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► **To cite this version:**

Zeyd Benseghier, P. Cuéllar, Li-Hua Luu, J.Y. Delenne, Stéphane Bonelli, et al.. Relevance of Free Jet Model for Soil Erosion by Impinging Jets. *Journal of Hydraulic Engineering*, 2020, 146 (1), pp.04019047. 10.1061/(ASCE)HY.1943-7900.0001652 . hal-02609790

**HAL Id: hal-02609790**

**<https://hal.inrae.fr/hal-02609790>**

Submitted on 5 Sep 2023

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# Relevance of the free jet model for the soil erosion by impinging jets

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## ABSTRACT

The surface erosion of soil samples caused by an impinging jet can be analyzed with the Jet  
Erosion Test (JET), a standard experimental test to characterize the erosion resistance of soils. Here

24 we specifically address the flow characteristics of a laminar impinging jet over the irregular surface  
25 of granular beds to discuss the pertinence and relevance of commonly used empirical estimations  
26 based on the self-similar model of a free jet. The JET is here investigated at the micro-scale with  
27 a coupled fluid-particle flow numerical model featuring the Lattice Boltzmann Method (LBM) for  
28 the fluid phase combined with the Discrete Element Method (DEM) for the mechanical behavior  
29 of the solid particles.

30 We confront the hydrodynamics of a laminar plane free jet with the results from a parametric  
31 study of the jet impingement, both on solid smooth and fixed granular surfaces, which take into  
32 account variations of the particle size, of the distance from the jet origin, and of the jet Reynolds  
33 number. The flow characteristics at the bed surface are here quantified, including the maximal  
34 values in tangential velocity and wall shear stress, which can be regarded as the major cause for  
35 particle detachments under hydrodynamic sollicitation.

36 We show that the maximal velocity at the impinged surface can be described by the free jet  
37 self-similar model, provided that a simple empirical coefficient is introduced. We further propose  
38 an expression for the maximal shear stress in laminar conditions including a Blasius-like friction  
39 coefficient that is inversely proportional to the square root of the jet Reynolds number.

40 To conclude, we finally analyze the JET erosion of different cohesionless granular samples,  
41 confirming that the threshold condition at the onset of granular motion is consistent with the  
42 Shields diagram and also in close agreement with previous experimental results.

43 **Keywords:** Free jet, Impingement jet, Lattice Boltzmann Method, Soil erosion, Laminar flow,  
44 Discrete Element Method.

## 45 INTRODUCTION

46 The impinging jet is widely encountered in numerous natural and industrial applications, such  
47 as heat transfer (heating, cooling and drying) (Martin 1977; Jambunathan et al. 1992), discharge  
48 of pollutants in rivers, lakes, and oceans (Gholamreza-Kashi et al. 2007), and headcut erosion  
49 (Bennett and Alonso 2005). The jet flow configuration has been profusely studied in the past  
50 both from theoretical and experimental perspectives (Beltaos and Rajaratnam 1973; Beltaos and

51 Rajaratnam 1974; Rajaratnam 1976; Hanson et al. 1990; Looney and Walsh 1984; Poreh et al.  
52 1967; Ghaneezad et al. 2015; Phares et al. 2000), often addressing the particular cases of the free  
53 jet and wall jet with a special focus on their self-similarity features.

54 Notably, the case of the impinging jet bears great interest in the field of civil engineering, where  
55 it is specifically used to quantify the resistance against erosion of cohesive soils. The erodibility of  
56 soils is a key parameter for the safety of earthen hydraulic structures such as earth-dams, levees,  
57 and dikes against the risk of erosion-induced failures (Foster et al. 2000; Bonelli 2012; Bonelli  
58 2013).

59 The JET testing device (Jet Erosion Test) was firstly introduced by Hanson and Cook (2004)  
60 in order to assess the erosion rate  $E$  of soils for given flow conditions, with particular devices  
61 developed both for laboratory and *in situ* conditions. The interpretation of the test is based on the  
62 assumption that the rate of erosion is proportional to the excess of hydraulic shear stress  $\tau$  exerted  
63 on the soil surface over a critical value  $\tau_c$  at which the erosion will initiate. In mathematical form,  
64 this assumption reads  $E = k_d(\tau - \tau_c)$ , where the parameters  $\tau_c$  and  $k_d$  are the critical shear stress  
65 and the erosion rate coefficient respectively, which define the soil's erodibility.

66 The rate of erosion is usually quantified by measuring the depth of the scour hole generated  
67 by the jet impingement on the soil surface over time. Then, the free jet theory can be applied  
68 to estimate the hydraulic shear stress on the soil surface, so that empirical values for the soil's  
69 erodibility (i.e. for  $\tau_c$  and  $k_d$ ) can be quantified by fitting the experimental data. It can be noted  
70 that such approach only considers the free jet theory (Schlichting 1960; Bickley 1937) and wall  
71 shear stress estimates on a smooth wall (Beltaos and Rajaratnam 1977; Beltaos and Rajaratnam  
72 1974; Hanson et al. 1990) without taking into account the possible recirculation of the flow inside  
73 the scour crater nor any irregularity or roughness of the impinged surface.

74 A relevant weakness of this testing procedure is that the interpretation of the results is still  
75 based on strong assumptions for the estimation of the excess shear stress. Furthermore, the  
76 complex hydrodynamics of the impinging jet flow, which itself depends on many parameters (e.g.  
77 nozzle diameter, jet Reynolds number -nature of the flow-, inlet velocity profile) as well as the type

78 of the impinged surface, make the determination of the flow characteristics at the bed surface not  
79 an easy task, especially in the presence of a scour crater.

80 Moreover, the estimation of the shear stress disregards any confinement and wall effects in the  
81 JET device, as if the experiments were performed in unconfined conditions. In this respect, Gha-  
82 neezad and co-workers have shown that the maximum shear stress under confined conditions can  
83 be 2.4 times higher than the value estimated in the original JET (Ghaneezad et al. 2015). Never-  
84 theless, the present study is here restricted to the unconfined condition that allows however a direct  
85 comparison to existing experimental data as explained hereafter. Phares and co-workers (2000)  
86 gave a theoretical prediction of the wall shear stress produced by an impinging jet over a flat surface  
87 for various jet configurations (axially-symmetric and two-dimensional jets, both for turbulent and  
88 laminar flows). They found that some of the theoretical results are not consistent with the measured  
89 wall shear stress reported by previous experimental investigations of impinging jets.

90 Recent experimental studies have investigated the erosion caused by an immersed impinging  
91 jet on a granular material, see e.g. the work of Badr et al. (2014) for turbulent and laminar  
92 planar impinging jets. They found that the flow characteristics can actually be estimated using  
93 the self-similar free jet model. Subsequently, Brunier-Coulin et al. (2017a) proposed an empirical  
94 expression to model the jet velocity inside the scoured crater of a cohesionless artificial granular  
95 material (refractive index matched glass beads) for a laminar round impinging jet.

96 On the other hand, the numerical models are nowadays gaining growing relevance for studying  
97 small to large-scale engineering applications. Some of the difficulties of the experimental tests  
98 can be overcome using customized numerical simulations, which can provide an insight into  
99 the local parameters of the flow that are hardly measurable in the experiments. Past numerical  
100 studies of the Jet Erosion Test have generally involved two main approaches. A first mono-phasic  
101 approach consists in solving directly the Navier-Stokes equations for the fluid flow. Several different  
102 turbulence models can be used for the axisymmetric jet condition. The water/soil interface is thereby  
103 considered as a Lagrangian boundary, which is updated using a suitable erosion law and an adaptive  
104 re-meshing technique (Mercier et al. 2014). The second possibility is to use a combined bi-phasic

105 approach, where the soil is modeled as a collection of discrete particles described by Newton's  
106 second law of motion (e.g. with the Discrete Element Method, or DEM) and the fluid flow is  
107 reproduced by a suitable computational fluid dynamics method (CFD), either solving the Navier-  
108 Stokes or the Boltzmann equations. Given that the pertinence and validity of the erosion law  
109 are the main concern here, the latter bi-phasic approach has several advantages for simulating Jet  
110 Erosion Test, since no erosion law needs to be assumed *a priori*. Among the studies that have been  
111 carried out using this approach, Kuang and co-workers (2013) presented a 3D CFD-DEM model  
112 of a turbulent round air jet impinging on a granular bed, focusing mainly on the crater formation  
113 induced by the air jet. Concerning the surface erosion of a cohesive soil, the Lattice Boltzmann  
114 Method (LBM) has been coupled with the DEM for a micro-mechanical simulation of the 2D  
115 laminar impinging jet in (Cuéllar et al. 2015; Cuéllar et al. 2017), which constitutes the basis for  
116 the present study with an extended model. Such combination of the LBM-DEM methods appears  
117 as a promising technique for simulating a wide range of geomechanical problems, including soil  
118 erosion (Cuéllar et al. 2017; Lominé et al. 2013), various porous flows (Han and Cundall 2013),  
119 the fluidization of soils (Cui et al. 2014; Ngoma et al. 2018), and immersed granular avalanches  
120 (Mutabaruka et al. 2014) for instance.

