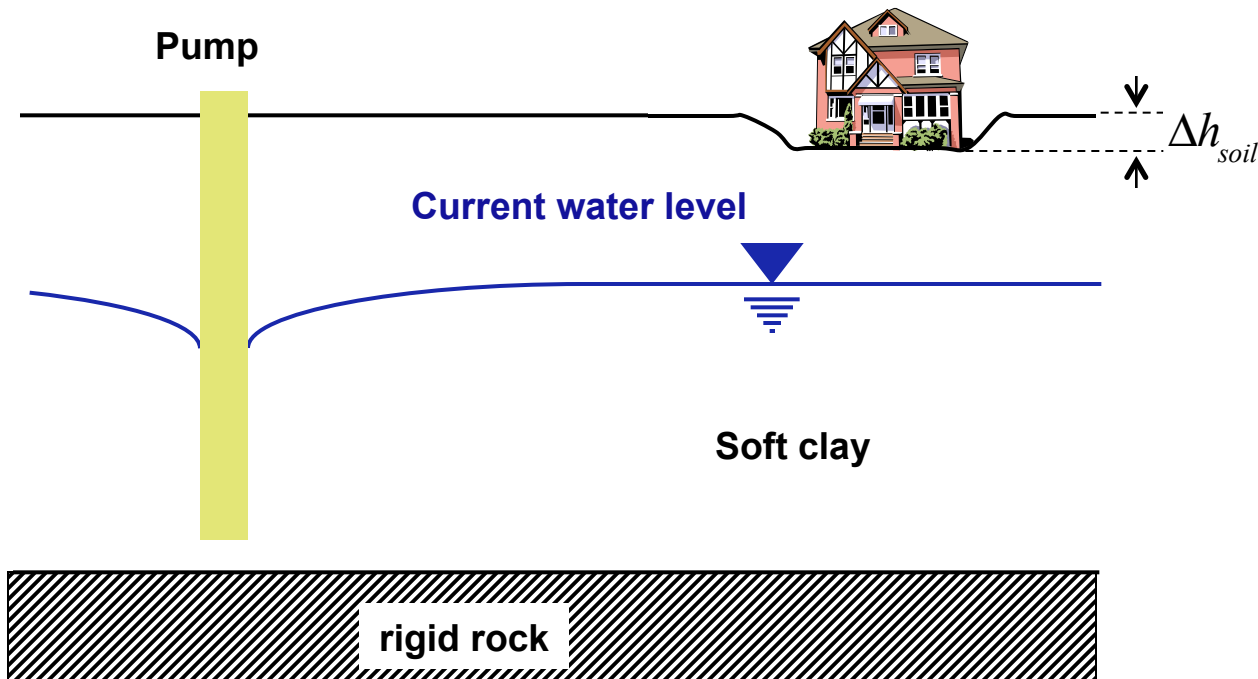
3. Somebody install a huge pump not too far from the building. The water level falls down. The clay is assumed saturated, and have a constant density. In addition, this clay behaves elastically as follows

$$\sigma'_{zz} = E_{\alpha d} \varepsilon_{zz} \quad E_{\alpha d} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

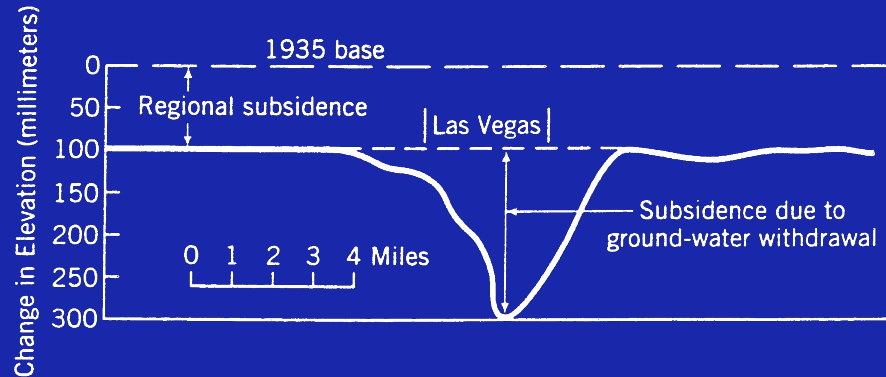
where ε_{zz} is the vertical strain, and $E_{\alpha d}$ is the oedometric modulus.

