ResiWater deliverable report D4.2: Stable and reliable Modelling of Control Devices

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RESIWATER

Deliverable Report D4.2

INNOVATIVE SECURE SENSOR NETWORKS AND MODEL-BASED ASSESSMENT TOOLS FOR INCREASED RESILIENCE OF WATER INFRASTRUCTURES

Deliverable 4.2

Stable and reliable Modelling of Control Devices

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Robust hydraulic simulation tools

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WP 4 – Robust hydraulic simulation tools

D4.2 Stable and reliable Modelling of Control Devices

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Objectives The objective of this report is to develop a suitable framework for dealing with large disconnected subnetwork undergoing a partially or full system collapse with focus on the behaviour of control devices.			

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1 SUMMARY

This report describes different methods for modelling of pressurized water supply networks with consideration of pressure dependent demands (PDM: Pressure Driven Modelling) in combination with flow constraints. For that purpose, the PDM model described in D 4.1 is extended by additional bounds for the link flows that refer to closed or failed links (equality conditions) and the operation of more sophisticated and automated flow control devices such as check valves and flow control valves (inequality constraints). In contrast to the pure PDM model where the proof of the Linear Constraint Qualification (LICQ) to hold was simple (only assumption is connectivity of the network) the same is not true for general flow bounded problems. A simple example is given by the case where two valves are closed in a path disconnecting at least one node from the rest of the system. If the disconnected node is a demand node then in DDM the problem would have no solution. In contrast, PDM allows to reduce the outflow to zero such that a solution exists. Adding the two flow constraints with $q = 0$ for the closed valves to the continuity equation results in linear dependency of the constraints, which in turn leads to a singular Jacobian. Because of the singularity, there is non-uniqueness of equation, with infinity of solutions for nodal heads and minor losses of active flow control devices (the Lagrangian multipliers associated with the problem constraints). In this report, it will be shown that almost all problems of non-convergence are related to the problem of linear dependent constraints and the resulting singularity of the system. Consequently, the central problem of the robust solver is the prevention of linear dependent constraints or the derivation of methods that deal explicitly with non-unique multipliers.

The content of the deliverable is as follows. After a brief introduction that summarizes the problem of modelling disconnected systems a relevant Literature about flow and pressure control modelling in hydraulic system analysis is reviewed. In the main part the extended PDM model for flow controlled problems is derived and the appearance of non-unique pressure heads is discussed, Then, different approaches for avoiding singularities in the network equations are summarized and compared as to their robustness, efficiency and practical applicability.

2 INTRODUCTION

The stable and robust calculation of WDS hydraulics and water quality under anomalous operational conditions as they appear under extreme events like natural disasters, (terrorist) attacks or electrical power black outs is a basic requirement for all model-based decisions. Existing simulation techniques are not prepared for these situations and often fail at calculating a solution or to converge. There is a strong need for improved mathematical methods that successfully deal with ill-posed systems and other situations where existing modelling techniques reach the limits of their theoretical basis.
One important step towards robust and realistic modelling of extreme situations is the development of a robust hydraulic system solver that can deal, amongst others, with insufficient pressure conditions. Considering a scenario with numerous failures of system devices like pumps, control valves or pipe breaks the network is decomposed into different parts that might be connected to the sources only by pipes with insufficient diameter or not connected at all. In this case, the state of the art demand driven models fail to converge or to calculate reliable results. First published attempts of implementation of pressure dependent modelling still have problems to calculate the correct results for highly interrupted systems.
In addition, extreme operational conditions have a strong impact on the hydraulic performance of control devices and pumping stations. It is of course state of the art at all the water utilities to be prepared for energy blackouts and failure of single pumps. However, for example area-wide interruption of power supply for a longer period or massive damage due to flooding or attacks put such a tremendous stress on the system that is not included in existing emergency plans. Reasons are the strong interdependencies and the lack of existing modelling tools that can simulate those situations that are all characterized by disconnecting parts of the system. From a modelling point of view this is a challenge since the system is changed by disruptions and the hydraulic steady-state of the disconnected part may not be well defined. It must be noted here that such events are not steady-state in nature but cause transient flows and pressures within the system that can also result in situations where the basic assumptions of modelling pressurized pipe systems do not hold anymore (e. g. emptying of pipes). The time of transient behaviour is not addressed in this research. It is assumed that after a certain time of adaption that includes for example closing of valves to isolate damaged pipes and to stop large leakage losses the system reaches a stable state whose mathematical representation by steady-state modelling is justifiable.

The objective of this deliverable is to develop a mathematical framework that is able to calculate the steady-state for (stabilized) disrupted, possibly disconnected systems. The problem will be studied from a theoretical (existence and uniqueness of solutions) as well as from a practical (numerical solution) point of view. As a key question will appear how to deal with singularities in the system matrix in a Newton type solution procedure. It will be shown that singular and ill-posed systems are always the consequence of the appearance of disconnected components. For motivation, in the following part of this chapter the situation is visualized for a simple example system.

In real water supply systems, both, routine network operations as well as operations that are necessary as a response to extreme events (pipe burst, contamination) may require disconnection of parts of the system by valve closure. Consider the following simple system that consist of a supply are (nodes b, c) that gets water from two reservoirs R1 and R2. If for some reasons the supply from one reservoirs shall be stopped the valves 2 and 4 can be shut off. In the very rare case that both valves are closed at the same time the supply are is disconnected from any water source and consumption demands cannot be satisfied.

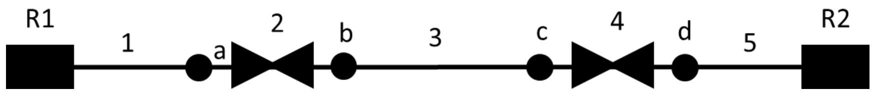


Figure 1: Example system with two control devices in series.

A similar situation can occur if the valves are automated flow control valves that restrict the flow to be in-between given bounds. Consider the case were R1 supplies again nodes b and c and should fill tank R2. The flow from R2 into c and b is prohibited by a non-return valve. Then, if valve 2 is closed, nodes b and c are again disconnected since also the non-return valves closes automatically. In the last case, the valves are flow control valves that restrict the flow to a upper bound. If the demand of b and c is greater than the possible inflow to the supply area the consumption is reduced accordingly.

Mathematically, all three cases have in common that the decomposition of the network graph results in singularity of the system of equations that are necessary for solution. Hence, the numerical solution by Newton-type method fails. In the following we refer to cases:

1.) Fully disconnected system (valve closure after pipe burst, for inspection, isolation of contaminant modelled by equality conditions)
2.) System disconnected by automatic operation of flow control devices (inequality conditions)
3.) System connected with restricted flow (active flow control)

Case 1 is distinguished from case 2 by the fact that valve closures are known a priori whereas in the second case the state of the valve is dependent on the hydraulic state of the entire system. In mathematical language, the first case is expressed by equality conditions and the second by inequality conditions. In the third case the system is physically connected however, mathematically it is also disconnected as in the two cases before by active upper bound for the flow.

Existence (and uniqueness) of a solution depends on the model type to be used. In general, of the disconnected subnet has at least one demand node with desired consumption ($d > 0$) then there exists no solution for DDM models in case 1 and 2. If the total demand of the disconnected network part is bigger than the sum of flow bounds into the subnet there is also no solution in DDM models. For PDM the situation is different. For all three cases, there exists a solution because the consumption of disconnected demand nodes is reduced to zero by the model. The only reason for non-existence is the choice of contradicting flow constraints imposed by control devices. This is possible only of zero flow is not inside the feasible range of flows for the device. This case is possible only for combination of pump control (maintain minimum flow) with flow control (do not exceed maximum flow) and not of practical relevance. In this deliverable, the three cases will be investigated from a mathematical point of view. As it was indicated by the simple example, PDM modelling greatly enhances the possibilities for modelling flow control. However, it will be also seen that singularity due to system decomposition is still a challenge for all type of numerical solution methods.

3 LITERATURE REVIEW

Computer hydraulic solvers such as Porteau, SIR 3S or EPANET allow complex water distribution system networks to be simulated by design engineers. Modelling of systems containing combinations of pressure reducing valves (PRVs), pressure sustaining valves (PSVs) and flow control valves (FCVs) can sometimes be problematical. The status of PRVs, PSVs and FCVs is usually solved for based on a heuristic approach. Starting from an assumed state of the control device, a check is performed during the next step of the iteration procedure, as to whether the original assumption of each assumed valve status still holds. If the assumption proves to be no longer true, a correction of the state of the control device is made. It has been observed in the past that this approach may sometimes fail to converge at all, or even worse may converge to incorrect solutions when simulating networks that include pressure regulating devices. In most cases for which no correct solution was obtained, the solver converged quickly to an incorrect solution. Hence, no warning message was generated to indicate there was a problem with the solution. Some simple networks incorporating PRVs and FCVs have been considered in this research. The networks have multiple pressure regulating and flow regulating devices that are placed in series and separated by substantial lengths of pipe. Even for simple networks comprising two pressure regulating devices (one pressure regulating device and one flow regulating device) in series, current hydraulic modelling software fails to converge to the correct solution for particular configurations and settings. This paper aims to consider the circumstances in which such difficulties occur and the reasons for the failure to solve. The problem of non-uniqueness of nodal pressures is discussed also in (Gorev, Kodzhespirova, & Sivakum, 2016).

The ResiWater project deals with challenges that result from situations where the connectivity of the network is lost due to massive system failures caused by extreme events that often lead to insufficient pressure conditions even in the remaining system.

Some limitations of DDM and PDM Modelling for large deficient networks was presented at the CCWI 2016 conference (Braun et al. 2016). From literature, the notion of deficient networks can take several different definitions. These definitions may be divided into model, mathematical and physical deficiencies. Model deficiencies are errors in the creation, conversion or transfer of the network graph. A mathematical deficiency can be defined as a maximal connected network where, due to some boundary condition the set of feasible solutions is reduced to the empty set or the solution is not unique. In contrast to mathematical deficiencies, in the case of a hydraulic deficiency a unique solution exists, but it is physically incorrect. In the following several deficiency phenomena of special interest for the ResiWater project are presented and evaluated with respect to demand and pressure driven modelling:

- **Conflicting constraints:** The first scenario consist of boundary conditions in conflict for certain parts of the network. This may occur if flow regulating devices are incorporated into the model and introduce additional constraints to the mathematical model. In unfortunate cases, these constraints may conflict with the demand request of the consumption nodes. Simply put, the flow entering a region of the network is not satisfying the required demand. In demand driven modelling this reduces the set of feasible solutions for the Content optimization problem to empty set as demonstrated by Deuerlein et al. (2012). Deuerlein also suggests an algorithm to determine if a feasible solution exists for this scenario. Looking at the pressure dependent calculation of the same system, it can be shown that by loosening the demand boundary conditions the system becomes solvable again, but the consumers will be supplied with a reduced flow.

- **Ambiguous constraints:** Another example for a mathematical deficiency is given if the boundary conditions allow for an infinite number of solutions. In their article, Gorev et al. (2016) describe a scenario where two flow control valves (FCV) are installed in series. In this case, the two FCVs create a combined head-loss, but due to the ambiguous nature of this problem an infinite number of solutions exist and it is impossible to determine which of the two FCVs contributes how much. This phenomenon is neither addressed by DDM nor by PDM approaches.
- **Pipe rupture:** In respect to resilience, phenomena like pipe ruptures (or bursts) are of special interest. In this case, the massive water loss dominates the flow in the network. Recent research has shown that the Fixed and Varied Area Discharge (FAVAD) model for leakage outflow provides a good description for leakage behaviour of elastic materials Van Zyl & Cassa (2014). Due to the pressure-dependent nature of the phenomenon, in demand-driven modelling it is not possible to adequately handle the problem. In contrast, like the pressure driven demand, it is possible to solve these problems in the PDM framework.
- **Presence of high-lying nodes supplying a demand zone:** The fourth scenario is correlated with the occurrence of low pressure zones in the network. This may for instance be triggered by a pipe burst and the subsequent pressure loss. Looking at current demand and pressure driven models this behaviour is not considered. In the case of zero or negative pressure, software packages like Porteau, SIR 3S and Epanet will give a warning notifying the user that pressure dropped below zero, but the hydraulic connection is still intact and disconnected network parts will still be supplied. A conceptually simple way to solve this problem in the PDM framework may be implemented by an iterative approach that analyses the pressure on every node and deletes all links connected to the deficient ones. A different approach has been proposed by Piller and Van Zyl (2009). They introduce artificial pressure valves that reduces the flow passing high-lying nodes to zero.

The focus of this deliverable is on the impact of system deficiencies control valve behaviour. For that purpose, a mathematical modelling framework is derived that is suited for modelling deficient systems with consideration of pressure dependent demands and general flow and pressure control. Two Lemmas are formulated that guarantee the existence and uniqueness of a solution. The numerical calculation of the solution is discussed and challenges arising from the risk of running into singular systems during the iterative process are highlighted.

4 MATERIALS AND METHODS

4.1 Motivation and problem formulation

4.1.1 Background and objective

Real water supply networks often include several flow control devices. There are different types of flow control. The simplest type of flow control valves are the throttle control valves. Completely closed they are used for isolation of pipes during rehabilitation work. They can be also partly closed to reduce the flow through the valve but no not stop it completely. The state of the valves is fixed and can changed normally only manually. More sophisticated hydraulic behavior can be found in the group of check valves (unilateral flow control) for backflow prevention. They allow flow in one direction only and close automatically if flow direction would change due to hydraulic conditions in the system. This kind of valves is required for protection of pumps in case of an emergency (power blackout or similar). More complicated are flow control valves to restrict the flow through the valve to a certain maximum flow. For operation of such valves, a closed control loop is implemented consisting of flow measurement in combination with a motor valve. From a modelling point of view, flow control devices can be subdivided into valves with fixed status (open, closed, fixed partly opened state) and those whose hydraulic behavior or status is described by inequality conditions for the flow.

The modeling of flow control devices has been widely studied. However, there is still a lack of robust simulation algorithms that can deal with extreme situations where several pipes are in failure mode and, as a result, the system might be decomposed into several disconnected parts. One possible approach to analyze such situations (with use of existing tools) is to remove the closed (or broken) pipes from the system, check connectivity and analyze the different resulting components separately with a remaining risk that flow control devices with inequality conditions result in further decomposition. However, this approach is not practicable especially for online calculations since the Jacobian matrix of the system must be changed (multiple incidence matrix manipulations, …). The objective of the robust system solver development (deliverable 4.4) is to overcome existing limitations. The desired outcome is a comprehensive mathematical model that includes all the above-mentioned constraints and a corresponding stable algorithm that finds a solution in any case. The task of the mathematical model is to proof existence and uniqueness of a solution, the algorithm must be able to converge also in degenerate cases.

In this chapter, first, the mathematical model of the steady-state solution for pressurized pipe systems with consideration of pressure dependent demands in combination with flow control devices is derived from the PDM Model of deliverable 4.1. It will be shown that if a solution exists the flows are always unique due to the strict convexity of the System Content function. The range of models for which such solution exists is strongly increased by relaxation of demand constraints in PDM modeling. There is no risk of isolated demands anymore since the PDM model can reduce the outflow to zero. The only situation for nonexistence can result from improper choice of flow constraints of flow control devices causing that the polyhedral set that is composed of the continuity equation and the flow constraints of the control devices and outflow conditions is empty. It will be shown that under a certain constraint qualification the KKT conditions are necessary for a unique solution of the problem.

