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# Sustainability of an economy relying on two reproducible assets

Robert D. Cairns\*    Stello Del Campo†    Vincent Martinet‡

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\*Department of Economics and CIREQ, McGill University, Montreal, Quebec, Canada H3A 2T7, and CESifo, Munich, Germany. Email: robert.cairns@mcgill.ca

†EconomiX (UMR CNRS 7235), Université Paris Nanterre, and Économie Publique, AgroParisTech, INRA, Université Paris Saclay, France. E-mail: sdelcampo@parisnanterre.fr

‡Corresponding author. Économie Publique, AgroParisTech, INRA, Université Paris Saclay, 78850 Thiverval-Grignon, France, and CESifo, Munich, Germany. Email: vincent.martinet@inra.fr

# Sustainability of an economy relying on two reproducible assets

## Abstract

The highest utility level that can be sustained in an economy is given by the *maximin value*. To be able to use this value for sustainability accounting, the corresponding maximin problem must be solved. This paper studies the sustainability of an economy composed of two reproducible assets, each producing one of two consumption goods which are substitutes in utility. We characterize the maximin path of the economy, and associated maximin shadow values. We discuss how these shadow values could be used as accounting prices for development paths that differ from the maximin trajectory. The corresponding genuine savings indicator informs on the improvement or decline of the sustainable level of utility and the prospects of future generations.

**Key words:** sustainable development; maximin; sustainability accounting; substitutability.

**JEL Code:** O44; Q56.

# 1 Introduction

The growing impacts of human activity on the environment have increased concern for sustainability and call for the definition of tools to assess it. To Solow (1993), sustainability means the ability to support a standard of living for the very long-run, and requires conserving a “generalized capacity to produce economic well-being,” accounting for all components of human well-being, including the consumption of manufactured goods, the flow of services from the environment, etc. A growing body of work proposes metrics for sustainability accounting (Neumayer, 2010), among which genuine savings indicators are prominent. Genuine savings measures the evolution of the productive capacities of the economy through net investment in a comprehensive set of capital stocks.

If the concerns for sustainability come from the hypothesis that society’s current decisions are not sustainable, it can hardly be held that observed, market prices can be used for sustainability accounting. Most of the genuine savings literature is based on the maximization of a welfare function, which defines a value  $V(X)$  for any economic state  $X$  (vector of capital stocks). *Shadow values*  $\partial V(X)/\partial X_i$  are then used to compute genuine savings as  $\sum_i \frac{\partial V(X)}{\partial X_i} \frac{dX_i}{dt}$  (Asheim, 2007; Dasgupta, 2009).<sup>1</sup> Genuine savings then measures the net investment in the *capacity* to produce the chosen measure of welfare.

The literature on genuine savings mostly adopts discounted utility as a measure of welfare. While it is the customary measure of intertemporal value in economics, discounted utility is criticized in the sustainability literature as being inequitable (Heal, 1998; Martinet, 2012). An alternative measure is the maximin value, which is related to “intergenerational equity” (Solow, 1974) and defines the highest egalitarian path that could be implemented from the current state. This criterion motivated Hartwick’s work on nil net investment (Hartwick, 1977), which is the backbone of genuine savings measures. As an egalitarian maximin path may be inefficient (Asheim and Zuber, 2013), an important stream of the lit-

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<sup>1</sup>There are some notable exceptions. Aronsson and Löfgren (1998) study green accounting for imperfect economies with pollution, focusing on welfare measurement more than on sustainability. Dasgupta, Mäler, and colleagues (Dasgupta and Mäler, 2000; Arrow et al., 2003) use a general, possibly non-optimal resource allocation mechanism (*ram*) instead of maximizing welfare to define genuine savings. As in the optimization models, the accounting price of each capital stock corresponds to the marginal contribution of that stock to the value (discounted utility in their case) associated to the trajectory determined by the *ram*. Integrating the dynamic path and computing the associated value as a function of all capital stocks can be done only for simple models with strong assumptions on the *ram*.

erature, mainly axiomatic, has focused on the definition of alternative social welfare functions (SWF) that encompass both economic efficiency and intergenerational equity (Chichilnisky, 1996; Alvarez-Cuadrado and Long, 2009; Asheim et al., 2012; Asheim and Zuber, 2013). This literature tries to overcome impossibility theorems stating that there is no SWF satisfying both the axiom of *strong anonymity* and the axiom of *strong Pareto efficiency*. A criterion relaxes either the axiom for efficiency (e.g., the maximin criterion is fully anonymous but does not satisfy strong Pareto efficiency) or the axiom for equity (such as Chichilnisky's criterion, which replaces anonymity by the axioms of *non-dictatorship of the present* and *non-dictatorship of the future*), or it has to be incomplete (such as overtaking criteria) or non-constructible.<sup>2</sup> This literature raises interesting normative issues. Even if some of the SWFs it has produced have interesting properties for sustainability accounting, the associated genuine savings indicators have not been studied yet, to the best of our knowledge.<sup>3</sup> We here focus on the maximin.

The maximin value has a clear, positive interpretation in terms of sustainability, as soon as one defines sustainability as the “ability to sustain utility.” This value is the highest level of utility that can be sustained forever given the current state of the economy (Cairns and Long, 2006; Cairns, 2011, 2013; Fleurbaey, 2015). It is our measure of *sustainability* herein. A genuine savings indicator can be defined for maximin, the shadow value of a stock being its marginal contribution to the maximin value. Net investment accounted using these shadow values, for any given dynamic path –efficient or not, and whether or not maximin is the pursued social objective–, represents the evolution over time of the highest sustainable level of utility and is interpreted as a measure of sustainability improvement or decline (Cairns and Martinet, 2014). Computing net maximin investment is thus meaningful for sustainability accounting as it informs on the effect of current decisions on the ability to sustain utility that is bequeathed to future generations.

The possibility of developing a sustainability accounting system based on maximin values requires defining the various capital stocks' shadow values. As with any other measure of

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<sup>2</sup>See Basu and Mitra (2003); Zame (2007); Asheim (2010); Lauwers (2012), among others.

<sup>3</sup>A genuine savings indicator can be defined for any dynamic, forward-looking welfare function satisfying the property of *independent future* (Asheim, 2007). The *sustainable recursive social welfare functions* characterized by Asheim et al. (2012), and the particular case of *sustainable discounted utilitarianism* (Asheim and Mitra, 2010), satisfy this axiom and has been implemented in the DICE integrated assessment model for the evaluation of climate policies, emphasizing its tractability (Dietz and Asheim, 2012).

welfare, the shadow values are determined through differentiating the corresponding value function. The computation of maximin shadow values for any actual economy, with all its various assets, consumption goods, production techniques, etc., is presently out of reach. A sensible way to proceed is to build up from the few solved maximin problems, and to try to gain a greater understanding of the economic issues involved, as was done for discounted utility (Arrow et al., 2003). For the general idea of genuine savings to make sense, it is necessary that there be at least two capital stocks (Hartwick, 1977). We postulate an economy with two productive sectors which interact only through utility. By solving the maximin problem for this economy, we provide some insights for the future development of a system of accounting based on maximin shadow values.

A key constituent of our examination is the question of substitutability. Neumayer (2010) stresses that substitutability in production as well as in utility plays a central role in the study of sustainability. The influence of substitutability in production on the maximin solution has been emphasized since the work of Solow (1974) and Dasgupta and Heal (1979), who study interactions between sectors in the form of a sector extracting a nonrenewable resource used as an input to a manufacturing sector. Some authors (e.g. Asako, 1980; Stollery, 1998; d'Autume and Schubert, 2008; d'Autume et al., 2010) study maximin problems with two substitutes in utility, one of which is a decision variable (consumption) and the other a state variable (the ambient temperature or the stock of a non-renewable resource). Substitutability of consumption goods in utility has received less scrutiny but is as important a question for sustainability as substitutability in production.<sup>4</sup>

Keeping the economy at a steady state by maintaining current capital stocks has been proposed as a way to achieve sustainability (Daly, 1974). By solving the maximin problem for our economy, we show in Section 2 that, whenever the two goods are substitutes and one of the sectors is more productive at the margin, it is possible to sustain utility at a higher level than at that steady state, through a higher consumption of the less productive stock and a lower consumption of the more productive one. The depletion of the less productive stock is compensated for by investment in the more productive one, in line with Hartwick's nil net investment rule (Hartwick, 1977; Cairns and Long, 2006). This investment pattern

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<sup>4</sup>Substitutability of consumption goods in utility has been studied in the discounted utilitarian framework (Quaas et al., 2013; Baumgärtner et al., 2017) and shown to strongly influence optimal development paths and their ability to sustain utility.

is driven by the shadow values of the stocks, i.e., the sustainability accounting prices. What is maintained is a general capacity to sustain utility, not levels of particular stocks.

In single-sector models such as the Solow (1956) growth model with capital depreciation or the simple fishery, stocks that are beyond the golden-rule level or the maximum sustainable yield have negative marginal products and are redundant for maximin value (Solow, 1974; Asako, 1980). The maximin shadow values are then nil and of little information for sustainability accounting (Cairns and Martinet, 2014). It is thus important to identify the conditions under which such a case occurs. In our framework, stock redundancy arises only if *all* technologies have a single productivity peak. Whenever abundant stocks can be used in the investment pattern to build up a scarce resource, all capital stocks have a positive sustainability accounting value. Stock redundancy is thus less likely to occur in a multi-sector model with substitutability.

The consequences of our results for sustainability accounting with maximin shadow values are discussed in Section 3. In particular, we determine two conditions for current decisions to improve the level of utility that can be sustained. First, current utility has to be lower than the maximin value. Second, the resource thus freed-up must be invested in order to get a positive maximin net investment. Llavador et al. (2011) stress that the year 2000 consumption in the USA was lower than the sustainable, maximin value. Such a lower utility can be consistent with long-run growth as long as investment decisions result in an increase of the maximin value. Conversely, our results illustrate the possibility that, in multiple capital good models, ill-considered consumption and investment decisions can result in a decrease of the sustainable level of utility even though current utility is lower than the sustainable, maximin level. Our conclusions and prospects for future research are given in Section 4.

