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Modelling under ambiguity with dynamically consistent Choquet random walks and Choquet–Brownian motions

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ABSTRACT

Ambiguity is pervasive in many environments and is increasingly being introduced into economic and financial models. This paper characterises ambiguity in the form of newly defined Choquet random walks: discrete-time binomial trees with capacities instead of exact probabilities on their branches. We describe the axiomatic basis of Choquet random walks, including dynamic consistency. We also discuss the convergence of Choquet random walks to Choquet–Brownian motion in continuous time. In contrast to previous literature, we derive tractable stochastic processes that allow for a wide range of ambiguity preferences to be represented in continuous time (including ambiguity-seeking preferences). Finally, we apply Choquet–Brownian ambiguity to a model of stationary inter-temporal portfolio choice. We find that both the mean and the variance of the underlying stochastic process are modified. This result opens the way for qualitative and quantitative results that differ from those of standard expected utility models and other models that feature ambiguity.
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1. Introduction

Models that involve uncertainty typically assume that uncertainty parameters should take the familiar form of risk (i.e., a probability measure). Consequently, in continuous-time applications, the dynamics of a risky variable (such as the cash flows of a project) are often described by some known stochastic processes such as Brownian motion. However, uncertainty is a rich and complex concept that may not be limited to risk.

Indeed, alternative ways of characterising uncertainty are sometimes required in model construction, especially when the nature of the uncertainty prevalent in a given environment is not fully captured by standard risk parameters. This need has led to the development of models that incorporate ambiguity parametrically. Ambiguity may, for instance, result from different perceptions among individuals of the sources of uncertainty or from incomplete information/transparency. In the economic and financial modelling literature, the significance of ambiguity is increasingly recognised, with ambiguity parameters being introduced into a growing body of fruitful models.[1]

Nevertheless, how best to model ambiguity remains controversial at the axiomatic and theoretical level, as well as in practice. Let us recall that in the presence of risk, it is possible to use the objective expected utility maximisation model or, in the absence of a computable or observable probability distribution, its subjective version. Expected utility models are extremely tractable and allow for the construction of flexible yet sound models. However, in contexts that feature ambiguity, these standards models cannot be used without ignoring a whole range of behaviours and preferences. This limit has been repeatedly identified experimentally and in many empirical studies (since Ellsberg, 1961).

To address this challenge, ambiguity has, since the 1980s, been represented by mathematical concepts in decision theory. Several axiomatic

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[1] Incorporating ambiguous rather than just risky uncertainty parameters into models was shown to substantially impact individual choices and the resulting macro-economic equilibrium conditions (Dow and Verlang, 1992; Epstein and Wang, 1994). This has already led to reinterpretations of well-known financial puzzles, such as the "equity premium" puzzle or the "home bias" puzzle (on the reinterpretations of financial puzzles, see the review on ambiguity and asset pricing in Guidolin and Rinaldi, 2011).

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bases have been proposed and applied. In particular, the multiple-priors model of Gilboa and Schmeidler (1989) was a significant breakthrough, especially as it was successfully expanded to a continuous time setting (Chen and Epstein, 2002). The multiple-priors model is based on the *maxmin* criterion, implying optimisation in a worst case scenario.

However, it is remarkable that pioneer models of ambiguity, such as the multiple-priors model, focus exclusively on ambiguity-adverse agents. This is indeed a reasonable assumption in many situations where agents focus on worst-case scenarios to determine their optimal decisions. However, even if ambiguity aversion may usually prevail among agents, it does not fully account for the variety of observed individual behaviours and choices that occur in situations that feature ambiguity.

Although several models expand the domain of acceptable preferences to account for a wider range of attitudes towards ambiguity,[2] most of these alternative models, such as those in Schroder (2008) or Kim et al. (2009), are not dynamically consistent.[3] Even the very promising model of smooth ambiguity suffers from constraints that may significantly undermine its usefulness in modelling.[4] This impairs, to a certain extent, its potential application to model construction and it may be complemented by other representations of ambiguity such as ours.

The methodology of our paper derives from two preliminary observations: a) while far from perfect, Brownian motion is still widely used in economic and financial models (for instance, to describe the dynamics of cash flows in the real investment literature), and b) existing ambiguity models almost always focus exclusively on worst-case scenarios by adopting the multiple-priors model. In many contexts, this represents an excessive limitation on the domain of admissible individual preferences in the presence of ambiguity.

In light of these two observations, the goal of this paper is twofold. First, we preserve the representation of ambiguity in the Brownian motion model[5] to derive a tool that should be tractable enough to be applied in many settings (as initiated by Epstein's multiple-priors models). Adopting Brownian motion also helps us compare our findings with those of multiple-priors models. To construct ambiguous geometric Brownian motion, we start with a simple binomial process measured by a capacity[6] (rather than a probability). We then show that this process converges to a type of Brownian motion. This approach differs from the multiple-priors approach, which directly considers a family of Brownian motion processes.

Second, we allow for a wider range of individual ambiguity preferences to be explicitly included (i.e. ambiguity seeking as well as aversion). To do so, we use capacities, rather than standard probabilities, to weight the perceived likelihood of ambiguous outcomes. The use of capacities allows us to expand the domain of acceptable preferences regarding ambiguity to a wider range than is permitted by such models as the multiple-priors model.

Let us clarify what we mean by uncertainty, as subjectively measured by the decision maker: we mean that there is not necessarily an objective level of ambiguity that is observed independently of one's preferences with respect to objective ambiguity, as in the smooth ambiguity models. We rely on discounted Choquet expected utility[7] to value alternatives and decide between ambiguous outcomes.

This paper thus offers a tractable alternative to the existing powerful ambiguity aversion models by relying on some properties of Choquet integrals and capacities, as adapted to stochastic processes. Our paper defines ambiguous random walks and Brownian processes in a Choquet expected utility framework. An earlier unpublished version of this work (Kast and Lapied, 2010a) has been applied to real investment decisions (Roubaud et al., 2010) and to assessments of environmental policy options (Agliardi and Sereno, 2011).

In this paper, we also show that the binomial processes constructed are dynamically consistent, which is an interesting (and often necessary!) property of economic and financial models. Without this property, the model would be seriously limited. However, even with it, if one is not careful, the model may still collapse into an additive model (Sarin and Wakker, 1998) and hence lose its relevance to the issue of ambiguity. One way out of this difficulty is to consider only binomial recursive models (one period ahead) instead of the full dynamic model. This is the method adopted in this paper, in accordance with the *maxmin* recursive expected utility approach, as it enables us to address most of the modelling problems that arise.

The remainder of the paper is organised as follows. In Section 2, we build the axiomatic basis for Choquet random walks. Specifically, uncertainty is described by a binomial tree measured by the same capacity on the up and down branches. We show how individual preferences over payoff processes may be represented by a discounted Choquet expectation that satisfies model consistency and a weakened version of dynamic consistency. This opens the way for discrete-time applications of Choquet ambiguity. In Section 3, we investigate the limit behaviour of the joint capacity when time intervals converge to zero to allow for continuous-time modelling. We obtain a type of Brownian motion as the limit of the Choquet binomial random walk. In Section 4, we apply Choquet–Brownian ambiguity to a model of stationary inter-temporal portfolio choice. In Section 5, we relate our results to similar results in the literature and conclude.

