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Asymptotic Behavior in a Salmonella Infection Model

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Abstract. Salmonella is one of the major sources of toxi-infection in humans in France and United States. The incidence of human salmonellosis has considerably increased over the past 20 years and this can be largely attributed to epidemics of S. enteritidis phage type 4 in poultry in numerous countries. In this article, we formulate and analyse a model in which the transmission of the disease is determined by contact between hens and Salmonella in the environment.

Key words: Salmonella infection, ordinary differential equations, spatially structured model
AMS subject classification: 92D30

1. Introduction

Salmonella is one of the major sources of toxic infection in humans in France and in United states (Bouvet *et al*. [3] and Patrick *et al.* [26]). The incidence of human salmonellosis has increased greatly over the last 20 years and this can largely be attributed to an epidemics of *S. enteritidis* phage types 4 in poultry in many countries (Barrow *et al.* [1], Guard-Petter [13], CDC [6, 7], and Patrick *et al.* [26]). Transmission to hens may originate from contaminated food, water, or from transmission by wild animals. However, one of the main concerns is the existence of silent carriers, *i.e.* animals harboring *Salmonella* without expressing any visible symptoms. Since they can hardly be distinguished from healthy animals, these animals can, in turn, transmit the bacterium to their flock-mates through horizontal transmission or to their offspring by vertical transmission. They are also responsible for transmission to human beings. The analysis of different experimental infection has permitted to develop a number of prophylactis: vaccination (Zhang-Barber [32]), competitive

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exclusion (Rantala and Nurmi [25], Rabsch *et al.* [24]), and genetic methods (in increasing resistance to systemic disease (Bumstead and Barrow [4]) or in carrier-states (Beaumont *et al*. [2]), thus reducing the need for antibiotic treatments and the risk of resistance to antibiotics). However, the efficiency of these methods was evaluated after experimental inoculation, thus comparing *Salmonella* contamination rates at a given interval after inoculation and neglecting the dynamics of bacterial dissemination within the flock.

Here, our objective is to develop and to analyze a mathematical model for *Salmonella* infection which takes into account the dynamics of bacterial dissemination. Hence, we present a model incorporating a spatial structure for the dynamics of *Salmonella* infection within a egg laying hen flock. To formulate our model, we consider the population of hens and the bacterial environmental contamination, and we assume that the transmission of the disease is determined by contact between hens and bacteria in the environment. The model presented here is not a direct contact epidemic model. Epidemic models considering direct contact have been reviewed by Hethcote [16] as well as Diekmann and Heesterbeek [8]. In Fitzgibbon *et al.* [9, 10] a model with indirect transmission was also considered, but the authors consider a population of cats which diffuses in a static contaminated environment, whereas here the population of hens is static and the *Salmonella* diffuses in the environment. Models of *Salmonella* infection in dairy herds which incorporate the bacterial environmental contamination have been presented by Xiao *et al.* [30, 31].

In the present article we formulate a spatially structured model for which two extreme cases as well as the diffusion rate of *Salmonella* in the environment are numerically compared. Since for real hens houses the diffusion rate is high, a non-spatially structured model, which is a system of ordinary differential equations (ODE), can be considered as an alternative to the spatially structured model. This ODE model was presented previously by Prevost *et al.* [23] together with a model for the production of eggs contaminated with Salmonella. The ODE model was compared to real data in [23]. To improve the understanding of the parameters in the described epizooty, here we analyze the asymptotic behavior of the ODE system, as the results obtained for the asymptotic behavior were found to coincide with numerical simulation in [23].

The paper is organized as follows. In section 2, we present the model and some numerical simulations. In section 3 we investigate the existence and uniqueness of solutions for the spatially structured model. In section 4, we study the uniform persistence, the local (and the global) stability of endemic equilibrium of the ODE model.

2. Mathematical model

Let us consider a population of hens and the bacterial environmental contamination within an industrial house hens. We assume that the population of hens is motionless (because hens remain within their cages) and that *Salmonella* population disperses via a diffusion process in the environment. The diffusion process is well adapted here for *Salmonella* dispersing in the environment through dust, air flux, etc.... We assume that the population of hens is confined into the hen house, which is represented here by the spatial domain $\Omega \subset \mathbb{R}^2$, in which Ω represents the area covered by the hen house. We also consider three steps of contamination for the hens: di-

gestive contamination, systemic contamination (when systemic organs such as liver or spleen are contaminated after translocation of bacterium through the digestive barrier), and bacterial clearance leading to recovery. We denote by $s(t,x), i_D(t,x), i_S(t,x), r(t,x)$ the density of population at time t and position $x \in \Omega$ of susceptible hens (*i.e.* those capable to contract the disease), hens suffering from digestive contamination (*D*-infectious), hens suffering from systemic contamination (*S*-infectious), and recovered hens (*i.e.* having eliminated all bacteria), respectively. The total number of hens in each class $S(t), I_D(t), I_S(t)$ and $R(t)$ may be computed by integration over the habitat Ω, *i.e.* $S(t) = \int_\Omega s(t,x)dx$, $I_D(t) = \int_\Omega i_D(t,x)dx$, $I_S(t) = \int_\Omega i_S(t,x)dx$, $R(t) = \int_\Omega r(t,x)dx$. Moreover, we assume that the total number of hens remains constant and equal to $\overline{N}$ (because hens don't die from *Salmonella*). This leads to

$$S(t) + I_D(t) + I_S(t) + R(t) = \overline{N}, \forall t \geq 0.$$

Let $c(t,x)$ be the density of bacterial environmental contamination. As above, the total number of bacteria in the environment $C(t)$ is obtained by integration over Ω *i.e.* $C(t) = \int_\Omega c(t,x)dx, \forall t \geq 0$. The transmission rate (*i.e.* the rate at which susceptible hens become *D*-infectious) is assumed to have following form

$$\sigma(t,x) = \int_\Omega \gamma(x,y,C(t))c(t,y)dy,$$

where $\gamma(x,y,C(t))$ is the infection rate at position x by bacteria at position y and depends on bacterial load $C(t)$. We also assume that *D*-infectious and *S*-infectious hens release bacteria in the environment via an excretion process. This flux of excreted bacteria is represented in the model by $\beta_D i_D(t,x) + \beta_S i_S(t,x)$.