## 2 An economy with two reproducible assets

Consider an economy with two reproducible assets  $X_1$  and  $X_2$ , produced by separate sectors according to technologies  $F_i(X_i)$ ,  $i = 1, 2$ , which depend only on the stock  $X_i$ . The production functions are assumed to be continuous, twice differentiable and strictly concave ( $F_i'' < 0$ ), and to satisfy  $F_i(0) = 0$  and  $F_i'(0) < \infty$ . This last condition ensures that the marginal productivity of both capital stocks is bounded from above.

Because many possible technologies attain a maximum for a finite level of the state variable, we consider the possibility of production peaks. In the Solow (1956) growth model with capital decay at a constant rate, output reaches a maximum, called the golden-rule level, so that marginal net product turns negative for a large capital stock. In the study of resource economics, to which sustainability is intimately related, the simple fishery has a maximum natural production level, called the maximum sustainable yield (MSY), and for larger resource stocks productivity turns downward.<sup>5</sup> Such technologies have interesting features for the study of sustainability, and have been studied in one-sector models (Asheim and Ekeland, 2016). We characterize these technologies with the following definition.

**Definition 1** (Single-peaked technology). *A technology  $F_i(X_i)$  is single-peaked if there exists  $\bar{X}_i$  such that  $F'_i(X_i) > 0$  for  $X_i < \bar{X}_i$  and  $F'_i(X_i) < 0$  for  $X_i > \bar{X}_i$ .*

The stock  $\bar{X}_i$  is implicitly defined by the condition  $F'_i(\bar{X}_i) = 0$ . Capital stock  $\bar{X}_i$  is the stock which yields the highest production level (golden-rule level or maximum sustainable yield). The usual neoclassical assumption of a strictly increasing production function ( $F'_i(X_i) > 0$  for all  $X_i$ ) corresponds to the limiting case  $\bar{X}_i \rightarrow \infty$ , that we discuss below.<sup>6</sup> Our model can thus represent an economy with two manufactured stocks or two natural resources without interactions, or one of each.

Production is either consumed ( $c_i$ ) or invested ( $\dot{X}_i$ ), and capital dynamics are

$$\dot{X}_i(t) \equiv \frac{dX_i(t)}{dt} = F_i(X_i(t)) - c_i(t), \quad i = 1, 2. \quad (1)$$

The economy is composed of infinitely many generations of identical consumers, each living for an instant in continuous time. They have ordinal preferences over the two goods, represented by a twice-differentiable, strictly quasi-concave utility function  $U(c_1, c_2)$ , such that both goods have a positive marginal utility and are essential in consumption.<sup>7</sup> For convenience, we assume that  $U(0, 0) = 0$  and that  $U(c_1, c_2) \geq 0$ .

<sup>5</sup>As we do not include stock dependent harvesting costs, and consider concave production functions, the model may not be realistic for some resources. Our purpose is to study a stylized economy.

<sup>6</sup>We consider bounded production, in the sense that  $\lim_{X_i \rightarrow \infty} F_i(X_i) = \bar{F}_i < \infty$ .

<sup>7</sup>Formally, this means that  $U_{c_i} > 0$  for  $i = 1, 2$  and  $\lim_{c_i \rightarrow 0} U_{c_i}|_{U=u} = \infty$ . Strict quasi-concavity implies that marginal rates of substitution are strictly decreasing, a property that is used below in the proofs of Propositions 4 and 5.



**Maximin value: some preliminary results** The maximin value  $m$  of a state  $(X_1, X_2)$ , provided that it exists, is the highest level of utility that can be sustained forever from that state (from any time  $t \geq 0$ ):

$$\begin{aligned}
 m(X_1, X_2) &= \max_{u, c_1(\cdot), c_2(\cdot)} u, & (2) \\
 \text{s.t. } (X_1(t), X_2(t)) &= (X_1, X_2); \\
 \dot{X}_i(s) &= F_i(X_i(s)) - c_i(s), \quad i = 1, 2 \text{ and} \\
 U(c_1(s), c_2(s)) &\geq u \text{ for all } s \geq t. & (3)
 \end{aligned}$$

Herein, the term *value* refers to maximin value. Below, we omit the time argument in the expressions where no confusion is possible.

Differentiation of the maximin value with respect to time yields current net maximin investment (Cairns and Martinet, 2014, Lemma 1):

$$M(X_1(t), X_2(t), c_1(t), c_2(t)) = \frac{dm(X_1, X_2)}{dt} = \frac{\partial m(X_1, X_2)}{\partial X_1} \dot{X}_1 + \frac{\partial m(X_1, X_2)}{\partial X_2} \dot{X}_2. \quad (4)$$

The links between the maximin problem and net investment have been studied since the work of Hartwick (1977), with recent contributions by Doyen and Martinet (2012) and Fleurbaey (2015). We shall discuss the links between net maximin investment and sustainability accounting in Section 3.

Before solving the maximin problem for this economy, we establish the following results.

First, we show that the maximin value is bounded from above. With a single-peaked technology, production is bounded from above by level  $F_i(\bar{X}_i)$ , and so is the sustainable utility in the single-sector case (Cairns and Martinet, 2014). This is also the case in our two-sector economy.

**Proposition 1** (Bounded maximin value). *If each technology has a single peak,<sup>8</sup> the maximin value is bounded from above by  $\bar{m} = U(F_1(\bar{X}_1), F_2(\bar{X}_2))$ .*

<sup>8</sup>This result holds also for the neoclassical case with bounded productions  $\bar{F}_i = \lim_{X_i \rightarrow \infty} F_i(X_i) < \infty$ , defining the limiting upper value  $\bar{m} = U(\bar{F}_1, \bar{F}_2)$ , as well as for the case in which only one technology has a single peak, with, for example,  $\bar{m} = U(\bar{F}_1, F_2(\bar{X}_2))$ .

*Proof of Proposition 1.* Consider two single-peaked technologies with  $F'_i(\bar{X}_i) = 0$ ,  $i = 1, 2$  and the value  $\bar{m} = U(F_1(\bar{X}_1), F_2(\bar{X}_2))$ . Assume that the utility level  $\bar{m} + \varepsilon$  for some  $\varepsilon > 0$  is sustainable. A maximin path sustaining  $\bar{m} + \varepsilon$  would have no steady state, as none can sustain this level. Such a dynamic path cannot have consumption from one stock decreasing to zero and from the other increasing to infinity either, as production is bounded from above. The maximin path would either correspond to a limit cycle or to a back-and-forth along a curve. Along a limit cycle, there would be a part of the cycle where the two stocks increase at the same time. This requires  $c_1 < F_1(X_1) \leq F_1(\bar{X}_1)$  and  $c_2 < F_2(X_2) \leq F_2(\bar{X}_2)$ , which would imply  $U(c_1, c_2) \leq U(F_1(\bar{X}_1), F_2(\bar{X}_2)) \leq \bar{m} + \varepsilon$ , a contradiction. For a back-and-forth, at switching times, both stocks are at a steady state. Here again,  $\bar{m} + \varepsilon$  cannot be sustained. The highest sustainable level of utility is  $\bar{m}$ .  $\square$

The maximin value is also “bounded from below” in the following sense.

**Lemma 1** (Stationary fallback). *For any state  $(X_1, X_2)$ , the maximin value is at least equal to the utility derived from consumption at the corresponding steady state:*

$$m(X_1, X_2) \geq U(F_1(X_1), F_2(X_2)).$$

*Proof of Lemma 1.* The dynamic path  $\dot{X}_i = 0$  driven by decisions  $c_i = F_i(X_i)$  is feasible and yields the constant utility  $U(F_1(X_1), F_2(X_2))$ . This provides a lower bound for the maximin value.  $\square$

Lemma 1 relies on the fact that consuming the whole production, keeping the economy in a steady state, makes it possible to sustain  $U(F_1(X_1), F_2(X_2))$ . A dynamic path may, however, yield a higher sustainable utility.

**Lemma 2** (Dynamic maximin path). *If the maximin value of a state  $(X_1, X_2)$  is greater than the utility derived at the steady state, i.e., if  $m(X_1, X_2) > U(F_1(X_1), F_2(X_2))$ , then, along a maximin path starting from that state (i) the consumption of at least one good is greater than the production of the corresponding stock and (ii) that stock decreases.*

*Proof of Lemma 2.* This is a direct result from Lemma 1 and the dynamics.  $\square$

The existence of such a dynamic path means that keeping the economy at a steady state (Daly, 1974) is not the only sustainable option. Solving the maximin problem (2) may provide a superior path. To do so, we rely on optimal control.

**Optimal control** Cairns and Long (2006) propose a direct optimal control approach to the maximin problem.<sup>9</sup> We follow the same approach, but detail the way the different optimality conditions provided in Cairns and Long (2006) are derived from standard optimal control problems. In their approach, the sustained utility level  $u$  is a control parameter to be optimized.<sup>10</sup> The standard trick in control theory is to consider an additional state variable  $u(t)$ , with  $\dot{u} = 0$  and a free initial state  $u(0)$ .<sup>11</sup> An initial value (simply  $u(0)$  in our case) is associated to the optimization problem, which is formulated as follows:

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<sup>9</sup>Another approach would be to study egalitarian and efficient paths in a competitive economy framework (see Dixit et al., 1980; Withagen and Asheim, 1998; Mitra, 2002, for example). The two approaches rely on different formalisms but provide the same insights regarding maximin paths in most cases: whenever an egalitarian and efficient path exists, it is the (unique) maximin path. And in most cases, an optimal control problem can characterize such a path, with the usual correspondence with a decentralized competitive equilibrium. This correspondence was already central in Burmeister and Hammond (1977), who defined necessary conditions for a maximin path using a “Lagrangian” integral, and proved correspondence with an efficient path. The two approaches lead, however, to different perspectives on sustainability accounting. The efficiency pricing approach is used to discuss the significance of Hartwick’s investment rule for sustainability. The maximin approach we use here provides another perspective, focusing on the future maximin value (Cairns and Martinet, 2014; Fleurbaey, 2015). At each step of the analysis, we will emphasize correspondences and differences between the two approaches.

<sup>10</sup>Different optimal control approaches have been used to solve maximin problems. For example, Léonard and Long (1992) propose to maximize the discounted sum of the sustained utility  $u$ , i.e.,  $\max \int_0^\infty \rho e^{-\rho t} u dt = u$  for an arbitrary discount rate  $\rho$ . Using a different approach, d’Autume and Schubert (2008) solve an usual discounted utilitarian program of the form  $\max \int_0^\infty e^{-\rho t} \frac{U(t)^{1-1/\sigma}}{1-1/\sigma} dt$ , where  $\sigma$  is an arbitrary intertemporal elasticity of substitution, and take the limit  $\sigma \rightarrow 0$  to get the maximin path.