121 The purpose of the present contribution is to provide a numerical insight into both the free and  
122 impinging laminar plane free jet on either smooth or granular surfaces, thereby complementing  
123 the previous experimental work of Badr et al. (2014) and Brunier-Coulin et al. (2017a). We place  
124 a particular focus here on quantifying the flow characteristics at the granular surface, namely the  
125 maximum shear stress  $\tau_m$  and the maximum tangential fluid velocity  $V$ , and discuss their relation to  
126 the free plane jet model in laminar regime, as illustrated in Fig. 1. Finally, we also address here the  
127 onset of jet erosion of a cohesionless granular sample and provide an interpretation of the results  
128 in terms of a Shields diagram.

129 The remainder of this paper is organized as follows: Firstly, we describe the numerical methods  
130 employed for this study in Sec. 2. In Sec. 3, the laminar two-dimensional free jet theory is presented  
131 and the corresponding numerical setup is described, followed by a parametric study of the free

132 jet for various flow conditions. Afterwards, we analyze in detail the jet impingement on both a  
 133 smooth wall and on a fixed granular surface. Sec. 4 finally deals with the full Jet Erosion Test  
 134 on a cohesionless granular sample, discussing and comparing the numerical data with existing  
 135 experimental and theoretical results from the literature. As a closure, Sec. 5 discusses the present  
 136 findings and possible comparison with previous works before Sec. 6 provides a brief conclusion  
 137 and outlines some open perspectives for future research.

## 138 NUMERICAL METHODS

### 139 Lattice Boltzmann Method

140 The LBM is used to simulate the fluid phase (jet flow) based on the solution of the discrete  
 141 Boltzmann equation, as an alternative to other conventional CFD techniques that rely on the direct  
 142 solution of the Navier-Stokes equations. It is usually solved in two main steps, namely a collision  
 143 and a streaming step performed on an Eulerian spatial grid of nodes featuring a limited number of  
 144 discrete velocity vectors for fluid particle populations. We use here the D2Q9 model (Qian et al.  
 145 1992), involving a two-dimensional space and nine velocity vectors  $\mathbf{c}_\alpha$  defined in Eq. 3. For the  
 146 collision model, we employ the multiple relaxation time approach (MRT) (Lallemand and Luo  
 147 2000) to overcome some well-known deficiencies (e.g. numerical stability issues) of the standard  
 148 single relaxation time collision model of Bhatnagar-Gross-Krook (BGK) (Bhatnagar et al. 1954).

149 The multi-relaxation-time lattice Boltzmann equation can be written as:

$$150 \quad f_\alpha(\mathbf{x} + \mathbf{c}_\alpha \Delta t, t + \Delta t) = f_\alpha(\mathbf{x}, t) - M^{-1} \mathbf{S} [m_\alpha(\mathbf{x}, t) - m_\alpha^{eq}(\mathbf{x}, t)] \quad (1)$$

151 where  $f_\alpha$  are the discrete distribution functions (i.e. the probability density of fluid molecules  
 152 with velocity  $\mathbf{c}_\alpha$  in position  $\mathbf{x}$  and at time  $t$ ) and  $\mathbf{S}$  is the diagonal relaxation matrix,  $\mathbf{S} =$   
 153  $diag(0, s_1, s_2, 0, s_4, 0, s_6, s_7, s_8)$ . For the D2Q9 model, the coefficients  $s_{1,2,4}$  are constants to be  
 154 chosen in the range  $0 < s < 2$  (for stability reasons) and  $s_7 = s_8 = 1/\tau$ , where  $\tau$  is the relaxation

155 time related to the fluid kinematic viscosity as follows:

$$156 \quad \nu = \frac{1}{3} \left( \tau - \frac{1}{2} \right) \frac{(\Delta x)^2}{\Delta t} \quad (2)$$

157 We employ here the following relaxation rates after (Lallemand and Luo 2000):  $s_1 = 1.63$ ;  
 158  $s_2 = 1.14$ ;  $s_4 = s_6 = 1.92$ .

159  $M$  is the invertible transformation matrix which links the distribution functions to their moment  
 160 vectors  $m$  by  $m = Mf$ .

161 The macroscopic fluid variables for density  $\rho$  and velocity  $u$  can be obtained from the distribution  
 162 functions as:  $\rho = \sum_{\alpha=0}^8 f_{\alpha}$ ,  $\mathbf{u} = \frac{1}{\rho} \sum_{\alpha=0}^8 f_{\alpha} \mathbf{c}_{\alpha}$ .

163 The fluid pressure is directly given by the following state equation:  $p = c_s^2 \rho$ , where  $c_s = c/\sqrt{3}$   
 164 is the speed of sound in the lattice system,  $c = \Delta x/\Delta t$  is the characteristic lattice speed,  $\Delta x$  and  $\Delta t$   
 165 are the discretization units in space and time respectively.

166 It can be noted that we employ here the swap algorithm for the propagation step proposed in  
 167 (Mattila et al. 2007; Latt 2007), which has the advantage of using and storing only one single copy  
 168 of the density distribution functions  $f$  in memory. This algorithm requires the following special  
 169 ordering:

$$170 \quad \mathbf{c}_{\alpha} \begin{cases} (0, 0) & \alpha = 0 \\ (-1, 1), (-1, 0), (-1, -1), (0, -1) & \alpha = 1, 2, 3, 4 \\ (1, -1), (1, 0), (1, 1), (0, 1) & \alpha = 5, 6, 7, 8 \end{cases} \quad (3)$$

171 It is also worth noting that the Lattice Boltzmann models can recover the incompressible Navier-  
 172 Stokes equation through the Chapman-Enskog expansion (Chapman and Cowling 1970) when  
 173 the density fluctuations are assumed to be negligible (Chen et al. 1992). Therefore, to correctly  
 174 simulate an incompressible flow and reduce the density fluctuations of the model which lead to  
 175 computational errors, the Mach number  $Ma = |u_{max}|/c$  must be kept small as compared to unity:  
 176 the maximum velocity in the system  $u_{max}$  should be much smaller than the sound speed  $c_s$ , usually  
 177 with a limit value of  $Ma < 0.1$  as found in the literature. For this reason, we maintained a constant



178 characteristic speed of  $c = \Delta x / \Delta t = 10$  m/s in our simulations to keep the Mach number smaller  
179 than 0.1.

180 The magnitude of the time-step  $\Delta t$  can then be calculated based on the choice of fluid viscosity  
181  $\nu$  and spatial discretization parameter  $\Delta x$  from the known solid particles (see the section on LBM-  
182 DEM coupling later on), while the relaxation parameter  $\tau$  can be derived based on Eq. (2).

### 183 *Boundary conditions*

184 In the LBM, neither pressure nor velocity boundary conditions can be directly imposed, since  
185 these variables derive from the particle distribution functions  $f_\alpha$ . Therefore, the unknown dis-  
186 tribution functions must be properly defined to retrieve the desired values of the hydrodynamic  
187 quantities at the boundary nodes. Pressure and velocity boundary conditions can, for instance, be  
188 set by using the method proposed by Zou and He (1997). However, the alternative regularized  
189 method proposed by Latt and Chopard (2008) has been used here for the sake of stability to impose  
190 the pressure or velocity boundary conditions for the inlet velocity, since this method was found to  
191 be more stable in all of our simulation cases.

192 Alternatively, we also use here for comparison another boundary condition in which all the  
193 particle populations are constructed by considering only the equilibrium part (the so-called equilib-  
194 rium boundary). The equilibrium boundary is somehow less accurate than the regularized method  
195 (see e.g. Fig. 3 later on), but has the merit of being much easier to implement.

196 The nonslip boundary condition between the fluid and a stationary solid wall can be imposed  
197 through the so-called bounce-back scheme. We use here the half-way bounce-back model, which  
198 assumes that the wall is located right in the middle between solid and fluid nodes and has been  
199 shown to feature a second-order numerical accuracy (Zou and He 1997).

### 200 *Wall shear stress*

201 The hydrodynamic shear stress exerted by the fluid on the wall can generally be derived using  
202 the shear stress tensor  $\tau_{xy}$ , which is given by the following expression for a two-dimensional  
203 incompressible flow:  $\tau_{xy} = \rho\nu(\partial_x u_y + \partial_y u_x)$ .

204 To calculate the wall shear stress, an extrapolation is needed in order to evaluate the velocity

205 gradient at the wall. For the case of a horizontal wall located at  $y = 0$  and where  $\partial_x u_y$  is assumed  
 206 to be negligible, the wall shear stress will be  $\tau_{xy,w} = \rho\nu \left. \frac{du_x}{dy} \right|_{y=0}$ .