Then, the degenerate case of a solution with non-unique Lagrangian multipliers is discussed. This case appears if at the solution point the LICQ (linear constraint qualification) does not hold. It will be shown that linear dependency of flow constraints is equivalent with the decomposition of the distribution network into several components. Remembering that the Lagrangian multipliers of the equality constraint for network continuity refer to nodal heads this means that also the nodal heads are not uniquely defined in this situation.
Moving from the theoretical discussions to the development of a robust numerical algorithm that can implemented for practical solution of the steady-state equations some regularization techniques will be proposed for avoiding singularity of the system equations.

4.2 Mathematical model

4.2.1 Content of flow control devices

The content of a flow control device is described by a similar model that was used for the PDM nodes. There is a range of flows between the lower and upper bound where the headloss along the link is described by a nonlinear function of the flow (chosen pipe headloss function for pipes and minor headloss for devices). For example, the task of a Flow Control Valve (FCV) is to restrict the flow to a maximum set flow q_{set} independent from the current head difference between initial node and last node (see Figure 2). If the flow is below the maximum set flow q_{set}, the valve is fully opened and behaves as a minor loss element. If the flow would exceed the set flow the head loss coefficient is increased (by reducing the opening of the valve until the set flow is reached again. In steady-state modelling this control behavior can be represented by the multivalued mapping shown in Figure 2. If the valve is in active control mode, it operates on the vertical line in Figure 2 left. In this case, the head-difference between upstream and downstream node of the valve is composed by the headloss h_{set} of the open valve for the set flow q_{set} and an additional minor headloss that is required for restricting the flow to q_{set}. In the mathematical model the additional minor headloss is represented by the Lagrangian multiplier of the active flow constraint.

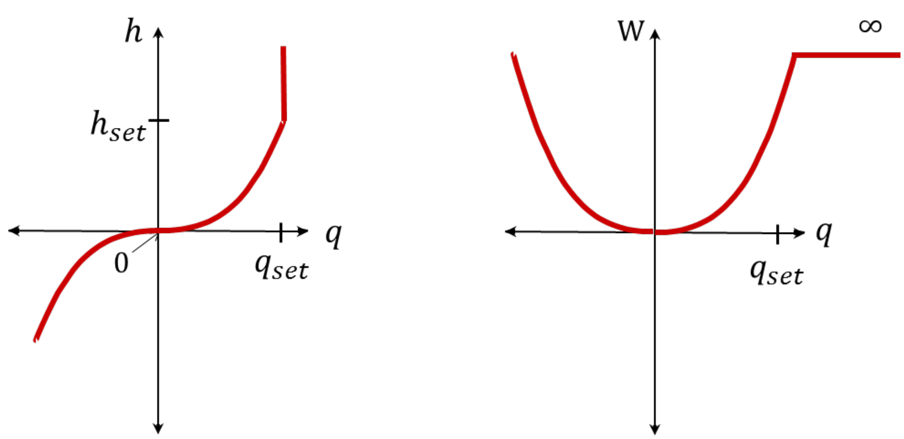


Figure 2: Multivalued subdifferential mapping and Content for Flow Control Valve (FCV).

The content of the multivalued subdifferential mapping $q \mapsto h$ of the FCV is defined by the lower semi-continuous function W on the right-hand site of Figure 2.

A similar relationship can be formulated for a non-return valve (check valve). Here, the flow is possible only in one direction leading to the inequality condition $q \geq 0$. If the constraint is binding (active, fulfilled with equality) the head difference between upstream and downstream node operates on the vertical line for negative h. In contrast to the FCV where the Lagrangian multiplier that is linked with the active flow constraint represents a minor headloss, the Lagrangian multiplier of the active CHV constraint refers to the head difference of the to nodes that are not connected any more due to valve closure. The multivalued (subdifferential) hydraulic relation of the CV and its corresponding Content function is visualized in Figure 3.

There may be other controls on network links or combinations of them with similar multivalued hydraulic mappings. Or for planning purposes one might be interested in defining a maximum capacity of a pipe. In the following the problem of flow control is generalized. Therefore, it is assumed that for each link of the network graph there could be a lower and upper flow bound. Based on this assumption the Content minimization problem for restricted flow networks is formulated. As it will be seen later the PDM Model (as described I deliverable 4.1) has much better performance as the DDM model due to the increased feasible flow set by allowing the reduction of the consumption for insufficient pressures.

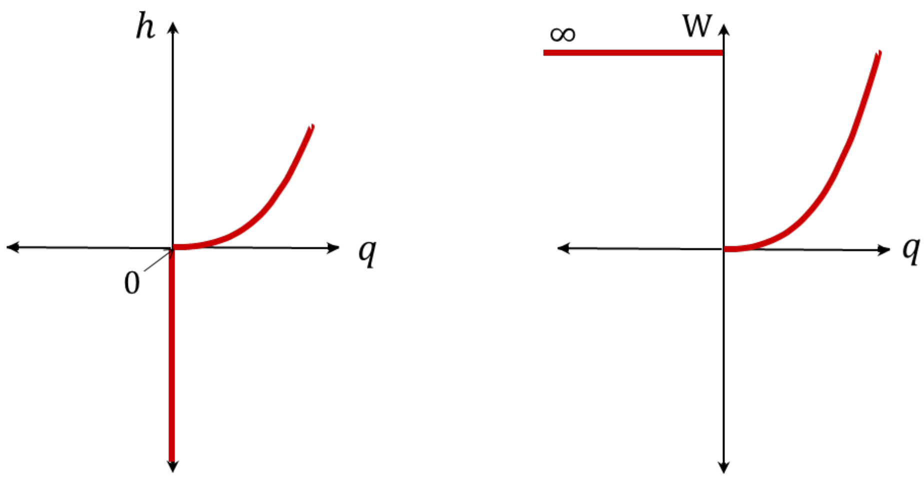


Figure 3: Multivalued (subdifferential) mapping and Content for Check Valve (CV).

Certainly, in real systems, a certain equipment is required to control the flow. Here, for simplicity of notation but without losing the generality it is assumed that every link of the model has an upper and a lower bound for the flow. Then, like Eq. (11) in Deliverable 4.1 the unconstrained (meaning no inequality constraints) minimization problem of the convex and lower semi-continuous Content function is:

$$\min_{[\mathbf{q},\mathbf{c}]\in R^{np+nj}} C(\mathbf{q},\mathbf{c}) = \sum_{j=1}^{np} \overline{\mathrm{W}}_j(q_j) + \sum_{i=1}^{nj} \overline{\mathrm{W}}_i(c_i) - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0$$
$$s.t.\ -\mathbf{A}_1^T\mathbf{q} - \mathbf{c} = \mathbf{0}_{nj} \tag{1}$$

If the Wagner function is used for the POR[1] function, the corresponding constrained minimization of the continuous Content function follows as (see Eq. (13) in D 4.1):

$$\min_{[\mathbf{q},\mathbf{c}]\in R^{np+nj}} C(\mathbf{q},\mathbf{c}) = \mathbf{q}^T\overline{\boldsymbol{\Delta}\mathbf{h}} - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 + \frac{1}{3}\mathbf{c}^T\,\mathbf{N}\mathbf{c} + \mathbf{c}^T\mathbf{h}_{\min}$$
$$\begin{aligned} s.t.\ -\mathbf{A}_1^T\mathbf{q} - \mathbf{c} &= \mathbf{0}_{nj} \\ -\mathbf{c} &\le \mathbf{0}_{nj} \\ \mathbf{c} &\le \mathbf{d} \\ \mathbf{q} - \mathbf{q}_{\max} &\le \mathbf{0} \\ -\mathbf{q} + \mathbf{q}_{\min} &\le \mathbf{0} \end{aligned} \tag{2}$$

The only difference to the PDM model consists in the additional flow constraints. Please note that also equality constraints can be considered in Eq. (2) by adding the same lower and upper bound for the respective link flow. The Lagrangian of the minimization problem of Eq. (2) is defined as:

$$L(\mathbf{q},\mathbf{c},\mathbf{h},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\kappa},\boldsymbol{\upsilon}) = \mathbf{q}^T\overline{\boldsymbol{\Delta}\mathbf{h}} - \mathbf{q}^T\mathbf{A}_0\mathbf{h}_0 + \frac{1}{3}\mathbf{c}^T\,\mathbf{N}\mathbf{c} + \mathbf{c}^T\mathbf{h}_{\min} - \mathbf{h}^T(\mathbf{A}_1^T\mathbf{q} + \mathbf{c}) + \boldsymbol{\mu}^T(\mathbf{c} - \mathbf{d}) - \boldsymbol{\lambda}^T\mathbf{c} +$$
$$\boldsymbol{\kappa}^T(-\mathbf{q} + \mathbf{q}_{\min}) + \boldsymbol{\nu}^T(\mathbf{q} - \mathbf{q}_{\max})$$
$$\boldsymbol{\lambda} \ge \mathbf{0}_{nj}, \boldsymbol{\mu} \ge \mathbf{0}_{nj}, \boldsymbol{\kappa} \ge \mathbf{0}_{np}, \boldsymbol{\nu} \ge \mathbf{0}_{np} \tag{3}$$

In addition to the Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ that refer to active constraints of the nodal consumptions there are also the additional multipliers $\boldsymbol{\kappa}$ and $\boldsymbol{\nu}$ for the active lower and upper bounds of link flows. In this case, the necessary KKT-conditions are:

$$\begin{aligned} \mathbf{G}(\mathbf{q})\mathbf{q} - \mathbf{A}_1\mathbf{h} - \mathbf{A}_0\mathbf{h}_0 - \mathbf{V}_{\mathrm{L}}\boldsymbol{\kappa}^* + \mathbf{V}_{\mathrm{U}}\boldsymbol{\nu}^* &= \mathbf{0}_{\mathrm{np}} \\ \mathbf{N}(\mathbf{c})\mathbf{c} + \mathbf{h}_{\min} - \mathbf{U}_{\mathrm{L}}{}^T\boldsymbol{\lambda}^* + \mathbf{U}_{\mathrm{U}}{}^T\boldsymbol{\mu}^* &= \mathbf{h} \\ -A_1^T\mathbf{q} - \mathbf{c} &= \mathbf{0}_{\mathrm{nj}} \\ -\mathbf{U}_{\mathrm{L}}\mathbf{c} &= \mathbf{0}_m \\ \mathbf{U}_{\mathrm{U}}(\mathbf{c} - \mathbf{d}) &= \mathbf{0}_s \\ -\mathbf{V}_{\mathrm{L}}(\mathbf{q} - \mathbf{q}_{\min}) &= \mathbf{0} \\ \mathbf{V}_{\mathrm{U}}(\mathbf{q} - \mathbf{q}_{\max}) &= \mathbf{0} \end{aligned} \tag{4}$$

The new vectors $\boldsymbol{\kappa}^*$ and $\boldsymbol{\nu}^*$are the Lagrangian Multipliers of active flow bounds at the solution point. They can be interpreted as additional minor head loss penalties to maintain the flow at the setting. The last two equations represent the complementary slackness condition with the index sets $\mathbf{V}_{\mathrm{L}}$ and $\mathbf{V}_{\mathrm{U}}$. The value is 1 for active bounds and 0 else. For solution, a Newton-Raphson type active set projection

[1] POR : Pressure Outflow Relationship ; the Wagner function is described in D4.1/

method in the primal space is proposed. The algorithms start with a feasible flow vector $[\mathbf{q}\ \ \mathbf{c}]^T$ and continues with iteratively solving the linearized system:

$$
\begin{pmatrix}
\mathbf{F}^{(k)} & \mathbf{0}_{np,nj} & -\mathbf{A}_1 & \mathbf{0}_{np,nl} & \mathbf{0}_{np,nu} & -\mathbf{V}_L^{T\,(k)} & \mathbf{V}_U^{T\,(k)} \\
\mathbf{0}_{nj,np} & \mathbf{M}^{(k)} & -\mathbf{I}_{nj} & -\mathbf{U}_L^{T\,(k)} & \mathbf{U}_U^{T\,(k)} & \mathbf{0}_{nj,npl} & \mathbf{0}_{nj,npu} \\
-\mathbf{A}_1^T & -\mathbf{I}_{nj} & \mathbf{0}_{nj,nj} & \mathbf{0}_{nj,nl} & \mathbf{0}_{nj,nu} & \mathbf{0}_{nj,npl} & \mathbf{0}_{nj,npu} \\
\mathbf{0}_{nl,np} & -\mathbf{U}_L{}^{(k)} & \mathbf{0}_{nl,nj} & \mathbf{0}_{nl,nl} & \mathbf{0}_{nl,nu} & \mathbf{0}_{nl,npl} & \mathbf{0}_{nl,npu} \\
\mathbf{0}_{nu,np} & \mathbf{U}_U{}^{(k)} & \mathbf{0}_{nu,nj} & \mathbf{0}_{nu,nl} & \mathbf{0}_{nu,nu} & \mathbf{0}_{nl,npl} & \mathbf{0}_{nl,npu} \\
-\mathbf{V}_L{}^{(k)} & \mathbf{0}_{npl,nj} & \mathbf{0}_{npl,nj} & \mathbf{0}_{npl,nl} & \mathbf{0}_{npl,nu} & \mathbf{0}_{npl,npl} & \mathbf{0}_{npl,npu} \\
\mathbf{V}_U{}^{(k)} & \mathbf{0}_{npu,nj} & \mathbf{0}_{npu,nj} & \mathbf{0}_{npu,nl} & \mathbf{0}_{npu,nu} & \mathbf{0}_{npu,npl} & \mathbf{0}_{npu,npu}
\end{pmatrix}
\begin{pmatrix}
\mathbf{q}^{(k+1)} - \mathbf{q}^{(k)} \\
\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)} \\
\mathbf{h}^{(k+1)} - \mathbf{h}^{(k)} \\
\boldsymbol{\lambda}^{*(k+1)} - \boldsymbol{\lambda}^{*k} \\
\boldsymbol{\mu}^{*(k+1)} - \boldsymbol{\mu}^{*k} \\
\boldsymbol{\kappa}^{*(k+1)} - \boldsymbol{\kappa}^{*k} \\
\boldsymbol{\nu}^{*(k+1)} - \boldsymbol{\nu}^{*k}
\end{pmatrix}
$$

$$
= -\begin{pmatrix}
\mathbf{G}\left(\mathbf{q}^{(k)}\right)\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - \mathbf{A}_0\mathbf{h}_0 - \mathbf{V}_L^{T\,(k)}\boldsymbol{\kappa}^{*k} + \mathbf{V}_U^{T\,(k)}\boldsymbol{\nu}^{*k} \\
\mathbf{N}^{(k)}\mathbf{c}^{(k)} - \mathbf{h}^{(k)} + \mathbf{h}_{min} - \mathbf{U}_{\mathrm{L}}{}^{\boldsymbol{T}}\boldsymbol{\lambda}^{*\boldsymbol{k}} + \mathbf{U}_{\mathrm{U}}{}^{\boldsymbol{T}}\boldsymbol{\mu}^{*\boldsymbol{k}} \\
-\mathbf{A}_1^T\mathbf{q}^{(k)} - \mathbf{c}^{(\boldsymbol{k})} \\
-\mathbf{U}_L{}^{(k)}\mathbf{c}^{(k)} \\
\mathbf{U}_U{}^{(k)}\left(\mathbf{c}^{(k)} - \mathbf{d}\right) \\
-\mathbf{V}_L{}^{(k)}\left(\mathbf{q}^{(\boldsymbol{k})} - \mathbf{q}_{\min}\right) \\
\mathbf{V}_U{}^{(k)}\left(\mathbf{q}^{(\boldsymbol{k})} - \mathbf{q}_{\max}\right)
\end{pmatrix}
\tag{5}
$$

