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MAXIMUM PRINCIPLES, SLIDING TECHNIQUES AND APPLICATIONS TO NONLOCAL EQUATIONS

JÉRÔME COVILLE

Abstract. This paper is devoted to the study of maximum principles holding for some nonlocal diffusion operators defined in (half-) bounded domains and its applications to obtain qualitative behaviors of solutions of some nonlinear problems. It is shown that, as in the classical case, the nonlocal diffusion considered satisfies a weak and a strong maximum principle. Uniqueness and monotonicity of solutions of nonlinear equations are therefore expected as in the classical case. It is first presented a simple proof of this qualitative behavior and the weak/strong maximum principle. An optimal condition to have a strong maximum for operator $\mathcal{M}[u] := J \star u - u$ is also obtained. The proofs of the uniqueness and monotonicity essentially rely on the sliding method and the strong maximum principle.

1. Introduction and Main results

This article is devoted to maximum principles and sliding techniques to obtain the uniqueness and the monotone behavior of the positive solution of the following problem

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \text{in } \Omega \\ u = u_0 \quad \text{in } \mathbb{R} \setminus \Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a domain, J is a continuous non negative function such that $\int_{\mathbb{R}} J(z)\,dz = 1$ and f is a Lipschitz continuous function.

Such problem arises in the study of so-called *Traveling Fronts* (solutions of the form $u(x,t) = \phi(x+ct)$) of the following nonlocal phase-transition problem

$$\frac{\partial u}{\partial t} - (J \star u - u) = f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \tag{1.2}$$

The constant c is called the speed of the front and is usually unknown. In such model, $J(x-y)dy$ represent the probability of an individual at the position y to migrate to the position x, then the operator $J \star u - u$ can be viewed as a diffusion operator. This kind of equation was initially introduced in 1937 by Kolmogorov,

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Petrovskii and Piskunov [23, 18] as a way to derive the Fisher equation (i.e (1.3) below with $f(s) = s(1-s)$)

$$\frac{\partial U}{\partial t} = U_{xx} + f(U) \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1.3}$$

In the literature, much attention has been drawn to reaction-diffusion equations like (1.3), as they have proved to give a robust and accurate description of a wide variety of phenomena, ranging from combustion to bacterial growth, nerve propagation or epidemiology. For more information, we point the interested reader to the following articles and reference therein: [2, 4, 5, 18, 20, 22, 23, 24, 30].

Equation (1.1) can be seen as a nonlocal version of the well known semi-linear elliptic equation

$$\begin{gathered} u'' - cu' + f(u) = 0 \quad \text{in } \Omega, \\ u = u_0 \quad \text{in } \partial\Omega. \end{gathered} \tag{1.4}$$

When $\Omega = (r, R)$, it is well known [6, 7, 28] that the positive solution of (1.4) is unique and monotone provided that $u_0(r) \neq u_0(R)$ are zeros of f. More precisely, assume that $u_0(r) = 0$ and $u_0(R) = 1$ are respectively a sub and a super-solution of (1.4), then

Theorem 1.1 ([6, 7, 28])**.** *Any smooth solution u of*

$$\begin{gathered} u'' - cu' + f(u) = 0 \quad \textit{in } (r,R) \\ u(r) = 0, \quad u(R) = 1. \end{gathered} \tag{1.5}$$

is unique and monotone.

Remark 1.2. The above theorem holds as well, if you replace 0 and 1 by any constant sub and super-solution of (1.5).

Remark 1.3. Obviously, by interchanging 0 and 1, u will be a decreasing function.

Since, (1.1) shares many properties with (1.4), we expect to obtain similar result. Indeed, assume that $u_0(r) = 0$ and $u_0(R) = 1$ are respectively a sub and a super-solution of (1.1), then one has

Theorem 1.4. *Let $\Omega = (r, R)$ for some real $r < 0 < R$ and let J be such that $[-b, -a] \cup [a, b] \subset \operatorname{supp}(J) \cap \Omega$ for some constant $0 \le a < b$. Then any smooth solution u of*

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \textit{in } (r,R) \\ u(x) = 0 \quad \textit{for } x \le r, \\ u(x) = 1 \quad \textit{for } x \ge R. \end{gathered} \tag{1.6}$$

is unique and monotone.

Observe that in the nonlocal situation, we require more information on the function u since u is explicit outside Ω. This is due to the nature of the considered convolution operator.

For unbounded domain Ω, the situation is more delicate and according to Ω, we need further assumptions on f to characterize positive solutions u of (1.1). Two situations can occur, either $\Omega = \mathbb{R}$ or Ω is a semi infinite interval (i.e. $\Omega = (-\infty, r)$ or $\Omega = (r, +\infty)$ for some real r). In the latter case, assuming that 0 and 1 are a sub and a super-solution of (1.1) and f is non-increasing near the value of 1, the positive solution u of (1.1) is unique and monotone. More precisely, we have the following result.

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Theorem 1.5. *Assume that J satisfies $J(a) > 0$ and $J(b) > 0$ for some reals $a < 0 < b$. Let $\Omega = (r, +\infty)$ for some r and let f be such that f non-increasing near 1. Then any smooth solution u of*

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \text{in } \Omega \\ u(x) = 0 \quad \text{for } x \le r, \\ u(+\infty) = 1, \end{gathered} \tag{1.7}$$

is unique and monotone.

By the notation $u(+\infty)$, I mean $lim_{x\to+\infty}u$. Observe that in this situation, there is no further assumption on Ω. Obviously, as for a bounded domain, sweeping 0 and 1 will changes the monotonic behavior of u provided f is non decreasing near 0.

Remark 1.6. With the adequate assumption on f, Theorem 1.5 holds as well for unbounded domain of the form $\Omega = (-\infty, r)$.

When $\Omega = \mathbb{R}$, problem (1.1) is reduced to the well known convolution equation

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R} \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{gathered} \tag{1.8}$$

which has been intensively studied see [1, 3, 8, 9, 12, 13, 14, 15, 16, 19, 26, 27, 29] and references therein. Observe that in this case, from the translation invariance, we can only expect uniqueness up to translation of the solution. Monotonicity and uniqueness issue has been fully investigate for general bistable and monostable nonlinearities f with prescribed behavior nears 0 and 1 see [1, 3, 8, 9, 13, 14]. We sum up these results in the two following theorems:

Theorem 1.7 ([3, 9, 13] ("bistable"))**.** *Let f be such that $f(0) = 0 = f(1)$ and f non-increasing near 0 and 1. Assume that J is even. Then any smooth solution u of*

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R} \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{gathered} \tag{1.9}$$

is unique (up to translation) and monotone. Furthermore there exists unique couple (u, c) solution of (1.9).

Theorem 1.8 ([8, 26]("monostable"))**.** *Let f be such that $f(0) = f(1) = 0$, $f(s) > 0$ in $(0, 1)$, $f'(0) > 0$ and f non-increasing near 1. Assume that J has compact support and even. Then any smooth solution u of*

$$\begin{gathered} J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R} \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{gathered} \tag{1.10}$$

is unique (up to translation) and monotone.

Remark 1.9. In both situations, bistable and monostable, the behavior of u is governed by the assumption made on f near the value $u(\pm\infty)$.

Remark 1.10. All the above theorems stand if we replace 0 and 1 by any constant α and β which are respectively a sub and super-solution of (1.1).

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1.1. **General comments.** Equation (1.2) appears also in other contexts, in particular in Ising model and in some Lattice model involving discrete diffusion operator. I point the interested reader to the following references for deeper explanations [3, 10, 24, 26, 29].

A significantly part of this paper is devoted to maximum and comparison principles holding for (1.5), (1.6) and some nonlinear operators. I obtain weak and strong maximum principle for those problems. These maximum principles are analogue of the classical maximum principles for elliptic problem that we find in [21, 25].

I have so far only investigate the one dimensional case. Maximum and comparison principles in multi-dimension for various type of nonlocal operators are currently under investigation and appears to be largely be an open question.

As a first consequence of this investigation on maximum principles, I obtain a generalized version of Theorem 1.4. More precisely, I prove the following result.

Theorem 1.11. *Let $\Omega = (r, R)$ for some real $r < 0 < R$, g an increasing function and J be such that $[-b, -a] \cup [a, b] \subset \operatorname{supp}(J) \cap \Omega$ for some constant $0 \le a < b$. Then any smooth solution u of*

$$\begin{gathered} J \star g(u) - cu' + f(u) = 0 \quad \text{in } (r, R) \\ u(x) = 0 \quad \text{for } x \le r, \\ u(x) = 1 \quad \text{for } x \ge R, \end{gathered} \tag{1.11}$$

satisfying $0 < u < 1$ is unique and monotone.

In this analysis, I also observe that provided an extra assumption on J, the proof of Theorem 1.11 holds as well for the nonlinear density depending nonlocal operator

$$\int_{\mathbb{R}} J(\frac{x-y}{u(y)})\, dy - u(x),$$

recently introduced by Cortazar, Elgueta and Rossi [11].