<sup>11</sup>See Seierstad and Sydsæter (1987, p.192) and Léonard and Long (1992, p.255).

$$\begin{aligned}
& \max \quad \{u(0)\} & (5) \\
\text{s.t.} \quad & f_i(X, u, c) = \dot{X}_i(t) = F_i(X_i(t)) - c_i(t), \quad i = 1, 2 \\
& f_3(X, u, c) = \dot{u}(t) = 0 \\
& X_1(0) - X_1^0 = 0 \\
& X_2(0) - X_2^0 = 0 \\
& u(0) \geq 0 \\
& (c_1(t), c_2(t)) \in \mathcal{U} = [0; \infty) \times [0; \infty) \\
& (X_1(t), X_2(t), u(t)) \in [0, \infty) \times [0, \infty) \times [0, \bar{m}] \\
& g(X, u, c) = U(c_1(t), c_2(t)) - u(t) \geq 0.
\end{aligned}$$

In an interesting problem, both initial stocks are strictly positive ( $X_i(0) > 0$ ,  $i = 1, 2$ ); otherwise, one is back to the single sector problem. Under the condition that both goods are essential to consumption, and given Lemma 1, we can say that consumption of both goods is positive ( $c_i > 0$ ,  $i = 1, 2$ ) at any time along a maximin path. As such, none of the stocks is exhausted, and we can avoid imposing positivity constraints on the stocks.<sup>12</sup> We start by proving the existence of an optimal solution.

**Proposition 2** (Existence). *There exists an optimal solution  $(X_1^*(t), X_2^*(t), u^*(t), c_1^*(t), c_2^*(t))$  to the optimal control problem (5).*

*Proof of Proposition 2.* This problem is a particular case of the very general optimal control problem presented in Seierstad and Sydsæter (1987, p.390), extended to the infinite horizon as in Seierstad and Sydsæter (1987, p.406), with no integral objective ( $f_0 = 0$ ),<sup>13</sup> time-autonomous functional forms, given initial states for  $(X_1, X_2)$ , a free initial state for  $u$ , free final state for all states variables, no pure-state constraints, and a single mixed constraint. To prove existence, we refer to Theorem 21 of Chapter 6 in Seierstad and Sydsæter (1987,

<sup>12</sup>Pure state constraints are difficult to deal with in optimal control problems. See Seierstad and Sydsæter (1987, Chapter 5).

<sup>13</sup>Such a particular case occurs not only for maximin problems, but also for target problems (see, e.g., the way Takayama, 1974, introduces optimal control) or optimal terminal time problems (see, e.g., Example 2 in Seierstad and Sydsæter, 1987, p. 184).

p. 406).

The functions  $f_1, f_2, f_3$  and  $g$  are continuous. The set of controls  $\mathcal{U}$  is closed. As (i)  $\mathcal{U}$  is convex, (ii)  $g$  is quasi-concave in  $(c_1, c_2)$  since  $U$  is quasi-concave, and (iii) dynamics  $f_1, f_2$  and  $f_3$  are concave in the controls for all  $X = (X_1, X_2, u) \in [0, \infty) \times [0, \infty) \times [0, \bar{m}]$ , the set

$$N(X, \mathcal{U}) = \{(F_1(X_1) - c_1, F_2(X_2) - c_2, 0) | U(c_1, c_2) \geq u, (c_1, c_2) \in \mathcal{U}\}$$

is convex for all  $(X_1, X_2, u)$  (See Note 23 in Seierstad and Sydsæter, 1987, p. 403). Therefore, condition (159) in the Theorem is satisfied. Moreover,  $N(X, \mathcal{U})$  has a closed graph as a function of the state since the dynamics  $f_1, f_2$  and  $f_3$  are continuous (see Note 24 in Seierstad and Sydsæter, 1987, p. 403). Therefore, condition (166) in the Theorem is satisfied. Also, there is an admissible trajectory  $(X_1(t), X_2(t), u(t), c_1(t), c_2(t))$ , for example the “stationary state” trajectory defined by  $c_1(t) = F_1(X_1^0)$ ,  $c_2(t) = F_2(X_2^0)$  and  $u(t) = U(F_1(X_1^0), F_2(X_2^0))$ . Finally, the condition (164) in the Theorem is satisfied as, for any  $p \neq 0$  and any  $(X_1, X_2, u, c_1, c_2)$  such that  $U(c_1, c_2) - u \geq 0$ , we have  $p \cdot [F_1(X_1) - c_1, F_2(X_2) - c_2, 0]' \leq p \cdot [F_1(\bar{X}_1), F_2(\bar{X}_2), 0]' + \psi \cdot \|(X_1, X_2, u)\|$ , where  $\phi_p(t) = p \cdot [F_1(\bar{X}_1), F_2(\bar{X}_2), 0]'$ , a constant, and  $\psi_p(t) = 1$  are locally integrable (i.e., integrable on each finite time interval).<sup>14</sup> The Theorem applies.<sup>15</sup>  $\square$

The optimal control problem (5) thus defines optimal paths and the associated finite sustained utility for any initial state  $(X_1^0, X_2^0)$ . Therefore, the following optimal value function is well-defined over the set of possible initial states:

$$m(X_1^0, X_2^0) = \sup\{u(0) | (X_1(t), X_2(t), u(t), c_1(t), c_2(t)) \text{ admissible}\} \geq 0 \quad (6)$$

In what follows, we omit superscripts “0” when referring to the value function itself. We shall assume that the value function  $m(X_1, X_2)$  is differentiable.<sup>16</sup>

<sup>14</sup>For neoclassical production functions with bounded production but no single-peak, replace  $F_i(\bar{X}_i)$  by  $\bar{F}_i$  in this step of the proof.

<sup>15</sup>As there is no terminal condition in our problem, the other conditions in the Theorem are not required. Note that if we had imposed positivity of the capital stocks as a terminal constraint, condition (177a) in the Theorem would have been satisfied, as for  $i = 1, 2$ , the  $f_i$  are bounded from above by  $F_i(\bar{X}_i)$  (or  $\bar{F}_i$  in the neoclassical case), a constant. Not having expressed non-negativity constraints for the stocks in the problem formulation is not an issue.

<sup>16</sup>See Theorem 9 in Seierstad and Sydsæter (1987, p.213) as well as Seierstad (1982) for a discussion on

**Optimal paths characterization** The *maximum principle* can be used to characterize the necessary conditions that an optimal solution to problem (5) must satisfy. We first give these conditions before discussing when they are useful.<sup>17</sup>

These necessary conditions are given by Seierstad and Sydsæter (1987, Theorem 9, p. 381), completed by necessary transversality conditions for free terminal conditions problems formulated in Seierstad and Sydsæter (1987, Theorem 16, p. 244-245).<sup>18</sup> They lead to the necessary conditions given in Cairns and Long (2006).

Let  $(X_1^*(t), X_2^*(t), u^*(t), c_1^*(t), c_2^*(t))$  be an optimal solution, and assume that it is piecewise continuous. According to the above mentioned Theorems, there exist (i) a number  $p_0$  (with  $p_0 = 0$  or  $p_0 = 1$ ), (ii) functions  $\mu_1(t)$ ,  $\mu_2(t)$  and  $\mu_3(t)$  that are the costate variables associated respectively to stocks  $X_1$  and  $X_2$ , and to the control parameter  $u$ , as well as (iii) a Lagrange multiplier  $\omega(t)$  associated with the constraint (3) such that, given the Hamiltonian

$$\mathcal{H}(X, u, c, \mu) = \mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 + \mu_3 \dot{u} = \mu_1 (F_1(X_1) - c_1) + \mu_2 (F_2(X_2) - c_2) , \quad (7)$$

and the Lagrangian

$$\mathcal{L}(X, u, c, \mu, \omega) = \mathcal{H}(X, c, \mu, u) + \omega (U(c_1, c_2) - u) , \quad (8)$$

the differentiability of the optimal value function in control theory. We emphasize that the Hamiltonian (7) used below is concave and that for non-trivial cases, the shadow values are unique.

<sup>17</sup>In some cases, the necessary conditions provide no information at all. This is the case when all shadow values are equal to zero, so that the Hamiltonian vanishes. Usually, (sufficient) *constraint qualification* conditions are used to rule out this case. For example, a constraint qualification of the full rank type (see Seierstad and Sydsæter, 1987, p. 380) can be used to show that the control variables enter the constraint functions in an essential way. Unfortunately, for maximin-type problems, as explained in Cairns and Long (2006, pp. 279 and 291-295), no constraint qualification has been formalized yet to rule out triviality. In such a case, the strategy is to solve the problem and check afterward that the solution is non-trivial, and in particular that the shadow values are non-nil (see the discussions on non-triviality in Seierstad and Sydsæter, 1987, e.g., p. 278-279 and their Note 4 p. 334). We discuss this point further later on.

<sup>18</sup>These transversality conditions are proved to be necessary *only* for optimal trajectories satisfying (i)  $\int_0^\infty |f_i| dt < \infty$  (with  $i = 1, 2, 3$ ), which will be satisfied for trajectories converging fast enough to a stationary state, and (ii) some “growth conditions” that reduce for our problem to  $F'_i(X_i)$  being bounded from above. We shall thus assume that the optimal trajectory converges fast enough, a point we discuss later on.

the following conditions<sup>19</sup> hold for all time  $t$ :

$$\frac{\partial \mathcal{L}}{\partial c_i} = 0 \quad \text{for } i = 1, 2 \quad (9)$$

$$\frac{\partial \mathcal{L}}{\partial X_i} = -\dot{\mu}_i \quad \text{for } i = 1, 2 \quad (10)$$

$$\frac{\partial \mathcal{L}}{\partial u} = -\dot{\mu}_3 \quad (11)$$

along with the usual complementary slackness conditions

$$U(c_1, c_2) - u \geq 0, \quad \omega \geq 0, \quad \omega (U(c_1, c_2) - u) = 0, \quad (12)$$

and the transversality conditions for free terminal states<sup>20</sup>

$$\lim_{t \rightarrow \infty} \mu_i = 0 \quad \text{for } i = 1, 2, 3. \quad (13)$$

Regarding the optimality condition on the choice of the free initial state  $u(0)$ , we rely on Seierstad and Sydsæter (1987, Theorem 15, p. 396). In our case, the initial value is simply  $u(0)$  and there is no restriction on the choice of  $u(0)$ , so that the following condition has to hold:

$$\mu_3(0) = -p_0. \quad (14)$$

We can derive the following results from these necessary conditions.