The binding constraints can be used for reducing the number of variables. For that purpose, first, after reordering of the nodes, the rows and columns of the matrices $\mathbf{M}$, $\mathbf{I}, \mathbf{A}_1$ are subdivided into three partitions: the index $_A$ in $(\mathbf{M}_{A,}\mathbf{I}_A)$ and second index in $\mathbf{A}_{*,A}$ refers to nodes, for which the current outflow is dependent on the current pressure. The index $_L$ refers to nodes where no outflow is possible because the pressure is below the minimum pressure and index $_U$ refers to nodes with full supply (pressure above minimum service pressure). Similarly, and in contrast to the pure PDM formulation without flow constraints, the rows of $\mathbf{A}_1$ are reordered and subdivided into three parts. First index A refers to links with no active flow constraints, first index L denotes links for which the lower bound is active and first index U means that the links have active upper bounds. The modified system is shown in the following Eq. (6). The dimension on the zero matrices are omitted for sake of simplicity and improved readability. Since we are dealing with simple bound constraints only the matrices $\mathbf{U}$ and $\mathbf{V}$ are identity matrices and have been removed from the right-hand side.

$$
\begin{pmatrix}
\mathbf{F}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{A,A} & -\mathbf{A}_{A,L} & -\mathbf{A}_{A,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{F}_L & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{L,A} & -\mathbf{A}_{L,L} & -\mathbf{A}_{L,U} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{ql}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{F}_U & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{U,A} & -\mathbf{A}_{U,L} & -\mathbf{A}_{U,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{qu}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_A & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_L & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_U & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,A}{}^T & -\mathbf{A}_{L,A}{}^T & -\mathbf{A}_{U,A}{}^T & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,\mathrm{L}}{}^T & -\mathbf{A}_{L,L}{}^T & -\mathbf{A}_{U,A}{}^T & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{\mathrm{A},U}{}^T & -\mathbf{A}_{\mathrm{L},U}{}^T & -\mathbf{A}_{\mathrm{U},U}{}^T & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\mathbf{I}_{\mathrm{ql}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{qu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mathbf{q}_A{}^{(k+1)} - \mathbf{q}_A{}^{(k)} \\
\mathbf{q}_L{}^{(k+1)} - \mathbf{q}_L{}^{(k)} \\
\mathbf{q}_U{}^{(k+1)} - \mathbf{q}_U{}^{(k)} \\
\mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\
\mathbf{c}_L{}^{(k+1)} - \mathbf{c}_L{}^{(k)} \\
\mathbf{c}_U{}^{(k+1)} - \mathbf{c}_U{}^{(k)} \\
\mathbf{h}_A{}^{(k+1)} - \mathbf{h}_A{}^{(k)} \\
\mathbf{h}_L{}^{(k+1)} - \mathbf{h}_L{}^{(k)} \\
\mathbf{h}_U{}^{(k+1)} - \mathbf{h}_U{}^{(k)} \\
\boldsymbol{\lambda}^{*(k+1)} - \boldsymbol{\lambda}^{*(k)} \\
\boldsymbol{\mu}^{*(\boldsymbol{k}+\mathbf{1})} - \boldsymbol{\mu}^{*(k)} \\
\boldsymbol{\kappa}^{*(k+1)} - \boldsymbol{\kappa}^{*(k)} \\
\boldsymbol{\nu}^{*(k+1)} - \boldsymbol{\nu}^{*(k)}
\end{pmatrix}
= -
\begin{pmatrix}
\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{A,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{A,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{\mathrm{A},0}\mathbf{h}_\mathbf{0} \\
\mathbf{G}_L\left(\mathbf{q}_L^{(k)}\right)\mathbf{q}_L^{(k)} - \mathbf{A}_{L,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{L,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{L,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{\mathrm{L},0}\mathbf{h}_\mathbf{0} - \boldsymbol{\kappa}^{*k} \\
\mathbf{G}_U\left(\mathbf{q}_U^{(k)}\right)\mathbf{q}_U^{(k)} - \mathbf{A}_{U,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{U,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{U,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{\mathrm{U},0}\mathbf{h}_\mathbf{0} + \boldsymbol{\nu}^{*k} \\
\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\
-\mathbf{h}_L{}^{(k)} + \mathbf{h}_{L,min} - \boldsymbol{\lambda}^{*(\boldsymbol{k})} \\
\mathbf{h}_{U,S} - \mathbf{h}_U{}^{(k)} + \boldsymbol{\mu}^{*(\boldsymbol{k})} \\
-\mathbf{A}_{A,A}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,A}{}^T\mathbf{q}_L^{(k)} - \mathbf{A}_{U,A}{}^T\mathbf{q}_U^{(k)} - \mathbf{c}_A{}^{(k)} \\
-\mathbf{A}_{A,\mathrm{L}}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,L}{}^T\mathbf{q}_L^{(k)} - \mathbf{A}_{U,L}{}^T\mathbf{q}_U^{(k)} - \mathbf{c}_L{}^{(k)} \\
-\mathbf{A}_{\mathrm{A},U}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{\mathrm{L},U}{}^T\mathbf{q}_L^{(k)} - \mathbf{A}_{\mathrm{U},U}{}^T\mathbf{q}_U^{(k)} - \mathbf{c}_U{}^{(k)} \\
-\mathbf{c}_L{}^{(k)} \\
\mathbf{c}_U{}^{(k)} - \boldsymbol{d}_U \\
-(\mathbf{q}_L^{(k)} - \mathbf{q}_{\mathrm{L,min}}) \\
(\mathbf{q}_U^{(k)} - \mathbf{q}_{\mathrm{U,max}})
\end{pmatrix}
\tag{6}
$$

The system in Eq. (6) can be simplified by elimination of variables with active bounds. First, elimination of $\mathbf{q}_L$ and $\mathbf{q}_U$ by solving the last two rows delivers:

$$
\begin{pmatrix}
\mathbf{F}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{A,A} & -\mathbf{A}_{A,L} & -\mathbf{A}_{A,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{L,A} & -\mathbf{A}_{L,L} & -\mathbf{A}_{L,U} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{ql}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{U,A} & -\mathbf{A}_{U,L} & -\mathbf{A}_{U,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{qu}} \\
\mathbf{0} & \mathbf{M}_A & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{M}_L & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_U & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,A}{}^T & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,\mathrm{L}}{}^T & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{\mathrm{A},U}{}^T & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbf{q}_A{}^{(k+1)} - \mathbf{q}_A{}^{(k)} \\
\mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\
\mathbf{c}_L{}^{(k+1)} - \mathbf{c}_L{}^{(k)} \\
\mathbf{c}_U{}^{(k+1)} - \mathbf{c}_U{}^{(k)} \\
\mathbf{h}_A{}^{(k+1)} - \mathbf{h}_A{}^{(k)} \\
\mathbf{h}_L{}^{(k+1)} - \mathbf{h}_L{}^{(k)} \\
\mathbf{h}_U{}^{(k+1)} - \mathbf{h}_U{}^{(k)} \\
\boldsymbol{\lambda}^{*(k+1)} - \boldsymbol{\lambda}^{*(k)} \\
\boldsymbol{\mu}^{*(\boldsymbol{k}+\mathbf{1})} - \boldsymbol{\mu}^{*(k)} \\
\boldsymbol{\kappa}^{*(k+1)} - \boldsymbol{\kappa}^{*(k)} \\
\boldsymbol{\nu}^{*(k+1)} - \boldsymbol{\nu}^{*(k)}
\end{pmatrix}
\tag{7}
$$

$$
= -\begin{pmatrix}
\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{A,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{A,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{A,0}\mathbf{h}_0 \\
\mathbf{F}_L\left(\mathbf{q}_L^{(k)}\right)\left(\mathbf{q}_{L,min} - \mathbf{q}_L^{(k)}\right) + \mathbf{G}_L\left(\mathbf{q}_L^{(k)}\right)\mathbf{q}_L^{(k)} - \mathbf{A}_{L,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{L,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{L,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{L,0}\mathbf{h}_0 - \boldsymbol{\kappa}^{*k} \\
\mathbf{F}_U\left(\mathbf{q}_U^{(k)}\right)\left(\mathbf{q}_{U,max} - \mathbf{q}_U^{(k)}\right) + \mathbf{G}_U\left(\mathbf{q}_U^{(k)}\right)\mathbf{q}_U^{(k)} - \mathbf{A}_{U,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{U,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{U,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{U,0}\mathbf{h}_0 + \boldsymbol{\nu}^{*k} \\
\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\
-\mathbf{h}_L{}^{(k)} + \mathbf{h}_{L,min} - \boldsymbol{\lambda}^{*(k)} \\
\mathbf{h}_{U,S} - \mathbf{h}_U{}^{(k)} + \boldsymbol{\mu}^{*(k)} \\
-\mathbf{A}_{A,A}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,A}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,A}{}^T\mathbf{q}_{U,max} - \mathbf{c}_A{}^{(k)} \\
-\mathbf{A}_{A,L}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,L}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,L}{}^T\mathbf{q}_{U,max} - \mathbf{c}_L{}^{(k)} \\
-\mathbf{A}_{A,U}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,U}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,U}{}^T\mathbf{q}_{U,max} - \mathbf{c}_U{}^{(k)} \\
-\mathbf{c}_L{}^{(k)} \\
\mathbf{c}_U{}^{(k)} - \mathbf{d}_U
\end{pmatrix}
$$

Then, the known outflows that belong to the active POR constraints (last two rows in the matrix Eq. (7)) are removed and the known consumptions $\mathbf{c}_L$ and $\mathbf{c}_U$ (columns with index "L" and index "U") are put on the right-hand side of the system of equations.

$$
\begin{pmatrix}
\mathbf{F}_A & \mathbf{0} & -\mathbf{A}_{A,A} & -\mathbf{A}_{A,L} & -\mathbf{A}_{A,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{A}_{L,A} & -\mathbf{A}_{L,L} & -\mathbf{A}_{L,U} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{ql}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{A}_{U,A} & -\mathbf{A}_{U,L} & -\mathbf{A}_{U,U} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathrm{qu}} \\
\mathbf{0} & \mathbf{M}_A & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & -\mathbf{I}_{\mathrm{cl}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{I}_{\mathrm{cu}} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,A}{}^T & -\mathbf{I}_{\mathrm{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,L}{}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\mathbf{A}_{A,U}{}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbf{q}_A{}^{(k+1)} - \mathbf{q}_A{}^{(k)} \\
\mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\
\mathbf{h}_A{}^{(k+1)} - \mathbf{h}_A{}^{(k)} \\
\mathbf{h}_L{}^{(k+1)} - \mathbf{h}_L{}^{(k)} \\
\mathbf{h}_U{}^{(k+1)} - \mathbf{h}_U{}^{(k)} \\
\boldsymbol{\lambda}^{*(k+1)} - \boldsymbol{\lambda}^{*(k)} \\
\boldsymbol{\mu}^{*(k+1)} - \boldsymbol{\mu}^{*(k)} \\
\boldsymbol{\kappa}^{*(k+1)} - \boldsymbol{\kappa}^{*(k)} \\
\boldsymbol{\nu}^{*(k+1)} - \boldsymbol{\nu}^{*(k)}
\end{pmatrix}
$$

$$
= -\begin{pmatrix}
\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{A,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{A,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{A,0}\mathbf{h}_0 \\
\mathbf{F}_L\left(\mathbf{q}_L^{(k)}\right)\left(\mathbf{q}_{L,min} - \mathbf{q}_L^{(k)}\right) + \mathbf{G}_L\left(\mathbf{q}_L^{(k)}\right)\mathbf{q}_L^{(k)} - \mathbf{A}_{L,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{L,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{L,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{L,0}\mathbf{h}_0 - \boldsymbol{\kappa}^{*k} \\
\mathbf{F}_U\left(\mathbf{q}_U^{(k)}\right)\left(\mathbf{q}_{U,max} - \mathbf{q}_U^{(k)}\right) + \mathbf{G}_U\left(\mathbf{q}_U^{(k)}\right)\mathbf{q}_U^{(k)} - \mathbf{A}_{U,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{U,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{U,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{U,0}\mathbf{h}_0 + \boldsymbol{\nu}^{*k} \\
\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\
-\mathbf{M}_L\left(\mathbf{c}_L{}^{(k)}\right) - \mathbf{h}_L{}^{(k)} + \mathbf{h}_{L,min} - \boldsymbol{\lambda}^{*(k)} \\
-\mathbf{M}_U\left(\mathbf{c}_U{}^{(k)} - \boldsymbol{d}_U\right) + \mathbf{h}_{U,S} - \mathbf{h}_U{}^{(k)} + \boldsymbol{\mu}^{*(k)} \\
-\mathbf{A}_{A,A}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,A}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,A}{}^T\mathbf{q}_{U,max} - \mathbf{c}_A{}^{(k)} \\
-\mathbf{A}_{A,L}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,L}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,L}{}^T\mathbf{q}_{U,max} \\
-\mathbf{A}_{A,U}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,U}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,U}{}^T\mathbf{q}_{U,max} - \boldsymbol{d}_U
\end{pmatrix}
\quad (8)
$$

In the previous equation, the last four columns that belong to the Lagrangian multipliers are decoupled from the rest of the system. After elimination of the corresponding rows and columns the remaining system is:

$$\begin{pmatrix} \mathbf{F}_A & \mathbf{0} & -\mathbf{A}_{A,A} & -\mathbf{A}_{A,L} & -\mathbf{A}_{A,U} \\ \mathbf{0} & \mathbf{M}_A & -\mathbf{I}_{\text{ca}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{A,A}{}^T & -\mathbf{I}_{\text{ca}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{A,\text{L}}{}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{\text{A},U}{}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{q}_A{}^{(k+1)} - \mathbf{q}_A{}^{(k)} \\ \mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\ \mathbf{h}_A{}^{(k+1)} - \mathbf{h}_A{}^{(k)} \\ \mathbf{h}_L{}^{(k+1)} - \mathbf{h}_L{}^{(k)} \\ \mathbf{h}_U{}^{(k+1)} - \mathbf{h}_U{}^{(k)} \end{pmatrix} = - \begin{pmatrix} \mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,A}\mathbf{h}_A{}^{(k)} - \mathbf{A}_{A,L}\mathbf{h}_L{}^{(k)} - \mathbf{A}_{A,U}\mathbf{h}_U{}^{(k)} - \mathbf{A}_{\text{A},0}\mathbf{h}_\mathbf{0} \\ \mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\ -\mathbf{A}_{A,A}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,A}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,A}{}^T\mathbf{q}_{U,max} - \mathbf{c}_A{}^{(k)} \\ -\mathbf{A}_{A,\text{L}}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,L}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,L}{}^T\mathbf{q}_{U,max} \\ -\mathbf{A}_{\text{A},U}{}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{\text{L},U}{}^T\mathbf{q}_{L,min} - \mathbf{A}_{\text{U},U}{}^T\mathbf{q}_{U,max} - \boldsymbol{d}_U \end{pmatrix} \tag{9}$$

Once the above system is solved the multipliers can be calculated as follows:

$$\begin{gathered} \boldsymbol{\lambda}^{(k+1)} = \mathbf{h}_{L,min} - \mathbf{h}_L{}^{(k+1)} - \mathbf{M}_L\left(\mathbf{c}_L{}^{(k)}\right) \\ \boldsymbol{\mu}^{(k+1)} = \mathbf{M}_U\left(\mathbf{c}_U{}^{(k)} - \boldsymbol{d}_U\right) + \mathbf{h}_U{}^{(k+1)} - \mathbf{h}_{U,S} \\ \boldsymbol{\kappa}^{(k+1)} = \left(\mathbf{F}_L\left(\mathbf{q}_L^{(k)}\right)\left(\mathbf{q}_{L,min} - \mathbf{q}_L^{(k)}\right) + \mathbf{G}_L\left(\mathbf{q}_L^{(k)}\right)\mathbf{q}_L^{(k)} - \mathbf{A}_{L,A}\mathbf{h}_A{}^{(k+1)} - \mathbf{A}_{L,L}\mathbf{h}_L{}^{(k+1)} - \mathbf{A}_{L,U}\mathbf{h}_U{}^{(k+1)} - \mathbf{A}_{\text{L},0}\mathbf{h}_\mathbf{0}\right) \\ \boldsymbol{\nu}^{(k+1)} = -\left(\mathbf{F}_U\left(\mathbf{q}_U^{(k)}\right)\left(\mathbf{q}_{U,max} - \mathbf{q}_U^{(k)}\right) + \mathbf{G}_U\left(\mathbf{q}_U^{(k)}\right)\mathbf{q}_U^{(k)} - \mathbf{A}_{U,A}\mathbf{h}_A{}^{(k+1)} - \mathbf{A}_{U,L}\mathbf{h}_L{}^{(k+1)} - \mathbf{A}_{U,U}\mathbf{h}_U{}^{(k+1)} - \mathbf{A}_{\text{U},0}\mathbf{h}_\mathbf{0}\right) \end{gathered} \tag{10}$$

The red terms can be removed if the constraint is already saturated/active at iteration k.