2. Dynamically consistent binomial Choquet random walks

2.1. Method

As a starting point, we consider a particular belief represented by a non-additive measure, the Choquet capacity. Because it is non-additive, this measure may be interpreted as revealing the attitude of a decision maker towards the evolution over time of the value of a particular process that is perceived as ambiguous. As noted above, our approach is axiomatic and subjective (the measure derives exclusively from the decision maker's preferences), without reference to an objective probability distribution that would be subjectively distorted. There is no need for an "objective" source of ambiguity, in response to which the decision maker would take a particular stance. We directly introduce the ambiguity preference into the dynamics of the ambiguity parameter.

We first consider a familiar discrete time dynamic model that we assume to be ambiguous, an ambiguous random walk (or Choquet random walk). In Section 3, we show that it converges to a continuous time model, an ambiguous type of Wiener process that we call a Choquet–Brownian motion. First, however, how we define a Choquet random walk.

Consider a standard random walk that may be represented in a typical binomial decision tree. Suppose also that 'up' and 'down' movements across the decision tree have the same magnitude and likelihood

[2] See the α-MEU (Ghirardato et al., 2004) or the neo-additive capacities (Chateauneuf et al., 2007).

[3] Dynamic consistency implies that the decision maker, once he commits to a contingent plan, does not later change his plans. To meet this condition, it is sufficient to allow the use of a dynamic programming principle (Riedel, 2009). On the normative side, such a condition also intuitively appears to be a key requirement for rational behaviour over time.

[4] The smooth ambiguity model (Klibanoff et al., 2005, 2009) assumes separation between beliefs and tastes through a second-order functional. But its extension to continuous time has not yet been clearly established. Skiadas (2014) shows that in continuous time, the smooth ambiguity adjustment may vanish and fail to preserve the ambiguity preferences.

[5] Another alternative approach where vagueness is incorporated in Geometric Brownian motions in financial models (through fuzzy sets) was proposed in Agliardi and Agliardi (2009).

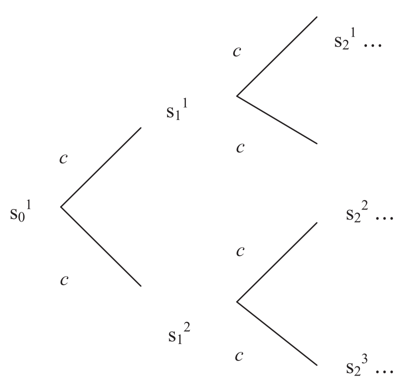
[6] The key characteristic of a capacity is that it is non-additive (Choquet, 1954).

[7] See Schmeidler (1986, 1989), Gilboa (1987), and Sarin and Wakker (1992). Let us recall that when beliefs are represented by capacities, a specific notion of integration, the Choquet integral, is required, which takes into account the rank of outcomes.

of occurrence (denoted as c). This is the basic description of a symmetric Choquet random walk.[8]

To characterise a Choquet random walk, we rely on the use of capacities rather than probabilities.[9] A capacity summarises preferences regarding ambiguity by measuring the likelihood of up and down movements (rather than by assigning standard probability measures). In a Choquet random walk, if the capacity is convex[10] (ambiguity aversion[11]), then the value attributed to a fair game (the same likelihood for up and down movements, where the movements are of the same magnitude in absolute terms) is negative, due to the use of Choquet integrals to compute the value, which is sufficient to account for ambiguity preferences (in contrast with the probabilistic case, where the value of a fair game is zero). The opposite holds if the capacity is concave (ambiguity seeking).

Graphically, a symmetric Choquet random walk may be represented as follows:



A binomial Choquet random walk based on capacity c.

Ambiguity is introduced at each node of the binomial tree, determining the likelihood of movement up or down. Without loss of generality, we take the increments to be unity $(-1, +1)$ and the departure point to be zero. The uncertain states, $s_1, \ldots, s_n$ in S are trajectories or sequences of nodes in the tree.

For any node s_t at date t $(0 \le t < T)$, if ${s^u}_{t+1}$ and ${s^d}_{t+1}$ are the two possible successors of s_t at date $t + 1$ (for, respectively, an 'up' or a 'down' movement in the binomial tree), the conditional capacity is a constant: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$, with $0 < c < 1$.

The value of the parameter c is fundamental, as it summarises the decision maker's attitude towards ambiguity. It represents an index of the individual's preferences regarding ambiguity. If $c = \frac{1}{2}$, then we are back to the probabilistic case.

In summary, we consider a modified version of a standard binomial tree, where uncertain states are represented through sequences of nodes describing the trajectories of the random variable, and where *capacities* rather than probabilities represent preferences. Thus, the dynamics of such an uncertain random variable may be described by a discrete time Brownian motion process, in which probability 1/2 is replaced by a constant[12] c (the ambiguous weight that the decision maker places both on the event 'up' and the event 'down' instead of the unambiguous 1/2).

2.2. Insuring dynamic consistency

The construction of the axiomatic basis for Choquet random walks is described in detail in Appendices. It first includes the dynamic consistency axioms and the conditions expressing consistency between conditional and unconditional expectations, as characterised in Proposition 1 and relation (1) in Appendix 1.

Proposition 1. *Under the representation of preferences satisfying our six axioms, for any $X \in R^{S \times T}$, $\forall \tau \in \{0, \ldots, T\}$, $\forall i \in I$, $\forall [Y_\tau = i] \subset F_\tau$, $\forall t, \tau \le t \le T$, $E_\nu(X_t) = E_\nu\left[E_\nu^{[Y_\tau = i]}(X_t)\right]$.*

Proof. See Appendix 1.

From Proposition 1, dynamic consistency implies (see Appendix 1):

$$\forall \tau = 1, \ldots, T-1, \forall t = \tau, \ldots, T, \sum_{s_\tau \in S_\tau} [\sum_{s_t \in S_t} X_t(s_t) \Delta\nu(s_t/s_\tau)] \Delta\nu(s_\tau) = \sum_{s_t \in S_t} X_t(s_t) \Delta\nu(s_t). \quad (1)$$

The axiomatic basis allows for the characterisation of subjective capacity as follows:

Proposition 2. *A dynamically consistent Choquet random walk that satisfies relation (1) is completely defined by a unique capacity ν satisfying:*

$$\nu(s^u_{t+1}/s_t) = \nu\left(s^d_{t+1}/s_t\right) = c. \quad (2)$$

Proof. See Appendix 2.

The idea is that, in our binomial lattice, the filtration is fixed and then, we can only consider comonotonic payoffs (on a comonotonic cone). Therefore, on the binomial tree, the capacity is equivalent to a unique probability. This identification is correct because we define non-conditional capacities from given conditional capacities and not the reverse.