**Nil net investment** From conditions (13), we get the usual condition that<sup>21</sup>

$$\lim_{t \rightarrow \infty} \mathcal{H}(X, c, \mu) = 0. \quad (15)$$

<sup>19</sup>Alternatively, one could apply directly the necessary conditions provided in Cairns and Long (2006), which rely on Hestenes' Theorem (see Takayama, 1974, Theorem 8.C.4, p. 658-660). We retrieve all these conditions.

<sup>20</sup>These conditions rely on the fact that there is no condition on  $\lim_{t \rightarrow \infty} X_i(t)$ . See eq. (215c) in Seierstad and Sydsæter (1987, Theorem 16, p. 244-245).

<sup>21</sup>Eq. (15) is introduced as a necessary condition in Cairns and Long (2006, eq. 15). They prove it is necessary using the Theorem by Michel (1990).

Moreover, as the problem is time autonomous, we have  $\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} = 0$ , and by complementary slackness

$$\mathcal{H}(X, u, c, \mu) = 0 = \mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 . \quad (16)$$

This result corresponds to Proposition 1 in Cairns and Long (2006) and is related to Hartwick's rule and its converse. Cairns and Long (2006) show that along a maximin path,  $\mu_1$  and  $\mu_2$  are the shadow values of stocks at current state, i.e.,  $\mu_i = \frac{\partial m(X_1, X_2)}{\partial X_i}$ , and that the Hamiltonian, which thus represents net investment at maximin shadow values (eq. 4) is (possibly trivially) nil over time. The maximin value is constant over time.

**Regularity and non-regularity: the shadow value of equity** Let us now study the shadow value  $\mu_3$  associated to the minimal utility parameter  $u$ . As  $\frac{\partial \mathcal{L}}{\partial u} = -\omega$ , integrating eq. (11) gives  $\int_0^\infty \omega(t)dt = \mu_3(\infty) - \mu_3(0)$ . Given the transversality condition (13), we have  $\mu_3(\infty) = 0$ , while the optimality condition (14) imposes  $\mu_3(0) = -p_0$ . We thus get<sup>22</sup>

$$\int_0^\infty \omega(t)dt = p_0 . \quad (17)$$

The multiplier  $\omega$  provides information on the difficulty of satisfying the minimal utility constraint at time  $t$ , i.e., how much would be gained in value if the constraint was locally relaxed. This opportunity cost of meeting the equity constraint at the current state is interpreted as the shadow value of equity by Cairns and Long (2006). The shadow value  $\mu_3(0)$  thus captures the aggregate of the marginal costs of the equity constraint over the program.

This shadow value of equity is linked to the shadow value of the stocks. From eq. (9), we get that the shadow value of stock  $X_i$  is equal to the marginal utility of consumption  $c_i$  weighted by the shadow value of equity

$$\mu_i = \omega U_{c_i} , \quad i = 1, 2 . \quad (18)$$

Note that, as  $U_{c_i} > 0$ , as soon as one of the shadow value is nil, the two others are nil

<sup>22</sup>Eq. (17) is introduced as a necessary condition in Cairns and Long (2006, eq. 14), with the value  $p_0 = 1$ . It corresponds to the transversality condition in Hestenes' Theorem (see Theorem 8.C.4 in Takayama, 1974, p. 658-660, and more precisely eq. 65).



too, through eq. (18), i.e.,  $(\mu_1, \mu_2, \omega) = (0, 0, 0)$ .<sup>23</sup> The Hamiltonian is trivially zero because of nil maximin shadow values and not because of conditions governing the changes in the stocks. The necessary conditions then provide no restrictions on the path (i.e., the maximum principle is not informative to characterize the optimal solution). This will be the case in particular (but not only) whenever the constraint (3) is not binding, according to the complementary slackness condition (12). It happens in what are called *non-regular* cases.

Maximizing the minimal utility over time can be perceived as successively raising the level of the least well-off to the extent possible (Solow, 1974). The end result of this sequence of redistributions can be an equalization of utility, if this leads to a (strongly) Paretian allocation of utility among generations. Intergenerational equality may be the outcome of the maximin problem but is not its objective. Equity does not necessarily mean equality. Even if a maximin solution exists, a redistribution to achieve equality is not always possible or efficient. When it is not efficient to distribute well being equally over time, the solution is *non-regular* and constraint (3) may not be binding at some times. As such, a necessary (but not sufficient) condition for the shadow values to be non-nil is that the equity constraint is binding at all times, i.e., that the maximin path is egalitarian. The condition that  $\omega(t) > 0$  at all times is central in the definition of *regularity* for a maximin path, which is basically defined as  $\omega(t) > 0$  and  $\int_0^\infty \omega(t)dt = 1$  in Cairns and Long (2006, Assumption R).<sup>24</sup>

In Cairns and Martinet (2014, Proposition 3), it is shown that the form of non-regularity that can emerge in our model<sup>25</sup> occurs when decisions do not influence net maximin in-

<sup>23</sup>Cairns and Long (2006) show in their Appendix that, in a fishery model related to ours, when  $\omega = 0$  at some time in the program,  $\omega(t)$  is nil at all times, so that  $p_0 = 0$  too. This case of *triviality* is termed *non-regularity* in the maximin literature. Non-regularities in maximin problems have been found in the models by Solow (1974) and Asako (1980), and are discussed in Doyen and Martinet (2012). Cairns and Martinet (2014) stress its consequences for maximin shadow values, and thus for sustainability accounting.

<sup>24</sup>In the literature studying efficient egalitarian paths, *regularity* is defined following Burmeister and Hammond (1977) as a requirement that the path is egalitarian and there exist (i) a path  $\omega(\cdot)$  of positive present value price of utility satisfying  $\int_0^\infty \omega(t)dt = 1$  (by normalization) and (ii) competitive prices  $\mu_i$  for all capital goods, such that consumption maximizes utility (in the sense of  $\omega U(c_1, c_2) - \mu_1 c_1 - \mu_2 c_2$ ) and production maximizes profit (in the sense of  $\mu_1 F_1(X_1) + \mu_2 F_2(X_2) - q_1 X_1 - q_2 X_2$ , where the  $q_i = -\dot{\mu}_i$  are the rental prices of the various capital goods). These conditions lead exactly to our eqs. (18) and (20). The two approaches rely on the same conditions.

<sup>25</sup> Another type of non-regularity may occur when the consumption vector maximizes utility over all feasible consumption decisions given current state (or if  $U_{c_i} = 0$  for all consumption goods). This possibility was noted in Burmeister and Hammond (1977, p. 854) and corresponds to the case in which current capital goods limit the production of consumption goods (see, e.g., Cairns and Tian, 2010, in which current labor force limits harvesting and thus consumption). This is not the case in our model where consumption is not

vestment (eq. 4), i.e.,  $\frac{\partial M}{\partial c_i} = 0$  for all decisions. As  $M(X_1, X_2, c_1, c_2) = \frac{\partial m(X_1, X_2)}{\partial X_1}(F_1(X_1) - c_1) + \frac{\partial m(X_1, X_2)}{\partial X_2}(F_2(X_2) - c_2)$ , we have  $\frac{\partial M}{\partial c_i} = -\frac{\partial m(X_1, X_2)}{\partial X_i}$ . In our framework, non-regularity occurs when the capital stocks have no marginal value, i.e., when  $\mu_1 = \mu_2 = 0$ . This induces  $\omega = 0$  (from eq. 18) and thus there is no opportunity cost of satisfying the minimal consumption constraint (and in some cases, utility can be larger than the maximin value without jeopardizing sustainability).

This type of non-regularity is a central property in a one-sector model with a production peak, where a stock is *redundant* if it is beyond the golden-rule level (Asako, 1980). As non-regularity emerges even for simple problems and is a concern for accounting purposes, it is important to determine if, and under what conditions, non-regularity occurs in our two-sector models.

**Single-peakedness and stock redundancy: a source of non-regularity** Proposition 1 generalizes to two dimensions (and obviously can be extended to more than two dimensions) the idea of maximum sustainable yield. It also generalizes to two dimensions the non-regularity associated with the golden rule in an economy with a single asset. Redundancy occurs in our two-sector economy when capital stocks are sufficient to sustain the highest possible maximin value  $\bar{m}$ . For such capital stocks, the shadow values of stocks are nil, as well as the shadow cost of equity  $\omega$ . The maximum principle is trivially satisfied, and there may be an infinity of candidates for optimality. In fact, any path satisfying  $U(c_1(t), c_2(t)) \geq \bar{m}$  and converging to the steady state  $(\bar{X}_1, \bar{X}_2)$  is optimal. This steady state is characterized by  $F'_1(\bar{X}_1) = F'_2(\bar{X}_2) = 0$ .

The set of states from which it is possible to sustain utility  $\bar{m}$  is a viability kernel and can be determined through a viability problem (Martinet and Doyen, 2007; Doyen and Martinet, 2012). Characterizing explicitly this set is beyond the objective of this paper.<sup>26</sup> We thus provide the following remark to stress that non-regular cases occur in a delimited area of the state domain.

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bounded from above and marginal utility is positive.

<sup>26</sup>The method to do so would follow the steps described in Martinet (2012, Section 8.5.2). The Hamilton-Jacobi-Bellman equation that the boundary of the viability kernel must satisfy is easy to determine, but has no closed-form solution. It is, however, possible to show from this boundary condition that this set is convex.

**Remark 1** (Stock redundancy). *If both technologies are single-peaked, states for which  $m(X_1, X_2) = \bar{m}$  belong to a convex set of redundant stocks, with maximin value  $\bar{m}$  and shadow values of zero.*

Conversely, when one technology has no single peak (and thus, a strictly positive marginal productivity), stock redundancy cannot occur because, even if the other stock is above its production peak (with a negative marginal productivity), it is always possible to use it to build up the first stock. As soon as one of the technologies has no production peak,  $\bar{m}$  is defined only as a limit (see footnote 8) and no capital stock allow to actually sustain utility at level  $\bar{m}$ . The previously mentioned set of redundant stocks is empty.