Give an estimation of the subsidence Δh_{soil} , considering that

$$E = 1\,000 \text{ kPa}, \nu = \frac{1}{3}, h_w^0 = 2 \text{ m}, h_w^1 = 4 \text{ m}, H = 12 \text{ m}$$



Question (4/4): subsidence by pumping



Subsidence profile, Las Vegas Valley, showing differential subsidence due to pumping superposed on regional subsidence (from Malmberg, 1960).

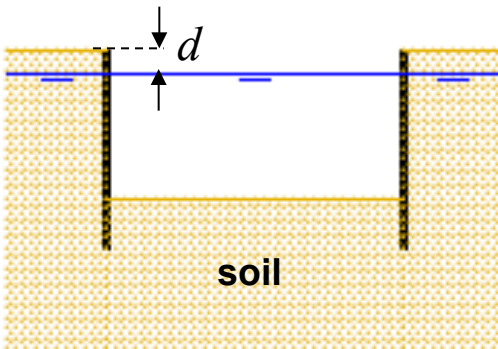


Question(1/3): floatation

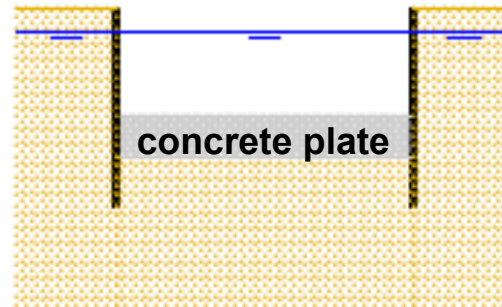
1. A concrete floor under water.

Examples: foundations of basements, or pavements of the access road of a tunnel.

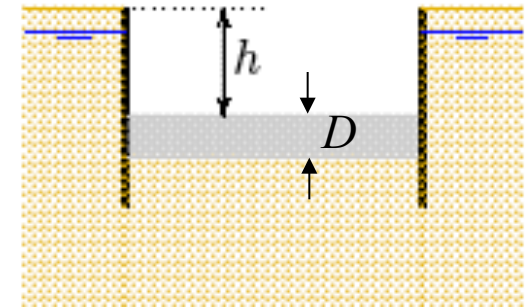
One of the function of the concrete plate is to give additional weight to the soil, in order to prevent the soil to float.



Excavation of the pit under water, with dredging equipment



Construction of the concrete floor under water



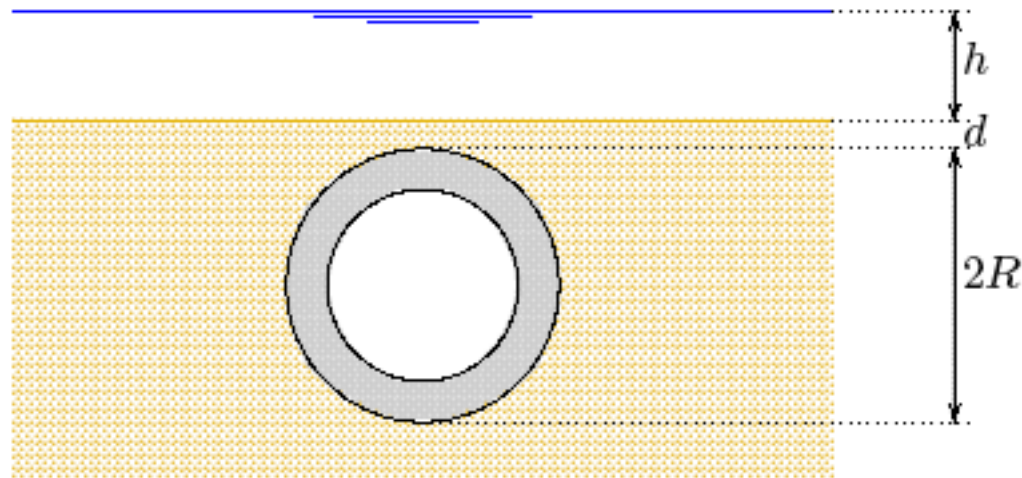
Lowering of water

What is the minimum thickness D of the concrete layer ensuring that it will not float itself (and therefore it will be able to provide additional weight to the soil) ?