Proposition 3. *In a dynamically consistent Choquet random walk, the capacity ν is sub-linear if and only if $c \le 1/2$. Moreover, it does not reduce to a probability if and only if $c \ne 1/2$.*

Proof. See Appendix 3.

Sublinear is implied by convexity but not equivalent.

$$\forall A \in 2^S, v(S) = 1 \ge v(A) + v\left(A^C\right).$$

The decision maker consider a fair game when $c = 1/2$.

If $c < 1/2$, the sum of one \$ if A is drawn and one \$ if A^C is drawn is less than one \$ with certainty (Schmeidler's uncertainty aversion).

2.3. Choquet expectation of symmetric Choquet random walks

To compute the Choquet expectation of such a process (which is required to provide a decision criterion), we must characterise the

[8] Symmetric random walks are binomial processes where 'up' and 'down' movements correspond to the same weight. In the probabilistic model, i.e., the case where $c = 1/2$, this process is a discrete time Brownian motion.

[9] We are not trying here to model distortions of probabilities *stricto sensu*, which is another vision of the potential impact of ambiguity. Such distortions, such as the tendency to overweight small probabilities, have been widely documented in the literature.

[10] A convex capacity on a finite set of states of nature S is a real-valued set function on the subsets of S such that: $\forall A, B \in 2^S, \nu(A \cup B) + \nu(A \cap B) \ge \nu(A) + \nu(B)$. Choquet defined the concept in 1953, with some additional continuity requirements, that are actually satisfied when S is finite.

[11] A behavioural interpretation of capacities (Schmeidler, 1989) leads to representing a wide range of attitudes towards ambiguity (as concavity with $\le$ instead of $\ge$ in the expression above may represent ambiguity loving).

[12] The model is robust to a non-constant c if the conditional capacities have given values. The non-conditional capacities can be deduced from these values. But, if c is non constant, the symmetry required from a fair game disappear and then the formalization of the random walk. Moreover the calculus is no more tractable!

decumulative distribution function of capacity ν. The decumulative function of capacity ν is obtained by iteration:

$$\forall t = 2, \ldots, T, \forall n = 1, \ldots, t, \nu\left(s_t^1, \ldots, s_t^n\right) = c\nu\left(s_{t-1}^1, \ldots, s_{t-1}^n\right) + (1-c)\nu\left(s_{t-1}^1, \ldots, s_{t-1}^{n-1}\right) \tag{3.1}$$

and $\nu(s_1^1) = c$.

The closed form of the decumulative function is:

$$\forall t = 1, \ldots, T, \forall n = 1, \ldots, t\text{–}1, \qquad \nu\left(s_t^1, \ldots s_t^n\right) = c^{t-n+1}\sum_{j=0}^{n-1}\binom{j}{t-n+j}(1-c)^j. \tag{3.2}$$

Proof. See Appendix 4.

Remark. The binomial tree should be "path independent" (for example, "up then down" nodes correspond to "down then up" nodes). Consequently $n = t$ (i.e., the number of nodes equals the date at each period). Indeed with a path dependent lattice, it is not possible to determine a unique capacity as in Proposition 2, because, in this case, the dynamic consistency formula leads to different values at time t from the following *up* and *down* nodes.

Proposition 4. *The Choquet expectation of the payoffs at date t of a symmetric Choquet random walk is:*

$$\forall t = 0, \ldots, T, \qquad E(X_t) = t(2c\text{–}1)\,. \tag{4}$$

Proof. See Appendix 5.

Remarks. $c < 1/2 \Rightarrow E(X_t) < 0$. This is consistent with ambiguity aversion as it results in the attribution of a negative expected value to a fair game. Furthermore, other symmetric random walks can be obtained from this one by a positive affine transformation. The Choquet integral is linear with respect to such a transformation.

For $\forall\, t = 0,\ldots,T, Y_t = a\,X_t + b, a > 0$, we have: $E(Y_t) = a\,t\,(2c - 1) + b$.

As a result, if required by modelling considerations, we can address cases where the mean is nonzero and allow volatility to vary.

It may be useful to clarify the link between dynamically consistent Choquet random walks and the recursive multiple-priors model of Epstein and Schneider (2003). For a convex or concave capacity, as shown above, it is easy to exhibit the possible conditional one-step-ahead priors, making our model recursive. Moreover, we have shown that if c $\leq$ 1/2, the convex core of capacity ν is the convex and rectangular set of priors in the multiple-priors model.

The literature on dynamic risk measures is also very close to this method. Risk is measured, at time t, by the maximum of the expectations minus the sum of discounted cash-flows, with respect to the probabilities in a closed and convex set, conditional on information at time t. This result is obtained, given certain properties: coherence, dynamic consistency and relevancy (see Riedel, 2004, Theorem 1). The crucial one, consistency, is equivalent to rectangularity in Epstein and Schneider's model (2003).

3. Convergence to Choquet–Brownian motion

Choquet random walks can be used in discrete-time modelling in the presence of ambiguity, but we consider their expansion to continuous time. In this section, we show that when the time interval converges to zero, the Choquet binomial model converges to a deformed Brownian motion, where both the drift and the volatility are modified, in contrast with the Chen and Epstein (2002) recursive multiple-priors model, where only the volatility is modified.

Let us first characterise the variations of the *decumulative* function of the previously defined dynamically consistent Choquet random walk.

Proposition 5.

$$\forall t = 1, \ldots, T, \forall n = 1, \ldots, t+1,$$

$$\begin{aligned}&\Delta\nu_{t+1}^n = c\Delta\nu_t^n + (1-c)\Delta\nu_t^{n-1}\\&\Delta\nu_t^n \equiv \nu\left(s_t^1, \ldots s_t^n\right) - \nu\left(s_t^1, \ldots s_t^{n-1}\right) = \binom{n-1}{t}c^{t-n+1}(1-c)^{n-1}\end{aligned} \tag{5}$$

where we set: $\nu(s_t^0) = 0$.

Proof. See Appendix 6.

Proposition 6. *When the time interval converges to 0, the symmetric random walk, defined by Eq. (5), converges to a general Wiener process, with mean m = 2c − 1 and variance $s^2 = 4c(1-c)$.*

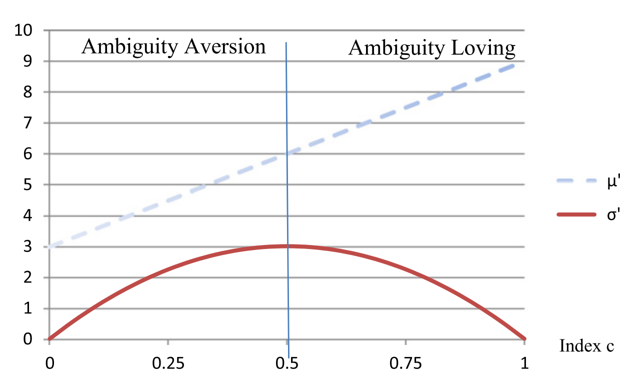
Proof. See Appendix 7.