207 On the other hand, the shear stress tensor also can be obtained in the LBM out of the non-  
 208 equilibrium part of the distribution functions as proposed by Mei et al. (2002):

$$209 \quad \tau_{xy} = \left(1 - \frac{1}{2\tau}\right) \sum_{\alpha} f_{\alpha}^{neq}(\mathbf{x}, t) (c_{\alpha,x} c_{\alpha,y}) \quad (4)$$

210 where  $f_{\alpha}^{neq} = f_{\alpha} - f_{\alpha}^{eq}$ , and  $c_{\alpha,x}$  and  $c_{\alpha,y}$  are the  $x$  and  $y$  components of the velocity vector  $\alpha$   
 211 respectively.

212 Since we assume here that the wall surface is not directly located on a fluid node, we extrapolate  
 213 linearly the shear stress on the wall boundary so that  $\tau_{xy,w} = \tau_{xy}^{y=1} + \Delta(\tau_{xy}^{y=1} - \tau_{xy}^{y=2})$ , where  $y = 1$   
 214 and  $y = 2$  are the locations in lattice units of the two next neighboring fluid nodes along the normal  
 215 direction with respect to the wall.  $\Delta$  is thereby the relative distance from the fluid node to the wall.  
 216 Since the half-way bounce-back scheme is used here,  $\Delta$  is assumed to be equal to 0.5. According to  
 217 Mei et al. (2002), the estimation of the shear stress using the non-equilibrium part of the distribution  
 218 functions (Eq. 4) is more accurate than that by using the velocity gradient.

## 219 **Discrete Element Method**

220 The DEM originally proposed by Cundall and Strack (1979) is used here to describe both the  
 221 interaction and motion of the solid particles as governed by Newton's equations. The total force  
 222 acting on a given particle is the summation of the interaction forces with other particles plus the  
 223 hydraulic forces. The interaction force exerted by grain  $j$  on grain  $i$  is  $\mathbf{F}_{ij} = f_n \mathbf{n} + f_t \mathbf{t}$  and  $\mathbf{n}$  and  $\mathbf{t}$   
 224 represent the unit normal and tangential vectors along the line of particle centers.

225 The contact forces (normal and tangential), as well as the torque, arise only whenever two  
 226 particles overlap. The normal contact force  $f_n$  can then be calculated based on a linear viscoelastic  
 227 model:

$$228 \quad f_n = -k_n \delta_n - \gamma_n v_n \quad (5)$$

229 where  $v_n$  is the relative velocity in the normal direction,  $k_n$  is the normal contact stiffness and  $\gamma_n$  is

230 the normal viscous damping.

231 The tangential contact force  $f_t$  can also be computed using a linear viscoelastic model as:

$$232 \quad f_t = -k_t \delta_t - \gamma_t v_t \quad (6)$$

233 where  $k_t$  is the tangential contact stiffness,  $\delta_t$  is the relative shear displacement during each contact,  
234  $\gamma_t$  is the tangential viscous damping and  $v_t$  is the relative tangential velocity.

235 The tangential "shear" force is here limited by considering Coulomb's friction law:

$$236 \quad |f_t| \leq \mu_s f_n \quad (7)$$

237 where  $\mu_s$  is the friction coefficient.

238 The torque acting on the particle due to the contact frictional force is then given by

$$239 \quad \mathbf{T}_i = -(\mathbf{r}_i \mathbf{n}) \times (f_t \mathbf{t}) \quad (8)$$

240 We add here a rolling resistance at the contact  $\mathbf{T}_{roll} = -\frac{v_r}{|v_r|} \mu_r r_{eff} f_n$ , with the rolling velocity  
241  $v_r = \omega_i - \omega_j$  defined by the difference in the angular velocities  $\omega_i$  and  $\omega_j$  of grains  $i$  and  $j$   
242 respectively.  $r_{eff} = \frac{r_i r_j}{(r_i + r_j)}$  is thereby the effective radius and  $\mu_r$  is the rolling coefficient.

243 In order to obtain a stable simulation and integrate correctly the equations of motion, the  
244 time step  $\Delta t_{DEM}$  must be chosen below a critical value  $\Delta t_{cr}$  which represents the oscillation  
245 duration of the spring-mass system used to model two contacting particles.  $\Delta t_{cr}$  is thus calculated  
246 taking into account the smallest mass in the granular system  $m_{min}$  and the normal stiffness  $k_n$   
247 as  $\Delta t_{cr} = 2\pi \sqrt{m_{min}/k_n}$ . The DEM time-step is then usually adopted as  $\Delta t_{DEM} = \lambda \Delta t_{cr}$  with a  
248 time-step factor  $\lambda$  chosen around 0.1.

249 The coefficient of normal viscous damping  $\gamma_n$  can be derived from the coefficient of restitution  
250  $e$  (Ting and Corkum 1992).

251 Once all the external forces, including the contact and hydraulic forces, are computed at a time

252  $t$ , an integration algorithm of the velocity-Verlet type can be used to compute the new kinematic  
253 variables of the grain at time  $t + \Delta t$  (Swope et al. 1982).

## 254 **Coupling of the LBM and DEM**

255 The fluid-solid interaction is here introduced with the model proposed by Bouzidi et al. (2001),  
256 which assumes a non-slip bounce-back condition at the solid boundary nodes. This model is  
257 adapted to the curvature of the particle's boundary through a linear interpolation of the post-  
258 collision distribution functions. As a consequence of the particle's translation, some of the solid  
259 boundary nodes may convert to fluid nodes, so that the unknown distribution functions must be  
260 recovered. Several techniques to solve this issue can be found for instance in (Lallemand and Luo  
261 2003), where the unknown distributions functions of the fresh fluid node are simply approximated  
262 using the equilibrium distribution functions (Mansouri et al. 2016).

263 The total force and torque exerted by the fluid on a given particle can then be calculated with  
264 the momentum-exchange algorithm (Ladd 1994) and be introduced into the DEM calculation after  
265 conversion to physical units. The buoyancy effect (i.e. the submerged weight) is also considered  
266 by reducing the gravitational acceleration with the factor  $(1 - \rho_f/\rho_s)$ .

267 To overcome the unphysical situation in 2D simulations where no fluid paths exist through a  
268 densely packed sample of disks, we introduce a reduced "hydraulic" radius  $r_h$  of the grains in the  
269 LBM domain while keeping the particle's real radius  $r$  in the DEM domain (Cui et al. 2012; Boutt  
270 et al. 2007). The ratio  $r_h/r$  is generally set to values around 0.8, as suggested in (Cui et al. 2012).

271 We adopt a fixed spatial resolution ratio  $2r_{min}/\Delta x$  of 10 as recommended by Yu et al. (2003),  
272 where  $r_{min}$  is the radius of the smallest particle. This ratio therefore defines the lattice discretization  
273 parameter  $\Delta x$  for a given sample of solid particles.

274 Since the LBM and DEM often require different time-step sizes (the DEM time-step being  
275 usually smaller than the LBM time-step), an efficient coupling between both methods can be  
276 adopted by introducing sub-cycles for the DEM algorithm (Han et al. 2007). To this end we fix  
277 here an integer sub-cycle number  $n_p = \Delta t_{LBM}/\Delta t_{DEM}$ , ratio of the LBM and DEM time-steps  
278 respectively. In this study,  $n_p$  is chosen equal either to 1 or 2 by adjusting the DEM time-step factor

279  $\lambda$ .

280 The presented LBM-DEM coupling technique has been implemented in our in-house code and  
281 thoroughly validated in several previous works, see for instance (Ngoma et al. 2018). In Sec. 3 we  
282 present a further validation of the LBM model with the well-known laminar free jet theory.

### 283 **Jet configurations**

284 The methods and techniques presented so far are now used to study various jet flow configu-  
285 rations, namely the free jet and the impinging jet on either a smooth wall (case a) or on a fixed  
286 granular surface (case b). These configurations are illustrated in Fig. 2. We impose a bounce-back  
287 boundary condition (Chen et al. 1996) for the solid walls (i.e. for the nozzle boundaries and the  
288 smooth impinged surface), while we assume a Zou/He outlet condition with zero pressure (Zou and  
289 He 1997) for the exterior boundaries. For the jet's nozzle (nozzle width  $b$ ) we consider the velocity  
290 inlet condition with either the regularized or the equilibrium methods as introduced in Sec. 2 and  
291 featuring a Poiseuille velocity profile. The mean velocity of the Poiseuille injection is thereby  
292  $u_j = \frac{2}{3}U_0$  with  $U_0$  being the maximal velocity of the inlet. In these conditions, the jet Reynolds  
293 number can be defined as  $Re_j = u_j b / \nu$ . We consider here relatively high values of  $\nu$  to keep the  
294 flow within the laminar regime, allowing thereby a direct comparison with recent experimental data  
295 from (Badr et al. 2014; Brunier-Coulin et al. 2017a).