Another consequence of this investigation is the generalization of Theorem 1.7. Indeed, in a previous work [13], I have observe that Theorem 1.7 holds for linear operator L satisfying the following properties:

(1) For all positive functions U, let $U_h(.) := U(. + h)$. Then for all $h > 0$ we have $L[U_h](x) \le L[U](x + h)$ for all $x \in \mathbb{R}$.
(2) Let v a positive constant then we have $L[v] \le 0$.
(3) If u achieves a global minimum (resp.a global maximum) at some point ξ then the following holds:
   - Either $L[u](\xi) > 0$ (resp. $L[u](\xi) < 0$)
   - Or $L[u](\xi) = 0$ and u is identically constant.

Such condition are easily verified by the operator $J \star u - u$ when J is even. In this present note, I establish a necessary and sufficient condition on J to have the above conditions. This may therefore generalized Theorem 1.7 for a new class of kernel.

1.2. **Methods and plan.** The techniques used to prove Theorems 1.4 and 1.5 are mainly based on an adaption to nonlocal situation of the sliding techniques introduced by Berestycki and Nirenberg [6] to obtain the uniqueness and monotonicity of solutions of (1.4). These techniques crucially rely on maximum and comparison principles which hold for the considered operators. In the first two sections, I study

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some maximum principles and comparison principles satisfied by operators of the form:

$$\int_\Omega J(x-y)u(y)\,dy - u \tag{1.12}$$

Then in the last two section, using sliding methods, I deal with the proof of Theorem 1.4 and 1.5.

2. Maximum principles

In this section, I prove several maximum principles holding for integrodifferential operators defined respectively in bounded and unbounded domain. I have divided this section into two subsections each of them devoted to Maximum principles in bounded domains and unbounded domains. We start with some notation that we will constantly use along this paper. Let $\mathcal{L}, \mathcal{S}, \mathcal{M}$ be the following operators:

$$\mathcal{L}[u] := \int_\Omega J(x-y)u(y)dy - u + c(x)u, \quad \text{when } \Omega = (r,R), \tag{2.1}$$

$$\mathcal{S}[u] := \int_\Omega J(x-y)u(y)dy - u, \quad \text{when } \Omega = (r,+\infty) \text{ or } \Omega = (-\infty, r), \tag{2.2}$$

$$\mathcal{M}[u] := \int_{\mathbb{R}} J(x-y)u(y)dy - u := J \star u - u, \tag{2.3}$$

where $J \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\int_{\mathbb{R}} J = 1$ and $c(x) \in C^0(\Omega)$ so that $c(x) \le 0$.

2.1. **Maximum principles in bounded domains.** Along this subsection, Ω will always refer to $\Omega = (r,R)$ for some $r < R$ and $\mathcal{L}$ is defined by (2.1). Let first introduce some functions that use along this subsection. Let α and β be two reals and let h_α^- and h_β^+ be defined by

$$h_\alpha^- := \alpha \int_{-\infty}^r J(x-y)dy, \quad h_\beta^+ := \beta \int_R^\infty J(x-y)dy.$$

My first result is a weak maximum principle for $\mathcal{L}$.

Theorem 2.1 (Weak Maximum Principle)**.** *Let $u \in C^0(\bar\Omega)$ be such that*

$$\mathcal{L}[u] + h_\alpha^- + h_\beta^+ \ge 0 \quad \text{in} \quad \Omega, \tag{2.4}$$

$$u(r) \ge \alpha, \tag{2.5}$$

$$u(R) \ge \beta. \tag{2.6}$$

Then $\max_{\bar\Omega} u \le \max_{\partial\Omega} u^+$. *Furthermore, if* $\max_{\bar\Omega} u \ge 0$ *then* $\max_{\bar\Omega} u = \max_{\partial\Omega} u$.

Remark 2.2. Similarly if

$$\mathcal{L}[u] + h_\alpha^- + h_\beta^+ \le 0 \quad \text{in } \Omega,$$
$$u(r) \le \alpha,$$
$$u(R) \le \beta.$$

Then $\min_{\bar\Omega} u \ge \min_{\partial\Omega} u^-$. And if $\min_{\bar\Omega} u \le 0$ then $\min_{\bar\Omega} u = \min_{\partial\Omega} u$.

Proof of Theorem 2.1. First, let h^- and h^+ be defined by

$$h^- := u(r) \int_{-\infty}^r J(x-y)dy, \quad h^+ := u(R) \int_R^\infty J(x-y)dy.$$

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Next, extend u outside Ω the following way

$$\tilde{u}(x) := \begin{cases} u(x) & \text{in } \Omega, \\ u(r) & \text{in } (-\infty, r), \\ u(R) & \text{in } (r, \infty). \end{cases} \tag{2.7}$$

Observe that in Ω we have:

$$\mathcal{M}[\tilde{u}] + c(x)\tilde{u} = \mathcal{L}[u] + h^- + h^+ \geq (h^- - h^-_\alpha) + (h^+ - h^+_\beta) \geq 0. \tag{2.8}$$

Now observe that if the following inequality holds

$$\max_{\bar{\Omega}} \tilde{u} \leq \max_{\mathbb{R}\setminus\Omega} \tilde{u}^+, \tag{2.9}$$

then from the definition of $\tilde{u}$ we get

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+.$$

So to prove Theorem 2.1, we are reduce to show (2.9). Define

$$\gamma^+ := \max\{u(r), u(R)\},$$

then one has $\max_{\mathbb{R}\setminus\Omega} \tilde{u}^+ \geq \gamma^+$. We argue now by contradiction. Assume that (2.9) does not hold, then $\tilde{u}$ achieves a positive maximum at some point $x_0 \in \Omega$ and $\tilde{u}(x_0) = \max_{\bar{\Omega}} \tilde{u} > \gamma^+$. By definition of $\tilde{u}$, we have $\tilde{u}(x_0) = \max_{\mathbb{R}} \tilde{u} > \gamma^+$. Therefore, at x_0, $\tilde{u}$ satisfies

$$\int_{\mathbb{R}} J(x_0 - y)[\tilde{u}(y) - \tilde{u}(x_0)]\, dy \leq 0, \tag{2.10}$$

$$c(x_0)\tilde{u}(x_0) \leq 0. \tag{2.11}$$

Combining now (2.10)-(2.11) with (2.8) we end up with

$$0 \leq \underbrace{J \star \tilde{u}(x_0) - \tilde{u}(x_0)}_{\leq 0} + \underbrace{c(x_0)\tilde{u}(x_0)}_{\leq 0} \leq 0$$

Therefore,

$$J \star \tilde{u}(x_0) - \tilde{u}(x_0) = \int_{\mathbb{R}} J(x_0 - y)[\tilde{u}(y) - \tilde{u}(x_0)]\, dy = 0.$$

Hence, for all $y \in x_0 - \operatorname{supp}(J)$, $\tilde{u}(y) = \tilde{u}(x_0)$. In particular for all $y \in x_0 - [a, b]$, $\tilde{u}(y) = \tilde{u}(x_0)$ for some $a < b$. We have now the following alternative:

- Either $(\mathbb{R} \setminus \Omega) \cap (x_0 - [a, b]) \neq \emptyset$ and then we have a contradiction since there exits $y \in \mathbb{R}$ such that either $\gamma^+ < \tilde{u}(x_0) = \tilde{u}(y) = u(R) \leq \gamma^+$ or $\gamma^+ < \tilde{u}(x_0) = \tilde{u}(y) = u(r) \leq \gamma^+$.
- Or $(\mathbb{R} \setminus \Omega) \cap (x_0 - [a, b]) = \emptyset$ and then $(x_0 - [a, b]) \subset\subset \Omega$.

In the later case, we can repeat the previous computation at the points $x_0 + b$ and $x_0 + a$ to obtain for all $y \in x_0 - [2a, 2b]$, $\tilde{u}(y) = \tilde{u}(x_0)$. Again we have the alternative:

- Either $(\mathbb{R} \setminus \Omega) \cap (x_0 - [2a, 2b]) \neq \emptyset$ and then we have a contradiction.
- Or $(\mathbb{R} \setminus \Omega) \cap (x_0 - [2a, 2b]) = \emptyset$ and then $(x_0 - [2a, 2b]) \subset\subset \Omega$.

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By iterating this process, since Ω is bounded, we achieve for some positive integer n,

$$(\mathbb{R} \setminus \Omega) \cap (x_0 - [na, nb]) \neq \emptyset$$

and

$$\forall\, y \in x_0 - [na, nb], \quad \tilde{u}(y) = \tilde{u}(x_0),$$

which yields to a contradiction.

In the case, $\max_{\bar{\Omega}} u > 0$, following the above argumentation, we can prove that

$$\max_{\bar{\Omega}} \tilde{u} \leq \max_{\mathbb{R}\setminus\Omega} \tilde{u}. \tag{2.12}$$

Hence,

$$\max_{\partial\Omega} u \leq \max_{\bar{\Omega}} u \leq \max_{\mathbb{R}\setminus\Omega} \tilde{u} = \max_{\partial\Omega} u.$$

□

Remark 2.3. Note that the weak maximum principle will also holds when h_α^- and h_β^+ are replace by any function g^- and g^+ satisfying $h_\alpha^- \geq g^-$ and $h_\beta^+ \geq g^+$.