3.1. Discussion of properties and sensitivity analysis

The resulting Brownian motion exhibits some interesting properties. Note that if $c < 1/2$, then $m < 0$ and $s^2 < 1$: both the mean and the variance are smaller than in the probabilistic model. Indeed, ambiguity aversion yields smaller weights on the up and down movements ($c < 1/2$), for given value increments $(+1, -1)$; hence, the variance is smaller.

The sensitivity of the Choquet–Brownian motion with respect to the main value drivers may be further illustrated through a numerical application. Let us recall first that an *increase* in aversion (or "love") towards perceived ambiguity in our setting means that the value of parameter c is going further away from its central key anchor 1/2 (corresponding to the limit probabilistic case, that of an absence of ambiguity). Possible deviations are consequently confined in a range and c represents the index of the intensity and nature of the attitude towards perceived ambiguity.

Suppose B is a standard Brownian motion and μ' and σ' are some real numbers, with $\mu' = \mu + m\,\sigma, \sigma' = s\,\sigma, m = 2\,c - 1, s^2 = 4\,c\,(1-c), \mu > 0$ and $\sigma > 0$. In the graph below, we see how a change in the value of parameter c impacts both the drift μ' and volatility σ' of the Choquet–Brownian motion (where μ and σ correspond to the initial values for the drift and volatility before the introduction of ambiguity). The contrast between the nature of the impact on the drift and on the volatility appears clearly.



Sensitivity of key drivers to a change in ambiguity preferences (based on initial values $\mu = 2$ and $\sigma = 5$).

Remark. The absence of ambiguity (or the neutrality towards ambiguity) is the case where: $c = 0.5$ (obviously in that case in the numerical illustration above $\mu' = \mu = 2$ and $\sigma' = \sigma = 5$).

Before applying this model (in Section 4) and comparing it with the multiple-priors model (in Section 5), we conclude this section with a remark on the convergence of Choquet–Brownian motions. The notion of "stochastic convergence" formalises the idea that a sequence of essentially random or unpredictable events can sometimes be expected to settle into a pattern. However, the notion of convergence of a distribution is very frequently used in practice; most often, it arises from application of the central limit theorem. See Billingsley (1999) and Jacod and Shiryaev (2002) for detailed discussions of convergence concepts.

Let us emphasise that, in our representation, in a cone with comonotonic variables, we obtain a probabilistic representation and thus can use the traditional results on statistical convergence. Furthermore, observe that, in the multiple-priors model, Chen and Epstein avoid addressing convergence directly by employing, in continuous-time, an analogy with discrete-time. This has not prevented widespread use of the multiple-priors approach in economic and financial modelling. Finally, our representation is close to that of the multiple-priors model, with the symmetrical case of concavity of capacities added to the usual case of convexity.

4. Applications

Following earlier versions of this paper, a few applications of Choquet–Brownian motion have been proposed, showing significant and contrasting effects of ambiguity. They have addressed real investment decisions (the optimal timing and valuation of real options, as in Roubaud et al. (2010)) and the optimal timing of environmental policies (Agliardi and Sereno, 2011). The former extends the real options theory and method (Dixit and Pindyck, 1994; Trigeorgis, 1996) by modelling ambiguous cash flows expected from an investment project.

Agliardi and Sereno use Choquet–Brownian ambiguity to explore the optimal timing of environmental policies, such as taxes and non-tradable quotas, in the presence of ambiguity about the future costs and benefits of such policies.

In this paper, we now suggest the application of our model to the optimal portfolio choice of traded assets. We consider a stationary version of the inter-temporal Capital Asset Pricing Model (Merton, 1969, 1971, 1973).

Let $(w(t))_{0 \le t \le T}$, the wealth of the investor, which has to be allocated between a riskless asset with constant instantaneous rate of return r, $r > 0$, and a risky asset the price of whom follows a Choquet–Brownian motion:

$$dP(t) = m' P(t)dt + \sigma' P(t)dB(t) \tag{6}$$

with $P(0) > 0$, where B is a standard Brownian motion and μ' and σ' are some real numbers, with $\mu' = \mu + m\,\sigma$, $\sigma' = s\,\sigma$, $m = 2\,c - 1$, $s^2 = 4\,c\,(1 - c)$, $\mu > 0$ and $\sigma > 0$.

$(x(t))$ is the part of the capital invested in the risky asset at date t and then the following Stochastic differential equation characterises the agent's wealth:

$$\frac{\mathrm{d}w(t)}{w(t)} = x(t)\frac{\mathrm{d}P(t)}{P(t)} + [1-x(t)]r\mathrm{d}t. \tag{7}$$

The programme of the agent for a time horizon T is to maximise the expected utility of his final wealth:

$$\underset{(x(t))_{0 \le t \le T}}{Max} \quad E_t[u(w(T))], w(0)>0 \tag{8}$$

where $u(.)$ is an increasing and concave utility function, with respect to the following SDE straightforwardly obtained with Eqs. (6) and (7):

$$\mathrm{d}w(t) = \left[r + x(t)\left(\mu' - r\right)\right]w(t)\mathrm{d}t + \sigma' x(t)w(t)\mathrm{d}B(t). \tag{9}$$

In the well-known iso-elastic case: $u(w) = \frac{w^{1-\alpha}}{1-\alpha}$, $\alpha > 0$, $\alpha \neq 1$, where the constant α is the relative risk aversion coefficient, the optimal solution is a constant:

$$x^*(t, w) = x' = \frac{1}{\alpha}\left(\frac{\mu' - r}{\sigma'^2}\right). \tag{10}$$

If $0 < c < 1/2$, we have $\mu - \sigma < \mu' < \mu$ and $0 < \sigma' < \sigma$, and if $1/2 < c < 1$, we have $\mu < \mu' < \mu + \sigma$ and $0 < \sigma' < \sigma$.

If x characterises the optimal solution without ambiguity ($c = 1/2$), it is easy to check that:

$$x' < x \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < -\frac{m}{1 - s^2} = \frac{1}{1 - 2c}.$$

The value of the market price of risk λ relatively to the ambiguity parameter c gives the hierarchy between investment in the risky asset with or without ambiguity.

Similarly:

$$\frac{\partial x'}{\partial c} > 0 \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < \frac{1 - 2c + 2c^2}{1 - 2c}.$$

Let us first underline that the impact of risk aversion itself on the optimal choice is straightforward: the quantity of risky asset is inversely proportional to the relative risk aversion coefficient α in relation (10).

Next, and in contrast with this linear impact of risk aversion, the impact of the attitude towards ambiguity is not so direct. Indeed, for an ambiguity averse decision maker, the fact that investment in the risky asset is increasing with the reduction of ambiguity (when c increases towards 1/2, the ambiguity decreases) depends on the value of λ, the market price of risk.