If we replace $(-\mathbf{I}_{\text{ca}} \quad \mathbf{0} \quad \mathbf{0})$ by $-\tilde{\mathbf{I}}_{ca}$ and $(\mathbf{0} \quad \mathbf{0} \quad -\mathbf{I}_{,\text{u}})$ by $-\tilde{\mathbf{I}}_{\text{cu}}$ Eq (9) can be written in the more compact form:

$$\begin{aligned} \begin{pmatrix} \mathbf{F}_A & \mathbf{0} & -\mathbf{A}_{A,1} \\ \mathbf{0} & \mathbf{M}_A & -\tilde{\mathbf{I}}_{ca} \\ -\mathbf{A}_{A,1}^T & -\tilde{\mathbf{I}}_{ca}{}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_A{}^{(k+1)} - \mathbf{q}_A{}^{(k)} \\ \mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\ \mathbf{h}^{(k+1)} - \mathbf{h}^{(k)} \end{pmatrix} \\ = - \begin{pmatrix} \mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,1}\mathbf{h}^{(k)} - \mathbf{A}_{\text{A},0}\mathbf{h}_\mathbf{0} \\ \mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\ -\mathbf{A}_{A,1}^T\mathbf{q}_A^{(k)} - \mathbf{A}_{L,1}^T\mathbf{q}_{L,min} - \mathbf{A}_{U,1}^T\mathbf{q}_{U,max} - \tilde{\mathbf{I}}_{ca}{}^T\mathbf{c}_A{}^{(k)} - \tilde{\mathbf{I}}_{cu}{}^T\mathbf{d}_U \end{pmatrix} \end{aligned} \tag{11}$$

For comparison, the system for PDM without flow constraints was:

$$\begin{pmatrix} \mathbf{F} & \mathbf{0} & -\mathbf{A}_1 \\ \mathbf{0} & \mathbf{M}_A & -\tilde{\mathbf{I}} \\ -\mathbf{A}_1^T & -\tilde{\mathbf{I}}^{\text{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}^{(k+1)} - \mathbf{q}^{(k)} \\ \mathbf{c}_A{}^{(k+1)} - \mathbf{c}_A{}^{(k)} \\ \mathbf{h}^{(k+1)} - \mathbf{h}^{(k)} \end{pmatrix} = - \begin{pmatrix} \mathbf{G}\left(\mathbf{q}^{(k)}\right)\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - \mathbf{A}_0\mathbf{h}_\mathbf{0} \\ \mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min} \\ -\mathbf{A}_1^T\mathbf{q}^{(k)} - \mathbf{c}(\mathbf{h}^{(k)}) \end{pmatrix} \tag{12}$$

As it can be seen the second rows referring to the active POR functions are identical. The energy balance includes only the links that have no active flow constraints and the mass balance includes additional terms for the links with active flow constraints. In essence, the system is equivalent with the system that follows from the original network where all links with active flow constraints are replaced by positive and negative fixed demands at the initial and end node that are equal to the known flow of the link.

If the diagonal matrices $\mathbf{F}_A$ and $\mathbf{M}_A$ have full rank, the flow vector and consumption vector can be expressed by the unknown heads:

$$\begin{pmatrix} \mathbf{q}_A^{(k+1)} - \mathbf{q}_A^{(k)} \\ \mathbf{c}_A^{(k+1)} - \mathbf{c}_A^{(k)} \end{pmatrix} = -\begin{pmatrix} \mathbf{F}_A^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_A{}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,0}\mathbf{h_0} - \mathbf{A}_{A,1}\mathbf{h}^{(k+1)} \\ \mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k+1)} + \mathbf{h}_{A,min} \end{pmatrix} \tag{13}$$

Inserting Eq. (13) in the last row of Eq. (12) delivers the Schur complement of the flow controlled hydraulic steady-state equations:

$$\begin{aligned} &\left(\mathbf{A}_{A,1}^T\mathbf{F}_A^{-1}\mathbf{A}_{A,1} + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\tilde{\mathbf{I}}_{ca}\right)\mathbf{h}^{(k+1)} \\ &\qquad = \mathbf{A}_{A,1}^T\mathbf{F}_A^{-1}\left(\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,0}\mathbf{h}_0\right) + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\left(\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} + \mathbf{h}_{A,min}\right) - \mathbf{A}_{A,1}^T\mathbf{q}_A^{(k)} \\ &\qquad - \mathbf{A}_{L,1}^T\mathbf{q}_{min} - \mathbf{A}_{U,1}^T\mathbf{q}_{max} - \tilde{\mathbf{I}}_{ca}{}^{\boldsymbol{T}}\mathbf{c}_A{}^{(k)} - \tilde{\mathbf{I}}_{cu}{}^{\boldsymbol{T}}\mathbf{d}_U \end{aligned} \tag{14}$$

The global gradient solution consists of sequential solving of:

$$\begin{aligned} &\left(\mathbf{A}_{A,1}^T\mathbf{F}_A^{-1}\mathbf{A}_{A,1} + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\tilde{\mathbf{I}}_{ca}\right)\mathbf{h}^{(k+1)} \\ &\qquad = \mathbf{A}_{A,1}^T\mathbf{F}_A^{-1}\left(\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,0}\mathbf{h}_0\right) + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\left(\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} + \mathbf{h}_{A,min}\right) - \mathbf{A}_{A,1}^T\mathbf{q}_A^{(k)} \\ &\qquad - \mathbf{A}_{L,1}^T\mathbf{q}_{min} - \mathbf{A}_{U,1}^T\mathbf{q}_{max} - \tilde{\mathbf{I}}_{ca}{}^{\boldsymbol{T}}\mathbf{c}_A{}^{(k)} - \tilde{\mathbf{I}}_{cu}{}^{\boldsymbol{T}}\mathbf{d}_U \\ &\mathbf{q}_A^{(k+1)} = \mathbf{q}_A^{(k)} - \mathbf{F}_A^{-1}\left(\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,0}\mathbf{h_0} - \mathbf{A}_{A,1}\mathbf{h}_A{}^{(k+1)}\right) \text{-} \\ &\mathbf{c}_A^{(k+1)} = \mathbf{c}_A^{(k)} - \mathbf{M}_A{}^{-1}\left(-\tilde{\mathbf{I}}_{ca}\left(\mathbf{h}^{(k+1)} - \mathbf{h}^{(k)}\right) + \mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} - \mathbf{h}_A{}^{(k)} + \mathbf{h}_{A,min}\right) \\ &\boldsymbol{\lambda}^{(k+1)} = \mathbf{h}_{L,min} - \mathbf{h}_L{}^{(k+1)} \\ &\boldsymbol{\mu}^{(k+1)} = \mathbf{h}_U{}^{(k+1)} - \mathbf{h}_{U,S} \\ &\boldsymbol{\kappa}^{(k+1)} = \left(\mathbf{G}_L\left(\mathbf{q}_L^{(k)}\right)\mathbf{q}_L^{(k)} - \mathbf{A}_{L,A}\mathbf{h}_A{}^{(k+1)} - \mathbf{A}_{L,L}\mathbf{h}_L{}^{(k+1)} - \mathbf{A}_{L,U}\mathbf{h}_U{}^{(k+1)} - \mathbf{A}_{L,0}\mathbf{h_0}\right) \\ &\boldsymbol{\nu}^{(k+1)} = -\left(\mathbf{G}_U\left(\mathbf{q}_U^{(k)}\right)\mathbf{q}_U^{(k)} - \mathbf{A}_{U,A}\mathbf{h}_A{}^{(k+1)} - \mathbf{A}_{U,L}\mathbf{h}_L{}^{(k+1)} - \mathbf{A}_{U,U}\mathbf{h}_U{}^{(k+1)} - \mathbf{A}_{U,0}\mathbf{h_0}\right) \end{aligned} \tag{15}$$

The difference to Eq. (26) of D 4.1 is that the incidence matrix $\mathbf{A}_A$ includes only the rows that belong to links with non-active flow constraints. The additional multipliers $\boldsymbol{\kappa}$ and $\boldsymbol{\nu}$ are required for checking the validity of the KKT-conditions. Once a multiplier becomes negative the decomposition of matrix A must be changed accordingly. For practical implementation, it is only necessary that matrix $\mathbf{A}_A$ is updated by adding or removing rows. The calculation of the multiplier is straight forward by calculation of the difference between nodal heads and the headloss along the link.

4.2.2 Existence and uniqueness of a solution

To prove the existence and uniqueness of a hydraulic steady-state, the content formulation is advantageous. If all the content functions are strictly convex and norm-coercive ($\left|C_j(x_j)\right| \longrightarrow +\infty$ if $\left|x_j\right| \to +\infty$), then the total system content is a strictly convex and norm-coercive function of $(\mathbf{q}, \mathbf{c})$. The strict convexity of the content can be proven by using the monotonicity of the mappings $q_i \mapsto h_i$ of network elements. From the norm-coercivity it follows that the not restricted system content has a minimum. So far, there is no difference between the PDM model in D 4.1 and the flow constrained extension. To prove the existence of a solution it is sufficient to show that the polyhedron that is described by the constraints is nonempty.

$$\boldsymbol{P} = \{\mathbf{x} \in \mathbf{R}^{n_p+n_j} \mid [\mathbf{A}_1^T \ \ \mathbf{I}]\mathbf{x} = \mathbf{0};\ \ \mathbf{x} \le \mathbf{x}_{min}\,; \mathbf{x} \ge \mathbf{x}_{max}\} \ \neq \emptyset \tag{16}$$

with $\mathbf{x}^{\boldsymbol{T}} = [\boldsymbol{q}^{\boldsymbol{T}}\ \boldsymbol{c}^{\boldsymbol{T}}]$, $\mathbf{x}_{min}^{T} = [\mathbf{q}_{min}^{T}\ \mathbf{0}]$, $\mathbf{x}_{m\,ax}^{T} = [\mathbf{q}_{m\,ax}^{T}\ \mathbf{d}]$. If $\mathbf{q}_{min} \le \mathbf{0}$, the pipe flow rates and nodal outflows $\mathbf{q} = \mathbf{0}_{np}$ and $\mathbf{c} = \mathbf{0}_{nj}$ are trivially feasible solutions for the set of constraints (Eq. (16)). It is proven that there always exists a solution to the problem. The more theoretical case where $\boldsymbol{q}_{i\,,min} > \mathbf{0}$ for some $\boldsymbol{i}$ refers to flow constraints that would require pumping is not considered here. Together with the strict convexity of the system content, the existence of a unique flow distribution is proved.

However, this does not guarantee that also the pressure heads are unique. Please remember that the KKT conditions are necessary and sufficient for (strictly) convex problems. Whereas in the pure PDM case without flow constraints, the linear independency condition was automatically fulfilled for connected networks this is not true in the general case with flow bounds.

Proposition 1:

Let $\widehat{\mathbf{A}}_A = \begin{bmatrix} \mathbf{A}_{A,1} \\ \mathbf{I}_A \end{bmatrix}$ be the reduced incidence matrix of the augmented graph where all links and virtual links (referring to consumption nodes) with active flow/consumption bounds have been removed. If the reduced network graph is connected then $\widehat{\mathbf{A}}_A$ has full column rank.

$A_{A,1}$ has as many columns as junction nodes and the rows correspond to links with non-active flow control, including simple links. The following Lemma proves Existence and Uniqueness based on proposition 1.

Lemma 1:

The norm-coercivity was already proven above. Together with the strict convexity of the Content function it follows that there is almost one minimum that corresponds with a unique flow distribution and consumption vector. With Proposition 1 it is also shown that the LICQ holds. As a consequence, the Lagrangian multipliers are unique (the set of Lagrange multiplier vector is a singleton).

In practice, problems with singular system matrices and resulting non-uniqueness are often reported. How is this consistent with the statement of uniqueness? The answer is that in practice it is often difficult to guarantee that the Linear Independency Constraint Qualification (LICQ) is always fulfilled during the iterative solution process. If the set of active flow constraints is changed it must be checked that the linear system that includes the continuity equation and all active flow and outflow constraints has no linearly dependent rows (has full row rank). Moreover, in practical applications it is sometimes difficult to decide which of the redundant constraints should be activated. Non-uniqueness of heads (the Lagrange multipliers) is always a consequence of linearly dependent equations of active flow constraints combined with the continuity equations. Such situations can only be avoided by careful pre-analysis that detects infeasible configurations (as described in Appendix 7.1) or a check when constraints are

activated. In general, however, this approach is not suitable due to the requirement of frequently changing the system. Especially if the model is used for automatized online simulations the maintenance of the same system (same number of rows in and columns in the system matrix) is highly desirable.
As it will be seen in the following unlike as for the PDM model in D 4.1 where the proof of linear interdependency of active flow constraints by maintaining the original system size was shown to be trivial if the system is connected, the LICQ may not be fulfilled in the case of general flow control. In principle, any link may have an active flow bound, which can result in disconnected network graph. Therefore, in this case the most challenging problem is to avoid such linear dependency and connected to that to avoid singular or ill-posed systems of equations. Such system deficiencies may occur at the final solution that indicated instability of the real physical system but also during the iterative process when the true solution is far from instable states.
In the following possible solution techniques are presented and discussed in terms of robustness and efficiency.