Question(2/3): floatation

2. A pipe or a tunnel under water.

For the risk of floatation, the most dangerous situation will be when the structure is empty.



What is the minimum thickness d of the soil above the structure ensuring that the structure it will not float ?

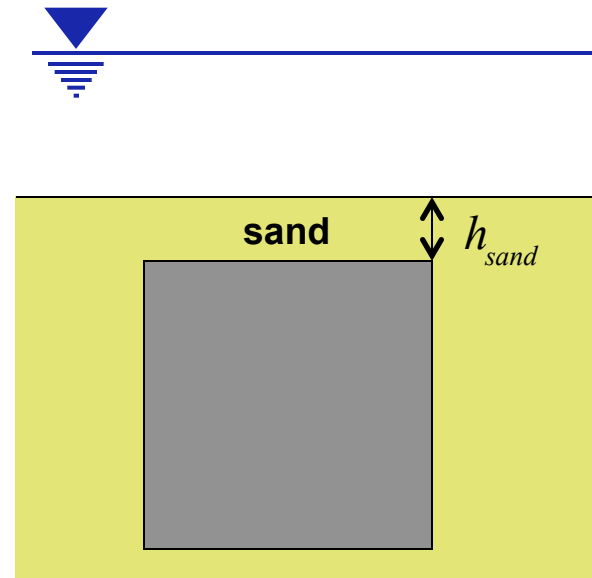
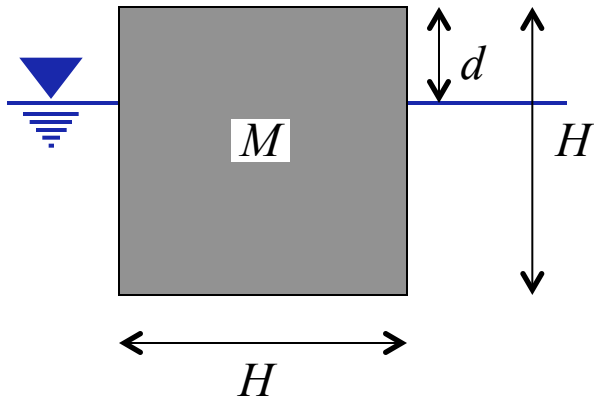
Question(3/3): floatation

3. Miscealinous.

A tunnel os square cross section H^2 has a weight (above water) M per meter length. The tunnel is beeing floated to its destination. Calculate the draught d .

The tunnel is now sunk into a trench that has been dredged in the sand at the bottom of the river, and then covered with sand of volumic weight γ_{sand} . Determine the minimum cover of sand h_{sand} necessary to prevent floatation of the tunnel.

Numerical values: $H = 8 \text{ m}$, $M = 50 \text{ t/mL}$, $\gamma_{sand} = 20 \text{ kN/m}^3$