Indeed, in the case of ambiguity aversion, if the market price of risk λ is relatively small, the quantity of risky (and ambiguous) asset is also decreasing with the ambiguity aversion. But if the market price of risk is relatively important, the quantity of risky asset is greater than in the absence of ambiguity and, furthermore, increasing with the ambiguity aversion. In this case, there is some trade-off between risk and ambiguity.

Finally, models where ambiguity has only one type of effect may be somewhat restrictive. The results of the various applications of Choquet–Brownian ambiguity differ from those of the recursive multiple-priors model of ambiguity to real options (Nishimura and Osaki, 2007) and to the continuous-time capital asset pricing model (Chen and Epstein, 2002). The effect of ambiguity is not straightforward, as reflected in the deformations of both the mean and the variance in our Choquet expectations model.

5. Conclusion

In summary, we have shown that with a less-restrictive model than the multiple-priors model, and even in the case of ambiguity aversion ($c < 1/2$), the effects of ambiguity on optimal decisions may vary. We thereby contribute to a growing body of literature focusing on the ambiguity that characterises uncertain prospects.

Expanding economic and financial models to incorporate the many substantial sources of uncertainty that characterise them remains challenging, in spite of significant breakthroughs in the decision sciences over the last two decades. Even if a typical investor may often be described as conservative and cautious, ambiguity-seeking is also a reality. Adopting Choquet–Brownian ambiguity avoids an early reduction of the range of possible preferences regarding ambiguity. This may be an interesting feature in contexts where aversion may not be the only acceptable attitude toward ambiguity.

In addition to the applications presented in Section 4 to portfolio choice, real options theory and environmental policy, our approach has possible applications to any model in which ambiguity is present, for example, to macroeconomic "uncertainty models," where central banks do not know the "true" economic model. Expansion of preferences to a wider range of preferences also opens the way for applications to entrepreneurial models or to the study of contractual arrangements under ambiguity. To conclude, Choquet–Brownian ambiguity can be applied to a rich range of issues where beliefs cannot be represented by standard probability measures. Choquet–Brownian ambiguity may thus usefully complement other models of utility in stochastic continuous-time settings.

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Appendix 1. Axiomatic basis for Choquet random walks

In this appendix, we establish the key axioms and properties of Choquet random walks (for a more extensive presentation please refer to Kast and Lapied, 2010a). Suppose a decision maker is aware of its preferences over payoffs contingent on uncertain states, i.e., measurable functions from S to R. These preferences may satisfy Dieccidue and Wakker (2002) axioms and be represented by a value function on the set of uncertain payoffs:

Axiom 1. Preferences define a *complete pre-order* on the set of measurable functions.

Axiom 2. For any measurable function X, there exists a constant number (constant equivalent), for which the decision maker is indifferent to the function, i.e.,

$$X \approx E(X) (\text{where} \approx \text{represents indifference}).$$

Axiom 3. Preferences allow no comonotonic Dutch Books.

A comonotonic Dutch Book is a finite sequence of random variable pairs f_i, g_i with the f_i's and the g_i's in a comonotonic set and such that: $\forall i, f_i \underset{\approx}{\succ} g_i$ and $\forall s \in S, \sum_i f_i(s) < \sum_i g_i(s)$.

1.1. Interpretation

At each "step", $X \succcurlyeq X'$ but at the end of all steps the result is actually the following: $X' \succ X$. Mimicking the case of standard Dutch books, after a series of "successive" bets, the DM is losing in all possible situations.

Then, according to Dieccidue and Wakker (2002):For a preference relation on R^S satisfying Axioms 1 and 2, for all X in R^S, there exists a constant equivalent $E(X) \in R$, such that the following three statements are equivalent:

(i) $E(.)$: $R^S \to R$ are strictly monotonic, additive on comonotonic vectors (but not necessarily additive on non-comonotonic vectors).
(ii) There exists a unique capacity such that $E(X)$ is the Choquet integral of X with respect to this measure.
(iii) $E(.)$ is such that Axiom 3 is satisfied.
Otherwise stated: $\exists v$, a unique capacity on $(S, 2^S)$, $X_t \approx E(X_t) = \int_S X_t dv$.

Let us note that, in this representation theorem, the (subjective) capacity aggregates all the decision maker's behaviour: there is no need for utility on payoffs, and the certainty equivalent is merely an expected (subjective) value. Similarly, a decision maker has preferences for present over future consumption (payoffs, here), i.e., preferences over payoffs are contingent on dates. We assume that the decision maker's preferences over certain future payoffs satisfy Koopman's (1972) axioms and are represented by a linear function.

Axiom 4. Preferences define a complete continuous pre-order on the set of measurable functions. Preferences are also strictly monotonic and satisfy separability over dates.

Then, according to Koopman's theorem (1972):
For a preference relation on R^T satisfying Axioms 4 for all X in R^T, there exists a present equivalent $D(X) \in R$ such that:$\exists\pi$, a unique bounded additive measure on $(T, 2^T)$, $X_s \approx D(X_s) = \int_T X_s d\pi$.
Or in terms of preferences:

Axiom 5. Dynamic consistency axiom

$\forall t = 1,\dots, T-1, \forall X, X'$ such that:

$$X_\tau(s) = X'_\tau(s), \forall \tau = 0, \dots, t, \forall s \in S, \left[\forall i_t \in I_t, X \underset{\approx i_t}{\succ} X'\right] \Rightarrow X \underset{\approx}{\succ} X'.$$

Let us clarify the relation between our approach and papers where a weakened axiom of dynamic consistency and of model consistency is considered (e.g Karni and Schmeidler, 1991; Sarin and Waker, 1998).

On the one hand, Sarin and Wakker express dynamic consistency together with the compound lotteries axiom: The valuation of a two-stage lottery and the valuation of the equivalent one-stage lottery is the same. They also add model consistency (named *sequential consistency*).

On the other hand, for Karni–Schmeidler dynamic consistency is the preservation of preferences between the first period and second period, when the counterfactual payoffs are the same for the two lotteries.

In contrast, our expression of dynamic consistency is the same as in Nishimura and Ozaki (2003): when a lottery is preferred to another at the first stage, *for any possible information set*, then, it is preferred at the second stage.

Consequently our hypothesis is not weaker nor stronger than Karni and Schmeidler's hypothesis: it is not weaker because the set of lotteries considered by Karni and Schmeidler (with identical counterfactual payoffs) is included in the set of lotteries we consider. It is not stronger because there is not equivalence between the preferences at period one and period two, but an implication from period one to period two.

Axiom 6. Model consistency axiom

Preferences conditional on information satisfy Axioms 1–3 and the hierarchy axiom between time and uncertainty.

If the criterions for time and/or uncertainty are not linear, the discounted expectation (DE) and the expected discounting (ED) are not equivalent. Here, discounting is linear and not expectation (Choquet expectation). In this case, the appropriate method is DE (Cf. Kast and Lapied, 2010b).[13]

We can now state the following condition expressing consistency between conditional and unconditional expectations.