Remark 2.4. When $c(x) \equiv 0$, the assumption $\max_{\bar{\Omega}} u \geq 0$ is not needed to have

$$\max_{\partial\Omega} u \leq \max_{\bar{\Omega}} u \leq \max_{\mathbb{R}\setminus\Omega} \tilde{u} = \max_{\partial\Omega} u.$$

Indeed it is needed to guaranties that (2.11) holds. When $c(x) \equiv 0$, (2.11) trivially holds.

Next, we give a sufficient condition on J and Ω such that $\mathcal{L}$ satisfies a strong maximum principle. Assume that J satisfies the following conditions

(H1) $\Omega \cap R^+ \neq \emptyset$ and $\Omega \cap \mathbb{R}^- \neq \emptyset$
(H2) There exists $b > a \geq 0$ such that $[-b, -a] \cup [a, b] \subset \operatorname{supp}(J) \cap \Omega$

Then we have the following strong maximum principle

Theorem 2.5 (Strong Maximum principle)**.** *Let $u \in C^0(\bar{\Omega})$ be such that*

$$\begin{gathered} \mathcal{L}[u] + h_\alpha^- + h_\beta^+ \geq 0 \quad \text{in } \Omega \quad (\text{resp. } \mathcal{L}[u] + h_\alpha^- + h_\beta^+ \leq 0 \text{ in } \Omega), \\ u(r) \geq \alpha \quad (\text{resp. } u(r) \leq \alpha), \\ u(R) \geq \beta \quad (\text{resp. } u(R) \leq \beta). \end{gathered}$$

Assume that J satisfies (H1)–(H2) then u may not achieve a non-negative maximum (resp. non-positive minimum) in Ω without being constant and $u(r) = u(R)$.

From these two maximum principles we obtain immediately the following practical corollary.

Corollary 2.6. *Assume that J satisfies (H1)–(H2). Let $u \in C^0(\bar{\Omega})$ be such that*

$$\begin{gathered} \mathcal{L}[u] + h_\alpha^- + h_\beta^+ \geq 0 \quad \text{in } \Omega, \\ u(r) = \alpha \leq 0, \\ u(R) = \beta \leq 0. \end{gathered}$$

Then: Either $u < 0$, Or $u \equiv 0$.

Remark 2.7. Similarly if $\mathcal{L}[u] \leq 0$, $u(r) = \alpha \geq 0$ and $u(R) = \beta \geq 0$ then u is either positive or identically 0.

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The proof of the corollary is a straightforward application of theses two theorems. Now let us prove the strong maximum principle.

Proof of Theorem 2.5. The proof in the other cases being similar, I only treat the case of continuous function u satisfying

$$\begin{gathered}\mathcal{L}[u] + h_\alpha^- + h_\beta^+ \geq 0 \text{ in } \Omega \\ u(r) \geq \alpha \\ u(R) \geq \beta.\end{gathered}$$

Assume that u achieves a non-negative maximum in Ω at x_0. Using the weak maximum principle yields

$$\begin{gathered}u(x_0) = \max\{u(r), u(R)\}, \\ \tilde{u}(x_0) = u(x_0) = \max_{\bar{\Omega}} u = \max_{\partial\Omega} u = \max_{\mathbb{R}\setminus\Omega} \tilde{u},\end{gathered}$$

where $\tilde{u}$ is defined by (2.7). Therefore, $\tilde{u}$ achieves a global non-negative maximum at x_0. To obtain $u \equiv u(x_0)$, we show that $\tilde{u} \equiv \tilde{u}(x_0)$. The later is obtained via a connexity argument.

Let Γ be the set

$$\Gamma = \{x \in \Omega | \tilde{u}(x) = \tilde{u}(x_0)\}.$$

We will show that it is a nonempty open and closed subset of Ω for the induced topology.

Since $\tilde{u}$ is a continuous function then Γ is a closed subset of Ω. Let us now show that Γ is a open subset of Ω. Let $x_1 \in \Gamma$ then $\tilde{u}$ achieves a global non-negative maximum at x_1. Arguing as in the proof of the weak maximum principle, we get

$$J \star \tilde{u}(x_1) - \tilde{u}(x_1) = \int_{\mathbb{R}} J(x_1 - y)[\tilde{u}(y) - \tilde{u}(x_1)]\, dy = 0.$$

Since $\tilde{u}$ achieves a global maximum at x_1, we have for all $y \in \mathbb{R}$, $\tilde{u}(y) - \tilde{u}(x_1) \leq 0$. Therefore, for all $y \in x_1 + \operatorname{supp}(J)$, $\tilde{u}(y) = \tilde{u}(x_1)$. In particular for all $y \in x_1 + [-b, -a] \cup [a, b]$, $\tilde{u}(y) = \tilde{u}(x_1)$. We are lead to consider the following two cases:

- $x_1 + b \in \Omega$: In this case, we repeat the previous computation with $(x_1 + b)$ instead of x_1 to get for all $y \in (x_1 + b) + [-b, -a] \cup [a, b]$, $\tilde{u}(y) = \tilde{u}(x_1)$. Now from the assumption on J and Ω, we have $x_1 + a \in \Omega$. Repeating the previous computation with $x_1 + a$ instead of x_1, it follows that for all $y \in (x_1 + a) + [-b, -a] \cup [a, b]$, $\tilde{u}(y) = \tilde{u}(x_1)$. Combining these two results, yields for all $y \in x_1 + [-b + a, b - a]$, $\tilde{u}(y) = \tilde{u}(x_1)$.
- $x_1 + b \notin \Omega$: In this case, using the assumption on a and b, it easy to see that $x_1 - b$ and $x_1 - a$ are in Ω. Using the above arguments, we end up with for all $y \in (x_1 - b) + [-b, -a] \cup [a, b]$, $\tilde{u}(y) = \tilde{u}(x_1)$ and for all $y \in (x_1 - a) + [-b, -a] \cup [a, b]$, $\tilde{u}(y) = \tilde{u}(x_1)$. Again combining these two results yields to for all $y \in x_1 + [-b + a, b - a]$, $\tilde{u}(y) = \tilde{u}(x_1)$.

From both cases we have $\tilde{u}(y) = \tilde{u}(x_1)$ on $(x_1 + (-(b - a), (b + a))) \cap \Omega$, which implies that Γ is an open subset of Ω. $\square$

Remark 2.8. Observe that the strong maximum principle relies on the possibility of "covering" Ω with closed sets.

When $h_\alpha^- + h_\beta^+$ has a sign, we can improve the strong maximum principle. Indeed, in that case we have the following result.

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Theorem 2.9. *Let $u \in C^0(\bar{\Omega})$ be such that*

$$\mathcal{L}[u] \geq 0 \quad \text{in } \Omega \quad (\text{resp. } \mathcal{L}[u] \leq 0 \text{ in } \Omega). \tag{2.13}$$

Assume that J satisfies (H1)–(H2) then u cannot achieve a non-negative maximum (resp. non-positive minimum) in Ω without being constant.

Proof. The proof follows the lines of Theorem 2.5. Since $\int_{\mathbb{R}} J(z)\,dz = 1$, we can rewrite $\mathcal{L}[u]$ the following way

$$\mathcal{L}[u] = \int_r^R J(x-y)[u(y)-u(x)]dy + \widetilde{c}(x)u, \tag{2.14}$$

where $\widetilde{c}(x) = c(x) - h_1^- - h_1^+ \leq c(x) \leq 0$. Therefore, if u achieves a non-negative maximum at x_0 in Ω, then we have at this maximum

$$0 \leq \underbrace{\int_r^R J(x_0-y)[u(y)-u(x_0)]\,dy}_{\leq 0} + \underbrace{\widetilde{c}(x_0)u(x_0)}_{\leq 0} \leq 0$$

and in particular

$$\int_r^R J(x_0-y)[u(y)-u(x_0)]\,dy = 0. \tag{2.15}$$

We now argue as in Theorem 2.5 to obtain $u \equiv u(x_0)$. □

2.2. **Maximum principles in unbounded domains.** In this subsection, I deal with maximum principles in unbounded domains. Along this section, Ω will refer to $(r, +\infty)$ or $(-\infty, r)$ for some $r \in \mathbb{R}$. We also assume that $supp(J) \cap \Omega \neq \emptyset$.

Provided that J satisfies the following condition.

(H3) $\text{supp}(J) \cap \mathbb{R}^+ \neq \emptyset$ and $\text{supp}(J) \cap \mathbb{R}^- \neq \emptyset$.

One can show that the strong maximum principles (Theorems 2.5) holds as well for operators $\mathcal{S}$ and $\mathcal{M}$. More precisely , let $\Omega := (r, +\infty)$ or $(-\infty, r)$, we have the following result.

Theorem 2.10. *Let $u \in C^0(\mathbb{R})$ be such that*

$$\mathcal{M}[u] \geq 0 \quad \text{in } \Omega \quad (\text{resp. } \mathcal{M}[u] \leq 0 \text{ in } \Omega).$$

Assume that J satisfies (H3) then u cannot achieve a global maximum (resp. global minimum) in Ω without being constant.

As a special case of Theorem 2.10, we have the following theorem.