We know from Lemma 1 that  $m(X_1, X_2) \geq U(F_1(X_1), F_2(X_2)) > 0$ , and from Proposition 1 that  $m(X_1, X_2) \leq U(F_1(\bar{X}_1), F_2(\bar{X}_2))$ . For any stock that does not allow the sustaining of the golden rule utility  $\bar{m}$ , the maximin path is actually regular.

**Proposition 3** (Regularity). *If the value function is differentiable and non-decreasing, for any  $(X_1, X_2)$  such that  $m(X_1, X_2) < \bar{m}$ , the maximin path is egalitarian and efficient, with  $U(c_1^*(t), c_2^*(t)) = m(X_1^*(t), X_2^*(t))$  and  $\frac{dm(X_1, X_2)}{dt} = 0$  for all  $t$ .*

*Proof of Proposition 3.* For any  $(X_1, X_2)$  such that  $m(X_1, X_2) < \bar{m}$ , we have  $\frac{\partial m(X_1, X_2)}{\partial X_i} > 0$  and  $U_{c_i} > 0$ . Therefore  $\frac{\partial M}{\partial c_i} = -\frac{\partial m(X_1, X_2)}{\partial X_i} \neq 0$ . According to Cairns and Martinet (2014, Proposition 4), non-regularities due to locally bounded investment cannot emerge.<sup>27</sup> The maximin decisions are such that  $U(c_1^*(t), c_2^*(t)) = m(X_1^*(t), X_2^*(t))$  and  $\frac{dm(X_1, X_2)}{dt} = 0$ . The path is regular.  $\square$

For any state such that  $m(X_1, X_2) < \bar{m}$ , one has  $(\mu_1, \mu_2, \omega) \gg (0, 0, 0)$ , and the maximum principle provides information on the optimal path and has economic interpretations.

**Regular path characterization** As long as  $\omega > 0$ , given that  $U_{c_i} > 0$ , all shadow values are positive. From conditions (18), we derive the relative shadow value of the stocks, which

<sup>27</sup>For our model, utility is sensitive to the controls and marginal utility is always strictly positive, so that it is always possible to increase utility. Non-regularities due to locally bounded utility cannot emerge (see footnote 25).

is equal to the marginal rate of substitution in consumption:

$$\frac{\mu_1}{\mu_2} = \frac{U_{c_1}}{U_{c_2}}. \quad (19)$$

From eq. (10), we get  $\dot{\mu}_i = -\mu_i F'_i(X_i)$ , so that each shadow value decreases at a rate equal to the current marginal product of the corresponding stock:

$$-\frac{\dot{\mu}_i}{\mu_i} = F'_i(X_i), \quad i = 1, 2. \quad (20)$$

This depreciation rate is the cost of postponing an investment over a short period of time (see Dorfman, 1969, p. 821). The lower a stock, the higher its marginal product and the more costly in terms of maximin value it is to postpone investment in the stock.

The relative shadow value  $\frac{\mu_1}{\mu_2}$  decreases at a rate equal to the current difference between the stocks' marginal products:

$$\frac{1}{\mu_1/\mu_2} \frac{d(\mu_1/\mu_2)}{dt} = \frac{\dot{\mu}_1}{\mu_1} - \frac{\dot{\mu}_2}{\mu_2} = -(F'_1(X_1) - F'_2(X_2)). \quad (21)$$

Taking the logarithmic derivative of eq. (18) gives  $\frac{\dot{\mu}_i}{\mu_i} = \frac{\dot{\omega}}{\omega} + \frac{\dot{U}_{c_i}}{U_{c_i}}$ . Substituting  $\frac{\dot{\mu}_i}{\mu_i}$  by  $-F'_i(X_i)$ , we obtain, for  $i = 1, 2$

$$-\frac{\dot{\omega}}{\omega} = F'_i(X_i) + \frac{\dot{U}_{c_i}}{U_{c_i}}. \quad (22)$$

The shadow value of equity decreases at a rate equal to the sum of a stock's marginal product and the rate of change of the marginal utility of consumption for the associated good. Eq. (22) is analogous to the Keynes-Ramsey rule. The rate  $\delta \equiv -\frac{\dot{\omega}}{\omega}$  has features of a utility discount rate and  $\omega$  can be interpreted as a virtual discount factor along the maximin path (Cairns and Long, 2006). A maximin path thus has analogies to a discounted-utility path with this discount factor.<sup>28</sup> The shadow value  $\omega$ , however, is endogenous, and so is the virtual discount rate  $\delta$ , which is unlikely to be constant (except at a steady state), unlike in

<sup>28</sup>If the marginal utility  $U_{c_i}$  is the shadow current price of consumption for good  $i$ , by eq. (18), the shadow values  $\mu_i$  are analogous to present-value prices.

a discounted-utilitarian problem.<sup>29</sup>

The following Proposition characterizes optimal steady states. Since conditions (18)-(22) remain valid for  $n$  stocks, this condition is general to an economy with any number of separate sectors.<sup>30</sup>

**Proposition 4** (Steady state). *A state  $(X_1^*, X_2^*)$  is an optimal steady state if and only if the marginal products of all stocks are equal and positive*

$$F'_1(X_1^*) = F'_2(X_2^*) = \delta^* > 0 . \quad (23)$$

*Proof of Proposition 4. (Necessity)* At a steady state  $(X_1^*, X_2^*)$ ,  $\dot{X}_i = 0$ , i.e.,  $c_i = F_i(X_i)$ . Therefore,  $\dot{c}_i = 0$  and thus  $\dot{U}_{c_i} = 0, i = 1, 2$ . It follows from eq. (22) that  $\delta^* = -\frac{\dot{c}_i}{c_i} = F'_1(X_1^*) = F'_2(X_2^*)$  is a necessary condition for an optimal steady state.

*(Sufficiency)* Assume that  $F'_1(X_1^*) = F'_2(X_2^*) > 0$ . Then, according to eq. (22), one gets

$$\begin{aligned} \frac{\dot{U}_{c_1}}{U_{c_1}} = \frac{\dot{U}_{c_2}}{U_{c_2}} &\Leftrightarrow \frac{\dot{c}_1 U_{c_1 c_1} + \dot{c}_2 U_{c_1 c_2}}{U_{c_1}} = \frac{\dot{c}_1 U_{c_1 c_2} + \dot{c}_2 U_{c_2 c_2}}{U_{c_2}} \\ &\Leftrightarrow \dot{c}_1 \left( \frac{U_{c_1 c_1}}{U_{c_1}} - \frac{U_{c_1 c_2}}{U_{c_2}} \right) = \dot{c}_2 \left( \frac{U_{c_2 c_2}}{U_{c_2}} - \frac{U_{c_1 c_2}}{U_{c_1}} \right) \end{aligned} \quad (24)$$

Under strict quasi-concavity, marginal rates of substitution are decreasing, so that

$$\begin{aligned} \frac{\partial \text{MRS}_{c_j/c_i}}{\partial c_i} < 0 &\Leftrightarrow \frac{\partial (U_{c_i}/U_{c_j})}{\partial c_i} < 0 \\ &\Leftrightarrow U_{c_i c_i} U_{c_j} - U_{c_i} U_{c_j c_i} < 0 \\ &\Leftrightarrow \frac{U_{c_i c_i}}{U_{c_i}} - \frac{U_{c_j c_i}}{U_{c_j}} < 0 \end{aligned} \quad (25)$$

The two factors in brackets in eq. (24) are negative, which means that  $\dot{c}_1$  and  $\dot{c}_2$  have the same sign. As utility is constant along the maximin path, consumption levels cannot both increase or decrease. Therefore,  $\dot{c}_1 = \dot{c}_2 = 0$ .  $\square$

If the economy starts from any state satisfying  $F'_1(X_1^*) = F'_2(X_2^*) > 0$ , the stationary

<sup>29</sup>E.g., the discount rate in Withagen and Asheim (1998) varies over time.

<sup>30</sup>Weitzman (1976) originally used such a model with separate sectors to establish the formal links between national accounting and welfare in the discounted utility framework.

path  $c_i = F_i(X_i)$  for  $i = 1, 2$ , which yields utility  $U(F_1(X_1^*), F_2(X_2^*))$ , is egalitarian and efficient and corresponds to the maximin solution from that state.<sup>31</sup> At such steady states, the virtual discount rate  $\delta^*$  is endogenously set equal to the marginal productivity of all capital stocks so that there is no possibility of arbitrage by investing in or depleting the stocks.

Any state satisfying the condition  $F'_1(X_1^*) = F'_2(X_2^*) > 0$  is an optimal steady state. This is different from the optimality condition for an optimal steady state in a discounted utility problem with constant discount rate  $\rho$ , which is fully determined by the exogenous discount rate through the condition  $F'_1(X_1^*) = F'_2(X_2^*) = \rho$ . In the discounted utility framework, the optimal trajectory converges to that steady state for any initial state of the economy. We shall see that, in the maximin framework, a trajectory converges to a steady state that depends on the maximin value of its initial state.

When the economy is not at an optimal steady state, it follows a dynamic path characterized by nil net investment (eq. 16) and the following conditions.

**Proposition 5** (Transition path). *Along an optimal maximin path, when  $F'_i(X_i) > F'_j(X_j)$ , consumption and investment levels are such that:*

- *There is a positive investment in the stock with the higher marginal product (with  $c_i < F_i(X_i)$  and  $\dot{X}_i > 0$ ). Its marginal product decreases and its consumption increases.*
- *The stock with the lower marginal product is depleted (with  $c_j > F_j(X_j)$  and  $\dot{X}_j < 0$ ). Its marginal product increases and its consumption decreases.*
- *The maximin path leads to an optimal steady state, either in finite time or asymptotically, with the marginal products converging to equality.*

*Proof of Proposition 5.* Since both stocks have to satisfy condition (22), we have the equality  $F'_i(X_i) + \frac{\dot{U}_{c_i}}{U_{c_i}} = F'_j(X_j) + \frac{\dot{U}_{c_j}}{U_{c_j}}$ , which gives us, for  $F'_i(X_i) > F'_j(X_j)$

$$F'_i - F'_j = \dot{c}_i \left( \frac{U_{c_j c_i}}{U_{c_j}} - \frac{U_{c_i c_i}}{U_{c_i}} \right) - \dot{c}_j \left( \frac{U_{c_i c_j}}{U_{c_i}} - \frac{U_{c_j c_j}}{U_{c_j}} \right) > 0. \quad (26)$$

<sup>31</sup>Note that the golden rule state satisfying  $F'_1(X_1^*) = F'_2(X_2^*) = 0$  is also an optimal steady state, but corresponds to a non-regular case not covered by Proposition 4.