[13] The Hierarchy axiom between time and uncertainty may be expressed as such: preferences of the decision maker over payoffs contingent on $\Omega = S \times T$, are represented by $V = DE$. The subjective product measure $\upsilon \times \pi$ on $S \times T$ captures the decision maker's behaviour both on uncertainty and on time.

Proposition 1. *Under the representation of preferences satisfying our six axioms, for any* $X \in R^{S\times T}$, $\forall \tau \in \{0,\dots,T\}$, $\forall i \in I$, $\forall [Y_\tau = i] \subset F_\tau$, $\forall t, \tau \le t \le T$, $E_\nu(X_t) = E_\nu\left[E_\nu^{[Y_\tau=i]}(X_t)\right]$.

1.2. Proof of Proposition 1

Under our axioms, preferences are represented by the value functions $V = DE$ and $V^{[Y=i]} = D^{[Y=i]}E^{[Y=i]}$, such that for any $X \in R^{S\times T}$, we can write:

$$V(X) = DE(X) = \sum_{t=0}^{T} \pi(t)E_\nu(X_t) \text{ and}$$
$$\forall \tau \in \{0,\dots,T\}, \forall i \in I, \forall [Y_\tau = i] \subset F_\tau, V^{[Y_\tau=i]}(X) = \sum_{t=0}^{T} \pi^{[Y_\tau=i]}(t)E_\nu^{[Y_\tau=i]}(X_t)$$

with: $\forall t \in \{0,\dots,\tau-1\}, \pi^{[Y_\tau=i]}(t) = 0, \pi^{[Y_\tau=i]}(\tau) = 1, \forall t \in \{\tau,\dots,T\}, \pi(t) = \pi(\tau)\pi^{[Y_\tau=i]}(t)$.

Let the certainty equivalent payoffs process be: $EC(X) = (E_\nu(X_0),\dots, E_\nu(X_T))$ and $EC^{[Y_\tau=i]}(X) = \left(X_0,\dots,X_{\tau-1}, E_\nu^{[Y_\tau=i]}(X_\tau),\dots,E_\nu^{[Y_\tau=i]}(X_T)\right)$. We have, by definition: $V(X) = V(EC(X))$ and $\forall i \in I, V^{[Y_\tau=i]}(X) = V^{[Y_\tau=i]}\left(EC^{[Y_\tau=i]}(X)\right)$.

Under the dynamic consistency axiom, the last equality implies:

$$\forall i \in I, V(X) = V\left(EC^{[Y_\tau=i]}(X)\right)$$

Owing to the definition of $\pi^{[Y_\tau=i]}$, this equality simplifies to:

$$\forall \tau \in \{0,\dots,T\}, \forall i \in I, \forall [Y_\tau = i] \subset F_\tau, \forall t, \tau \le t \le T, E_\nu(X_t) = E_\nu\left[E_\nu^{[Y_\tau=i]}(X_t)\right]. \quad \text{QED}$$

We must also normalise the conditional capacities as follows:

$$\nu(\varnothing/s_t) = 0, \nu\left(\left\{s_{t+1}^u, s_{t+1}^d\right\}/s_t\right) = 1, \forall B \in A_{t+1}, \nu(B/s_t) = \nu\left(B \cap \left\{s_{t+1}^u, s_{t+1}^d\right\}/s_t\right).$$

Hence, from Proposition 1, dynamic consistency implies:

$$\forall \tau = 1,\dots,T-1, \forall t = \tau,\dots,T, \sum_{s_\tau \in S_\tau}\Big[\sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t/s_\tau)\Big]\Delta\nu(s_\tau) = \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t). \quad (1)$$

1.3. Interpretation

The bracketed expression on the left-hand side of expression (1) represents the conditional expectation of the possible payments of X in the states in t that are possible successors in the tree of events of a state s_τ in τ. They correspond to the values (given by the Choquet criterion) of X_t, seen from τ. Next, we integrate these values according to the possible values of s_τ in τ. We then have a value of 0 for all possible evaluations of X_t, seen from τ. If the decision maker is dynamically coherent, then this value is equal to that of the right-hand side of the expression (that is, the evaluation in 0 of the values of X_t).

Appendix 2. Proof of Proposition 2

We can concentrate without loss of generality on the characteristic functions $X_{\tau+n}$ of the sets in $A_{\tau+n}$ because any random variable has a unique decomposition into a non-negative linear combination of the characteristic functions in a cone containing $X_{\tau+n}$, and the Choquet expectation is linear on this cone.

Three cases are considered.

(i) If $t = \tau$, relation 2 is trivially satisfied.
(ii) If $t = \tau + 1$, relation 2 becomes:

$$\forall t = 1,\dots,T-1 \sum_{s_t \in S_t}\Big[\sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1})\Delta\nu(s_{t+1}/s_t)\Big]\Delta\nu(s_t) = \sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1})\Delta\nu(s_{t+1}). \quad (2.1)$$

For any $t = 1,\dots,T-1$, and any $B \in A_{t+1}$, the conditional capacity $\nu(B/s_t)$ can only take three different values: $(s^u_{t+1} \notin B$ and $s^d_{t+1} \notin B) \Rightarrow \nu(B/s_t) = 0$,

$$\left[\left(s^u_{t+1} \notin B \text{ and } s^d_{t+1} \in B\right) \text{ or } \left(s^u_{t+1} \in B \text{ and } s^d_{t+1} \notin B\right)\right] \Rightarrow \nu(B/s_t) = c,$$

$$\left(s^u_{t+1} \in B \text{ and } s^d_{t+1} \in B\right) \Rightarrow \nu(B/s_t) = 1.$$

For $X_{t+1} = 1_B$, relation (2.1) can be written as:

$$\nu(B) = c\nu\left(\left\{s_t : [s_{t+1}^u \in B] \vee \left[s_{t+1}^d \in B\right]\right\}\right) + (1-c)\nu\left(\left\{s_t : [s_{t+1}^u \in B] \wedge \left[s_{t+1}^d \in B\right]\right\}\right). \quad (2.2)$$

All the capacities at date $t + 1$ are then uniquely determined by capacities at date t.
With $\nu(s_1^1) = \nu(s_1^2) = c$, the set function ν is completely defined and hence unique.
Moreover: $B \subset D \Rightarrow \{s_t : [s^u_{t+1} \in B] \vee [s^d_{t+1} \in B]\} \subset \{s_t : [s^u_{t+1} \in D] \vee [s^d_{t+1} \in D]\}$.
Then, from Eq. (2.2): $B \subset D \Rightarrow \nu(B) \le \nu(D)$, ν is an increasing measure and then a capacity.