Question(1/3): nappe côtière phréatique

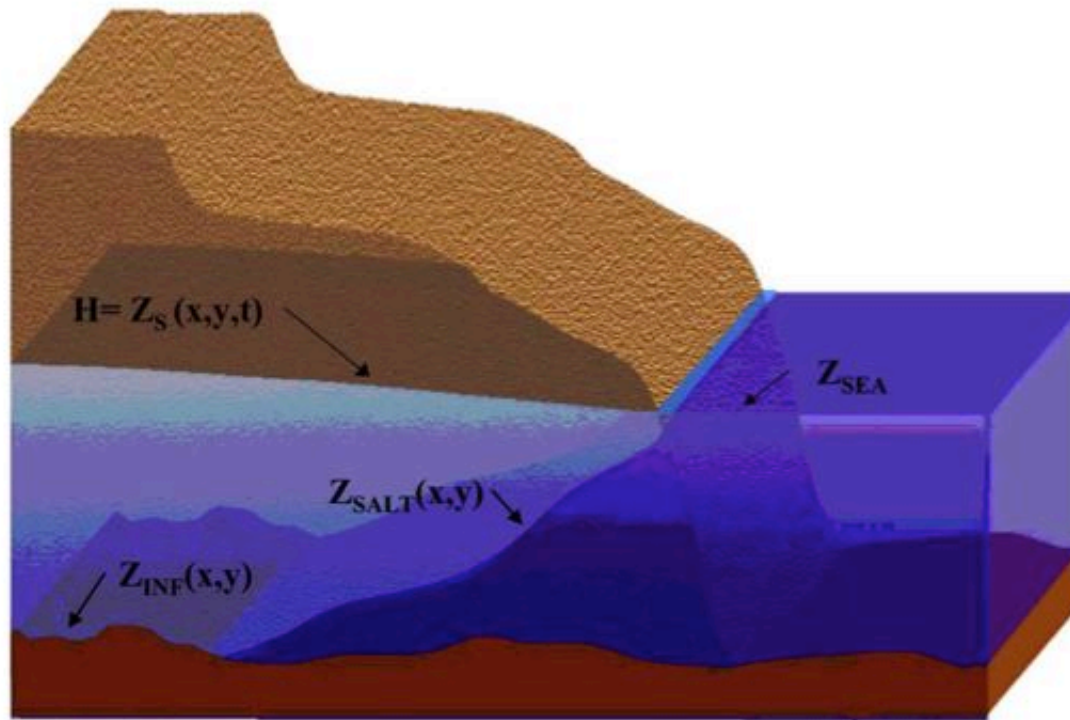


Schéma de l'intrusion d'un coin salé à l'équilibre dans une nappe côtière phréatique, sans recharge ni pompages (vue en perspective, milieux hétérogène sans symétrie plane).