4.3 Numerical solution of flow constrained PDM problems

4.3.1 Challenges

Problems in the numerical solution process that can arise not all exclusively for the flow constrained case are:

- Singular F matrix caused by zero flows
- Singular M matrix caused by zero consumption
- Singular Jacobian due to decomposed system as a consequence of (multiple) link removals (closed valves, active flow constraint)
- Empty feasible set due to conflicting flow constraints

The first two issues with singularity of the two diagonal matrices F and M can be resolved by simple regularization meaning that under a certain threshold for the flow or the consumption the derivative is calculated for the flow threshold and not for the actual flow. Alternatively, other techniques for dealing with zero flows can be found in literature.
The more challenging problem is the one of dealing with conflicting constraints and decomposed systems. By using PDM instead of DDM the feasible set is strongly increased. Even for disconnected parts that have a demand the feasible set is nonempty due to the ability of the PDM demand nodes to reduce the consumption to zero. In DDM disconnected demand nodes lead to empty feasible set. In PDM analysis empty feasible sets are rare but can results from poorly defined flow control device settings. In the following different methods for solution of the problem are emphasized and compared with respect to their benefits and shortcomings.

4.3.2 Methods for numerical solution of the Content Minimization Problem

The Content Minimization Problem Eq. (2) for flow constrained PDM analysis has the general form:

$$\min_{\mathbf{x}\in R^n} C(\mathbf{x})$$
$$s.t.\ \ \mathbf{Ax} = \mathbf{0}, \qquad \mathbf{Dx} \le \mathbf{b} \tag{17}$$

Where C is the strictly convex Content function, $\mathbf{A}$ is a $n_j \times n$ matrix[2], $\mathbf{D}$ is a $r \times n$ indicator matrix (entries are either 1, -1 or 0) that selects variables having lower and/or upper bounds, and $n = n_j + n_p$. The linear equality and inequality constraints define a polyhedron. Eq. (17) is the primal formulation, x the primal variable and other formulations exist (see for example, Elhay et al., 2016).

$$P = \{\mathbf{x} | \mathbf{Ax} = \mathbf{0}, \mathbf{Dx} \le \mathbf{b}\} \tag{18}$$

In the following possible solution techniques are briefly described.

4.3.2.1 Active set method (ASM)

A possible approach to solve the minimization problem of Eq. (17) is the active set method (ASM). The ASM method is subdivided into two phases. The first phase consists of the calculation of a primal feasible point. A point $\mathbf{x}$ is called primal feasible if $\mathbf{x} \in P$. A general algorithm for the analysis of the polyhedral set and the calculation of a primal feasible point is given in Appendix 7.1. The second phase of the ASM method includes the minimizing procedure. During the iterative process feasibility of the iterates is maintained. At a certain iteration, the inequality constraints that are fulfilled by equality are called active or binding. The set of active constraints at iteration k is called the active set $\mathcal{A}_k$. For the solution procedure, the working set $\mathcal{W}_k$ is of special importance. It consists of a subset of linearly independent constraints of $\mathcal{A}_k$:

$$\mathcal{W}_k \subseteq \mathcal{A}_k$$

The working set matrix $\mathbf{A}_W$ is composed by the matrix $\mathbf{A}$ and the rows of matrix $\mathbf{D}$ of the working set ($\mathbf{D}_W$):

$$\mathbf{A}_W = \begin{bmatrix} \mathbf{A} \\ \mathbf{D}_W \end{bmatrix} \tag{18}$$

As mentioned before, the rows of $\mathbf{D}_W$ must not be linearly dependent with $\mathbf{A}$. At each iteration, a new feasible point is calculated by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha \mathbf{d}_{k+1} \tag{19}$$

Where $\mathbf{d}_k$ is the direction vector and α is the step size. If the C function is twice continuously differentiable the direction can be calculated by a Newton Step, otherwise first order methods (gradient) can be used. For the first case

$$\begin{bmatrix} \mathbf{d}_k \\ \boldsymbol{M}_k \end{bmatrix} = -\begin{bmatrix} \mathbf{H} & A_W^T \\ \mathbf{A}_W & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \nabla f(\mathbf{x}_k) \\ \mathbf{A}_W \mathbf{x_k} - \mathbf{b}_W \end{bmatrix} = -\begin{bmatrix} \mathbf{H} & A_W^T \\ \mathbf{A}_W & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \nabla f(\mathbf{x}_k) \\ \mathbf{0} \end{bmatrix}. \tag{20}$$

[2] $\mathbf{A} = [\mathbf{A}_1{}^T \quad \mathbf{I}]$ is the incidence matrix of the augmented graph of the network.

or with distinction of active bounds and equality constraints:

$$\begin{bmatrix} \mathbf{d}_{k+1} \\ \boldsymbol{h}_{k+1} \\ \boldsymbol{n}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{H} & \mathbf{A}^{T} & \mathbf{D}_W^T \\ \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_W & \mathbf{0} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} -\nabla f(\mathbf{x}_k) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{21}$$

The matrix $\mathbf{H} = \nabla_{\mathbf{x}}^2 C(\mathbf{x}_k)$ denotes the reduced Hessian matrix of the twice differentiable function f. Since $\mathbf{x}_k$ is primal feasible the search direction must lay in the null space of $\mathbf{A}_W$, which is expressed by the second and third row.

4.3.3 Topological connectivity analysis based on active flow constraints

As mentioned above decomposed systems can result from different reasons. Indeed, due to disruptive events parts of the network may be destroyed. This case can be considered by an equality constraint (q = 0) for the pipes affected and/or for the valves that should be closed to isolate the damaged part from the rest of the system. Therefore, parts of the system (explicitly the parts to be isolated) and potentially other parts are disconnected from any source and can't be supplied anymore.

Decomposed system parts that are a consequence of closed links (equality constraint) can be treated in advance of the iterative solution process. A simple connectivity analysis by breath first search or depth first search delivers the different connectivity components in linear time ($O(|E| + |V|)$|where $|E|$ is the number of edges (links) and $|V|$ is the cardinality of the vertex (node) set. After identification, basically, two approaches are possible:

- Removal of all disconnected components from the model and modelling of the reduced systems
- Regularization of original system

The first approach is intuitively preferable since the disconnected parts are not treatable by standard steady-state analysis. If the consumers are still demanding water from the system they will open the taps resulting in emptying of the pipe system. This kind of physical process, however, is not tractable by the mathematical model for steady-state analysis of pressurized pipe systems since, first, it is a dynamic process and, second, the assumption of pressurized pipe flows does not hold any longer since the pipes may be only partly filled. It would require transient modelling of multiphase flow or free surface flow (open channel). Due to the unclear situation in such situation and the uncertainty about consumer behaviour and influence of system failures, it is reasonable to focus on the remaining working part of the system.

Sometimes, for example in hydraulic online simulation the size of the system cannot be modified. Therefore, it is required to regularize the system such that calculation give at least for the working part of the system reasonable results. So, the problem of possible singularity of the system matrix due to disconnected parts must be resolved. First, all disconnected components are identified and distributed into two sets. One set for components that include at least one resource node/pressure defining node (reservoir or tank) and components without resource node. As explained above for the latter, the pressure in the entire component is undefined and the system of equations is singular (there is infinity of pressure solutions). For regularization, an arbitrary node of the disconnected part is connected to a virtual reservoir with head chosen at the smallest elevation, which guarantees with the PDM system that there is no consumption in the disconnected component. This slight modification allows to maintain the original systems size. The calculation result for the working part is identical with the result of the first method. Whereas this approach is practical for equality constraints, it is pretty expensive for consideration of inequality constraints since the topological search has to be carried out after each iteration. Though very efficient, connectivity analysis adds an extra computational cost, especially in the case when the disconnection is caused by inequality constrains or combination of inequality constrains with equality constraints. Then, the connectivity check should be carried out at each iteration of the numerical solution process.

In the simulation software SIR 3S the described topological search method is implemented for avoiding irregular systems. In the Porteau software, the flow rate inequalities are solved by introducing external head loss penalties (and one equality is two inequalities). This is described in Piller and van Zyl (2014) and works well if some ad hoc initialisation of the constraint flow rate is made at the first iteration to avoid dramatic head loss penalties and numerical problem.

4.3.4 Projected active set Newton-type method

4.3.4.1 Outline of projected ASM method for PDM problems with flow bounds

From the above it is desirable to have an algorithm that can deal with all kind of disconnections by maintaining the same structure of the system (constant number of nodes) and without the requirement of numerous connectivity calculations. The solution of the KKT conditions by a projected active set Newton-type method aims at providing such a technique. As shown above the LICQ guarantees that the set of Lagrangian multipliers is a singleton. The task of finding a KKT-point described by Euler-Lagrange system Eq. (4) can be tackled by using the same strategy of a projected active set Newton-type method as it was described in D 4.1 for PDM solutions. With additional consideration of flow bounds for links the indicator set must be extended for the links in question (links that represent flow control devices, or if a maximum capacity shall be defined). In the pure PDM without flow constraints the main challenge was how to deal with zero flows (for Hazen-Williams). In the extended model as described here there is an additional difficulty that arises from possible disconnections of parts of the network by pipe failure or closed valves. This is particularly the case for situations where no or non-unique KKT point exists. One simple example is the separation of network parts by isolation valves as described in the Introduction. The numerical investigation of the constraints in Appendix 7.2 shows that the LICQ does not hold in this case.
To make the ASM method not only efficient but also robust, an additional regularization is required. The example in Appendix 7.2 brings about that maintaining at least one node in the set A of consumption nodes (nodes with reduced but nonzero consumption) could prevent the problem of deficiency. In principle this approach is possible, however, in reality, there might be disconnected parts with no consumption nodes. This is often the case with bypasses of pumps or control devices.

4.3.4.2 Tikhonov regularization

The situation of ill-conditioned or singular systems is widely treated in different fields of mathematical modelling, particularly in the field of inverse problems. Consequently, vast literature on regularization techniques exists. Probably the most famous is the so-called Tikhonov regularization. In brief, the solution of a possible singular linear system

$$\mathbf{Lx} = \mathbf{s}$$

can be calculated by the following expression

$$\mathbf{x_1} = \mathbf{x_0} - (\mathbf{L}^T\mathbf{L} + \alpha\mathbf{I})^{-1}\,\mathbf{L}^T[\mathbf{Lx_0} - \mathbf{s}]$$

Application to our linear to solve system in Eq. (15) gives:

$$\mathbf{h}^{k+1} = \mathbf{h}^k - (L^T\mathbf{L} + \alpha\mathbf{I})^{-1}\,\mathbf{L}^T[\mathbf{Lh}^k - \mathbf{s}],$$

with $\mathbf{L} = \mathbf{A}_A^T\mathbf{F}_A^{-1}\mathbf{A}_A + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\tilde{\mathbf{I}}_{ca}$ and $\mathbf{s} = \mathbf{A}_A^T\mathbf{F}_A^{-1}\left(\mathbf{G}_A\left(\mathbf{q}_A^{(k)}\right)\mathbf{q}_A^{(k)} - \mathbf{A}_{A,0}\mathbf{h}_0\right) + \tilde{\mathbf{I}}_{ca}^T\mathbf{M}_A^{-1}\left(\mathbf{N}_A{}^{(k)}\mathbf{c}_A{}^{(k)} + \mathbf{h}_{A,min}\right) - \mathbf{A}_A^T\mathbf{q}_A^{(k)} - \mathbf{A}_L^T\mathbf{q}_{min} - \mathbf{A}_U^T\mathbf{q}_{max} - \tilde{\mathbf{I}}_{ca}{}^{\boldsymbol{T}}\mathbf{c}_A{}^{(k)} - \tilde{\mathbf{I}}_{cu}{}^{\boldsymbol{T}}\mathbf{d}_U$

The constant α is the Tikhonov regularization parameter. A heuristic that is working well in the general purpose for identifying a suitable α, is to start with an initial α, then increase by a factor 10 if the linear system is singular (some pivot is non-positive in an appropriate decomposition algorithm) or if norm(Lx_1 –s) is not decreasing, else multiplying by a factor 0.4. Another shortcoming of this approach results from the additional computational burden induced by matrix multiplication, but this is possible to change a little bit the previous formulation to add α, or rather its square root, directly on the diagonal of $\mathbf{F}_A^{-1}$ and $\mathbf{M}_A^{-1}$ and solving system Eq. (15).

4.3.4.3 Using penalty function instead of ideal constraints.

One strategy for preventing deficient system states that are caused by valve closure or active flow control device is to replace ideal control with sharp or hard constraints by a head loss penalty function. As a consequence, the subdifferential mapping is replaced by a continuous, piecewise defined headloss function that is not restricted and valid for the whole range of flows.

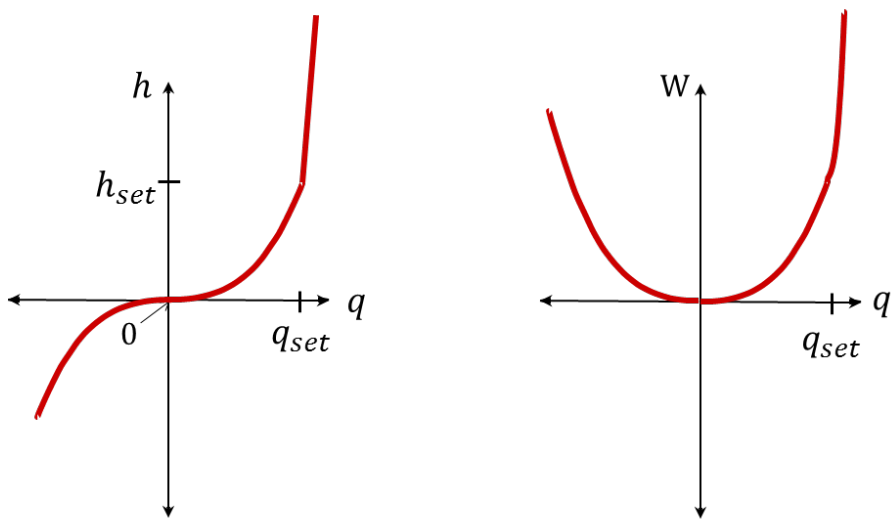


Figure 4: Regularized mapping and Content for Flow Control Valve (FCV).

A similar penalization approach is implemented in the software Porteau (Piller and van Zyl, 2014).

4.3.4.4 Regularization heuristics

The following method was found to be efficient for regularization: The flow constraints are considered as ideal control devices meaning that the matrix entries are completely removed if a flow constraint of that link is active. However, this can lead to singularity in case of disconnected systems. The idea is that if all nodes (including junctions without demand) get a small value on the main diagonal of the Jacobian the matrix must be non-singular even in the theoretical case where all links have active flow constraints. The resulting flows and consumptions are projected on the feasible box. Since the regularization parameter is not considered for updating the nodal consumptions the continuity equation is not affected. The method was implemented in Excel and Matlab and showed robust solution behaviour also for highly degenerate systems. However, the choice of the regularization parameter may affect the stability of the solution. So far, no rigorous approach for estimation of the regularization parameter was implemented and a method of Tikhonov can be implemented.