(iii) Finally, if $t = \tau + n$, $n > 1$, relation (3.1) partially characterises the conditional capacities $\nu(B/s_\tau)$, where $B \in A_{\tau+n}$. We have then $2^{\tau+n+1}$ equations for the $(\tau + 1) \times 2^{\tau+n+1}$ conditional capacities. Thus, these relations cannot constrain the capacity ν. *QED*

Appendix 3. Proof of Proposition 3

We need only prove that: $[\forall t = 1,\dots,T, \forall B \in A_t, \nu(B) + \nu(B^C) \le 1] \Leftrightarrow c \le 1/2$.

First, if the capacity is *sub-linear* at date 1 for $B = s_1^1$, and $B^C = s_1^2$, then $\nu(B) + \nu(B^C) = 2c \le 1$ implies $c \le 1/2$.

The reciprocal is obtained by induction. Let us assume that $c \le 1/2$ in the sequel:Then, at the first stage, $B \in A_1$. $B = \varnothing$ or $B = S_1$ implies $\nu(B) + \nu(B^C) = 1$, and $B = s_1^1$ or $B = s_1^2$ yields: $\nu(B) + \nu(B^C) = 2c$. The property is then established at date 1.

Suppose now that it is also true at date t and consider some $B \in A_{t+1}$. From relation (2.2), we have:

$$\begin{aligned}\nu(B) + \nu\left(B^C\right) &= c\nu\left(\left\{s_t : [s_{t+1}^u \in B] \vee [s_{t+1}^d \in B]\right\}\right) \\ &+ (1-c)\nu\left(\left\{s_t : [s_{t+1}^u \in B] \wedge [s_{t+1}^d \in B]\right\}\right) \\ &+ c\nu\left(\left\{s_t : \left[s_{t+1}^u \in B^C\right] \vee [s_{t+1}^d \in B^C]\right\}\right) \\ &+ (1-c)\nu\left(\left\{s_t : [s_{t+1}^u \in B^C] \wedge [s_{t+1}^d \in B^C]\right\}\right).\end{aligned}$$

With the notations: $\overline{D} = \left\{s_t : [s_{t+1}^u \in D] \vee [s_{t+1}^d \in D]\right\}$ and $\underline{D} = \left\{s_t : [s_{t+1}^u \in D] \wedge [s_{t+1}^d \in D]\right\}$, it follows that: $\nu(B) + \nu\left(B^C\right) = c\left[\nu(\overline{B}) + \nu\left(\overline{B^C}\right)\right] + (1-c)\left[\nu(\underline{B}) + \nu\left(\underline{B^C}\right)\right]$.

We have: $\overline{B^C} = \underline{B}^C$ and $\underline{B^C} = \overline{B}^C$ therefore: $\nu(B) + \nu\left(B^C\right) = c\left[\nu(\overline{B}) - \nu(\underline{B}) + \nu\left(\underline{B}^C\right) - \nu\left(\overline{B}^C\right)\right] + \nu(\underline{B}) + \nu\left(\overline{B}^C\right)$.

With $\underline{B} \subset \overline{B} \Rightarrow \nu(\overline{B}) - \nu(\underline{B}) \geq 0$, and $\overline{B}^C = \underline{B^C} \subset \overline{B^C} = \underline{B}^C \Rightarrow \nu\left(\underline{B}^C\right) - \nu\left(\overline{B}^C\right) \geq 0$, and because $c \leq 1/2$, it follows that: $\nu(B) + \nu\left(B^C\right) \leq \frac{1}{2}\left[\nu(\overline{B}) + \nu\left(\overline{B}^C\right) + \nu(\underline{B}) + \nu\left(\underline{B}^C\right)\right]$.

As $D \in A_{t+1} \Rightarrow [\overline{D} \in A_t \wedge \underline{D} \in A_t]$, by hypothesis: $\nu(\overline{B}) + \nu\left(\overline{B}^C\right) \leq 1, \nu(\underline{B}) + \nu\left(\underline{B}^C\right) \leq 1$, and then: $\nu(B) + \nu(B^C) \leq 1$.*QED*.

Appendix 4. Proof of relation 3.1

We have:

$$\forall i = 1, \ldots, n-1, \nu\left(s_t^1, \ldots, s_t^n / s_{t-1}^i\right) = 1, \nu\left(s_t^1, \ldots, s_t^n / s_{t-1}^n\right) = c, \forall j = n+1, \ldots, t, \nu\left(s_t^1, \ldots, s_t^n / s_{t-1}^i\right) = 0.$$

Now consider the following relation: $\forall t = 1, \ldots, T-1, \sum_{s_t \in S_t} [\sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1}) \Delta\nu(s_{t+1}/s_t)] \Delta\nu(s_t) = \sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1}) \Delta\nu(s_{t+1})$.

If we apply this relation to $X = 1_{s_t^1 \cup \ldots \cup s_t^n}$, it follows that:

$$\nu\left(s_t^1, \ldots, s_t^n\right) = c\nu\left(s_{t-1}^1, \ldots, s_{t-1}^n\right) + (1-c)\nu\left(s_{t-1}^1, \ldots, s_{t-1}^{n-1}\right). \tag{3.1}$$

The closed form of the decumulative function is:

$$\forall t = 1, \ldots, T, \forall n = 1, \ldots, t\text{-}1, \qquad \nu\left(s_t^1, \ldots s_t^n\right) = c^{t-n+1} \sum_{j=0}^{n-1} \binom{j}{t-n+j} (1-c)^j. \tag{3.2}$$

If we place the expression for the decumulative function given by Eq. (3.2) on the right-hand side of Eq. (3.1), we have:

$$\begin{aligned}
& c\nu\left(s_{t-1}^1, \ldots, s_{t-1}^n\right) + (1-c)\nu\left(s_{t-1}^1, \ldots, s_{t-1}^{n-1}\right) \\
&= c[c^{t-n} \sum_{j=0}^{n-1} \binom{j}{t-n+j-1} (1-c)^j] + (1-c)[c^{t-n+1} \sum_{j=0}^{n-2} \binom{j}{t-n+j} (1-c)^j] \\
&= c^{t-n+1} [\sum_{j=0}^{n-1} \binom{j}{t-n+j-1} (1-c)^j + \sum_{j=0}^{n-2} \binom{j}{t-n+j} (1-c)^{j+1}] \\
&= c^{t-n+1} [1 + \sum_{j=1}^{n-1} \binom{j}{t-n+j-1} (1-c)^j + \sum_{j=1}^{n-1} \binom{j-1}{t-n+j-1} (1-c)^j] \\
&= c^{t-n+1} \{1 + \sum_{j=1}^{n-1} \left[\frac{(t-n+j-1)!}{j!(t-n-1)!} + \frac{(t-n+j-1)!}{(j-1)!(t-n)!}\right] (1-c)^j\} \\
&= c^{t-n+1} [\binom{0}{t-n} (1-c)^0 + \sum_{j=1}^{n-1} \frac{(t-n+j)!}{j!(t-n)!} (1-c)^j] \\
&= c^{t-n+1} \sum_{j=0}^{n-1} \binom{j}{t-n+j} (1-c)^j \\
&= \nu\left(s_t^1, \ldots, s_t^n\right).
\end{aligned}$$

Therefore, relation (3.2) satisfies relation (3.1).