Question(2/3): nappe côtière phréatique

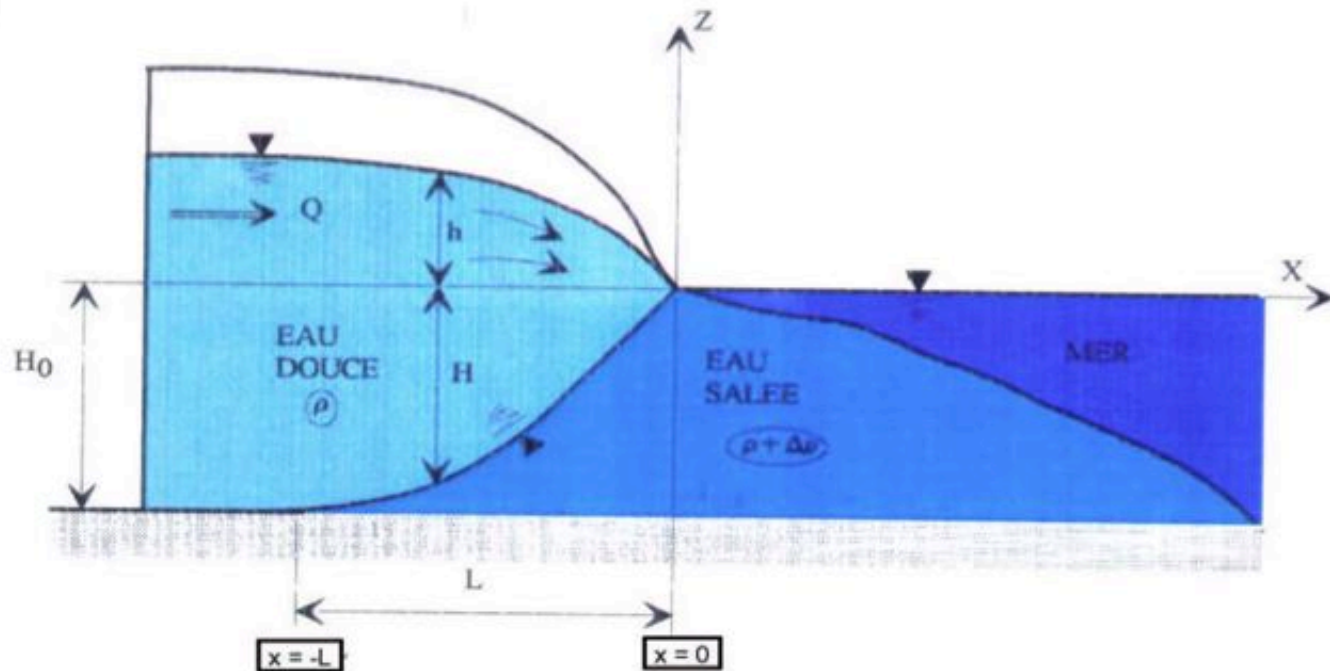


Schéma de l'intrusion d'un coin salé à l'équilibre dans une nappe côtière phréatique, sans recharge ni pompages, en symétrie plane (la coupe est transverse au trait de côte).

Question(3/3): nappe côtière phréatique

Principe de Ghyben-Herzberg

1) On suppose l'équilibre hydrostatique.

Ecrire la relation entre p_w et h_w .

Ecrire la relation entre p_f et h_f .

2) On suppose que la pression est continue à travers l'interface.

Ecrire la relation entre h_w et h_f .

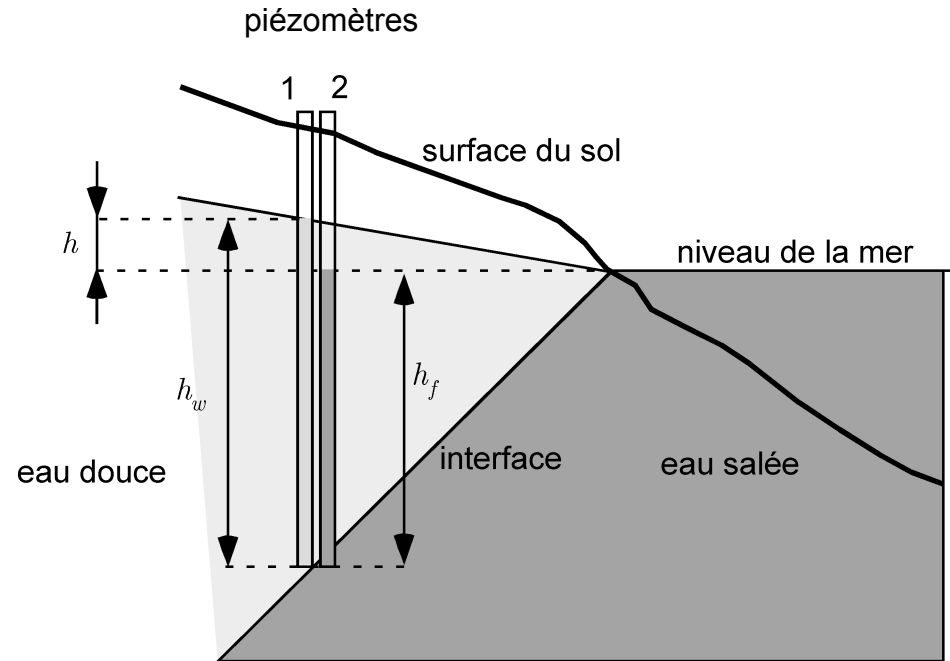
3) On note h l'élévation de la surface libre au-dessus du niveau de la mer.

Exprimer h_f en fonction de h .

Application numérique:

$$\rho_w = 1000 \text{ kg/m}^3$$

$$\rho_f = 1025 \text{ kg/m}^3$$



ρ_w : masse volumique de l'eau douce

p_w : pression dans l'eau douce

h_w : hauteur de colonne d'eau douce dans le piézo 1

ρ_f : masse volumique de l'eau salée

p_f : pression dans l'eau salée

h_f : hauteur de colonne d'eau salée dans le piézo 2