296 For the impingement case (a), the simulation procedure and conditions are the same as for the  
297 free jet except for the additional horizontal wall located at an axial distance  $H$  from the nozzle. For  
298 the study case (b), the smooth wall is replaced by a fixed granular surface at the same distance  $H$ .  
299 The granular surface is generated here with a particle size dispersity of  $d_{max}/d_{min} = 1.5$ . Note that  
300 the DEM and the coupling technique with LBM is only active in this configuration.

301 The input values adopted for the parametric study are summarized in Tables 1 and 2 for the free  
302 jet and impinging jets respectively.

## 303 **NUMERICAL RESULTS**

## Two-dimensional free jet

Multiple simulations were performed for various free jet conditions, with Reynolds numbers  $Re_j$  in the range of  $19 < Re_j < 130$ . Fig. 3 shows the numerical results as compared to the analytical solution of the centerline velocity with two different virtual origin adjustments (see the appendix I, which briefly summarizes the theoretical background for a 2D self-similar free jet in laminar regime). Note that, according to Eq. (18), the analytical solution does not originate from the maximal velocity at the nozzle exit. The slight departure from the analytical solution appears a bit more severe for the case with the equilibrium method inlet than for the regularized one. Nevertheless, the overall good agreement with the theory seems evident. The transversal profiles of fluid velocity depicted in Fig. 4a show that the velocity decreases continuously from its maximum value  $u_m$  at the centerline with growing lateral distance  $y$ . A normalized plot of the variables in Fig. 4b confirms that the velocity profiles are closely self-similar and well described by the theoretical prediction  $[ch^{-2}(y/\tilde{b}_u)]$  for all horizons farther than  $x > 1.5b$ , i.e. beyond the point where the free jet exits the potential core region. This region normally extends up to around  $6b$  for turbulent flows according to the literature (Hanson and Cook 2004; Beltaos and Rajaratnam 1977), and so it appears significantly reduced for the laminar conditions considered here. The normalization of the transversal coordinate  $y$  is here done employing the half-width  $b_u$ , which is the transversal distance where  $u(b_u) = \frac{1}{2}u_m$  applies.

Fig. 5a shows the variation of  $(u_j/u_m)^3$  and  $(b_u/b)^{3/2}$  with the normalized distance  $x/b$  from the nozzle for a jet Reynolds number equal to  $Re_j = 38.9$ . The profiles are linear with slopes  $\alpha$  and  $\beta$ , and negative  $x$ -intercepts denoted  $-\lambda_u/b$  and  $-\lambda_b/b$ , respectively. The proportionality between  $u_m$  and  $x^{-1/3}$ , as well as between  $b_u$  and  $x^{2/3}$ , predicted by the theory (see Eqs. 18 and 19 in the appendix) appears here clearly confirmed. A further implication, stemming from the fact that the  $x$ -intercepts are different from zero, is that the origin of the jet is not located right at the nozzle exit, but at a virtual point source at a distance  $\lambda$  from it. Here, by convention,  $\lambda$  is positive when the virtual origin is above the nozzle exit.

Fig. 5b shows the variation of the slopes  $\alpha$  and  $\beta$  with the jet Reynolds number in a log-log plot.

331 Here, the slope of the curve  $\alpha$  versus  $Re_j$  is  $-1.107$ , while its theoretical value is  $-1$  (see Appendix  
 332 I). This difference is not surprising in view of the slight departure in the centerline velocities  
 333  $u_m$  between the simulated and analytical results (see Fig. 3). Nevertheless, the proportionality  
 334  $\alpha \propto Re_j^{-1}$  implied by Eq. (18) is here verified for almost the whole range  $37 < Re_j < 120$ . In  
 335 the same vein, the proportionality  $\beta \propto Re_j^{-1}$  derived from Eq. (19) also appears to be well verified  
 336 here.

337 Concerning the virtual origin  $\lambda$ , the dimensionless quantity  $\lambda/b$  can be easily obtained from  
 338 the  $x$ -intercept of any of the two linear profiles  $(u_j/u_m)^3$  or  $(b_u/b)^{3/2}$  versus  $x/b$  (i.e. from either  
 339 A or B in Fig. 5a). The two different estimations of  $\lambda/b$  seem here to agree fairly (Fig. 6), with a  
 340 mean relative error of 13%, and can be fitted with a linear trend that appears slightly higher than the  
 341 existing solution  $\tilde{\lambda}/b = 0.029Re_j$  given in (Revuelta et al. 2002). The trendline of the numerical  
 342 data reads here:

$$343 \lambda = 0.036Re_j b \quad (9)$$

### 344 **Impinging jets**

345 Regarding now the impinging jet situation, the three main regions of fluid flow depicted in Fig. 2  
 346 can generally be distinguished (see e.g. (Beltaos and Rajaratnam 1973)): a free jet region (zone  
 347 1) in which the flow remains self-similar, an impingement region (zone 2) in which the impinged  
 348 surface affects the jet flow, decreasing the centreline velocity down to zero at the impingement  
 349 (stagnation) point and diverting the flow to the lateral directions, and finally a wall jet region (zone  
 350 3), where the flow becomes parallel to the impinged surface.

351 Many studies have addressed in detail the velocity, pressure, and shear stress fields for these  
 352 regions, e.g. (Rajaratnam 1976; Ghaneizad et al. 2015). However, the analysis of jet impingements  
 353 in the frame of soil erosion still remains largely empirical. No simple analytical approach has been  
 354 proposed so far for the prediction of the flow quantities at either the impingement region or at the  
 355 wall jet region (zones 2 and 3 respectively). In this respect, most of the estimations in the literature  
 356 are based on the free jet model (zone 1). This section therefore examines the influence of both the  
 357 jet Reynolds number and impingement height  $H$  on the distributions of fluid velocity and shear

358 stress at the impinged surfaces. Thereby, we explore firstly the relationship between the free-jet  
359 centerline velocity at the impingement height  $\tilde{u}_m(H)$  and the maximal velocity  $V$  of the impinging  
360 jet near the wall surface in zone 3, turning afterwards the attention to the maximal shear stress  $\tau_m$ .

361 In the simulations presented here, we observed transverse oscillations of the jet when impinging  
362 a granular surface. These oscillations were only hardly noticeable for small values of  $Re_j$  and tended  
363 to intensify progressively with an increasing inlet velocity. Therefore and for the sake of consistency,  
364 here we analyzed the velocity field only at the moments where the jet is exactly vertical. In contrast,  
365 such jet oscillations were never observed in the simulations with a smooth wall.

### 366 *Velocity field*

367 Figs. 7a and 7b show the profiles of transverse velocity  $v$  at different distances  $x_1$  from the  
368 impinged surface for the study cases (a) and (b) respectively. Note that the reference position  
369  $x_1 = 0$  for the fixed granular surface corresponds to the top of the uppermost particle. All profiles  
370 show a monotonic increase of velocity up to a maximum value  $v_m$  and a subsequent continuous  
371 decrease with growing distance  $y/H$  from the jet's axis. The local maximum of transverse velocity  
372  $v_m$  of each profile increases rapidly with  $x_1$  until reaching a global maximum  $V = \max(v_m)$  and  
373 then decays slowly.

374 The global maximum of fluid velocity  $V$  over the impinged surface can be extracted for different  
375 flow conditions and samples (different mean grain sizes), and then be plotted versus the free-jet  
376 maximum velocity  $\tilde{u}_m(H + \lambda)$  at the corresponding distance from the nozzle (Eq. 18), as shown in  
377 Fig. 8 in direct comparison to the smooth-wall results. We appreciate a close agreement of the data  
378 for the low velocity range, with growing deviations for higher fluid velocities and higher particle  
379 size due to the irregular form of the bed surface. We also notice that the trend is almost linearly  
380 proportional with a slope equal to 0.82 [case (a)], consistently for any given jet Reynolds number  
381  $Re_j$  and distance  $H$  within the range of our simulation sets. Thereby, we replace  $H$  by  $H + \lambda$  in  
382 order to take into account the virtual origin discussed previously (i.e.  $\lambda/b = 0.036Re_j$ ), although  
383 the effect of  $\lambda$  on the slope appears to be negligible. These results therefore confirm that, for the  
384 case of a smooth surface, the maximum impingement velocity can indeed be estimated by means



385 of the free jet theory. Concerning the impingement on a granular surface, it appears sensible to  
386 approximate the maximum velocity just as for the smooth wall case at least for situations with low  
387 Reynolds numbers.

388 In the case of granular surfaces, the sensitivity to the impact point location was also tested  
389 for the same inlet flow condition as presented in Fig. 7b. To this end, several calculations were  
390 performed after a slight lateral displacement of the nozzle (up to 3 times the minimal diameter) in  
391 either direction. In all cases, the maximal velocity  $V$  was found consistent to a mean value within  
392 a reasonable error bar estimated to less than 5% from the standard deviation values. The same  
393 relative error is used for the other inlet flow conditions.