Under strict quasi-concavity, marginal rates of substitution are decreasing, and we have  $\frac{U_{c_j c_i}}{U_{c_j}} - \frac{U_{c_i c_i}}{U_{c_i}} > 0$  and  $\frac{U_{c_i c_j}}{U_{c_i}} - \frac{U_{c_j c_j}}{U_{c_j}} > 0$  (see eq. 25) The expressions in both parenthesis in eq. (26) are positive, which allows us to rearrange the inequality as follows:

$$\dot{c}_i > \dot{c}_j \left( \frac{U_{c_i c_j}}{U_{c_i}} - \frac{U_{c_j c_j}}{U_{c_j}} \right) / \left( \frac{U_{c_j c_i}}{U_{c_j}} - \frac{U_{c_i c_i}}{U_{c_i}} \right) . \quad (27)$$

Levels of consumption cannot both increase or decrease at the same time along the maximin path, where utility is constant over time. As such,  $\dot{c}_j$  and  $\dot{c}_i$  must be of opposite sign. As  $\dot{c}_j > 0$  would imply  $\dot{c}_i > 0$ , one must have  $\dot{c}_j < 0$  and  $\dot{c}_i > 0$ .

From eq. (21), we know that when  $F'_i(X_i) > F'_j(X_j)$ , one has  $\frac{\dot{\mu}_i}{\mu_i} - \frac{\dot{\mu}_j}{\mu_j} = -F'_i(X_i) + F'_j(X_j) < 0$ . The relative price  $\frac{\mu_i}{\mu_j}$  decreases. From eq. (16), we have  $-\frac{dX_i}{dX_j} = \frac{\mu_i}{\mu_j}$ ; the tangents to the paths in the state map  $(X_i, X_j)$  have to decrease (in absolute value) as well, implying  $\dot{X}_i = F_i(X_i) - c_i > 0$  and  $\dot{X}_j = F_j(X_j) - c_j < 0$ . Therefore,  $F'_j(X_j)$  rises while  $F'_i(X_i)$  decreases.

Formally, we can study the stability of the steady states. To do so, sum up the necessary conditions into the following dynamic equations (with  $\pi \equiv \frac{\mu_1}{\mu_2}$ ):

$$\begin{cases} \dot{X}_i = F_i(X_i) - c_i , & i = 1, 2 ; \\ \dot{\pi} = \pi (F'_2(X_2) - F'_1(X_1)) . \end{cases}$$

Steady states are characterized by  $c_i = F_i(X_i^*)$ ,  $i = 1, 2$ , and  $F'_1(X_1^*) = F'_2(X_2^*)$ . It is shown in the Supplementary Materials that the Jacobian matrix of the linearized system, evaluated at the steady states, has eigenvalues with opposite sign. Therefore, a saddle-point steady state exists on the locus  $F'_1(X_1^*) = F'_2(X_2^*) > 0$ .  $\square$

The transition path to a steady state is such that utility is sustained at the maximin level (i.e.,  $U(c_1(t), c_2(t)) = m(X_1(t), X_2(t))$ ) through substitution of the less productive stock for the more productive one. There is a *disinvestment* in the stock with the lower marginal product, compensated for by a positive investment in the stock with the higher marginal product, so that maximin net investment is nil and the maximin value is constant over time. The two stocks evolve in opposite directions so long as the marginal products are unequal, toward an optimal steady state. The steady state reached depends on the initial

state, in the sense that a trajectory starting for an arbitrary state  $(X_1^0, X_2^0)$  with maximin value  $m(X_1^0, X_2^0)$  converges to the steady state  $(X_1^*, X_2^*)$  satisfying  $U(F_1(X_1^*), F_2(X_2^*)) = m(X_1^0, X_2^0)$  as well as the optimality condition  $F_1'(X_1^*) = F_2'(X_2^*)$  of Proposition 4.

The maximin trajectories are characterized by the shape of the associated iso-value curves as follows.

**Proposition 6** (Iso-value curves). *Along a maximin path with constant utility  $u$ , the current state  $(X_1, X_2)$  and optimal controls  $(c_1, c_2)$  satisfy*

$$\frac{\dot{X}_2}{\dot{X}_1} = \frac{\dot{c}_2}{\dot{c}_1} = -\frac{\mu_1}{\mu_2} \Leftrightarrow \left. \frac{dX_2}{dX_1} \right|_{m(X_1, X_2)=u} = \left. \frac{dc_2}{dc_1} \right|_{U(c_1, c_2)=u}. \quad (28)$$

*Proof of Proposition 6.* Along the optimal path, for any state  $(X_1(t), X_2(t))$ , the partial derivatives of the maximin value equal the shadow prices of the stocks, i.e.,  $\frac{\partial m}{\partial X_i} = \mu_i(t)$  (Cairns and Long, 2006). From condition (18), one gets  $\frac{\partial m}{\partial X_1} / \frac{\partial m}{\partial X_2} = \frac{\mu_1}{\mu_2} = \frac{U_{c_1}}{U_{c_2}}$ . At the optimum, the marginal rate of transformation equals the marginal rate of substitution. Along the optimal path,  $\mu_1 \dot{X}_1 + \mu_2 \dot{X}_2 = 0$  (eq. 16). By eq. (18), apart from a steady state

$$\frac{\dot{X}_2}{\dot{X}_1} = -\frac{\mu_1}{\mu_2} = -\frac{U_{c_1}}{U_{c_2}} < 0. \quad (29)$$

When utility is constant,  $\frac{dU(c_1, c_2)}{dt} = \dot{c}_1 U_{c_1} + \dot{c}_2 U_{c_2} = 0$  and, apart from a steady state

$$\frac{\dot{c}_2}{\dot{c}_1} = -\frac{U_{c_1}}{U_{c_2}} < 0. \quad (30)$$

Combining conditions (29) and (30), one gets

$$\frac{\dot{X}_2}{\dot{X}_1} = \frac{\dot{c}_2}{\dot{c}_1} \Leftrightarrow \frac{dX_2}{dX_1} = \frac{dc_2}{dc_1}. \quad (31)$$

At the steady state, i.e., when  $\dot{X}_i = 0$ , for  $i = 1, 2$ ,  $\frac{dc_i}{dX_i} = F_i'(X_i^*)$ . As  $F_1'(X_1^*) = F_2'(X_2^*)$ , we also have  $\frac{dc_2}{dc_1} = \frac{dX_2}{dX_1}$  at the steady state.  $\square$

A maximin trajectory thus follows an iso-value curve toward a steady state at which the marginal products of the two stocks are equal. A graphical representation in Fig. 1 below



illustrates the paths of consumption and stock levels starting at an arbitrary point  $A$  in the state space  $(X_1, X_2)$  and obeying eq. (28).<sup>32</sup>

The conditions derived in Propositions 4-6 are general, for any strictly quasi-concave, strictly increasing utility, and strictly concave production functions. Deriving further general results is difficult. In particular, determining explicitly the steady state reached from an arbitrary initial state would require integrating the trajectory implicitly characterized by Proposition 6. This requires specifying all functional forms and solving completely the particular maximin problem. The results would then be case-specific. A closed-form solution can be obtained for some problems, but for other problems, numerical approaches may be needed.

The implicit results of Proposition 6 allow us to provide general interpretations, however. By equation (28), the slope of each indifference curve in the decision map is equal to the slope of the corresponding path for the state variables. Therefore, the maximin paths are convex to the origin. The relative shadow value of the stocks governs the slopes  $\frac{dc_2}{dc_1}$  and  $\frac{dX_2}{dX_1}$ . In particular, it defines the relative value of the two stocks in terms of maximin investment. We examine in Section 3 how this result can be used to set up a sustainable accounting system based on maximin values.

**Graphical representation** We start by illustrating regular maximin paths. Fig. 1 is a plot of the solution, with neoclassical functions used to represent technologies and symmetric Cobb-Douglas utility. It is a four-quadrant graph in which the east axis represents  $X_1$ , the south axis  $X_2$ , the north axis  $c_1$  and the west axis  $c_2$ . The north-east quadrant represents production  $F_1(X_1)$  and the south-west quadrant production  $F_2(X_2)$ . The north-west quadrant plots indifference curves in the consumption map  $(c_1, c_2)$ , and the south-east quadrant is the state map  $(X_1, X_2)$  in which state trajectories can be drawn as well as iso-value curves.

The dashed curve starting at  $(0, 0)$  in the state map corresponds to optimal steady states satisfying  $F_1'(X_1) = F_2'(X_2)$ .<sup>33</sup> The corresponding optimal steady state consumption levels

<sup>32</sup>Such an iso-value curve is also defined for the limiting case  $m(X_1, X_2) = \bar{m}$ , even if its mathematical properties cannot be derived from our optimal control framework (see Corollary 1). Our analysis, based on maximin pricing, does not allow us to answer the question whether Hartwick's rule is followed along the Golden Rule paths of our model.