4.4 Pressure Control devices

4.4.1 Mathematical model: Nash Equilibrium of constraint nonlinear optimization problems

4.4.1.1 Modelling of pressure control devices

In the context of this manuscript pressure control refers to two groups of devices. The first group includes valves that maintain the upstream pressure (PSV: Pressure sustaining valve) or reduce the downstream pressure (PRV: Pressure Reducing Valve) to a given set value. The operational range of the valve is from fully open to closed (see Figure 5). Within the operational range the set pressure is maintained by adjusting the minor loss coefficient ζ of the valve by means of a feedback loop. There exist pressure control devices where the feedback loop is purely mechanical by means of a set value spring or electronically where the outlet pressure is measured and transmitted to a controller that adjust the valve opening accordingly.
The second group consists of variable speed pumps with pressure control. The control loop is similar as for the electronically controlled valves. In contrast to control valves the pressure is impacted by adjustment of the rotations per minute of the pump effectuating a shift of the pump curve. The two groups distinguish with respect to their impact on the system. Whereas the influence of control valves is by adjustment of the local headloss variable, speed pumps manipulate on the pressure gain of the pumping station.
Another difference exists by the fact that the setting pressure node for pump control is often at a certain distance of the pumping station. Mechanically operated valves control only the local pressure. In general, remote pressure control could be applied also to valves if the measured pressure is transmitted to the valve controller. As it will be seen later the design of the control loops is of great importance for the stability of the network.

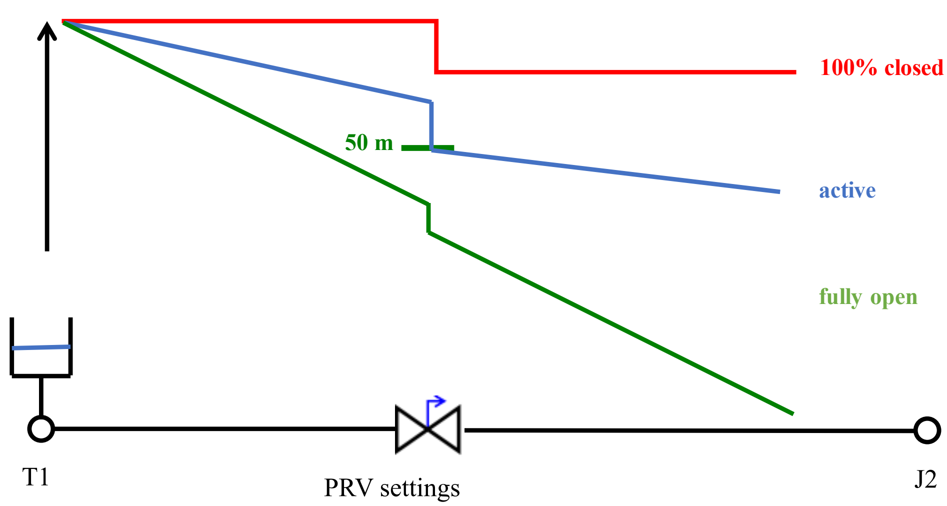


Figure 5: The three PRV operational statuses.

In the following the theoretical aspects of modelling pressure control devices will be discussed. Without loss of generality but for simplifying the notation the equations are shown only for PRV.

In addition to the hydraulic network equations presented in the previous sections, for each pressure control device, an additional optimization problem is formulated in the unknown control parameter **ζ** of the minor headloss function (example for PRV with local control):

$$\min_{\zeta_j \geq 0} \frac{1}{2}\left(h_i - \zeta_j \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - h_{j,set} \right)^2 \tag{22}$$

Where the j-th PRV consists of a control valve between nodes i and k, $h_{j,set}$ is the head setting at node k, q_j is the flow rate crossing the valve, and A_j is the cross-sectional area of the valve.

with the Lagrangian:

$$L_j(\zeta_j, q_j, \chi_j) = \frac{1}{2}\left(h_i - \zeta_j \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - h_{j,set} \right)^2 - \chi_j \zeta_j, \text{with } \zeta_j \geq 0 \; and \; \chi_j \geq 0 \tag{23}$$

Necessary for a minimum of the quadratic minimization problem is again the KKT condition with:

$$\frac{\partial L_j}{\partial \zeta_j} = -\left(h_i - \zeta_j \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - h_{j,set} \right) \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - \chi_j = 0 \tag{24}$$

From the complementary slackness condition for a stationary point of the Lagrangian it follows that either ζ_j=0 or χ_j=0 at optimum. In the first case, we get:

$$-\left(h_i - h_{j,set}\right) \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - \chi_j = 0 \quad or \quad \chi_j = \left(h_{j,set} - h_i\right) \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} \tag{25}$$

The second case has a stationary point for $q_j = 0$. This is the undefined case where no flow through the device is possible. Without flow no pressure control by creating additional turbulence is possible. Since the valve is not able to operate for a zero flow, we are interested only in the left factor in the parenthesis. For $|q_j| > 0$ the equation simplifies to

$$\left(h_i - \zeta_j \frac{\left(\frac{q_j}{A_j}\right)^2}{2g} - h_{j,set} \right) = 0 \; or \; \zeta_j = \left(h_i - h_{j,set}\right) \frac{2g}{\left(\frac{q_j}{A_j}\right)^2} \tag{26}$$

The derivative of the Lagrangian is affine in ζ_j.

4.4.1.2 Numerical solution of general PDM systems with flow and pressure control

As it can be seen from the previous section the pressure control problem is correlated with the content minimization problem by the flow through the pressure control link and the head at the non-setting pressure node. Vice versa, the head loss/head gain parameter must be considered in the Content minimization problem. There is a mutual feedback between the two problems expressed by variable -> parameter relationship. The two problems can be understood as a game where one player tries to minimize the Content for given ζ whereas the other player tries to minimize the deviation from the pressure setting for given **q, c.** In mathematical modeling, the concept of a convex non-cooperative differentiable game where the players have equal rights can be used. The concept is not limited to two players. Imagine that there are p pressure control devices the game includes p+1 players where the p pressure control players are not directly interacting and are connected through the Content player.

A solution of the game is the so-called Cournot-Nash equilibrium. For given values of the other players the necessary and sufficient conditions for each player can be formulated again as KKT conditions.

The previous formulation has the disadvantage that if $\mathbf{q}_{A,j}^{(k)} \dashrightarrow 0$ the minor loss coefficient $\zeta_{\mathrm{j}} \dashrightarrow \infty$ with the consequence that the headloss generated by pressure control device is not well defined. Therefore, based on the assumption that a local minor headloss $z_j \in [0, \infty)$ can be reached by the device independent from $\mathbf{q}_{A,j}^{(k)}$ the following substitution is made:

$$z_j = \zeta_{\mathrm{j}} \frac{\left(\frac{\mathrm{q_j}}{\mathrm{A_j}}\right)^2}{2g} \tag{27}$$

Another modification concerns the formulation of the control problem.

For the j-th pressure control device, a possibility is to solve the following problem as in Deuerlein et al. (2005):

$$\min_{\mathrm{z_j} \geq 0} \frac{1}{2} \left(\mathbf{h}_{0,\mathrm{j}} - \mathbf{P}_{A,j}^{T} \mathbf{G}_{\boldsymbol{A}} \mathbf{q}_{\boldsymbol{A}} - \mathbf{P}_{A,j}^{T} \mathbf{I}_{\boldsymbol{s} \setminus \mathrm{j}} \mathbf{z} - \mathrm{z}_j - \mathrm{h_{j,set}} \right)^2 \tag{28}$$

The matrix $\mathbf{P}_{\boldsymbol{A}}$ includes an independent path from an upstream resource node with head $\mathbf{h}_{0,\mathrm{j}}$ to the setting node k with expected setting $\mathrm{h_{j,set}}$. If the control valve is active, the sum of the minor and friction head losses along that path is zero.

Another possibility, is to consider the simpler case of a PRV with setting node at the downstream end of the valve; it reduces to:

$$\min_{\mathrm{z_j} \geq 0} \frac{1}{2} \left(\mathrm{h_i}(\mathbf{q}) - \mathrm{z_j} - \mathrm{h_{j,set}} \right)^2$$

These two formulations are equivalent as long as there exists a path between the upstream resource node to the first end of the valve at node i.

Please note that the path may be changing because they depend on the actual set of active flow constraints. To keep the Lagrangian multipliers of the Content minimization out of the pressure control problem formulation, the path must include only links that have no active flow constraint.

Let $\mathbf{A}_{A,n}^{T}$ be the incidence matrix reduced to junction nodes that are not setting nodes of PRDs and whose columns correspond to non-binding flow rates (non-active), and let $\mathbf{A}_{A,t}^{T}$ be the incidence matrix reduced to junction nodes that are the targeted setting nodes. Let $\mathbf{h}_{\boldsymbol{n}}$ and $\mathbf{h}_{\boldsymbol{t}}$ the corresponding head components such that $\mathbf{h}^{\boldsymbol{T}} = \left(\mathbf{h}_{\boldsymbol{n}}{}^{\boldsymbol{T}} \quad \mathbf{h}_{\boldsymbol{t}}{}^{\boldsymbol{T}}\right)$. With this specific decomposition, the energy balance at a non-binding pipe is:

$$\mathbf{G}_{\boldsymbol{A}}\mathbf{q}_{\boldsymbol{A}} + \mathbf{I}_{\boldsymbol{s}}\mathbf{z} - \mathbf{A}_{\boldsymbol{A},\boldsymbol{n}}\mathbf{h}_{\boldsymbol{n}} - \mathbf{A}_{\mathrm{A},0}\mathbf{h}_{0} - \mathbf{A}_{\boldsymbol{A},\boldsymbol{t}}\mathbf{h}_{\boldsymbol{t}} = \mathbf{0} \tag{29}$$

If the PRD is active the corresponding $\mathrm{h_t}$ component is fixed to the setting value $\mathrm{h_{j,set}}$ (see in **Figure 5**, the blue HGL).

The path matrix can be found by subdividing $\mathbf{A}_{A,n}^{T}$ into basis and non-basis columns, which is equivalent with finding the range space and the null space of the matrix. The null space corresponds to the pressure control valves.

The basis consists of a regular square matrix. The columns correspond to a spanning tree of the network graph without the links with binding flow constraints. It holds that with the separation $\mathbf{A}_{A,n}^{T} = \left[\mathbf{A}_{A,B}^{T}\ \mathbf{A}_{A,N}^{T}\right]$ the following basis and non-basis can be found:

$$\mathbf{Z} = [\mathbf{P}_{A}^{T}\ \mathbf{N}] = \left[-\mathbf{A}_{A,N}\mathbf{A}_{A,B}^{-1} \quad \mathbf{I}_{\boldsymbol{r}}\right] \tag{30}$$

By building we have $\mathbf{Z}\mathbf{A}_{A,n} = \mathbf{0}$, so that by pre-multiplying Eq. (29) by $\mathbf{e}_{\boldsymbol{j}}{}^{\boldsymbol{T}}\mathbf{M}$ we get:

$$\mathbf{e}_{\boldsymbol{j}}{}^{\boldsymbol{T}}\mathbf{Z}\left(\mathbf{G}_{\boldsymbol{A}}\mathbf{q}_{\boldsymbol{A}} + \mathbf{I}_{\boldsymbol{s}}\mathbf{z} - \mathbf{A}_{\mathrm{A},0}\mathbf{h}_{0} - \mathbf{A}_{A,t}\mathbf{h}_{t}\right) = \mathbf{0} \tag{31}$$

Which can be written very easily:

$$\mathbf{h}_{0,\mathrm{j}} - \mathbf{P}_{A,j}^{T}\mathbf{G}_{\boldsymbol{A}}\mathbf{q}_{\boldsymbol{A}} - \mathbf{P}_{A,j}^{T}\mathbf{I}_{\boldsymbol{s}\backslash\mathrm{j}}\mathbf{z} - \mathrm{z}_{j} - \mathrm{h_t} = 0$$

by choosing $\mathbf{h}_{0,\mathrm{j}} = \mathbf{e}_{\boldsymbol{j}}{}^{\boldsymbol{T}}\mathbf{Z}\left(\mathbf{A}_{\mathrm{A},0}\mathbf{h}_{0}\right)$ and assuming $\mathbf{e}_{\boldsymbol{j}}{}^{\boldsymbol{T}}\mathbf{Z}\left(\mathbf{A}_{\mathrm{A,t}}\mathbf{h}_{\mathrm{t}}\right) = -\mathbf{h}_{\mathrm{t}}$, which should be true as far there exists one path starting from one resource node and the local setting nodal for the j-th PRV that is equivalent assuming that the matrix is invertible.

The expression including the path matrix can be simplified based on the proposition that there is a regular basis (spanning tree) in $\mathbf{A}_{A,n}$ that includes a path to the initial nodes of all active pressure control devices. Therefore, it holds that:

$$\mathbf{h}_{0,\mathrm{j}} - \mathbf{P}_{A,j}^T \mathbf{G}_A \mathbf{q}_A - \mathbf{P}_{A,j}^T \mathbf{I}_{s\backslash \mathrm{j}} \mathbf{z} = \mathbf{h}_{\mathrm{i}}.$$

So Eq. (31) leads to:

$$\mathbf{h}_{\mathrm{i}} - \mathrm{z}_j - \mathrm{h}_{\mathrm{t}} = 0$$

In the Content model the active pressure control device is treated as head generator (a link with constant headloss). The monotone mapping and corresponding Content are shown in Figure 6..

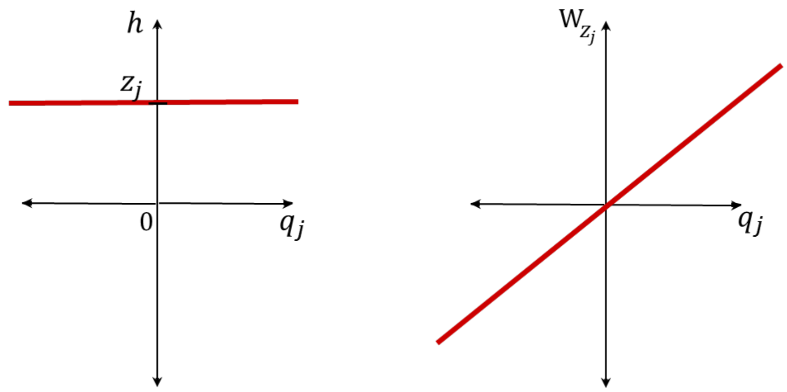


Figure 6: Monotone mapping and Content of head generator.