Theorem 2.11. *Let $u \in C^0(\bar{\Omega})$ be such that*

$$\mathcal{S}[u] + h_\alpha \geq 0 \quad \text{in } \Omega \quad (\text{resp. } \mathcal{S}[u] + h_\alpha \leq 0 \text{ in } \Omega)$$
$$u(r) \geq \alpha \quad (\text{resp. } u(r) \leq \alpha),$$

where $h_\alpha = \alpha \int_{\mathbb{R}\setminus\Omega} J(x-y)\,dy$. Assume that J satisfies (H3) then u cannot achieve a global maximum (resp. global minimum) in Ω without being constant.

Indeed, let us define $\tilde{u}$ by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{in } \Omega \\ u(r) & \text{in } \mathbb{R} \setminus \Omega \end{cases} \tag{2.16}$$

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and observe that in Ω, $\tilde{u}$ satisfies

$$\mathcal{M}[\tilde{u}] = \mathcal{S}[u] + u(r) \int_{\mathbb{R}\setminus\Omega} J(x-y)\,dy.$$

Hence

$$\mathcal{M}[\tilde{u}] \geq \Big(u(r) \int_{\mathbb{R}\setminus\Omega} J(x-y)\,dy - h_\alpha \Big) \geq 0$$

(resp.

$$\mathcal{M}[\tilde{u}] \leq \Big(u(r) \int_{\mathbb{R}\setminus\Omega} J(x-y)\,dy - h_\alpha \Big) \leq 0).$$

From Theorem 2.10, $\tilde{u}$ cannot achieve a global maximum (resp. global minimum) in Ω without being constant. Using the definition of $\tilde{u}$, we easily get that u cannot achieves a global maximum (resp. global minimum) in Ω without being constant.

When $\Omega = \mathbb{R}$, the following statement holds.

Theorem 2.12. *Let $u \in C^0(\mathbb{R})$ be such that*

$$\mathcal{M}[u] \geq 0 \quad \text{in } \mathbb{R} \quad (\text{resp. } \mathcal{M}[u] \leq 0 \text{ in } \mathbb{R}).$$

Assume that J satisfies (H3) then u cannot achieve a non-negative maximum (resp. non-positive minimum) in $\mathbb{R}$ without being constant.

In fact, (H3) is optimal to obtain a strong maximum principle for $\mathcal{M}$. Indeed, we have the following result.

Theorem 2.13. *Let $J \in C^0(\mathbb{R})$, then $\mathcal{M}$ satisfies the strong maximum principle (i.e. Theorem 2.12) if and only if (H3) is satisfied.*

Proof of Theorem 2.10. The argumentation being similar in the other cases, I only deal with $\Omega := (r, +\infty)$. Assume that u achieves a global maximum in Ω at some point x_0. At x_0, we have

$$0 \leq \mathcal{M}[u](x_0) \leq 0.$$

Hence, $u(y) = u(x_0)$ for all $y \in x_0 - supp(J)$. Using (H3), we have in particular,

$$u(y) = u(x_0) \quad \text{for all} y \in \Big(x_0 - [-d, -c] \cup [a, b] \Big) \cap \Omega, \tag{2.17}$$

for some positive reals a, b, c, d. We proceed now in two step. First, we show that there exists r_0 such that $u = u(x_0)$ in $[r_0, +\infty)$. Then, we show that $u \equiv u(x_0)$ in $\bar{\Omega}$.

Step 1. Since $x_0 \in \Omega$ then $x_0 + [c, d] \subset \Omega$ and $u(y) = u(x_0)$ for all $y \in x_0 + [c, d]$. We can repeat this argument with $x_0 + c$ and $x_0 + d$ to obtain $u(y) = u(x_0)$ for all $y \in x_0 + [nc, nd]$ with $n \in \{0, 1, 2\}$. By induction, we easily see that

$$u(y) = u(x_0) \quad \text{for all } y \in \cup_{n\in\mathbb{N}} \Big(x_0 + [nc, nd] \Big). \tag{2.18}$$

Choose n_0 so that $1 < n_0\big(\frac{d-c}{c}\big)$, then we have

$$u(y) = u(x_0) \quad \text{for all } y \in [x_0 + n_0 c, +\infty). \tag{2.19}$$

Indeed, since $1 < n_0\big(\frac{d-c}{c}\big)$, we have $x_0 + n(c+1) < x_0 + nd$ for all integer $n \geq n_0$. Hence,

$$[x_0 + n_0 c, +\infty) = \cup_{n\geq n_0} \Big(x_0 + [nc, (n+1)c] \Big) \subset \cup_{n\in\mathbb{N}} \Big(x_0 + [nc, nd] \Big). \tag{2.20}$$

We then achieve the first step by taking $r_0 := x_0 + n_0 c$.

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Step 2. Take any $x \in \bar{\Omega}$ and let $p \in \mathbb{N}$ so that $x+pb > r_0$. Such p exists since $b > 0$. From Step1, we have $u(x+pb) = u(x_0)$. Repeating the previous argumentation yields to

$$u(y) = u(x_0) \quad \text{for all } y \in \big(x + pb - [-d, -c] \cup [a, b]\big) \cap \Omega.$$

In particular, $u(x + (p-1)b) = u(x_0)$. Using induction, we easily get that $u(x) = u(x_0)$, thus

$$u(x) \equiv u(x_0) \quad \text{in } \bar{\Omega}.$$

□

Observe that up to minor change the previous argumentation holds as well to show Theorem 2.12. Let us now show Theorem 2.13. For sake of simplicity, we expose an alternative proof of Theorem 2.12 suggested by Pascal Autissier.

Proof of Theorems 2.12 and 2.13.

Necessary Condition. If this condition fails, then $\operatorname{supp}(J) \subset \mathbb{R}^-$ or $\operatorname{supp}(J) \subset \mathbb{R}^+$. Assume first that $\operatorname{supp}(J) \subset \mathbb{R}^-$. Let u be a non-decreasing function which is constant in $\mathbb{R}^+$. Then a simple computation shows that $\mathcal{M}[u] := J \star u - u \geq 0$. Hence, $\mathcal{M}[u] \geq 0$ and u achieves a global maximum without being constant. Hence $\mathcal{M}$ does not satisfy the strong maximum principle.

If $\operatorname{supp}(J) \subset \mathbb{R}^+$, a similar argument holds. By taking v a non-increasing function which is constant in $\mathbb{R}^-$, we obtain $\mathcal{M}[v] \geq 0$. Hence, $\mathcal{M}[v] \geq 0$ and v achieves a global maximum without being constant. This end the proof of the necessary condition.

Sufficient Condition. Since J is continuous, from (H3), there exists positive reals a, b, c, d such that $[-c, -d] \cup [a, b] \subset supp(J)$. Assume that $\mathcal{M}[u] \geq 0$ and $\tilde{u}$ achieves a global maximum at some point x_0. Let Γ be the following set

$$\Gamma = \{y \in \mathbb{R} | u(y) = u(x_0)\}.$$

Since u is continuous, Γ is a nonempty closed subset of $\mathbb{R}$. From $\mathcal{M}[u](x_0) \geq 0, J \geq 0$ and for all $y \in \mathbb{R}$ $u(y) - u(x_0) \leq 0$, at x_0, u satisfies

$$\mathcal{M}[u](x_0) = \int_{\mathbb{R}} J(x_0 - y)[u(y) - u(x_0)]\, dy = 0.$$

Hence, $(x_0 - [-c, -d] \cup [a, b]) \subset \Gamma$. Let choose $-C \in [-c, -d]$ and $A \in [a, b]$ such that $\frac{A}{C} \in \mathbb{R} \setminus \mathbb{Q}$. This is always possible since $[-c, -d]$ and $[a, b]$ have nonempty interiors. Therefore $x_0 - C \in \Gamma$ and $x_0 - A \in \Gamma$. Now repeating this argument at $x_0 + C$, $x_0 - A$, leads to $(x_0 - C - [-c, -d] \cup [a, b]) \subset \Gamma$ and $(x_0 - A - [-c, -d] \cup [a, b]) \subset \Gamma$. Thus,

$$\{x_0 + pC - qA | (p, q) \in \{0, 1, 2\}^2\} \subset \Gamma.$$

By induction, we then have

$$\{x_0 + pC - qA | (p, q) \in \mathbb{N}^2\} \subset \Gamma.$$

Since $\frac{A}{C} \in \mathbb{R} \setminus \mathbb{Q}$, $\{x_0 + pC - qA | (p, q) \in \mathbb{N}^2\}$ is a dense partition of $\mathbb{R}$. Hence, $\Gamma = \mathbb{R}$ since it is closed and contains a dense partition of $\mathbb{R}$. □

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2.3. **Some remarks and general comments.** We can easily extend all the above augmentations to operators of the form

$$\mathcal{L}+\mathcal{E}, \quad \mathcal{S}+\mathcal{E}, \quad \mathcal{M}+\mathcal{E}$$

where $\mathcal{E}$ is any elliptic operator, which can be degenerate. Thus $\mathcal{L}+\mathcal{E}$, $\mathcal{S}+\mathcal{E}$, $\mathcal{M}+\mathcal{E}$ verify also maximum principles.

Remark 2.14. In such case, the regularity required for u has to be adjusted with the considered operator.