<sup>33</sup>This curve starts from  $(0, 0)$  because the two production functions have the same marginal product at zero in this example. For different technologies, the curve could start from a point  $(\underline{X}_i, 0)$  satisfying

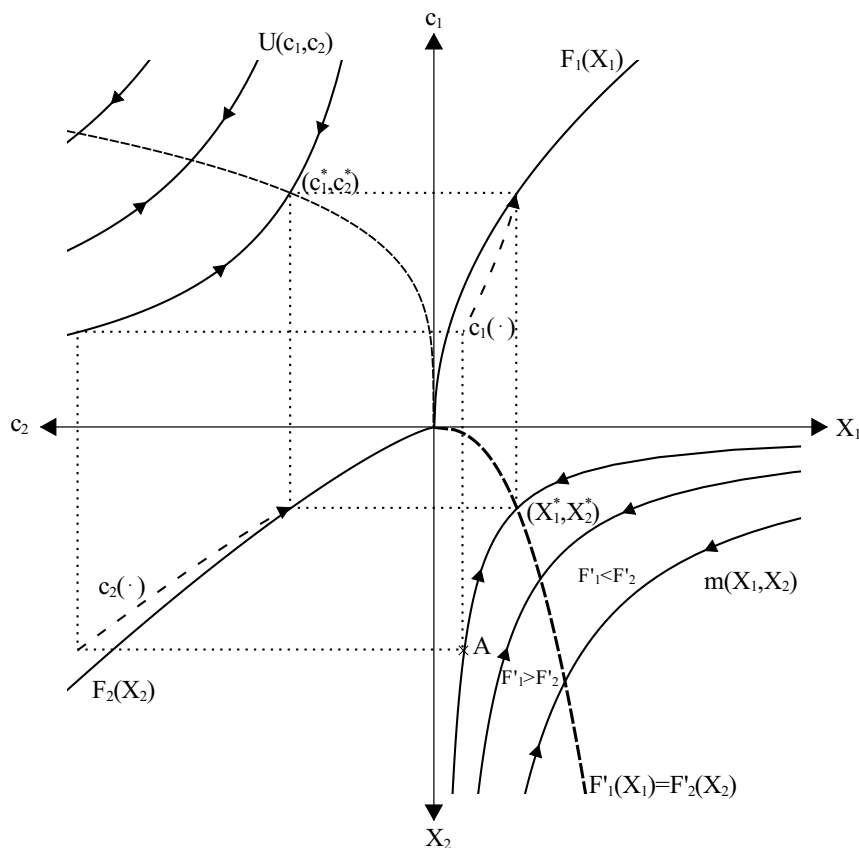


Figure 1: Graphical representation for the neoclassical benchmark

are represented by the dashed curve starting at  $(0, 0)$  in the consumption map, which makes it possible to relate the steady states to their maximin value on the indifference curves.

For any state south-west of the steady states curve (e.g., for state  $A$  on the figure), stock  $X_2$  is less productive at the margin ( $F'_1(X_1) > F'_2(X_2)$ ). Along the maximin path, consumption of stock 2 exceeds its production, while consumption of stock 1 is lower than its production. The trajectory goes north-east along the iso-value curve. Consumption levels and states converge toward the corresponding steady state. A similar pattern occurs north-east of the equilibrium line. For any state, maximin shadow values are positive.

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$$F'_i(\underline{X}_i) = F'_j(0).$$

We now illustrate non-regular cases. The boundary of the set of states satisfying  $m(X_1, X_2) = \bar{m}$  is the edge of a plateau of the maximin value function, represented as the hatched area in Fig. 2a. This figure is similar to Fig. 1, except that the production functions are single-peaked.<sup>34</sup> Without loss of generality we can assume that the stocks are indexed such that there is an  $\underline{X}_1 \geq 0$  where  $F'_1(\underline{X}_1) = F'_2(0)$ . Optimal steady states are along the line  $\underline{X}_1 M$ . Any state in the hatched area has maximin value  $m(\bar{X}_1, \bar{X}_2)$ . Any maximin path starting from a state in this region is non-regular in the sense that utility can be larger than the maximin value  $\bar{m}$  for some time, until a point on the boundary is reached, after which utility is constant at  $U(c_1, c_2) = \bar{m}$ . Both stocks are redundant and have nil shadow values, even a stock below its sector productive peak if the other stock is so abundant that  $m(X_1, X_2) = \bar{m}$ .<sup>35</sup> It does not mean, however, that capital above the production peak is necessarily redundant. For states such that  $m(X_1, X_2) < \bar{m}$ , a stock above the peak can be used intensively as a substitute for a more productive resource to build it up, just as in the neoclassical benchmark. Capital is not redundant.

Proposition 1 and Corollary 1 depend on the fact that both technologies have a production peak. If one technology does not, so that  $F'_i(X_i) > 0$  for all  $X_i$ , there is no stock redundancy, even if there is an upper bound on the maximin value. When one sector is always productive at the margin, the capital of another, single-peaked sector is never redundant if it can be used as a substitute for the productive sector to grow, as illustrated in Fig. 2b. All capital stocks have positive shadow values. This is a useful result for building a sustainability accounting system in a world with substitutable assets.

### 3 Accounting for changes in sustainability

Having characterized the maximin path in our two-sector economy, we can use it as a benchmark to assess the sustainability of current decisions, which may not correspond to maximin decisions. From previous results, we can examine the consequences of consumption

<sup>34</sup>We used quadratic growth functions to represent the technologies. As a consequence, the steady states curve is a straight line as the marginal products  $F'_i(X_i)$  are proportional to the stocks  $X_i$ , with  $i = 1, 2$ , giving a linear relationship between  $X_1^*$  and  $X_2^*$ , for which  $F'_1(X_1^*) = F'_2(X_2^*)$ .

<sup>35</sup>Along the boundary separating the area of redundant stocks from the states with positive shadow values, the shadow prices are also zero. As emphasized in footnote 32, the maximin framework is not suited to discuss the role of Hartwick's rule along such a Golden Rule path.

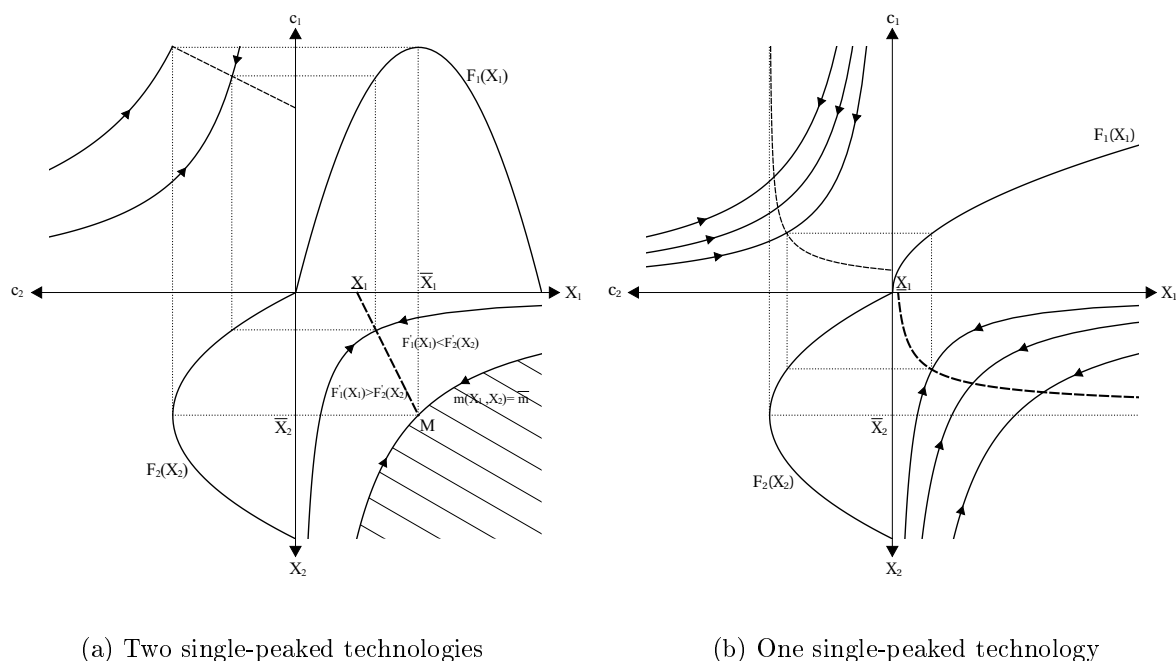


Figure 2: Graphical representation for single-peaked technologies

choices on the evolution of the sustainable level of utility, measured by the maximin value and its evolution. This examination provides insights for sustainability accounting for non-maximin paths.

Cairns and Martinet (2014) describe the interplay among consumption, the maximin value and changes in sustainability. Net investment at maximin shadow values is a measure of these changes. For any economic state, it is possible to define the consumption levels resulting in a positive maximin investment and an increasing of the level of utility that can be sustained, i.e., *sustainability improvement*. We consider the regular case in the following discussion. For non-regular cases, the maximin shadow values are nil, and so is sustainability improvement.

For a given economic state  $(X_1, X_2)$ , denote by  $(c_1^*, c_2^*)$  the maximin consumption levels. These decisions, which satisfy  $U(c_1^*, c_2^*) = m(X_1, X_2)$ , can be used as a reference point. To do so, consider the indifference curve  $U(c_1, c_2) = m(X_1, X_2)$ . At  $(c_1^*, c_2^*)$ , one has  $\frac{U_{c_1}}{U_{c_2}} = \frac{\mu_1}{\mu_2}$  (eq. 19). From the definition of net investment (eq. 4), we derive the condition for non-

negative net investment, for given levels of the stocks and shadow values:

$$\dot{m} = \mu_1[F_1(X_1) - c_1] + \mu_2[F_2(X_2) - c_2] \geq 0 \Leftrightarrow c_2 \leq \frac{\mu_1}{\mu_2}[F_1(X_1) - c_1] + F_2(X_2). \quad (32)$$

When  $\dot{m} = 0$ , there is a linear relationship between  $c_1$  and  $c_2$ .

Fig. 3 depicts the possible consumption decisions along with their consequences for changes in sustainability. In the consumption map, the line  $\dot{m} = 0$  is tangent to the indifference curve  $U(c_1, c_2) = m(X_1, X_2)$  at  $(c_1^*, c_2^*)$ . Three areas of interest are defined by the two curves. Area 1 corresponds to consumption decisions with a sustainable utility

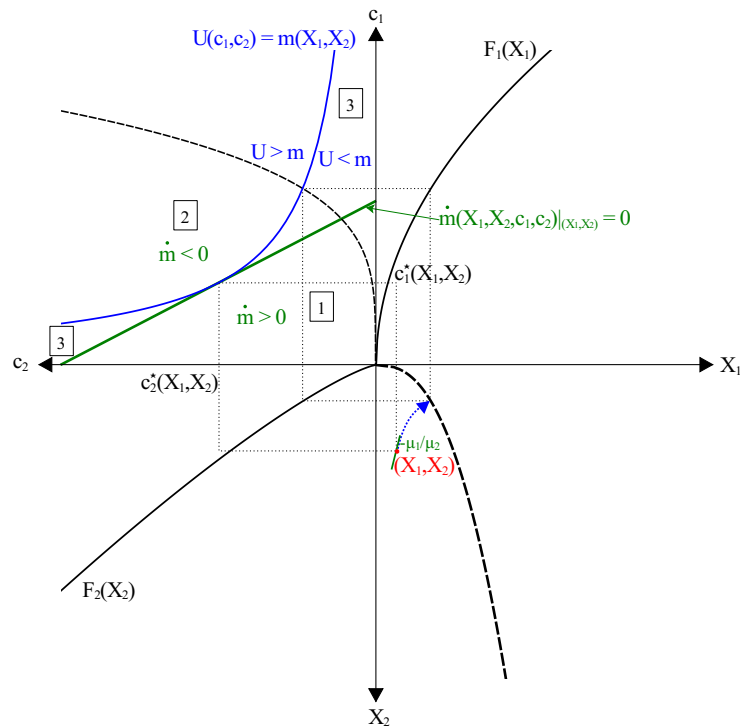


Figure 3: Utility, maximin and sustainability improvement

( $U(c_1, c_2) < m(X_1, X_2)$ ) and positive net maximin investment, i.e., to sustainability improvement (eq. 32). Area 2 corresponds to decisions with a unsustainable utility  $U(c_1, c_2) > m(X_1, X_2)$ , implying sustainability decline ( $\dot{m} < 0$ ). In the two regions of Area 3, consumption decisions induce a sustainability decline in spite of the fact that utility is lower than the

maximin value. With different decisions, the same utility could have been compatible with sustainability improvement. These areas illustrate how reducing utility below the maximin level improves sustainability only if there is an investment that increases the maximin value. Conversely, in our multiple capital good model, ill-considered consumption and investment decisions can result in a decline of the sustainable level of utility even though the current utility is below the sustainable, maximin level.