#### 394 *Wall shear stress*

395 The simulated distribution of dimensionless wall shear stress is plotted in Fig. 9a for different  
396 combinations of impingement height  $H/b$  and jet Reynolds number  $Re_j$ . Despite the difference  
397 in the flow configuration (2D laminar versus round turbulent jet), the dimensionless shear stress  
398 distribution agrees quite well with that given by Beltaos and Rajaratnam (1974). Here, it can  
399 also be noted that the maxima of shear stress are actually located closer to the jet's axis than the  
400 corresponding maxima of fluid velocity (see Fig. 7)

401 The maximal value of  $\tau$  is often assumed to be proportional to the square of the maximal  
402 velocity  $V$  (Beltaos and Rajaratnam 1977), namely in the form of  $\tau_m = \frac{1}{2}C_f\rho V^2$ , where  $C_f$  is the  
403 local friction coefficient and  $\rho$  is the fluid density.

404 The authors are not aware of any estimation of  $C_f$  for laminar impinging jets to be found in  
405 the literature. The typical value of  $C_f$  for turbulent flow conditions ranges around  $4 \times 10^{-3}$ , see  
406 e.g. (Beltaos and Rajaratnam 1974; Hanson and Cook 2004; Beltaos and Rajaratnam 1977). Based  
407 on the definition of  $\tau_m$ , now we can use our simulation results to estimate  $C_f$  for the different jet  
408 Reynolds numbers and impingement heights  $H$  shown in Table 2.

409 A plot of the maximum shear stress  $\tau_m$  versus  $\rho V^2/\sqrt{Re_j}$  is shown in Fig. 9b, suggesting a

410 linear dependency that permits to estimate  $C_f$  as:

$$411 \quad C_f = \frac{1.53}{\sqrt{Re_j}} \quad (10)$$

412 This way,  $\tau_m$  can be rewritten into:

$$413 \quad \tau_m = \frac{0.765\rho V^2}{\sqrt{Re_j}} = \frac{0.52\rho(\tilde{u}_m(H + \lambda))^2}{\sqrt{Re_j}} \quad (11)$$

414 By introducing Eqs. (18) and (9), it then reads as:

$$415 \quad \tau_m = \frac{0.137\rho u_j^2 Re_j^{1/6}}{(H/b + 0.036 Re_j)^{2/3}}. \quad (12)$$

416 Surprisingly, we found that the simulation results give close results compared to the estimation  
 417 based on the Blasius friction law for a laminar boundary layer over a flat plate:  $\bar{C}_f = 1.328/\sqrt{Re}$   
 418 (Streeter and Wylie 1975), where  $\bar{C}_f$  is the average friction coefficient over a plate of length  $L$  and  
 419  $Re = U_\infty L/\nu$ . Consistently with the development of a boundary layer, a scaling with  $Re_j^{1/2}$  can be  
 420 reasonably obtained for the  $x_1$ -value of location of the maximal velocity. Similar results were also  
 421 presented in (Phares et al. 2000) implying that the maximal shear stress within a laminar boundary  
 422 layer scales with  $Re_j^{-1/2} (H/b)^{-5/4}$  at  $y/H = 0.12$  for fully developed 2D jet impingements ( $H/b > 8$ ).

423 Summing up, these results show that the local friction coefficient at the maximum shear stress  
 424 seems to be proportional to  $1/\sqrt{Re_j}$  for laminar jets impinging on a smooth wall, just as predicted  
 425 by the laminar boundary layer theory on flat plates at zero incidence. The maximum shear stress  
 426 over a smooth wall can therefore be estimated using the approximation in Eq. (12) based on our  
 427 simulation results.

428 Concerning the wall shear stress distribution at the granular surface, the strong fluctuations of  
 429 shear stress related to the irregularity of the impinged surface generally preclude the appearance  
 430 of smooth distributions such as the one shown in Fig. 9a. Nevertheless, and in absence of more  
 431 specific estimations, it appears sensible to derive the wall shear stress over granular surfaces using

432 the previous approximation based on the maximum velocity  $V$  that was found for the smooth-wall  
433 case (Eq. 11).

## 434 ONSET OF EROSION FOR FRICTIONAL SAMPLES

435 We now turn the attention to the jet erosion, i.e. to the detachment of solid particles under the  
436 action of an impinging jet, for the case of a cohesionless granular bed. In this section, the solid  
437 particles are now let free to move based on purely frictional interactions (no cohesion) and under  
438 the hydrodynamic solicitations imposed by the impinging jet.

439 In general, the onset of erosion for cohesionless sediments can be described by the Shields  
440 number, which quantifies the erosion threshold as the ratio between the critical bed shear stress  
441  $\tau_s = \rho u_*^2$  and the submerged gravitational stress acting on the solid particles  $(\rho_g - \rho)gd$ , where  
442  $d$  is the particle diameter while  $\rho_g$  and  $\rho$  are the grain and fluid densities respectively (Shields  
443 1936). The abundant literature on the Shields diagrams shows that the erosion threshold can be  
444 well described by a critical Shields number  $Sh_\tau^*$  solely dependent on the particle Reynolds number  
445  $Re_\tau^* = u_*d/\nu$ , that is:

$$446 \quad Sh_\tau^* = \frac{\tau_s}{(\rho_g - \rho)gd} = f(Re_\tau^*) \quad (13)$$

447 where  $u_*$  is the so-called friction velocity, or shear velocity, at the bed surface, and  $\tau_s$  may be  
448 here approximated for a 2D laminar impinging jet from Eq. (11), or equivalently Eq. (12), as shown  
449 in the previous section.

## 450 Erosion threshold

451 In order to estimate the erosion threshold for a given cohesionless granular sample in our  
452 micromechanical JET simulations, we now let the particles move freely as we increase progressively  
453 the maximal inlet velocity  $U_0$  over time until reaching a fully developed erosive state, as shown in  
454 Fig. 10. Then, we identify the critical inlet velocity  $U_0^c$  based on the observation of the first grain  
455 motion.

456 We repeated this procedure for various jet Reynolds numbers and three different samples

457 ( $d_{mean}=2, 3, \text{ and } 5 \text{ mm}$  respectively, featuring a uniform size distribution ranging from  $d_{min} =$   
 458  $0.8d_{mean}$  to  $d_{max} = 1.2d_{mean}$ ), thus providing a range of conditions to be displayed in the Shields  
 459 diagram. The input data for this parametric study is summarized in Table 3.

460 We observe that the first granular motion generally takes place at a certain distance from the  
 461 impingement (stagnation) point and corresponding roughly with the location of the maximal shear  
 462 stress predicted in our analysis of the impinging jet over a smooth wall [see Fig. 9a]. However,  
 463 by increasing progressively the inlet velocity, we observe at some point the appearance of lateral  
 464 oscillations of the fluid jet caused by the irregularities of the bed surface, which in turn enhance the  
 465 on-going scouring process. Due to the oscillations of the jet and the increasing depth of the crater,  
 466 the location of the active erosion zone appears then to shift progressively towards the impingement  
 467 point, thereby creating a deeper crater right under the jet's axis (see the graphical sequence shown  
 468 in Fig. 10).

### 469 **Shields diagram**

470 The Shields diagram represents the relationship between the critical Shields number and the  
 471 particle Reynolds number, estimated from the friction velocity, as obtained for various particles  
 472 sizes and shapes in a wide range of flow conditions. Although the original Shields diagram presents  
 473 solely scatter data, several empirical approximations have been subsequently proposed to fit the  
 474 data. The explicit formulation of the Shields curve proposed by (Guo 1997; Guo 2002) reads for  
 475 instance:

$$476 \quad Sh_{\tau}^* = \frac{0.11}{Re_{\tau}^*} + 0.054 \left[ 1 - \exp \left( -\frac{4Re_{\tau}^{*0.52}}{25} \right) \right] \quad (14)$$

477 where  $Re_{\tau}^*$  is the particle Reynolds number as defined before.

478 Here it is worth noting that this expression is sometimes inconvenient for a practical use  
 479 (especially in complex flow configurations such as the impinging jet), since the shear velocity  
 480 actually appears in both  $x$  and  $y$  variables of the Shields diagram (see Eq. 13). However, this can  
 481 be circumvented with the alternative approach proposed by Badr et al. (2014) and later adopted  
 482 by Brunier-Coulin et al. (2017a), representing an equivalent form of the Shields number for the

483 impinging jet. The idea is to assume an inertial expression  $Sh_u$  for the Shields number, considering  
 484 that the shear stress is simply equal to  $\rho u^2$ , regardless of the flow regime. Here, following Badr  
 485 et al. (2014), the velocity  $u$  is chosen as a characteristic fluid velocity around the eroded particle  
 486 and can be directly estimated with the free jet model  $\tilde{u}_m(H)$  instead of the shear velocity  $u_*$  at the  
 487 impinged surface.