The combined system to be solved can be written as:

$$\begin{aligned}
\mathbf{G}(\mathbf{q})\mathbf{q} + \tilde{\mathbf{I}}_s\mathbf{z} - \mathbf{A}_1\mathbf{h} - \mathrm{A}_0\mathbf{h}_0 - \mathbf{V}_{\mathrm{L}}\boldsymbol{\kappa}^* + \mathbf{V}_{\mathrm{U}}\boldsymbol{\nu}^* &= \mathbf{0}_{\mathrm{np}} \\
\mathbf{N}(\mathbf{c})\mathbf{c} + \mathbf{h}_{\mathrm{min}} - \mathbf{U}_{\mathrm{L}}{}^T\boldsymbol{\lambda}^* + \mathbf{U}_{\mathrm{U}}{}^T\boldsymbol{\mu}^* &= \mathbf{h} \\
-\boldsymbol{A}_1^T\mathbf{q} - \mathbf{c} &= \mathbf{0}_{\mathrm{nj}} \\
-\mathbf{U}_{\mathrm{L}}\mathbf{c} &= \mathbf{0}_{\boldsymbol{m}} \\
\mathbf{U}_{\mathrm{U}}(\mathbf{c} - \mathbf{d}) &= \mathbf{0}_{\boldsymbol{s}} \\
-\mathbf{V}_{\mathrm{L}}(\mathbf{q} - \mathbf{q}_{\mathrm{min}}) &= \mathbf{0} \\
\mathbf{V}_{\mathrm{U}}(\mathbf{q} - \mathbf{q}_{\mathrm{max}}) &= \mathbf{0} \\
\mathbf{e}_j{}^T\mathbf{Z}\left(\mathbf{G}_A\mathbf{q}_A + \mathbf{I}_s\mathbf{z} - \mathbf{A}_{\mathrm{A},0}\mathbf{h}_0 - \mathbf{A}_{A,t}\mathbf{h}_{set}\right) - \chi_j^* &= 0, \quad j = 1, \dots, r \\
\chi_j^*\mathrm{z}_j &= 0, \quad j = 1, \dots, r
\end{aligned} \tag{32}$$

Considering the two last equations for PRVs in active mode ($\chi_j^* = 0, j = 1, \dots, ra$) delivers:

$$\mathbf{Z}_{\boldsymbol{a}}\left(\mathbf{G}_A\mathbf{q}_A + \mathbf{I}_{s,a}\mathbf{z}_{\boldsymbol{a}} - \mathbf{A}_{\mathrm{A},0}\mathbf{h}_0 - \mathbf{A}_{A,t}\mathbf{h}_{set}\right) = \mathbf{0}_{\boldsymbol{ra}}$$

Here we are not considering the PRVs in open-mode as their z_j is zero.

The size of matrix $\mathbf{I}_{s,a}$.is m x r_A. The r_A columns refer to the pressure control devices that are active. $\mathbf{Z}_a\mathbf{I}_{s,a}$ is a square matrix of size r_A, which is assumed to be invertible (most often it is reduced to identity when there is not another active pressure valve on the same path).

If variables with active bounds are eliminated, the following KKT system of the Nash Equilibrium between global Content minimization and local pressure control is to be solved. The reduced set of flows, consumption and head generator is indicated by index 'A' for **q, c** and **z** which means that only links with non-active flow constraints are included in $\mathbf{A}_{A,1}$ and only head generators that are in operational mode are included in $\mathbf{I}_{z,a}$. The system (35) to be solved reduces to:

$$\begin{pmatrix} \mathbf{G}_A & \mathbf{0} & -\mathbf{A}_{A,1} & \mathbf{I}_{s,a} \\ \mathbf{0} & \mathbf{N}_A & -\tilde{\mathbf{I}}_{ca} & \mathbf{0} \\ -\mathbf{A}_{A,1}^T & -\tilde{\mathbf{I}}_{ca}{}^{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_a\mathbf{G}_A & \mathbf{0} & \mathbf{0} & \mathbf{Z}_a\mathbf{I}_{s,a} \end{pmatrix} \begin{pmatrix} \mathbf{q}_A \\ \mathbf{c}_a \\ \mathbf{h} \\ \mathbf{z}_a \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{A,0}\mathbf{h}_0 \\ -\mathbf{h}_{A,min} \\ \mathbf{A}_{L,1}^T\mathbf{q}_{L,min} + \mathbf{A}_{U,1}^T\mathbf{q}_{U,max} + \tilde{\mathbf{I}}_{cu}{}^{T}\mathbf{d}_U \\ \mathbf{Z}_a\left(\mathbf{A}_{A,0}\mathbf{h}_0 + \mathbf{A}_{A,t}\mathbf{h}_{set}\right) \end{pmatrix} \tag{33}$$

We can eliminate $\mathbf{z}_a$ by:

$$\mathbf{z}_a = \left(\mathbf{Z}_a\mathbf{I}_{s,a}\right)^{-1}\left[\mathbf{H}_s - \mathbf{Z}_a\mathbf{G}_A\mathbf{q}_a\right] \tag{34}$$

With $\mathbf{H}_s = \mathbf{Z}_a\left(\mathbf{A}_{A,0}\mathbf{h}_0 + \mathbf{A}_{A,t}\mathbf{h}_{set}\right) = (\mathbf{h}_{0,j} - h_{j,set})$

Inserted into the first row we get the non-symmetrical system:

$$\begin{pmatrix} \mathbf{G}_A - \mathbf{I}_{s,a}\left(\mathbf{Z}_a\mathbf{I}_{s,a}\right)^{-1}\mathbf{Z}_a\mathbf{G}_A & \mathbf{0} & -\mathbf{A}_{A,1} \\ \mathbf{0} & \mathbf{N}_A & -\tilde{\mathbf{I}}_{ca} \\ -\mathbf{A}_{A,1}^T & -\tilde{\mathbf{I}}_{ca}{}^{T} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_a \\ \mathbf{c}_a \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{A,0}\mathbf{h}_0 - \mathbf{I}_{s,a}\left(\mathbf{Z}_a\mathbf{I}_{s,a}\right)^{-1}\mathbf{H}_s \\ -\mathbf{h}_{A,min} \\ \mathbf{A}_{L,1}^T\mathbf{q}_{L,min} + \mathbf{A}_{U,1}^T\mathbf{q}_{U,max} + \tilde{\mathbf{I}}_{cu}{}^{T}\mathbf{d}_U \end{pmatrix} \tag{35}$$

The additional terms in the first row, in fact, eliminate the contribution of $\mathbf{q}_a$ for the rows that belong to head generators and fix the head to the set pressure head. It follows that the first block matrix is not invertible.

This becomes more evident if the incidence matrix is split and reordered such that all pressure control devices are at the end of $\mathbf{A}_{A,1}$.

$$\begin{pmatrix} \mathbf{G}_{A,n} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{An,n} & -\mathbf{A}_{An,t} \\ -\mathbf{P}_{A,t}^T\mathbf{G}_A & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{At,n} & -\mathbf{A}_{At,t} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_A & -\tilde{\mathbf{I}}_{ca}\mathbf{I}_n & -\tilde{\mathbf{I}}_{ca}\mathbf{I}_t \\ -\mathbf{A}_{An,n}^T & -\mathbf{A}_{At,n}^T & -\mathbf{I}_n{}^{T}\tilde{\mathbf{I}}_{ca}{}^{T} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{An,t}^T & -\mathbf{A}_{At,t}^T & -\mathbf{I}_n{}^{T}\tilde{\mathbf{I}}_{ca}{}^{T} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_{a,n} \\ \mathbf{q}_{a,t} \\ \mathbf{c}_a \\ \mathbf{h}_n \\ \mathbf{h}_t \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{A,0}\mathbf{h}_0 \\ -\mathbf{H}_s \\ -\mathbf{h}_{A,min} \\ \mathbf{A}_{Ln,n}^T\mathbf{q}_{min} + \mathbf{A}_{Un,n}^T\mathbf{q}_{max} + \mathbf{U}_{n,nj}\mathbf{d} \\ \mathbf{A}_{Lt,t}^T\mathbf{q}_{min} + \mathbf{A}_{Ut,t}^T\mathbf{q}_{max} + \mathbf{U}_{t,nj}\mathbf{d} \end{pmatrix} \tag{36}$$

With the second row the head at set pressure nodes is fixed at their set value. Elimination of the fixed heads delivers:

$$\begin{pmatrix} \mathbf{G}_{A,n} & \mathbf{0} & \mathbf{0} & -\mathbf{A}_{An,n} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_A & -\tilde{\mathbf{I}}_{ca}\mathbf{I}_\mathrm{n} \\ -\mathbf{A}_{An,n}^T & -\mathbf{A}_{At,n}^T & -\mathbf{I}_\mathrm{n}{}^T\tilde{\mathbf{I}}_{ca}{}^T & \mathbf{0} \\ -\mathbf{A}_{An,t}^T & -\mathbf{A}_{At,t}^T & -\mathbf{I}_\mathrm{n}{}^T\tilde{\mathbf{I}}_{ca}{}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_\mathrm{a,n} \\ \mathbf{q}_\mathrm{a,t} \\ \mathbf{c}_\mathrm{a} \\ \mathbf{h}_\mathrm{n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{\mathrm{A},0}\mathbf{h_0} + \mathbf{A}_{An,t}\mathbf{h}_\mathrm{t} \\ -\mathbf{h}_{A,min} + \tilde{\mathbf{I}}_{ca}\mathbf{I}_\mathrm{t}\mathbf{h}_\mathrm{t} \\ \mathbf{A}_{Ln,n}^T\mathbf{q}_{min} + \mathbf{A}_{Un,n}^T\mathbf{q}_{max} + \mathbf{U}_{n,nj}\mathbf{d} \\ \mathbf{A}_{Lt,t}^T\mathbf{q}_{min} + \mathbf{A}_{Ut,t}^T\mathbf{q}_{max} + \mathbf{U}_{t,nj}\mathbf{d} \end{pmatrix} \quad (37)$$

From a practical point of view it can be assumed that a set pressure node has no consumption. Therefore $-\mathbf{I}_{ca,s}$ vanishes. On combination with the result that $\mathbf{A}_{At,t}^T$ is the identity matrix and that a pressure control device cannot have active flow constraints at the same time the last row can be solved for the flows through the pressure control links.

$$\mathbf{q}_\mathrm{a,t} = -\mathbf{A}_{An,t}^T\mathbf{q}_\mathrm{a,n} \quad (38)$$

It follows the reduced system:

$$\begin{pmatrix} \mathbf{G}_{A,n} & \mathbf{0} & -\mathbf{A}_{An,n} \\ \mathbf{0} & \mathbf{N}_A & -\tilde{\mathbf{I}}_{ca}\mathbf{I}_\mathrm{n} \\ -\mathbf{A}_{An,n}^T + \mathbf{A}_{At,n}^T\mathbf{A}_{An,t}^T & -\mathbf{I}_\mathrm{n}{}^T\tilde{\mathbf{I}}_{ca}{}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_\mathrm{a,n} \\ \mathbf{c}_\mathrm{a} \\ \mathbf{h}_\mathrm{n} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{\mathrm{A},0}\mathbf{h_0} + \mathbf{A}_{An,t}\mathbf{h}_\mathrm{t} \\ -\mathbf{h}_{A,min} \\ \mathbf{A}_{Ln,n}^T\mathbf{q}_{min} + \mathbf{A}_{Un,n}^T\mathbf{q}_{max} + \mathbf{U}_{n,nj}\mathbf{d} \end{pmatrix} \quad (39)$$

The solution can be calculated by application of Newton-Raphson:

$$\begin{pmatrix} \mathbf{F}_{A,n} & \mathbf{0} & -\mathbf{A}_{An,n} \\ \mathbf{0} & \mathbf{M}_A & -\tilde{\mathbf{I}}_{ca}\mathbf{I}_\mathrm{n} \\ -\mathbf{A}_{An,n}^T + \mathbf{A}_{At,n}^T\mathbf{A}_{An,t}^T & -\mathbf{I}_\mathrm{n}{}^T\tilde{\mathbf{I}}_{ca}{}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}_\mathrm{a,n}^{(k+1)} - \mathbf{q}_\mathrm{a,n}^{(k)} \\ \mathbf{c}_\mathrm{a}^{(k+1)} - \mathbf{c}_\mathrm{a}^{(k)} \\ \mathbf{h}_\mathrm{n}^{(k+1)} - \mathbf{h}_\mathrm{n}^{(k)} \end{pmatrix}$$
$$= -\begin{pmatrix} -\mathbf{A}_{\mathrm{A},0}\mathbf{h_0} - \mathbf{A}_{An,t}\mathbf{h}_\mathrm{t} + \mathbf{G}_{A,n}\mathbf{q}_\mathrm{a,n}^{(k)} - \mathbf{A}_{An,n}\mathbf{h}_\mathrm{n}^{(k)} \\ \mathbf{h}_{A,min} + \mathbf{N}_A\mathbf{c}_\mathrm{a}^{(k)} - \tilde{\mathbf{I}}_{ca}\mathbf{I}_\mathrm{n}\mathbf{h}_\mathrm{n}^{(k)} \\ -\mathbf{A}_{Ln,n}^T\mathbf{q}_{min} - \mathbf{A}_{Un,n}^T\mathbf{q}_{max} - \mathbf{U}_{n,nj}\mathbf{d} - \mathbf{A}_{An,n}^T\mathbf{q}_\mathrm{a,n}^{(k)} + \mathbf{A}_{At,n}^T\mathbf{A}_{An,t}^T\mathbf{q}_\mathrm{a,n}^{(k)} \end{pmatrix} \quad (40)$$

For numerical solution, the previous form is not convenient since the system matrix is not symmetric. One common approach to regain symmetry is to neglect the term $\mathbf{A}_{At,n}^T\mathbf{A}_{An,t}^T$ in the Jacobian. Therefore, the continuity is not forced at the initial nodes of the pressure control links. This approach is implemented in Epanet and looks having good convergence. Here, the Nash game is between the inflow calculated at the fixed setting pressure node and the outflow at the initial node of the pressure control device.

Please note that so far, we have assumed that the active flow control devices have been removed and that the remaining system has an incidence matrix with full column rank, which is equivalent that a spanning tree or basis exists that connects all nodes to a fixed head node. In the next section, conditions that guarantee this assumption will be derived.

Remarks:

- The closed status is treated by a flow constrained q >= 0. In that case, the associated Lagrange multipliers interprets as a minor head loss.
- For correct working of the pressure control valve, in practice, it is recommended that a path exists between a resource node and the control valve. If there is no such path the valve cannot be operating (isolated subsystem with non-unique solution as in the example with FCV and PRV in series where both are active at the same time).

4.4.1.3 Existence and uniqueness of the hydraulic steady-state in PDM analysis with general control

In the section about flow control problems, it was shown that a necessary and sufficient condition for the existence of a unique steady state of PDM problems with flow control (flow bounds for network links) is related to the following two conditions:

- The feasible set, which is the intersection of the affine continuity equations and the flow and consumption bounds (polyhedral cone), must not be empty;
- The system of active flow and consumption bounds together with the continuity equation is linearly independent. This is equivalent with the fact that the augmented incidence matrix that is reduced by all rows that belong to (virtual) links with active flow or consumption bounds has still full column rank. In other words, there exists a kernel (null space) that includes all links with active flow bounds. This in turn means that there is a spanning tree that spans all nodes of the system.

A similar criterion can be derived for pressure control devices. The proof that a unique Nash equilibrium exists is possible by showing that the self-mapping for the single pressure control devices is a contraction mapping. The extension to the whole system with possibly interacting devices is done then by induction as in Deuerlein (2002). The contraction property can be visualized as follows. Consider the two players Content and PRV. The PRV player selects a z such set the difference between hi-z and h_{set} vanishes. Given the fixed chosen z, the Content player calculates the pressures and flows of the system. With the new head at the initial node of the PRV the PRV player makes the new choice for z which leads to new flows and heads at the side of the Content player. If the function that is used by the PRV player creates decreasing adjustments for z the sequence converges to a unique limit that is the z of the Nash equilibrium. The corresponding flows and heads are calculated by the Content player for given (fixed) z. From this equilibrium point none of the players can improve its objective without worsening the objective of the other players.

Of course, the described approach is not suited for implementation since after every new choice by the PRV players a full hydraulic calculation must be carried out. A simultaneous solution as indicated above is desirable. Before details of possible numerical solutions are given the following Lemma gives a descriptive condition for the existence and uniqueness of a Nash Equilibrium.