The maximum principles can be also obtain for nonlinear operators of the form

$$\mathcal{L}[g(\cdot)], \quad \mathcal{S}[g(\cdot)], \quad \mathcal{M}[g(\cdot)],$$

where g is a smooth increasing function. In that case, we simply use the fact that $g[u(y)]-g[u(x)]=0$ implies $u(y)=u(x)$. For example, assume that

$$\mathcal{L}[g(u)] \geq 0 \quad \text{in } \Omega$$

If u achieves a global non-negative maximum at x_0 then u satisfies

$$0 \leq \underbrace{\int_r^R J(x_0-y)\big(g[u(y)]-g[u(x_0)]\big)\,dy}_{\leq 0} + \underbrace{g[u(x_0)](h_1^-(x_0)+h_1^+(x_0)-1)}_{\leq 0} \leq 0$$

Hence, $g[u(y)]-g[u(x_0)]=0$ for $y \in (x_0-\operatorname{supp} J)\cap\Omega$. Using the strict monotonicity of g, we achieve $u(y)=u(x_0)$ for $y \in (x_0-SuppJ)\cap\Omega$. Then, we are reduce to the linear case.

Remark 2.15. Nonlinear operator $\mathcal{M}[g(\cdot)]$ appears naturally in models of propagation of information in a Neural Networks see [17, 24].

Remark 2.16. When g is decreasing, the nonlinear operators $\mathcal{L}[g(\cdot)]$, $\mathcal{S}[g(\cdot)]$, and $\mathcal{M}[g(\cdot)]$ satisfy some strong maximum principle. For example, assume that

$$\mathcal{L}[g(u)] \geq 0 \quad \text{in } \Omega.$$

Then u cannot achieve a non-positive global minimum without being constant. Note that in this case, it is a global minimum rather than a global maximum which is required.

Recently, Cortazar et al. [11], introduce another type of nonlinear diffusion operator,

$$\mathcal{R}[u] := \int_{\mathbb{R}} J\Big(\frac{x-y}{u(y)}\Big)\,dy - u.$$

Assuming that J is increasing in $\mathbb{R}^- \cap supp(J)$ and decreasing in $\mathbb{R}^+ \cap supp(J)$, they prove that $\partial_t - \mathcal{R}$ satisfies a parabolic comparison principle. One can show that $\mathcal{R}[g(\cdot)]$ satisfies also a strong maximum principle provided that g is a positive increasing function. Indeed, assume that

$$\mathcal{R}[g(u)] \geq 0 \quad \text{in } \mathbb{R}$$

If u achieves a global positive maximum at x_0 then

$$\frac{x_0-y}{g(u(y))} > \frac{x_0-y}{g(u(x_0))} \quad \text{when } x_0-y>0$$
$$\frac{x_0-y}{g(u(y))} < \frac{x_0-y}{g(u(x_0))} \quad \text{when } x_0-y<0$$

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Using the assumption made on J, we have for every $y \in \mathbb{R}$,

$$\Big[J\Big(\frac{x_0 - y}{g(u(y))}\Big) - J\Big(\frac{x_0 - y}{g(u(x_0))}\Big)\Big] \leq 0.$$

Therefore u satisfies

$$0 \leq \int_{-\infty}^{+\infty} \Big[J\Big(\frac{x_0 - y}{g(u(y))}\Big) - J\Big(\frac{x_0 - y}{g(u(x_0))}\Big)\Big]\, dy \leq 0$$

Hence, $g[u(y)] - g[u(x_0)] = 0$ for $y \in x_0 - SuppJ$. Using the strict monotonicity of g, we achieve $u(y) = u(x_0)$ for $y \in x_0 - SuppJ$. Then, we are reduce to the linear case. These density dependant operator can be viewed as a nonlocal version of the classical porous media operator.

A consequence of the proofs of the strong maximum principle, is the characterization of global extremum of u. Namely, we can derive the following property.

Lemma 2.17. *Assume J satisfies (H3). Let u be a smooth (C^0) function. If u achieves a global minimum (resp.a global maximum) at some point ξ then the following holds:*

- *Either $\mathcal{M}[u](\xi) > 0$ (resp. $\mathcal{M}[u](\xi) < 0$)*
- *Or $\mathcal{M}[u](\xi) = 0$ and u is identically constant.*

Remark 2.18. An easy adaptation of the proof shows that Lemma 2.17 stands for u continuous by parts and with a finite number of discontinuities.

Remark 2.19. Lemma 2.17 holds as well for $\mathcal{M} + \mathcal{E}$, $\mathcal{L}$, $\mathcal{L} + \mathcal{E}$, $\mathcal{S}$, $\mathcal{S} + \mathcal{E}$ and $\mathcal{R}$, provided that the considered operator satisfies a strong maximum principle.

3. Comparison principles

In this section I deal with Comparison principles satisfied by operators $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{M}$. This property comes often as a corollary of a maximum principle. Here we present two comparison principles which are not a direct application of the maximum principle. The first is a linear comparison principle, the second concerns a nonlinear comparison principle satisfied by $\mathcal{S}$. This section is divided into two subsections, each one devoted to a comparison principle.

3.1. **Linear Comparison principle.**

Theorem 3.1 (Linear Comparison Principle)**.** *Let u and v be two smooth functions ($C^0(\mathbb{R})$) and ω a connected subset of $\mathbb{R}$. Assume that u and v satisfy the following conditions :*

- *$\mathcal{M}[v] \geq 0$ in $\omega \subset \mathbb{R}$*
- *$\mathcal{M}[u] \leq 0$ in $\omega \subset \mathbb{R}$*
- *$u \geq v$ in $\mathbb{R} - \omega$*
- *if ω is an unbounded domain, assume also that $\lim_\infty u - v \geq 0$.*

Then $u \geq v$ in $\mathbb{R}$.

Proof. Let first assume, that ω is bounded. Let $w = u - v$, so w will satisfy:

- $w \geq 0$, $w \not\equiv 0$ in $\mathbb{R} - \omega$,
- $\mathcal{M}[w] \leq 0$ in ω.

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Let us define the quantity $\gamma := \inf_{\mathbb{R}\setminus\omega} w$. Now, we argue by contradiction. Assume that w achieves a negative minimum at x_0. By assumption $x_0 \in \omega$ and is a global minimum of w. So, at this point, w satisfies

$$0 \geq \mathcal{M}[w(x_0)] = (J \star w - w)(x_0) = \int_{\mathbb{R}} J(x_0 - z)(w(z) - w(x_0))dz \leq 0.$$

It follows that $w(y) = w(x_0)$ on $y - \operatorname{supp}(J)$. Hence, for some reals a, b, we have the following alternative:

- Either $(\mathbb{R} \setminus \omega) \cap (x_0 - [a, b]) \neq \emptyset$ and then we have a contradiction since there exits $y \in \mathbb{R}$ such that $0 \leq \gamma \leq w(x_0) = w(y) < 0$.
- Or $(\mathbb{R} \setminus \omega) \cap (x_0 - [a, b]) = \emptyset$ and then $(x_0 - [a, b]) \subset\subset \omega$.

In the later case, arguing as for the proof of Theorem 2.1, we can repeat the previous computation at the points $x_0 - b$ and $x - a$ and using induction we achieve,

$$(\mathbb{R} \setminus \omega) \cap (x_0 - [na, nb]) \neq \emptyset,$$
$$\forall\, y \in x_0 - [na, nb], \quad w(y) = w(x_0),$$

for some positive $n \in \mathbb{N}$. Thus $0 \leq \gamma \leq w(x_0) < 0$, which is a contradiction.

In the case of ω unbounded, by assumption $\lim_{x\to\infty} w \geq 0$, then there exists a compact subset ω_1 such that $x_0 \in \omega_1$ and $w(x_0) < \inf_{\mathbb{R}\setminus\omega_1} w$. Then the above argument holds with $\mathbb{R} \setminus \omega_1$ instead of $\mathbb{R} \setminus \omega$. $\square$

3.2. **Nonlinear Comparison Principle.** In this subsection, I obtain the following nonlinear comparison principles.

Theorem 3.2 (Nonlinear comparison principle)**.** *Assume that $\mathcal{M}$ defined by 2.3 verifies (H3), $\Omega = (r, +\infty)$ for some $r \in \mathbb{R}$ and $f \in C^1(\mathbb{R})$, satisfies, $f'_{|(\beta,+\infty)} < 0$. Let z and v smooth ($C^0(\mathbb{R})$) functions satisfying,*

$$\mathcal{M}[z] + f(z) \geq 0 \quad \text{in } \Omega, \tag{3.1}$$
$$\mathcal{M}[v] + f(v) \leq 0 \quad \text{in } \Omega, \tag{3.2}$$
$$\lim_{x\to+\infty} z(x) \leq \beta, \quad \lim_{x\to+\infty} v(x) \geq \beta, \tag{3.3}$$
$$z(x) \leq \alpha, \quad v(x) \geq \alpha \quad \text{when } x \leq r. \tag{3.4}$$

If in $[r, +\infty)$, $z < \beta$ and $v > \alpha$, then there exists $\tau \in \mathbb{R}$ such that $z \leq v_\tau$ in $\mathbb{R}$. Moreover, either $z < v_\tau$ in Ω or $z \equiv v_\tau$ in $\bar{\Omega}$.