488 As a first test, we can compare quantitatively our simulation results to the experimental data of  
 489 Badr et al. (2014), since their quasi-2D configuration is closely consistent with the two-dimensional  
 490 conditions of our model. The equivalent Shields diagram proposed by Badr et al. (2014) relates the  
 491 critical value of the inertial Shields number,  $Sh_u^*$ , to the critical particle Reynolds number  $Re_p^* = \frac{ud}{\nu}$   
 492 as follows:

$$493 \quad Sh_u^* = \frac{\rho[\tilde{u}_m(H)]^2}{(\rho_g - \rho)gd} = f(Re_p^*) \quad (15)$$

494  $Sh_u^*$  is here evaluated from the expression of  $\tilde{u}_m(H)$  given in Eq. (18), where  $u_j$  is given by the  
 495 critical inlet velocity obtained for each simulation at the onset of erosion,  $u_j = 2/3U_0^c$ .

496 Here it is also important to note that, for a quantitative comparison, the expression of the inertial  
 497 Shields number  $Sh_u$  has to be modified for the simulated results to account for the dimensional  
 498 discrepancy of the solid particles, i.e. for the simulated disks in a plane instead of the solid spheres  
 499 in the quasi-2D experimental configuration. The correction employed here is explained as follows.  
 500 Firstly, we assume that the ratio of hydrodynamic drag force to the buoyant weight of a given  
 501 particle is the same both for disks and spheres. This ratio reads  $\frac{\tau_f S}{\Delta \rho g V} = Sh \frac{Sd}{V}$  where  $S$  and  $V$  are  
 502 the cross-section and volume of the particle respectively. For disks or cylindrical particles, this  
 503 expression leads to  $\frac{Sd}{V} = \frac{4}{\pi}$ , while for the case of a sphere  $\frac{Sd}{V} = \frac{3}{2}$  is obtained. As a consequence, the  
 504 inertial Shields number from the simulations is multiplied by  $\frac{3\pi}{8}$  to be quantitatively comparable  
 505 to the experimental data. Moreover, the reduced (hydraulic) diameter  $d_h = 0.8d$  is also taken into  
 506 account, as explained above.

507 Fig. 11 shows that a fair agreement of simulated and empirical data can be achieved this way.  
 508 Our numerical values of  $Sh_u^*$  are in the range  $1.16 \pm 0.33$ , comparing well with the results of Badr  
 509 et al. (2014) which appear to show an almost constant value of  $Sh_u^* = 1.2 \pm 0.6$  for a range of  $Re_p^*$

510 from laminar to turbulent flows. In this respect, our numerical data rather suggest a slight decrease  
511 of  $Sh_u^*$  with  $Re_p^*$ .

512 Not only the inertial Shields number  $Sh_u$  but also the usual Shields number defined by the real  
513 fluid shear stress,  $Sh_\tau$ , can be calculated out of the numerical results based on Eq. (12). This way,  
514 the previous results can be plotted in the classical Shields diagram through Eq. (13). This can also  
515 be done for the experimental data of Badr et al. (2014) if we assume the same friction coefficient  $C_f$   
516 as given in Eq. (10) from our LBM calculations with laminar impinging jets. The corresponding  
517 values of the critical Shields numbers  $Sh_\tau^*$  for both experimental and numerical results as compared  
518 to the explicit formulation in Eq. (14) are shown in Fig. 12 as a function of the particle boundary  
519 Reynolds number  $Re_\tau^*$ .

520 Here the quantitative agreement between the present numerical results and the experimental data  
521 by Badr et al. (2014) is slightly worse than by using the inertial Shields number  $Sh_u^*$ . Nevertheless,  
522 both data sets appear relatively close to the explicit Shields curve, with the simulated data laying  
523 slightly above it and the experimental one slightly below. Furthermore, a slight decrease of  $Sh_\tau^*$  with  
524  $Re_\tau^*$  can now be observed more clearly for this range of particle Reynolds number  $Re_\tau^*$ , generally  
525 consistent with the trend shown by the Shields curve.

## 526 DISCUSSION

527 The main results of the present study can be summarized as follows. Firstly, the results obtained  
528 with our LBM model of a 2D jet flow show high accuracy when compared to the laminar self-  
529 similar solution of a plane free jet for jet Reynolds numbers up to almost 130, thus extending  
530 previous comparisons with experimental data which were limited to jet Reynolds numbers under  
531 30 (Phares et al. 2000; Looney and Walsh 1984; Andrade 1939). Regarding the jet impingement  
532 on a smooth wall, our findings also confirm that a laminar boundary layer develops at the wall  
533 departing from the stagnation point and featuring a friction coefficient that scales with  $Re_j^{-1/2}$ , as  
534 already shown by Phares et al. (2000). On this basis, we propose here an accurate expression for  
535 the maximal bed shear-stress. On the other hand, and despite a greater scatter of the results, the  
536 data obtained for the impingement on a fixed granular wall appears broadly consistent with the

537 previous simulations on a smooth wall with respect to the maximal velocity reached by the flow  
538 in the impingement zone. To conclude, we finally propose a direct confrontation of our numerical  
539 results for granular erosion with the experimental data for three-dimensional planar jet erosion  
540 obtained by Badr and co-workers (2014). We find that the good quantitative agreement with the  
541 experimental data endorses the further use of our LBM-DEM modelling approach and appears to  
542 support our novel estimation of maximal shear-stress for the Shields diagram.

543 However, for several reasons the outcomes proposed within this study cannot be directly com-  
544 pared to real flow conditions, which are usually highly turbulent both in the context of soil erosion  
545 and particularly in complex applications such as the Jet erosion test.

546 To begin with, our LBM model is not yet adapted to natural turbulent water flows and therefore  
547 it was used here to simulate only laminar jets. Nevertheless, and beyond the fact that the flow  
548 impinging a granular bed is often laminar at the upper jet inlet, the redirection of the flow and  
549 the interaction with the sediment bed through intermittent and localized flow structures allow  
550 to explore not only the laminar flow domain but also the transitional one regarding the Shields  
551 diagram (i.e.  $Re_\tau$  up to about 20 in Fig. 12). Indeed, our results compare reasonably well with  
552 the experimental data by Badr and co-workers that was produced not only with laminar jets but  
553 also with turbulent jets, with jet Reynolds numbers up to around 1000 (Badr et al. 2014). On  
554 the other hand, in this respect it also appears important to note that the alternative use of a mean  
555 fully-turbulent modeling approach, possibly added to the LBM by means of a LES scheme (Large  
556 Eddy Simulation) as proposed in Feng et al. (2007), would probably not be able to reproduce the  
557 unsteady and short-lived flow bursts that can be observed in the simulations with our current model.

558 A second reason that prevents our findings from being directly applicable for a real JET test,  
559 whether in the lab or in the field, is the limitation of our models to 2D and plane jet conditions,  
560 whereas the jets that are commonly used in practice are naturally three-dimensional and round.  
561 Consequently, only qualitative comparisons can be expected between the 3D circular and the 2D  
562 planar impinging jets, particularly since the respective analytical free jet solutions show different  
563 scaling laws due to the increased lateral dispersion in 3D (Bickley 1937; Schlichting 1960).

564 The applicability of the present models would certainly increase substantially if both the 3D  
565 as well as the turbulent flow conditions were implemented in our algorithms. However, such  
566 an improvement would also involve a considerable increase of both the computational cost and  
567 necessary resources in terms of processing power and storage that at present we only envisage  
568 as a future objective. In this sense, the models presented here can therefore be considered as a  
569 first step on this route, but already with the major advantage of allowing a direct comparison with  
570 experimental data, which is relatively scarce in relation to jet erosion.