Proposition 2:

Let $\begin{bmatrix} \mathbf{A}_{A,1} \\ \mathbf{I}_A \end{bmatrix}$ be the reduced incidence matrix of the augmented graph where all links and virtual links (referring to consumption nodes) with active flow/consumption bounds have been removed. Let further

$\mathbf{A}_{An,n}$ be the submatrix of $\mathbf{A}_{A,1}$ where all pressure control links are replaced by virtual links that connect all set pressure nodes with a common virtual ground node (as in the case for the augmented graph for consumption nodes).

The following Lemma proves Existence and Uniqueness based on the proposition 2.

Lemma 2:
There exists a unique Nash-Equilibrium if and only if the matrix $\mathbf{A}_{An,n}$ defined in proposition 1 has full column rank.

Proof:
The necessary condition (only if) in Lemma 2 can by shown by contradiction. Imagine that such $\mathbf{A}_{An,n}$ does not exist. Then, there are parts of the network that are disconnected from the virtual ground node.
Distinction of case:
Case 1: The disconnected part is separated by active flow or pressure control devices: Like in the case with linearly dependent flow constraints there is an infinite number of combinations for the headlosses generated by the control devices (z for pressure control or Lagrangian multipliers for flow control), which in consequence leads to undefined pressures in the separated network part.
Case 2: The disconnected part is separated by one active pressure control device (the control device is in a tree). If the set pressure node is upstream of the control device there is no influence by the headloss generated by the control link. Therefore the pressure is undefined downstream. The second case is only relevant for DDM analysis since in PDM models there are one or more virtual links that connect the downstream nodes with the virtual ground node. If all consumption functions downstream have active bounds case 1 applies. In summary, disconnected parts always lead to non-uniqueness of the Nash-Equilibrium in terms of heads and z values which proves the necessity
Sufficiency is proved by the contraction property of the self mappings of the z values:

$$z_i = \mathbf{h}_{i,i}\big(z_i, \mathbf{q}(z_i), \mathbf{c}(z_i), \mathbf{z}_{\setminus i}(z_i)\big) - \mathrm{h}_{i,s} = F(z_i), \forall i = 1, \dots, r \tag{42}$$

The proof is by induction.
Induction start:
The first pressure control element is added. Therefore, the dependency of the other pressure control elements is removed:

$$z_i = \mathbf{h}_{i,i}(z_i, \mathbf{q}(z_i), \mathbf{c}(z_i)) - \mathrm{h}_{i,s} = F(z_i), \forall i = 1, \dots, r \tag{43}$$

The contraction property is proved if it can be shown that $|\nabla F(z_i)| < 1, \forall z_i \geq 0$. A formal proof can be derived from the sensitivity results developed in paper (Piller, Elhay, Deuerlein, & Simpson, 2017). A more descriptive version is outlined here. Imagine that the value of z is changed by 1: $\Delta z_i = 1$. Then the impact on the head at the initial node of the pressure control device is always $\Delta \mathbf{h}_{i,i} \leq 1$. This follows from proposition 1. In the case of DDM and the pressure control link laying in the forest equality holds. This case is not of interest here and will be neglected since in DDM the flows in the forest are fixed by the continuity equation and the calculation of the headloss z is straight forward using forest core decomposition. In the other case by proposition 1 there exists always a path between two fixed head nodes or a closed loop from the virtual ground node. The change in headloss $\Delta z_i = 1$ is distributed over all the links along this path. As a consequence, all heads upstream of link I are lifted whereas all heads downstream are decreased. Therefore, the head change $\Delta \mathbf{h}_{i,i}$ being the sum of upstream head loss changes is < 1 if the set pressure node is not a fixed head node which can be excluded without loss of generality. This proves the contraction property for one control device.

Induction step:

Now assume that there is a unique equilibrium for r-1 pressure control devices, then there exists also a unique Equilibrium for r pressure control devices. Based on Proposition 1 there exists matrix $\mathbf{A}_{An,n}$. For the location of the r-th pressure control device we should distinguish the two cases:

Case 1: there is a path that connects the set pressure node with a fixed head node without including no other pressure control device. Then the argumentation is as in the induction start. The upstream pressure may be defined by a fixed head node or the set pressure of another pressure controlling device.

Case 2: there is no path connecting the set pressure node with a fixed head node. Then a change in z does not affect the flow through the valve and $\Delta \mathbf{h}_{r,r} = 1$. However, this is the same case as for the valve in a tree and the z value is uniquely defined by $\boldsymbol{z}_{\mathrm{r}} = \mathbf{h}_{r,r} - \mathrm{h}_{r,s}$.

NOMENCLATURE

Table 6. Matrices and vector notations.

$\mathbf{A_1}$	Arc-node incidence matrix of junction nodes
$\mathbf{A_0}$	Arc-node incidence matrix of fixed pressure nodes
$\mathbf{h_0}$	Vector of fixed heads
$\mathbf{h}$	Vector of variable heads at junction nodes
$\mathbf{q}$	Flow vector
$\mathbf{G}$	Diagonal matrix for headloss $\Delta\mathbf{h} = \mathbf{G}(\mathbf{q})\mathbf{q}$
$\mathbf{F}$	Diagonal matrix of headloss derivatives $\mathbf{F} = \nabla_q(\Delta\mathbf{h})$
$\mathbf{d}$	Vector of nodal demands
$\mathbf{c}$	Vector of calculated external flows at nodes
$\mathbf{N}$	Diagonal matrix for inverse POR functions $\mathbf{h}(\mathbf{c}) = \mathbf{h}_{min} + \mathbf{N}(\mathbf{c})\mathbf{c}$
$\mathbf{M}$	Diagonal matrix of inverse POR derivatives $\mathbf{M} = \boldsymbol{\nabla}_c(\mathbf{h}(\boldsymbol{c}))$
$\mathbf{E}$	Diagonal matrix of POR derivatives $\mathbf{E} = \boldsymbol{\nabla}_h(\mathbf{c}(\mathbf{h}))$

5 SUMMARY AND CONCLUSION

A comprehensive mathematical framework for the calculation of the hydraulic steady-state of water supply networks with pressure dependent demands and general flow and pressure control conditions has been developed. The system may also be subdivided into several components.

The flow constraint problem can be modelled by a constraint convex minimization problem with simple bounds. For pressure control, additional minimization problems should be formulated for each pressure control device. In the general case the hydraulic steady state is modelled by a Nash-Equilibrium of the multiple optimization problems.

Conditions for the existence and uniqueness of flows and pressures have been proposed. It has been shown that even if it can be proved that there exists a unique Equilibrium, the numerical calculation is not straightforward. This is mainly because activation of interacting constraints during the iterative process may lead to singularity (ideal control) or ill-posedness (penalization) of the system matrix. As a consequence, the next iteration may be far away from the true solution that, eventually, results in non-convergence of the Newton-Raphson method.

Summarizing the recommendations for practical implementation we can state:

- Flow equality constraints should be treated in advance of the iterative process. Often, they result in disconnections that are much more difficult to deal with during the iterations. For identification, standard graph search algorithms such as depth first search or breath first search can be used. The algorithms run in linear time and are therefore very efficient. Also, for online simulations it is not necessary to modify the system matrix. The disconnected parts can be connected to a virtual set pressure node, thus, not affecting the rest of the system.
- For finding a feasible solution to start with in the ASM method, the phase I or pivot operations from the Simplex algorithm can be used (from linear programming theory).
- For stable iterations in flow controlled systems it is crucial to avoid linear dependency of the set of active flow and consumption bounds in combination with the continuity equation.
- Singularity caused by linearly dependent pressure control devices should be prevented by a preliminary analysis of the location of pressure control links and the corresponding set pressure nodes. A simple connectivity analysis of a modified graph has been proposed for that purpose.
- The connectivity analysis or similar techniques must be also applied to prevent from situations where the combination of flow and pressure control devices causes singularity of the system matrix.

More work should be done for developing efficient methods for checking the linear independency of modified set of active constraints. There exist straightforward methods such as basis change known from linear programming or graph search algorithms. However, running this algorithms at every iteration or adding constraints one by one would impact the performance negatively.

A more stable alternative that does not suffer from the risk of running into singular systems caused by linear dependent control devices is to treat the control problems outside the iterative process. There is experience with this approach at Irstea (Porteau) and 3S (SIR 3S). However, problems with convergence are also reported and the solution procedure is much slower. Future work could focus on a hybrid approach that combines the benefits of both methods.

The results were presented at the CCWI 2017 conference at Sheffield (Deuerlein et al., 2017).

This deliverable has focused on the mathematical background of modelling (deficient) PDM systems with general control conditions. The most appropriate method for implementation will be chosen as part of deliverable 4.4.

6 APPENDIX

6.1 Method for checking if the feasible set is non-empty

As it was shown above a unique flow distribution of the flow constraint PDM problem exists if the polyhedral set described by the continuity equation and the box constraints for consumption and link flows is non-empty. In the following, a method is described that can be used for checking the existence in a preprocessing step. The polyhedral set is given by:

$$\begin{aligned} A_1^T\mathbf{q} - \mathbf{c} &= \mathbf{0} \\ -\mathbf{c} &\leq \mathbf{0} \\ \mathbf{c} - \mathbf{d} &\leq \mathbf{0} \\ -\mathbf{q} + \mathbf{q}_{\min} &\leq \mathbf{0} \\ \mathbf{q} - \mathbf{q}_{\max} &\leq \mathbf{0} \end{aligned} \tag{A.1}$$

In matrix notation, we can write:

$$\begin{gathered} \begin{bmatrix} A_1^T & -\mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{c} \end{bmatrix} = \mathbf{0} \\ \begin{bmatrix} \mathbf{0} & -\mathbf{I}_n \\ \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_m & \mathbf{0} \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{c} \end{bmatrix} \leq \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \\ -\mathbf{q}_{\min} \\ \mathbf{q}_{\max} \end{bmatrix} \end{gathered} \tag{A.2}$$

with

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_n \\ \mathbf{0} & \mathbf{I}_n \\ -\mathbf{I}_m & \mathbf{0} \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_1^T & -\mathbf{I}_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \mathbf{c} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d} \\ -\mathbf{q}_{\min} \\ \mathbf{q}_{\max} \end{bmatrix}$$

We can write shorter:

$$\mathbf{Ax} = \mathbf{0}; \mathbf{Dx} \leq \mathbf{b} \tag{A.3}$$

From literature it is known that a possible approach to check if the polyhedral set is empty consists in the solution of the following LP:

$$\begin{gathered} \min_{(u,\xi)\in\Phi} \xi \\ \Phi = \left\{(\mathbf{x},\xi) \in \mathrm{R}^n \times \mathrm{R}\colon [\mathbf{Dx} - \mathbf{b}]_i \leq \xi, i = 1, \dots, m \wedge \mathbf{Ax} = \mathbf{0}\right\} \end{gathered} \tag{A.4}$$

The optimal value ξ indicates whether an interior point exists for the polyhedral set Φ:

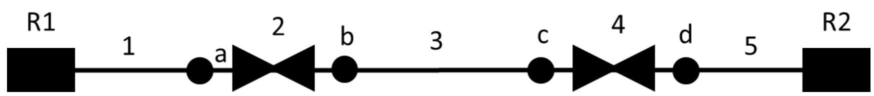
$\xi^* < 0$	The polyhedral set is non-empty. Thus, x* is an interior point of the feasible set. This implies that all the control devices are in an inactive state.

$\xi^* > 0$	The polyhedral set is empty. There is no feasible solution to the original problem.
$\xi^* = 0$	The polyhedral set devices is non-empty but an interior point does not exist.

Simpler, and for practical considerations, a sufficient condition is to assume that zero is in the feasible range for both consumptions and link flows. Then, in PDM analysis, the existence of a solution is already proven since the zero vector for c and q fulfils the mass residual and the Content constraints. Please note that this is not the case for DDM analysis.

6.2 Example for linearly dependent flow constraints and consequences

The following simple system that was already presented in the Introduction is considered:



The incidence matrix A1 is:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

First, let the consumption assumed to be fixed (DDM). The system that consists of the continuity equation and the active flow constraints is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} d_a \\ d_b \\ d_c \\ d_d \\ 0 \\ 0 \end{bmatrix}$$

As it can be easily seen in this case, rows 2,3 5 and 6 are linearly dependent (r2+r3-r5+r6=0). The last two rows can be used for elimination of q2 and q4. Then, it results the reduced modified incidence matrix $\mathbf{A}_{A,1}$:

$$\mathbf{A}_{A,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

It can be seen, that column 1, column 4 and columns 2, 3 are decoupled. As consequence, the Lagrangian multipliers (headloss in link 2 and link 4 and pressure heads at node b and c) are non-unique, which is obvious from the full system of equations:

$$\begin{bmatrix} f_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & f_3 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_4 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & f_5 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ h_a \\ h_b \\ h_c \\ h_d \\ \lambda_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} h_{R1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_{R2} \\ d_a \\ d_b \\ d_c \\ d_d \end{bmatrix}$$

It is easy to see that row 7+ row 8 – row 10 + row 11 = **0**, the rows are linearly dependent. From linear algebra, it is well known that there are two possibilities for rank deficient systems $\mathbf{Ax} = \mathbf{b}$ (with (n x n) matrix $\mathbf{A}$ and $rank(\mathbf{A}) < n$):

- $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b}) < n$: there is an infinite number of solutions[3]
- $rank(\mathbf{A}) \neq rank(\mathbf{A}|\mathbf{b})$: there is no solution

Which of the two cases applies to the example depends on the demands at nodes b and c. If $d_b \neq d_c$ it follows that $rank(\mathbf{J}) \neq rank(\mathbf{J}|\mathbf{rs})$, which proves that there is no solution to the problem. In the opposite case $d_b = d_c$ (practically this makes sense only if $d_b = d_c$=0) we get $rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b}) = n - 1$ from which we can conclude that there are infinite solutions.

$$\begin{bmatrix} f_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & f_3 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & f_5 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_3 \\ q_5 \\ h_a \\ h_b \\ h_c \\ h_d \\ \lambda_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} h_{R1} \\ 0 \\ 0 \\ 0 \\ h_{R2} \\ d_a \\ d_b \\ d_c \\ d_d \end{bmatrix}$$

Elimination of decoupled lambda delivers:

$$\begin{bmatrix} f_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & f_3 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & f_5 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_3 \\ q_5 \\ h_a \\ h_b \\ h_c \\ h_d \end{bmatrix} = \begin{bmatrix} h_{R1} \\ 0 \\ h_{R2} \\ d_a \\ d_b \\ d_c \\ d_d \end{bmatrix}$$

[3] $(\mathbf{A}|\mathbf{b})$ is the augmented matrix obtained by adding b at the last column.

The heads in the second row are completely decoupled from the rest of the system, which means that the equation is fulfilled by any combination h_b-$f_3 q_3 - h_c = 0$;
The Schur Matrix of the GGA in this case is:

$$\begin{bmatrix} f_1^{-1} & 0 & 0 & 0 \\ 0 & f_3^{-1} & -f_3^{-1} & 0 \\ 0 & -f_3^{-1} & f_3^{-1} & 0 \\ 0 & 0 & 0 & f_5^{-1} \end{bmatrix}$$

It is easy to see that row 2 and row3 are again linearly dependent.

7 REFERENCES

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