Before proving this theorem, we start with define some quantities that we will use in this subsection.

Let $\epsilon > 0$ be such that $f'(s) < 0$ for $s \geq \beta - \epsilon$. Choose $\delta \leq \frac{\epsilon}{4}$ positive, such that

$$f'(p) < -2\delta \quad \forall p \quad \text{such that } \beta - p < \delta. \tag{3.5}$$

If $\lim_{x\to+\infty} z(x) = \beta$, choose $M > 0$ such that:

$$\beta - v(x) < \frac{\delta}{2} \quad \forall x > M, \tag{3.6}$$
$$\beta - z(x) < \frac{\delta}{2} \quad \forall x > M. \tag{3.7}$$

Otherwise, we choose M such that

$$v(x) > z(x) \quad \forall x > M. \tag{3.8}$$

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The proof of this theorem follows ideas developed by the author in [13] for convolution operators. It essentially relies, on the following technical lemma which will be proved later on.

Lemma 3.3. *Let z and v be respectively smooth positive sub and supersolution satisfying* (3.1)-(3.4)*. If there exists positive constant $a \le \frac{\delta}{2}$ and b such that z and v satisfy:*

$$v(x+b) > z(x) \quad \forall x \in [r, M+1], \tag{3.9}$$

$$v(x+b) + a > z(x) \quad \forall x \in \Omega. \tag{3.10}$$

Then we have $v(x+b) \ge z(x)$ $\forall x \in \mathbb{R}$.

Proof of Theorem 3.2. Observe, that if $\inf_{\mathbb{R}} v \ge \sup_{\mathbb{R}} z$ then $v \ge z$ trivially holds. In the sequel, we assume that $\inf_{\mathbb{R}} v < \max_{\mathbb{R}} z$.

Assume for a moment that Lemma 3.3 holds. To prove Theorem 3.2, by construction of M and δ, we just have to find an appropriate constant b which satisfies (3.9) and (3.10) and showing that either $v_\tau > z$ in Ω or $z \equiv v_\tau$ in Ω.

Since v and z satisfy in $[r, +\infty)$: $z < \beta$ and $v > \alpha$, using (3.3)-(3.4) we can find a constant D such that on the compact set $[r, M+1]$, we have for every $b \ge D$

$$v(x+b) > z(x) \quad \forall x \in [r, M+1].$$

Now, we claim that there exists $b \ge D$ such that $v(x+b) + \frac{\delta}{2} > z(x)$ for all $x \in \mathbb{R}$. If not, then we have

$$\text{for all } b \ge D \text{ there exists } x(b) \text{ such that } v(x(b)+b) + \frac{\delta}{2} \le z(x(b)). \tag{3.11}$$

Since $v \ge \alpha$ and v satisfies (3.4) we have

$$v(x+b) + \frac{\delta}{2} > z(x) \quad \text{for all } b > 0 \text{ and } x \le r. \tag{3.12}$$

Take now a sequence $(b_n)_{n\in\mathbb{N}}$ which tends to $+\infty$. Let $x(b_n)$ be the point defined by (3.11). Thus we have for that sequence

$$v(x(b_n)+b_n) + \frac{\delta}{2} \le z(x(b_n)). \tag{3.13}$$

According to (3.12) we have $x(b_n) \ge M+1$. Therefore, the sequence $x(b_n)+b_n$ converges to $+\infty$. Passing to the limit in (3.13) to get

$$\beta + \frac{\delta}{2} \le \lim_{n\to+\infty} v(x(b_n)+b_n) + \frac{\delta}{2} \le \limsup_{n\to+\infty} z(x(b_n)) \le \beta,$$

which is a contradiction. Therefore there exists a $b > D$ such that

$$v(x+b) + \frac{\delta}{2} > z(x) \quad \forall x \in \Omega.$$

Since we have found our appropriate constants $a = \frac{\delta}{2}$ and b, we can apply Lemma 3.3 to obtain

$$v(x+\tau) \ge z(x) \quad \forall x \in \mathbb{R},$$

with $\tau = b$. It remains to prove that in Ω either $v_\tau > z$ or $u_\tau \equiv v$. We argue as follows. Let $w := v_\tau - z$, then either $w > 0$ in Ω or w achieves a non-negative minimum at some point $x_0 \in \Omega$. If such x_0 exists then at this point we have $w(x) \ge w(x_0) = 0$ and

$$0 \le \mathcal{M}[w(x_0)] \le f(z(x_0)) - f(v(x_0+\tau)) = f(z(x_0)) - f(z(x_0)) = 0. \tag{3.14}$$

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Then using the argumentation in the proof of Theorem 2.10, we obtain $w \equiv 0$ in $\bar{\Omega}$, which means $v_\tau \equiv z$ in $\bar{\Omega}$. This ends the proof of Theorem 3.2. $\square$

Let now turn our attention to the proof of the technical Lemma 3.3.

Proof of Lemma 3.3. Let v and z be respectively a super and a subsolution of (3.1)-(3.4) satisfying (3.6) and (3.7) or (3.8). Let $a > 0$ be such that

$$v(x+b) + a > z(x) \quad \forall x \in \Omega. \tag{3.15}$$

Note that for b defined by (3.9) and (3.10), any $a \geq \frac{\delta}{2}$ satisfies (3.15). Define

$$a^* = \inf\{a > 0 : v(x+b) + a > z(x)\ \forall x \in \Omega\}. \tag{3.16}$$

We claim that

Claim 3.4. $a^* = 0$.

Observe that Claim 3.4 implies that $v(x+b) \geq z(x)$ for all $x \in \Omega$, which is the desired conclusion. $\square$

Proof of claim 3.4. We argue by contradiction. If $a^* > 0$, since $\lim_{x\to+\infty} v(x+b) + a^* - z(x) \geq a^* > 0$ and $v(x+b) - z(x) + a^* \geq a^* > 0$ for $x \leq r$, there exists $x_0 \in \Omega$ such that $v(x_0+b) + a^* = z(x_0)$. Let $w(x) := v(x+b) + a^* - z(x)$, then

$$0 = w(x_0) = \min_{\mathbb{R}} w(x). \tag{3.17}$$

Observe that w also satisfies the following equations:

$$\mathcal{M}[w] \leq f(z(x)) - f(v(x+b)) \tag{3.18}$$

$$w(+\infty) \geq a^* \tag{3.19}$$

$$w(x) \geq a^* \quad \text{for} \quad x \leq r. \tag{3.20}$$

By assumption, $v(x+b) > z(x)$ in $(-\infty, M+1]$. Hence $x_0 > M+1$. Let us define

$$Q(x) := f(z(x)) - f(v(x+b)). \tag{3.21}$$

Computing $Q(x)$ at x_0, it follows

$$Q(x_0) = f(v(x_0+b) + a^*) - f(v(x_0+b)) \leq 0, \tag{3.22}$$

since $x_0 > M+1$ f is non-increasing for $s \geq \beta - \epsilon$, $a^* > 0$ and $\beta - \epsilon < \beta - \frac{\delta}{2} \leq v$ for $x > M$. Combining (3.18),(3.17) and (3.22) yields

$$0 \leq \mathcal{M}[w(x_0)] \leq Q(x_0) \leq 0.$$

Following the argumentation of Theorem 2.10, we end up with $w = 0$ in Ω which contradicts (3.19). Hence $a^* = 0$, which ends the proof of Claim 3.4. $\square$

Remark 3.5. The previous analysis only holds for linear operators. It fails for operators such as $\mathcal{M}[g(\cdot)]$ or $\mathcal{R}$.

Remark 3.6. The regularity assumption on f can be improved. Indeed, the above proof holds as well with f continuous and non-increasing in $(\beta - \epsilon, +\infty)$ for some positive ϵ.

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4. Sliding techniques and applications

In this section, using sliding techniques, I prove uniqueness and monotonicity of positive solution of the following problem:

$$\int_r^R J(x-y)g(u(y))\,dy + f(u) + t_\alpha^- + t_\beta^+ = 0 \quad \text{in } \Omega \tag{4.1}$$

$$u(r) = \alpha \tag{4.2}$$

$$u(R) = \beta, \tag{4.3}$$

where $t_\alpha^- = g(\alpha)\int_{-\infty}^r J(x-y)dy$, $t_\beta^+ = g(\beta)\int_R^\infty J(x-y)dy$, g is an increasing function. We also assume that f is continuous functions and that J satisfies (H1)–(H2). More precisely, I prove the following result.