571 And finally, further discussion can also be focused on the way to compare quantitatively the  
572 impinging planar jets and their related erosion onset between the 2D conditions in the present  
573 numerical study and the 3D reality in the experiments by Badr et al. (2014). In this respect  
574 we may first note that several aspects of the experimental conditions were different from their  
575 numerical counterparts: in the experiments, the grains were significantly smaller in size, from  
576 0.1 to 1mm, while the liquid used for the jet flow was either water or glycerol-water mixtures  
577 with a viscosity not greater than four times that of water. By contrast, the numerical simulations  
578 featured larger particles (with 2, 3 and 5 mm in mean grain size) and much more viscous fluids  
579 (30 to 50 times the water viscosity). Nevertheless, we show here that these differences do not  
580 prevent a quantitative comparison assuming the relevance of the Shields diagram approach based  
581 on dimensionless numbers, namely the Shields number and the particle Reynolds number, which  
582 cover approximately the same ranges. Therefore, the main issue here rather comes from the natural  
583 differences in terms of geometry: an assembly of disks with a 2D impinging flow versus a bed  
584 of spherical particles impinged by a 3D planar jet. As explained beforehand, the expression of  
585 the Shields number in 2D with disks can be modified based on the mechanical equilibrium at the  
586 particle scale. This feature leads to an accurate agreement with the experimental data when using  
587 the inertial Shields numbers as shown in Fig. 11. In contrast, the agreement appears less convincing  
588 when the correct dimensionless numbers needed in the Shields approach are used. However, this  
589 latter approach seems questionable since it is based on several major simplifications. For instance,  
590 the fluid friction coefficient in Eq. (10) is probably slightly different between the 2D jet and the 3D



591 planar jet. Furthermore, the relationship between the hydrodynamic drag force and the bed shear-  
592 stress may also be different between a sphere and a disk (or a cylinder), including the potential  
593 influence of the numerical hydraulic diameter. And finally, the effective maximal shear-stress  
594 exerted on the sediment bed is probably different from the one calculated on a smooth wall. All  
595 in all, it appears that our crude estimate of Shields number in 2D is probably compatible with the  
596 factor of about 2 that, according to Fig. 12, could account for the almost systematic discrepancy  
597 between the Shields curve and our numerical results.

## 598 CONCLUSIONS AND OUTLOOK

599 Some 2D laminar flow configurations of both free jet and impinging jet on a horizontal surface  
600 have been investigated here using the numerical Lattice Boltzmann Method (LBM). We show that  
601 the flow simulations are accurate for various jet Reynolds numbers by introducing the virtual origin  
602  $\lambda$ , as validated with the self-similarity theory of the free jet.

603 The simulations of the impinging jet for both a smooth wall and a fixed granular surface  
604 have further shown that the maximal tangential velocity in the vicinity of the surface is directly  
605 proportional to the free-jet velocity at a corresponding downstream distance as computed with the  
606 self-similar theory. The results therefore confirm that the free-jet velocity  $\tilde{u}_m(H)$  can be used as the  
607 characteristic impingement velocity when the virtual origin is taken into account. Furthermore, we  
608 have proposed here an expression for the maximal shear stress at the surface (Eq. 11) based as well  
609 on the free-jet theoretical velocity and including an additional friction coefficient of the Blasius  
610 type that is inversely proportional to the square root of the jet Reynolds number.

611 Finally, we have addressed the onset of jet erosion for frictional (cohesionless) granular samples  
612 by means of two-dimensional simulations of the Jet Erosion Test (JET) with a coupled LBM-DEM  
613 technique. The simulated results appear in fair agreement with the experimental data of Badr  
614 et al. (2014) for plane impinging jets regarding two different definitions of the Shields number,  
615 namely the inertial expression  $Sh_u^*$  and the usual one  $Sh_\tau^*$ . The classical Shields diagram was well  
616 reproduced in the latter case.

617 As a perspective, we are currently extending our JET simulations to the case of cohesive granular

618 samples with varying particle sizes and inter-particle cohesion, aiming to verify the generalization  
619 of the Shields number for cohesive granular materials proposed by Brunier-Coulin et al. (2017b).

#### 620 **DATA AVAILABILITY**

621 All data generated during the study are available from the corresponding author by request  
622 while the code used remains proprietary.

#### 623 **ACKNOWLEDGMENTS**

624 Z. Benseghier is grateful for the financial support provided by the "Région Sud, Provence-  
625 Alpes-Côte d'Azur" and the valuable exchanges with D. Chausseé (Suez Consulting Corp.). The  
626 authors would also like to thank F. Lominé, J. Duriez and J. Ngoma for fruitful discussions.

## APPENDIX I. TWO-DIMENSIONAL FREE JET

The two-dimensional free jet has been studied in the past by many researchers. Schlichting (1960) provided a solution describing a round jet based on the boundary-layer approximation, while Bickley (1937) gave an analytical solution for the two-dimensional case. The latter is based on the assumption that the momentum flux  $M$  remains constant so that the free jet flow remains self-similar with the downstream distance  $x$  from a source point. For the remainder, we introduce the index ( $\tilde{\cdot}$ ) to denote the analytical variables and distinguish them from the simulated ones.

The self-similarity of a plane free jet implies that the longitudinal velocity of the fluid at any point downstream of the nozzle can be described by:

$$\tilde{u}(x, y) = \tilde{u}_m(x)f(\eta) \quad (16)$$

where  $\tilde{u}_m$  is the fluid velocity along the jet axis (i.e. the maximal velocity at the horizon  $x$ ) and  $f$  is the similarity function in the following form:  $f(\eta) = 1/\text{ch}^2(\eta)$  with  $\eta = y/\tilde{\Delta}(x)$ . Here,  $\eta$  is the self-similar variable,  $\tilde{\Delta}(x)$  is the jet's half-width at the downstream distance  $x$  and  $y$  is the coordinate transversal to the jet's axis (see Fig. 13).

Defining  $\tilde{b}_u$  as the value of  $y$  where  $\tilde{u} = \frac{1}{2}\tilde{u}_m$ , the relationship between  $\tilde{\Delta}$  and  $\tilde{b}_u$  is simply given by  $\tilde{b}_u(x) = \text{ach}(\sqrt{2})\tilde{\Delta}(x)$ .

The analytical solutions for the jet's centerline velocity and half-width are provided by Bickley (1937) as:  $\tilde{u}_m(x) = \left(\frac{3M^2}{32\rho^2vx}\right)^{1/3}$  and  $\tilde{\Delta}(x) = \left(\frac{48\rho v^2x^2}{M}\right)^{1/3}$  respectively.

For the case of a 2D Poiseuille inlet flow, the constant momentum flux is  $M = \int_{-\infty}^{+\infty} \rho\tilde{u}^2 dy = (6/5)\rho u_j^2 b$ . Since the mass flux is assumed to be constant at any downstream location, the centerline velocity  $\tilde{u}_m$  can be expressed as:

$$\tilde{u}_m(x) = \frac{3}{10}u_j \left(\frac{5Re_j}{x/b}\right)^{1/3} \quad (17)$$

We note here that the analytical solution features a singular point at  $x = 0$  due to the assumption that the jet flow begins at a narrow orifice of infinitesimal width (Bickley 1937; Schlichting 1960).

651 Consequently, and in order to fit with the experimental and simulated data, the equations need to be  
 652 adjusted by introducing a virtual origin  $\tilde{\lambda}$  (see Fig. 13), which has been estimated in previous works  
 653 (Andrade and Tsien 1937; Andrade 1939; Revuelta et al. 2002). Revuelta and co-workers (2002)  
 654 gave for instance a numerical estimation of  $\tilde{\lambda}$  for both plane and round jets as a function of  $Re_j$  and  
 655  $b$ . The expression of  $\tilde{\lambda}$  for a laminar plane free jet with Poiseuille injection reads:  $\tilde{\lambda} = 0.029Re_j b$ .  
 656 After introducing the virtual origin, the centerline velocity and the jet's half-width become:

$$657 \quad \tilde{u}_m(x) = \frac{3}{10}u_j \left( \frac{5Re_j}{(x + \tilde{\lambda})/b} \right)^{1/3} \quad (18)$$

$$658 \quad \tilde{\Delta}(x) = 40^{1/3}bRe_j^{-2/3} \left( \frac{x + \tilde{\lambda}}{b} \right)^{2/3} \quad (19)$$

660 with  $x$  being now the downstream distance from the nozzle.

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**TABLE 1.** Input parameters used for the study cases with free jet configuration

$U_0$ (m/s)	$b$ (mm)	$\nu \times 10^{-6}$ (m <sup>2</sup> /s)	$Re_j$ (-)
0.37	5.2	33	38.9
0.37	5.2	66	19.4
0.185	10.8	16.5	80.7
0.185	5.2	16.5	38.9
0.185	10.8	33	40.4
0.185	5.2	33	19.4
0.74	5.2	33	77.7
0.37	10.8	33	80.7
0.37	6.8	33	50.8
0.37	8.4	33	62.8
0.37	12.4	33	92.7
0.62	5.2	16.5	130.3
0.53	5.2	16.5	111.4

**TABLE 2.** Input parameters used for the study cases with impinging jet configuration

Case	$U_0$ (m/s)	$b$ (mm)	$\nu \times 10^{-6}$ (m <sup>2</sup> /s)	$Re_j$ (-)	$H$ (mm)
a, b	0.37	5.2	33	38.9	90.4
a, b	0.74	5.2	33	77.7	90.4
a, b	0.5	5.2	33	52.5	90.4
a, b	0.6	5.2	33	63.03	90.4
a, b	0.37	6.8	33	50.8	90.4
a, b	0.74	6.8	33	101.7	90.4
a	0.37	5.2	33	38.9	73.2
a	0.74	5.2	33	77.7	73.2
a	0.37	6.8	33	50.8	73.2
a	0.37	5.2	33	38.9	108
a	0.74	5.2	33	77.7	108
a	0.37	6.8	33	50.8	108
a	0.74	5.2	10	256.5	90.4
a	0.37	5.2	10	128.3	90.4
a	0.74	5.2	100	25.7	90.4
a	0.37	5.2	100	12.8	90.4