Appendix 5. Proof of Proposition 4

The payoffs of X at date t are: $X(s_t^1) = t, X(s_t^2) = t-2, \ldots, X(s_t^t) = -t + 2, X(s_t^{t+1}) = -t$.

Their Choquet expectation is then:

$$\begin{aligned}
E(X_t) &= -t[1 - \nu\left(s_t^1, \ldots, s_t^t\right)] - (t-2)\left[\nu\left(s_t^1, \ldots, s_t^t\right) - \nu\left(s_t^1, \ldots, s_t^{t-1}\right)\right] \\
&\quad + \ldots + (t-2)[\nu\left(s_t^1, s_t^2\right) - \nu\left(s_t^1\right)] + t\nu\left(s_t^1\right) \\
&= -t + 2[\nu\left(s_t^1, \ldots, s_t^t\right) + \nu\left(s_t^1, \ldots, s_t^{t-1}\right) + \ldots + \nu\left(s_t^1\right)].
\end{aligned}$$

Relation (3.1) implies:

$$\begin{aligned}
\nu\left(s_t^1, \ldots, s_t^t\right) + \nu\left(s_t^1, \ldots, s_t^{t-1}\right) + \ldots + \nu\left(s_t^1\right) &= c[\nu\left(s_{t-1}^1, \ldots, s_{t-1}^t\right) \\
& - \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-1}\right) \\
& + \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-1}\right) - \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-2}\right) + \ldots \\
& + \nu\left(s_{t-1}^1, s_{t-1}^2\right) - \nu\left(s_{t-1}^1\right) + \nu\left(s_{t-1}^1\right)] \\
& + \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-1}\right) + \ldots + \nu\left(s_{t-1}^1\right) \\
& = c + \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-1}\right) + \ldots + \nu\left(s_{t-1}^1\right).
\end{aligned}$$

Remark. As $\nu(s_{t-1}^1, \ldots, s_{t-1}^t) = 1$, the right-hand side of the equation can be simplified as indicated.

It follows that:

$$\begin{aligned}
E(X_t) &= -t + 2[c + \nu\left(s_{t-1}^1, \ldots, s_{t-1}^{t-1}\right) + \ldots + \nu\left(s_{t-1}^1\right)] \\
&= -t + 2[c + c + \nu\left(s_{t-2}^1, \ldots, s_{t-2}^{t-2}\right) + \ldots + \nu\left(s_{t-2}^1\right)] \ldots \\
&= -t + 2[(t-1)c + \nu\left(s_1^1\right)] = t(2c-1). \qquad QED
\end{aligned}$$

Appendix 6. Proof of Proposition 5

Relation (5) is true for $t = 1$, and we suppose that it holds for a given t ($t \leq T$). With relation (3.1), we have:

$$\begin{aligned}
\Delta\nu_{t+1}^n &= c\nu\left(s_t^1, \ldots s_t^n\right) + (1-c)\nu\left(s_t^1, \ldots s_t^{n-1}\right) - c\nu\left(s_t^1, \ldots s_t^{n-1}\right) \\
&\quad - (1-c)\nu\left(s_t^1, \ldots s_t^{n-2}\right) \\
&= c[\nu\left(s_t^1, \ldots s_t^n\right) - \nu\left(s_t^1, \ldots s_t^{n-1}\right)] + (1-c)[\nu\left(s_t^1, \ldots s_t^{n-1}\right) \\
&\quad - \nu\left(s_t^1, \ldots s_t^{n-2}\right)] \\
&= c\Delta\nu_t^n + (1-c)\Delta\nu_t^{n-1} \\
&= c\binom{n-1}{t} c^{t-n+1} (1-c)^{n-1} + (1-c)\binom{n-2}{t} c^{t-n+2} (1-c)^{n-2} \\
&= \left[\binom{n-1}{t} + \binom{n-2}{t}\right] c^{t-n+2} (1-c)^{n-1} \\
&= \binom{n-1}{t+1} c^{t-n+2} (1-c)^{n-1}.
\end{aligned}$$

Relation (5.1) is then satisfied for $t + 1$ and, by induction, for any t.

Remark. To interpret formula (5.1), let us recall that on a comonotonic cone,[14] a capacity is represented by a particular probability distribution. In the present case, the distribution is a standard binomial distribution, with parameters such that $B(T, p)$ is in the core of capacity ν.

[14] A *comonotonic cone* is the set of the random variables obtained from the positive linear combinations of the set of characteristic functions that are comonotonic two by two.

There is then a clear link with the multiple-priors approach. Indeed, for a convex capacity($c < 1/2$), the core is given by:

$$\text{Core } \nu = \{\mu, \text{probability distribution}/\mu \geq \nu\}; \text{and over one period}: [p = \mu(s^u_{t+1}/s_t) \geq \nu(s^u_{t+1}/s_t) = c, 1-p = \mu\left(s^d_{t+1}/s_t\right) \geq \nu\left(s^d_{t+1}/s_t\right) = c] \Rightarrow p \in [c, 1-c].$$

For a symmetric random walk, $E_\mu[X_{t+1} - X_t/s_t] = 2p - 1$, so that the *maxmin* criterion yields: $\underset{\mu \in core(\nu)}{\text{ArgMin}}\ E_\mu(X_{t+1} - X_t/s_t) = (c, 1-c)$. This probability distribution, applied in each period, yields the binomial distribution corresponding to formula (5.1).

Appendix 7. Proof of Proposition 6

Take a time interval $[0, T]$, where the number of periods in the interval is N, and the length of each period is $h = T/N$. Recall that, in discrete time, X_n is independent of Y_n and that the Y_n's are independent[15]. Moreover, with Eq. (5.1), $X_n = X_{n-1} + Y_n$, in which Y_n takes a value of 1 with probability c and a value of -1 with probability $1 - c$.

Then: $E[Y_n] = 2c - 1 = m$, $Var[Y_n] = 4c(1 - c) = s^2$.

If we define a discrete time process W_n by: $W_n = m\,h + s\,h^{1/2}\,U_n$, where U_n takes a value of 1 with probability 1/2 and a value of -1 with a probability 1/2, then $E[W_n] = m\,h$, $Var[W_n] = s^2\,h$.

A standard result for Brownian motion is that W_n converges to a general Wiener process in continuous time $t \in [0, T]$: $W(t) = m\,t + s\,B(t)$, where $B(t)$ is the Brownian motion.

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[15] Note that we are here in the probabilistic version of the capacity (as we are on a comonotonic cone, which implies the existence of a unique probabilistic representation of the capacity). Therefore, in this proof, independence is understood in the classical (probabilistic) sense of the notion. We are dealing with a Markovian process.

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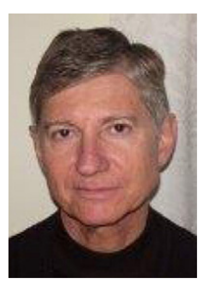

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