Theorem 4.1. *Let $\alpha < \beta$. Assume that $f \in C^0$. Then any solution u of* (4.1)-(4.3)*, satisfying $\alpha < u < \beta$, is monotone increasing. Furthermore, this solution if its exists is unique.*

Similarly, if $\alpha > \beta$, then any solution u of (4.1)-(4.3), satisfying $\beta < u < \alpha$, is monotone decreasing. Observe that Theorem 1.4 comes as a special case of Theorem 4.1. Indeed, choose $g = Id$, then a short computation shows that

$$\int_r^R J(x-y)g(u(y))\,dy = \int_r^R J(x-y)u(y)\,dy = \mathcal{L}[u] + u,$$
$$t_\alpha^- = h_\alpha^-, \quad t_\beta^+ = h_\beta^+,$$

where $\mathcal{L}$ is defined by (2.1) with $c(x) \equiv 0$. Hence, in this special cases (4.1)-(4.3) becomes

$$\mathcal{L}[u] + \bar{f}(u) + h_\alpha^- + h_\beta^+ = 0 \quad \text{in } \Omega$$
$$u(r) = \alpha, \quad u(R) = \beta,$$

where $\bar{f}(u) := f(u) + u$.

Before going to the proof, we define for convenience the nonlinear operator

$$\mathcal{N}[v] := \int_{-\infty}^{+\infty} J(x-y)g(v(y))dy \tag{4.4}$$

Proof of Theorem 4.1. We start by showing that u is monotone.

Monotonicity. Let us define the following continuous extension of u:

$$\tilde{u}(x) := \begin{cases} u(x) & \text{in } \Omega \\ u(r) & \text{in } (-\infty, r) \\ u(R) & \text{in } (R, +\infty). \end{cases} \tag{4.5}$$

Observe that in Ω, $\tilde{u}$ satisfies

$$\begin{gathered} \mathcal{N}[\tilde{u}] + f(\tilde{u}) = 0 \quad \text{in } \Omega \\ \tilde{u}(x) = \alpha \quad \text{for } x \in (-\infty, r] \\ \tilde{u}(x) = \beta \quad \text{for } x \in [R, +\infty) \end{gathered} \tag{4.6}$$

Showing that $\tilde{u}$ is monotone increasing in Ω will imply that u is monotone increasing. To obtain that $\tilde{u}$ is monotone increasing, we use a sliding technique developed by Berestycki and Nirenberg [6], which is based on comparison between $\tilde{u}$ and its

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translated $\tilde{u}_\tau := \tilde{u}(x+\tau)$. We show that for any positive τ we have $\tilde{u} < \tilde{u}_\tau$ in Ω. First, observe that $\tilde{u}_\tau$ satisfies

$$\begin{gathered} \mathcal{N}[\tilde{u}_\tau] + f(\tilde{u}_\tau) = 0 \quad \text{in } (r-\tau, R-\tau) \\ \tilde{u}_\tau(x) = \alpha \quad \text{for } x \in (-\infty, r-\tau] \\ \tilde{u}_\tau(x) = \beta \quad \text{for } x \in [R-\tau, +\infty) \end{gathered} \tag{4.7}$$

Now let us define

$$\tau^* = \inf\{\tau \geq 0 : \forall \tau' \geq \tau,\ \tilde{u}_{\tau'} > \tilde{u} \text{ in } \Omega\} \tag{4.8}$$

Observe that τ^* is well defined since for any $\tau > R-r$, by assumption and the definition of $\tilde{u}$, we have $\tilde{u} \leq \tilde{u}_\tau$ in $\mathbb{R}$ and $\tilde{u} < \tilde{u}_\tau$ in Ω. Hence $\tau^* \leq R-r$. We now show that $\tau^* = 0$. Observe that by proving the claim below we obtain the monotonicity of the solution u.

Claim 4.2. $\tau^* = 0$

Proof of the claim. We argue by contradiction. Assume that $\tau^* > 0$, then since $\tilde{u}$ is a continuous function, we will have $\tilde{u} \leq \tilde{u}_{\tau^*}$ in $\mathbb{R}$. Let $w := \tilde{u}_{\tau^*} - \tilde{u}$. From the definition of τ^* and the continuity of $\tilde{u}$, w must achieve a non positive minimum at some point x_0 in Ω. Namely, since $w \geq 0$, we have $w(x_0) = 0$. We are now lead to consider the following two cases:

- Either $x_0 \in [R-\tau^*, R)$
- Or $x_0 \in (r, R-\tau^*)$

We will see that in both case we end up with a contradiction.

First assume that $x_0 \in [R-\tau^*, R)$. Since $\tau^* > 0$, using the definition of $\tilde{u}$ we have $\tilde{u}_{\tau^*} \equiv \beta$ in $[R-\tau^*, R)$. We therefore get a contradiction since $0 = w(x_0) = \beta - \tilde{u}(x_0) > 0$.

In the other case, w achieves its minimum in $(r, R-\tau^*)$. Now, using (4.6) and (4.7), at x_0, we have

$$\mathcal{N}\tilde{u}_{\tau^*} - \mathcal{N}[\tilde{u}] = \int_{-\infty}^{+\infty} J(x_0-y)[g(\tilde{u}_{\tau^*}(y)) - g(\tilde{u}(y))]\,dy = 0 \tag{4.9}$$

Since, g is increasing and $\tilde{u}_{\tau^*} \geq \tilde{u}$, it follows that $g(\tilde{u}_{\tau^*}(y)) - g(\tilde{u}(y)) = 0$ for all $y \in x_0 - Supp(J)$. Using the monotone increasing property of g yields to $w(y) = \tilde{u}_{\tau^*}(y) - \tilde{u}(y) = 0$ for all $y \in x_0 - Supp(J)$. Arguing now as in Theorem 2.5, we end up with $w \equiv 0$ in all $[r, R-\tau^*]$. Hence, $0 = w(r) = \tilde{u}(r+\tau^*) - \alpha > 0$ since $\tau^* > 0$, which is our desired contradiction. Thus $\tau^* = 0$, which ends the proof of the claim and the proof of the monotonicity of $\tilde{u}$. $\square$

Uniqueness. We now prove that problem (4.1)-(4.3) has a unique solution. Let u and v be two solution of (4.1)-(4.3). From the previous subsection without loss of generality, we can assume that u and v are monotone increasing in Ω and we can extend by continuity u and v in all $\mathbb{R}$ by $\tilde{u}$ and $\widetilde{v}$. We prove that $\tilde{u} \equiv \widetilde{v}$ in $\mathbb{R}$, this give us $u \equiv v$ in Ω. As in the above subsection, we use sliding method to prove it. Let us define

$$\tau^{**} = \inf\{\tau \geq 0 : \widetilde{v}_\tau > \tilde{u} \text{ in } \Omega\} \tag{4.10}$$

Observe that τ^{**} is well defined since for any $\tau > R-r$, by assumption and the definition of $\tilde{u}$, we have $\tilde{u} \leq \widetilde{v}_\tau$ in $\mathbb{R}$ and $\tilde{u} < \widetilde{v}_\tau$ in Ω. Therefore $\tau^{**} \leq R-r$.

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Following now the argumentation of the above subsection with $\widetilde{v}_{\tau^{**}}$ instead of u_{τ^*}, it follows that $\tau^{**} = 0$. Hence, $\widetilde{v} \geq \tilde{u}$. Since u and v are solution of (4.1)-(4.3), the same analysis holds with $\tilde{u}$ replace by $\widetilde{v}$. Thus, $\widetilde{v} \leq \tilde{u}$ which yields to $\tilde{u} \equiv \widetilde{v}$. $\square$

Remark 4.3. Theorem 4.1 holds for the operator $\mathcal{R}$ introduced by Cortazar et al.

5. Qualitative properties of solutions of integrodifferential equation in unbounded domains

In this section, I study the properties of solutions to the problem

$$\begin{gathered} \mathcal{S}[u] + f(u) + h_\alpha(x) = 0 \quad \text{in } \Omega \\ u(r) = \alpha < \beta \\ u(x) \to \beta \quad \text{as } x \to +\infty \end{gathered} \tag{5.1}$$

where $\mathcal{S}$ is defined by (2.2) satisfies (H3), $h_\alpha(x) = \alpha \int_{-\infty}^{r} J(x-y)dy$, $\Omega := (r, +\infty)$ for some $r \in \mathbb{R}$ and $f \in C^1(\mathbb{R})$, satisfying $f'_{|[\beta,+\infty)} < 0$. For (5.1), I prove the following result.

Theorem 5.1. *Any smooth* (C^0) *solution of* (5.1) *satisfying* $\alpha < u < \beta$ *in* Ω *is monotone increasing. Furthermore, such solution is unique.*

Observe that Theorem 1.5 follows from Theorem 5.1 with $\alpha = 0$ and $\beta = 1$.

Before proving this theorem, let us observe that problem (5.1) is equivalent to the problem

$$\begin{gathered} \mathcal{M}[\tilde{u}] + f(\tilde{u}) = 0 \quad \text{in } \Omega \\ \tilde{u}(x) = \alpha \quad \text{as } x \leq r \\ \tilde{u}(x) \to \beta \quad \text{as } x \to +\infty, \end{gathered} \tag{5.2}$$

with $\mathcal{M}$ defined by (2.3) and $\tilde{u}$ is the following extension of u:

$$\tilde{u} := \begin{cases} u(x) & \text{when } x \in \Omega \\ \alpha & \text{when } x \in \mathbb{R} \setminus \Omega \end{cases}.$$

Theorem 5.1, easily follows from the following result.