**TABLE 3.** Input parameters for a parametric study of the erosion threshold

Solid particles	Fluid
Density $\rho_s$ : $2230 \text{ kg/m}^3$	Density $\rho_f$ : $847 \text{ kg/m}^3$
Normal stiffness $k_n$ : $1.1 \times 10^5$	Kinematic viscosity $\nu$ :
Shear stiffness $k_t$ : $1.1 \times 10^5$	$30 \text{ to } 50 \times 10^{-6} \text{ m}^2/\text{s}$
Friction coefficient $\mu$ : 0.3	Nozzle diameter $b$ : 5.2 mm
Rolling friction $\mu_r$ : 0.1	Impingement height $H$ : 90 mm
Restitution coefficient $e$ : 0.2	
Gravitational acceleration $g$ : $9.81 \text{ m/s}^2$	

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Fig1.pdf

**Fig. 1.** Paradigm of free jet and impinging jet on a granular sample. The hydrodynamic variable  $\tilde{u}_m(H)$  represents the free jet centerline velocity at a distance equal to the impingement height  $H$ , while  $V$  is the maximal fluid velocity over the impinged surface.

Fig2.pdf

**Fig. 2.** Sketch of study cases for jet analysis. Left: Free jet configuration; Right: Impinging jet on either a smooth wall (case a) or on a granular surface (case b) with distinction of the three characteristic jet flow regions.

Fig3.eps

**Fig. 3.** Simulation results for a laminar 2D free jet with Poiseuille inlet ( $U_0 = 0.37$  m/s,  $b = 5.2 \times 10^{-3}$  m,  $\nu = 33 \times 10^{-6}$  m<sup>2</sup>/s, and  $Re_j = 38.9$ ). The curves represent the dimensionless centerline velocity  $u_m/U_0$  along the dimensionless downstream distance  $x/b$  from the nozzle using the regularized (red circle line) and equilibrium (blue triangle line) boundary conditions for the inlet, as compared to the analytical solutions of the 2D laminar free jet using virtual origin given by (Revuelta et al. 2002) (dotted line) or by Eq. (9) (solid line), respectively. The dashed line represents the impinging jet on a smooth wall with  $H = 90.4$  mm for the same inlet conditions.

Fig4a.eps

(a)

Fig4b.eps

(b)

**Fig. 4.** Transversal profiles of fluid velocity for a laminar 2D free jet with Poiseuille inlet (same conditions as in Fig. 3). (a) Dimensionless fluid velocity  $u/U_0$  versus dimensionless transversal coordinate  $y/b$  at different downstream distances  $x/b$  from the jet's nozzle. (b) Profiles of fluid velocity  $u$  normalized by the maximum (centerline) value  $u_m$  at each downstream location  $x/b$ ; the coordinate  $y$  is now normalized by  $b_u$ , the transversal distance where  $u(b_u) = \frac{1}{2}u_m$ .

Fig5a.eps

(a)

Fig5b.eps

(b)

**Fig. 5.** (a) Variation of  $(u_j/u_m)^3$  (●) and  $(b_u/b)^{3/2}$  (○) with the normalized distance from the nozzle  $x/b$  for  $Re_j = 38.9$ ; the solid lines are linear fits:  $(u_j/u_m)^3 = \alpha(\frac{x+\lambda_u}{b})$  and  $(b_u/b)^{3/2} = \beta(\frac{x+\lambda_b}{b})$  with  $\alpha = 0.201$ ,  $\beta = 0.138$ ,  $\lambda_u/b = 1.29$ , and  $\lambda_b/b = 1.48$ . (b) Log-log representation of the slopes  $\alpha$  and  $\beta$  versus  $Re_j$ ; the lines represent the theoretical predictions extracted from Eqs. (18) and (19) in Appendix I.

Fig6.eps

**Fig. 6.** Variation of the dimensionless virtual origin  $\lambda/b$  versus the jet Reynolds number  $Re_j$  obtained from the linear regression of the profiles  $(u_j/u_m)^3$  ( $\bullet$ ) and  $(b_u/b)^{3/2}$  ( $\circ$ ). The linear trend fitting the data is shown as a dashed line and the solid line is the expression of  $\tilde{\lambda}$  provided by (Revuelta et al. 2002).

Fig7a.eps

(a) Smooth wall

Fig7b.eps

(b) Granular surface

**Fig. 7.** Profiles of transverse velocity  $v$  for different distances  $x_1$  from the impingement surface, both for the smooth wall configuration (a) and for a fixed granular surface with mean grain size  $d = 5$  mm (b) ( $b/d = 1.04$  and  $b_u/d = 1.93$  with  $b_u$  deduced from Eq. 19 with free jet model). The general simulation parameters are  $H = 90.4$  mm and  $U_0 = 0.37$  m/s. For symmetry reason, only the right part of the transverse velocity profiles are shown for the smooth wall configuration.



Fig8.eps

**Fig. 8.** Variation of maximum velocity  $V$  versus the free jet velocity  $\tilde{u}_m(H + \lambda)$  at the corresponding downstream distance  $x = H$  for a laminar jet impingement on either a smooth wall or on fixed granular surfaces with mean grain sizes  $d = 3$  mm and 5 mm respectively. The solid line is a linear fit from the smooth wall results.

Fig9a.eps

(a)

Fig9b.eps

(b)

**Fig. 9.** (a) Transversal profiles of dimensionless wall shear stress for different combinations of normalized impingement height  $H/b$  and jet Reynolds number  $Re_j$ . The solid line represents the estimation provided by (Beltaos and Rajaratnam 1974). (b) Maximum shear stress  $\tau_m$  on a smooth impinged surface versus  $\rho V^2 / \sqrt{Re_j}$ .

Fig10a.pdf

(a)  $t = 12.5$  s,  $Re_j = 40.21$ ,  $U_0 = 0.58$  m/s

Fig10b.pdf

(b)  $t = 15$  s,  $Re_j = 45.76$ ,  $U_0 = 0.66$  m/s

Fig10c.pdf

(c)  $t = 17.5$  s,  $Re_j = 50$ ,  $U_0 = 0.72$  m/s

Fig10d.pdf

(d)  $t = 20$  s,  $Re_j = 55.47$ ,  $U_0 = 0.8$  m/s

Fig10e.pdf

(e)  $t = 22.5$  s,  $Re_j = 61.01$ ,  $U_0 = 0.88$  m/s

Fig10f.pdf

(f)  $t = 25$  s,  $Re_j = 69.33$ ,  $U_0 = 1$  m/s

**Fig. 10.** Time sequence of jet erosion on a frictional granular sample composed of 3000 particles with  $d_{mean} = 2$  mm,  $\nu = 50 \times 10^{-6}$  m<sup>2</sup>/s,  $b = 5.2$  mm. A color scale is used for fluid velocity magnitude from zero (blue) to maximal inlet velocity  $U_0$  (red). Solid particles with kinetic energy above a critical threshold are classified as eroded (here depicted in red colour).

Fig11.eps

**Fig. 11.** Critical values of inertial Shields number  $Sh_u^*$  versus particle Reynolds number  $Re_p^*$  for the simulated jet erosion of frictional granular beds, as compared to the experimental results of (Badr et al. 2014) (●). The simulations were performed with different values of mean particle size  $d$  and fluid kinematic viscosity  $\nu$ :  $d = 2$  mm (▲),  $d = 3$  mm (○), and  $d = 5$ mm (□) with  $\nu = 4 \times 10^{-5}$   $m^2s^{-1}$ ;  $d = 2$  mm (△),  $d = 3$  mm (●), and  $d = 5$  mm (■) with  $\nu = 5 \times 10^{-5}$   $m^2s^{-1}$ ;  $d = 2$  mm (▽) with  $\nu = 3.3 \times 10^{-5}$   $m^2s^{-1}$ ;  $d = 3$  mm (▼) with  $\nu = 3.75 \times 10^{-5}$   $m^2s^{-1}$ ;  $d = 5$  mm (◆) with  $\nu = 3 \times 10^{-5}$   $m^2s^{-1}$ ;  $d = 5$  mm (◇) with  $\nu = 3.7 \times 10^{-5}$   $m^2s^{-1}$ .

Fig12.eps

**Fig. 12.** Critical Shields number  $Sh_{\tau}^*$  versus particle Reynolds number  $Re_{\tau}^*$  at the threshold. The solid line stands for the explicit Shields equation [Eq. (14)]. The symbols are the same as those used in Fig. 11.

Fig13.pdf

**Fig. 13.** Sketch of a plane free jet with virtual origin  $\lambda$  located above the nozzle exit.