Changes in sustainability can be measured by net maximin investment along any path. Therefore, maximin shadow values can be used as sustainability accounting prices. Basing an accounting system on shadow values requires that these values be well defined and computable. The analysis of this paper stresses that, as for any optimization problem, finding shadow values is a challenging task. The task for a maximin problem, however, is likely no more difficult than for other objectives.<sup>36</sup> The challenge can be met with proper numerical tools. Given the theoretical characterization of the maximin solution in this paper, and given the recursive structure of the maximin objective, a Bellman algorithm could be used to approximate maximin values and shadow values. Because there are strong links between maximin and viability (Doyen and Martinet, 2012), the numerical tools for set-valued analysis could also be used for that purpose.

We have found that substitutability in utility is important for the properties of the maximin solution and accounting values, as is substitutability in production (Solow, 1974; Hartwick, 1977; Mitra et al., 2013). One task is to estimate the substitutability among the different goods in the economy. For some sectors (manufactured goods and services), substitutability at the margin is a reasonable assumption and the elasticities of substitution estimated in the macroeconomic literature can provide a starting point. For environmental resources, the task is harder. Drupp (2018) surveys the empirical estimates of the substitutability between manufactured goods and ecosystem services. He relates this substitutability to the income elasticity of the willingness-to-pay for environmental goods, and emphasizes the variability of the substitutability parameter. Most studies find a relatively

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<sup>36</sup>The fishery problem is quite difficult to solve in the discounted utilitarian framework, and is comparatively simpler in a maximin setting. The Dasgupta-Heal-Solow model was solved immediately for a maximin problem (Solow, 1974), but not until much later for a discounted-utilitarian problem (Benckroun and Withagen, 2011).

high substitutability. However, the degree of substitutability may change as the environment becomes scarcer (Baumgärtner et al., 2017; Drupp, 2018). At some point, or for some resources or ecosystem services, substitution may not be possible.

Last, when all technologies are single-peaked, stock redundancy may occur. In that case, all maximin shadow values are zero. Society is faced with surplus stocks (from the maximin point of view) and sustainability accounting is trivial.

## 4 Conclusion

Sustaining utility is the very objective of a maximin problem. Even if maximin is not chosen as a social objective and society does not aim at following a maximin path from its current stage of development, the maximin value is an indicator of the highest utility level that can be sustained given current economic and environmental endowments (Cairns and Martinet, 2014; Fleurbaey, 2015). Maximin shadow values can be used *along any trajectory* to compute maximin net investment, a particular genuine savings indicator which measures the evolution of the capacity of the economy to sustain utility. These shadow values have to be derived from the resolution of the maximin problem.

Maximin has been applied to only a handful of problems. In this paper, we have characterized the maximin solution for an additional class of problems, corresponding to economies with two separate sectors that interact indirectly through the utility function. We have shown that maximin calls for a dynamic investment pattern that outperforms the static option of maintaining current productive stocks. Whenever a capital stock produces more at the margin than the other, positive investment in this sector is made possible through substitution in consumption and a decline of the less productive capital stock, according to Hartwick's rule of nil net investment (Hartwick, 1977; Solow, 1993; Cairns and Long, 2006). The maximin path ultimately leads to some optimal steady state, however, which depends on the initial state of the economy and its maximin value.

Maximin has been criticized as a social objective as possibly maintaining a poor economy in poverty. If growth out of poverty is pursued, it must be within sustainable limits. Computing the evolution of the maximin value informs us on the effect of current consumption and investment decisions on the level of sustainable utility. This accounting has to be done

with maximin shadow values. In the presence of a single-peaked technology, if the other technology is everywhere productive at the margin and consumption goods are substitutes, both capital stocks have positive accounting prices. If all technologies are single-peaked, capital stocks may be redundant, and accounting prices for sustainability are nil. In all cases, maximin accounting prices are well-defined and provide the relevant information about the relative marginal values of different stocks for net investment in the capacity to sustain utility.

It is clear that solving the maximin problem for a modern economy, with all its various assets, consumption goods, production techniques, etc., is conceptually and practically an extraordinarily difficult problem. Our study is only a tentative initial exploration of theoretic implications. Any complexity in determining accounting (shadow) prices in maximin, however, stems from the complexity of the interactions of capital and consumption goods in the economy, and hence is comparable in any accounting system based on shadow values. It would not do to use market prices for the shadow values, even for marketed assets. Whatever may be maximized in a market is almost surely not the sustainable level of utility. Even if the shadow values were found for our world, it is hard to conceive that they would be implemented in a global political economy that cannot agree on or implement a price for even a single good, atmospheric carbon. Still, the study of maximin solutions is useful because it provides insight into how to evaluate the relative contributions of the different types of capital to the ability of the economy to sustain utility. It systematically confronts the complicated interactions among all assets. Further research may indicate how shadow values differ from market prices and how the latter may induce unsustainable decisions in the economy.

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## A Appendix

### A.1 Stability of the steady state

Let us sum up the necessary conditions into three main equations to get the following dynamic system ( $\pi \equiv \frac{\mu_1}{\mu_2}$ )

$$\begin{cases} \dot{X}_1 = F_1(X_1) - c_1 ; \\ \dot{X}_2 = F_2(X_2) - c_2 ; \\ \dot{\pi} = \pi (F'_2(X_2) - F'_1(X_1)) . \end{cases} \quad (33)$$

Steady states are characterized by

$$\begin{cases} c_1^* = F_1(X_1^*) ; \\ c_2^* = F_2(X_2^*) ; \\ F'_1(X_1^*) = F'_2(X_2^*) . \end{cases} \quad (34)$$

Consider the Jacobian matrix of the linearized system, evaluated at the steady states<sup>37</sup>

$$J^* = \begin{pmatrix} F'_1(X_1^*) & 0 & -\frac{\partial c_1(\pi)}{\partial \pi} \\ 0 & F'_1(X_1^*) & -\frac{\partial c_2(\pi)}{\partial \pi} \\ -\pi F''_1(X_1^*) & \pi F''_2(X_2^*) & 0 \end{pmatrix} \quad (35)$$

<sup>37</sup>We use the equality  $F'_1(X_1^*) = F'_2(X_2^*)$  and express the Jacobian with respect to  $F'_1(X_1^*)$  only.

Let us compute the roots of the characteristic polynomial  $\mathcal{P}(\lambda) = \det(J^* - \lambda I_3)$ :

$$\begin{aligned}
 & \begin{vmatrix} F_1'(X_1^*) - \lambda & 0 & -\frac{\partial c_1(\pi)}{\partial \pi} \\ 0 & F_1'(X_1^*) - \lambda & -\frac{\partial c_2(\pi)}{\partial \pi} \\ -\pi F_1''(X_1^*) & \pi F_2''(X_2^*) & -\lambda \end{vmatrix} = 0 \\
 \Leftrightarrow & (F_1' - \lambda) \begin{vmatrix} F_1' - \lambda & -\frac{\partial c_2(\pi)}{\partial \pi} \\ \pi F_2'' & -\lambda \end{vmatrix} - \pi F_1'' \begin{vmatrix} 0 & -\frac{\partial c_1(\pi)}{\partial \pi} \\ F_1' - \lambda & -\frac{\partial c_2(\pi)}{\partial \pi} \end{vmatrix} = 0 \\
 \Leftrightarrow & (F_1' - \lambda) \left( -(F_1' - \lambda) \lambda + \pi F_2'' \frac{\partial c_2}{\partial \pi} \right) - \pi F_1'' (F_1' - \lambda) \frac{\partial c_1}{\partial \pi} = 0 \\
 \Leftrightarrow & (F_1' - \lambda) \left( -(F_1' - \lambda) \lambda - \pi \frac{\partial c_1}{\partial \pi} F_1'' + \pi \frac{\partial c_2}{\partial \pi} F_2'' \right) = 0. \quad (36)
 \end{aligned}$$

The first eigenvalue is  $\lambda_1 = F_1'$ . Also, due to the strict convexity of indifference curves (recall that at the optimum,  $\pi = \frac{U_{c_i}}{U_{c_j}}$ ), let  $\alpha_1 \equiv -\frac{\partial c_1}{\partial \pi} > 0$  and  $\alpha_2 \equiv \frac{\partial c_2}{\partial \pi} > 0$ . Let  $\Gamma \equiv -\pi(\alpha_1 F_1'' + \alpha_2 F_2'') > 0$ . We can reduce eq. (36) to  $\lambda^2 - F_1' \lambda - \Gamma = 0$ . Eigenvalues are then  $\lambda_1 = F_1' > 0$ ,  $\lambda_2 = \frac{F_1' - \sqrt{(F_1')^2 + 4\Gamma}}{2} < 0$ , and  $\lambda_3 = \frac{F_1' + \sqrt{(F_1')^2 + 4\Gamma}}{2} > 0$ .<sup>38</sup> The steady state is a saddle-point.

<sup>38</sup>Note that strict concavity of production functions rule out nil eigenvalues.