Theorem 5.2. *Let* $\tilde{u}$ *be a smooth solution of* (5.2) *satisfying* $\alpha < u < \beta$ *in* Ω*, then* $\tilde{u}$ *is monotone increasing in* Ω*. Moreover,* $\tilde{u}$ *is unique.*

Proof of Theorem 5.2. We break down the proof of Theorem 5.2 into two parts. First, we show the monotonicity of the solution of (5.2). Then we obtain the uniqueness of the solution. Each part of the proof will be subject of a subsection. In what follows, we only deal with problem (5.2) and for convenience we drop the tilde subscript on the function u. Recall that by assumption in Ω, one has

$$\alpha < u < \beta. \tag{5.3}$$

$\square$

Monotonicity. We obtain the monotonicity of u in the following three steps:

(1) We prove that for any solution u of (5.2) there exists a positive τ such that

$$u(x+\tau) \geq u(x) \quad \forall x \in \mathbb{R}.$$

(2) We show that for any $\widetilde{\tau} \geq \tau$, u satisfies

$$u(x+\widetilde{\tau}) \geq u(x) \quad \forall x \in \mathbb{R}.$$

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(3) We prove that

$$\inf\{\tau > 0 : \forall \widetilde{\tau} > \tau,\ u(x+\widetilde{\tau}) \geq u(x)\ \forall x \in \mathbb{R}\} = 0.$$

We easily see that the last step provided the conclusion.
Step One: The first step is a direct application of the nonlinear comparison principle, i.e. Theorem 3.2. Since u is a sub- and a super-solution of (5.2) one has $u_\tau \geq u$ for some positive τ.
Step Two: We achieve the second step with the following proposition.

Poroposition 5.3. *Let u be a solution of* (5.2)*. If there exists τ such that $u_\tau \geq u$. Then, for all $\widetilde{\tau} \geq \tau$ we have, $u_{\widetilde{\tau}} \geq u$.*

Indeed, using the first step we have $u_\tau \geq u$ for some $\tau > 0$. Step Two is then a direct application of Proposition 5.3.

The proof of Proposition 5.3 is based on the two following technical lemmas.

Lemma 5.4. *Let u be a solution of* (5.2) *and $\tau > 0$ be such that $u_\tau \geq u$. Then $u(x+\tau) > u(x)$ for all $x \in \bar{\Omega}$.*

Lemma 5.5. *Let u be a solution of* (5.2) *and $\tau > 0$ be such that*

$$u_\tau \geq u$$
$$u(x+\tau) > u(x) \quad \forall x \in \bar{\Omega}.$$

Then, there exists $\epsilon_0(\tau) > 0$ such that for all $\widetilde{\tau} \in [\tau, \tau+\epsilon_0]$, $u_{\widetilde{\tau}} \geq u$.

Proof of Proposition 5.3. Assume that the two technical lemmas hold and that we can find a positive τ, such that,

$$u(x+\tau) \geq u(x) \quad \forall x \in \mathbb{R}.$$

Using Lemmas 5.4 and 5.5, we can construct an interval $[\tau, \tau+\epsilon]$, such that $u_{\widetilde{\tau}} \geq u$ for all $\widetilde{\tau} \in [\tau, \tau+\epsilon]$. Let us defined the quantity

$$\bar{\gamma} = \sup\{\gamma : \forall \hat{\tau} \in [\tau, \gamma],\ u_{\hat{\tau}} \geq u\}. \tag{5.4}$$

We claim that $\bar{\gamma} = +\infty$, if not, $\bar{\gamma} < +\infty$ and by continuity we have $u_{\bar{\gamma}} \geq u$. Recall that from the definition of $\bar{\gamma}$, we have

$$\forall \hat{\tau} \in [\tau, \bar{\gamma}], \qquad u_{\hat{\tau}} \geq u. \tag{5.5}$$

Therefore to get a contradiction, it is sufficient to construct ϵ_0 such that

$$u_{\bar{\gamma}+\epsilon} \geq u, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{5.6}$$

Since $\bar{\gamma} > 0$ and $u_{\bar{\gamma}} \geq u$, we can apply Lemma 5.4 to have

$$u(x+\bar{\gamma}) > u(x) \ \ \forall \ \ x \in \bar{\Omega}. \tag{5.7}$$

Now apply Lemma 5.5, to find the desired $\epsilon_0 > 0$. Therefore, from the definition of $\bar{\gamma}$ we get

$$u_{\hat{\tau}} \geq u, \quad \forall \hat{\tau} \in [\tau, +\infty].$$

Which proves Proposition 5.3. □

Let us now turn our attention to the proofs of the technical lemmas.

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Proof of lemma 5.4. Using argumentation in the proof of the nonlinear comparison principle (Theorem 3.2) one has: Either

$$u(x+\tau) > u(x) \quad \forall x \in \bar{\Omega}, \tag{5.8}$$

or $u_\tau \equiv u$ in $\bar{\Omega}$. The latter is impossible, since for any positive τ,

$$\alpha = u(r) < u(r+\tau) = u_\tau(r).$$

Thus (5.8) holds. □

Proof of Lemma 5.5. Let u be a solution of (5.2) such that

$$u_\tau \geq u$$
$$u(x+\tau) > u(x) \quad \forall x \in \bar{\Omega},$$

for a given $\tau > 0$. Choose M, δ and ϵ such that (3.5)-(3.7) hold. Since u is continuous, we can find ϵ_0, such that for all $\epsilon \in [0, \epsilon_0]$, we have

$$u(x+\tau+\epsilon) > u(x) \quad \text{for } x \in [r, M+1].$$

Choose ϵ_1 such that for all $\epsilon \in [0, \epsilon_1]$, we have

$$u(x+\tau+\epsilon) + \frac{\delta}{2} > u(x) \quad \forall x \in \bar{\Omega}.$$

Let $\epsilon_3 = \min\{\epsilon_0, \epsilon_1\}$. Observe that for all $\epsilon \in [0, \epsilon_3]$, $b := \tau + \epsilon$ and $a = \frac{\delta}{2}$ satisfies assumptions (3.9) and (3.10) of Lemma 3.3. Applying now Lemma 3.3 for each $\epsilon \in [0, \epsilon_3]$, we get $u_{\tau+\epsilon} \geq u$. Thus, we end up with

$$u_{\widetilde{\tau}} \geq u, \quad \forall\, \widetilde{\tau} \in [\tau, \tau+\epsilon_3],$$

which completes the proof of Lemma 5.5. □

Step Three: From the first Step and Proposition 5.3, we can define t he quantity:

$$\tau^* = \inf\{\tau > 0 : \forall \widetilde{\tau} > \tau,\ u_{\widetilde{\tau}} \geq u\}. \tag{5.9}$$

We claim that

Claim 5.6. $\tau^* = 0$

Observe that this lemma implies the monotony of u, which concludes the proof of Theorem 5.2.

Proof of Claim 5.6. We argue by contradiction, suppose that $\tau^* > 0$. We will show that for ϵ small enough, we have,

$$u_{\tau^*-\epsilon} \geq u.$$

Using Proposition 5.3, we will have

$$u_{\widetilde{\tau}} \geq u \forall\, \widetilde{\tau} \geq \tau^* - \epsilon,$$

which contradicts the definition of τ^*. □

Now, we start the construction. By definition of τ^* and using continuity, we have $u_{\tau^*} \geq u$. Therefore, from Lemma 5.4, we have

$$u(x+\tau^*) > u(x), \quad \text{for all } x \in \bar{\Omega}.$$

Thus, in the compact set $[r, M+1]$, we can find $\epsilon_1 > 0$ such that

$$\forall \epsilon \in [0, \epsilon_1), \quad u(x+\tau^*-\epsilon) > u(x) \quad \text{in } [r, M+1].$$

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Since

$$u_{\tau^*} + \frac{\delta}{2} > u \quad \text{in } \bar{\Omega},$$

and $\lim_{x\to+\infty} u_{\tau^*} - u = 0$, we can choose ϵ_2 such that for all $\epsilon \in [0, \epsilon_2)$ we have

$$u(x + \tau^* - \epsilon) + \frac{\delta}{2} > u(x) \quad \text{for all } x \in \bar{\Omega}.$$

Let $\epsilon \in (0, \epsilon_3)$, where $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$, we can then apply Lemma 3.3 with $u_{\tau^*-\epsilon}$ and u to obtain the desired result.

5.1. **Uniqueness.** The uniqueness of the solution of (5.2) essentially follows from the argumentation in the above subsection, Step 3. Let u and v be to solutions of (5.2). Using the nonlinear comparison principle we can define the real number

$$\tau^{**} = \inf\{\tau \geq 0 | \ \ u_\tau \geq v\}. \tag{5.10}$$

and make the following claim.

Claim 5.7. $\tau^{**} = 0$.

Proof. In this context the argumentation in the above subsection (Step3) hold as well using $u_{\tau^{**}}$ and v instead of u_{τ^*} and u. □

Thus $u \geq v$. Since u and v are both solution, interchanging u and v in the above argumentation yields $v \geq u$. Hence, $u \equiv v$, which prove the uniqueness of the solution.

Remark 5.8. Since the proof of Theorem 5.2 mostly relies on the application of the nonlinear comparison principle, using Remark 3.6 the assumption made on f can be relaxed.

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