The different stages and bacteria in the environment are coupled into the following system

$$\left\{\begin{array}{lcl} \partial s(t,x)/\partial t &=& -\sigma(t,x)s(t,x) + \nu r(t,x), \\ \partial i_D(t,x)/\partial t &=& \sigma(t,x)s(t,x) - g i_D(t,x), \\ \partial i_S(t,x)/\partial t &=& g i_D(t,x) - \eta i_S(t,x), \\ \partial r(t,x)/\partial t &=& \eta i_S(t,x) - \nu r(t,x), \\ \partial c(t,x)/\partial t &=& \alpha^2 \Delta_x c(t,x) + \beta_D i_D(t,x) + \beta_S i_S(t,x) - \lambda c(t,x). \end{array}\right. \tag{2.1}$$

We also assume that bacteria remain confined into Ω. Hence, we impose a standard no-flux boundary condition

$$\partial c(t,x)/\partial \eta = 0, \text{ on } \partial\Omega.$$

The initial condition is assumed to be non-negative on Ω

$$(s(0,.), i_D(0,.), i_S(0,.), r(0,.), c(0,.)) = (s_0(.), i_{D0}(.), i_{S0}(.), r_0(.), c_0(.)),$$

and we also assume that

$$g \geq 0, \nu \geq 0, \eta \geq 0, \beta_D \geq 0, \beta_S \geq 0, \lambda \geq 0,$$

where g is the rate at which digestive contaminated hens become systemic contaminated, α^2 is the diffusion rate of bacteria, η the recovery rate, β_D (resp. β_i) the excretion rate of *D* (resp.

S)-infectious hens, $-\lambda$ the growth rate of bacteria in the environment and, ν the recontamination rate.

In the sequel, we only consider two extremes cases: 1) $\sigma(t,x) = \kappa C(t)$ (global case); 2) $\sigma(t,x) = \kappa c(t,x)$ (local case). The case 1) corresponds to a situation where the distance between the hens and the contaminant does not influence the infection rate. The case 2) corresponds to a situation where a hen can only be infected by *Salmonella* which are located at the same position as the hen. Below we numerically compare both situations by using a Crank-Nicholson scheme to compute the solutions. For the numerical simulations, we only modify the diffusion rate α. All the remaining parameters are fixed, and equal to $\bar{N} = 20000, \kappa = 0.1, g = 0.2, \eta = 0.1, \beta_D = 0, \beta_S = 0.1, \lambda = 0.1$, and $\nu = 0$. Moreover at time $t = 0$, all the hens are assumed to be susceptible. For the figures below the time $t = 800$ corresponds to a time without residual D-infectious, S-infectious, and bacteria in the environment. In consequence, on the figures below, the distribution of susceptible at time $t = 800$ is a stationary distribution.

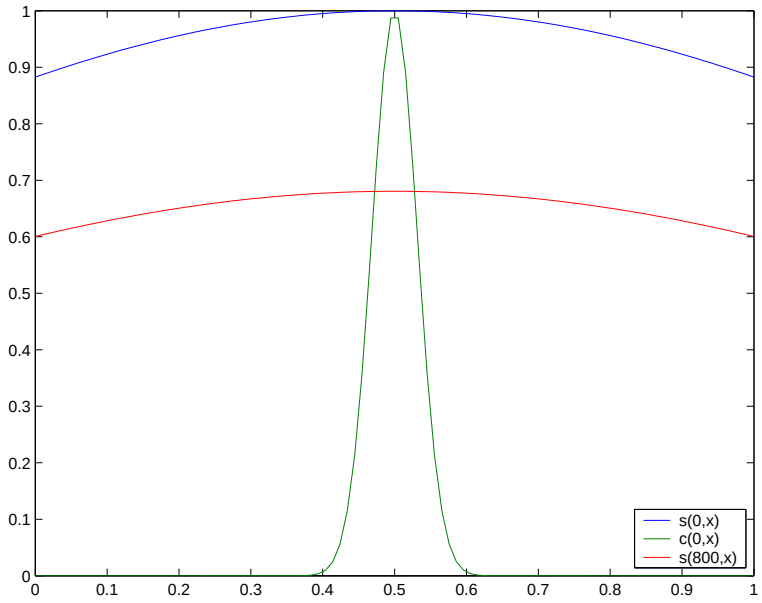

Figure 1: This figure corresponds to the global case. The green curve corresponds to the initial distribution of bacteria, the blue curve corresponds the initial distribution of susceptible hens, and red curve corresponds to distribution of S-infectious at time t=800.

For the global case 1), the solutions are independent of α. Therefore, in figure 1, we do not specify the diffusion rate α. Although the support of the initial density of bacteria in the environment is concentrated in the interval $[0.4, 0.6]$, we observe numerically that hens are uniformly contaminated. However, if we consider the local case 2), the diffusion rate influences the contamination process. In Figure 2, we have the same initial distribution of hens and bacterial population as in Figure 1, but in this case, we observe that the contamination of hens is located around the initial support of bacterial distribution.

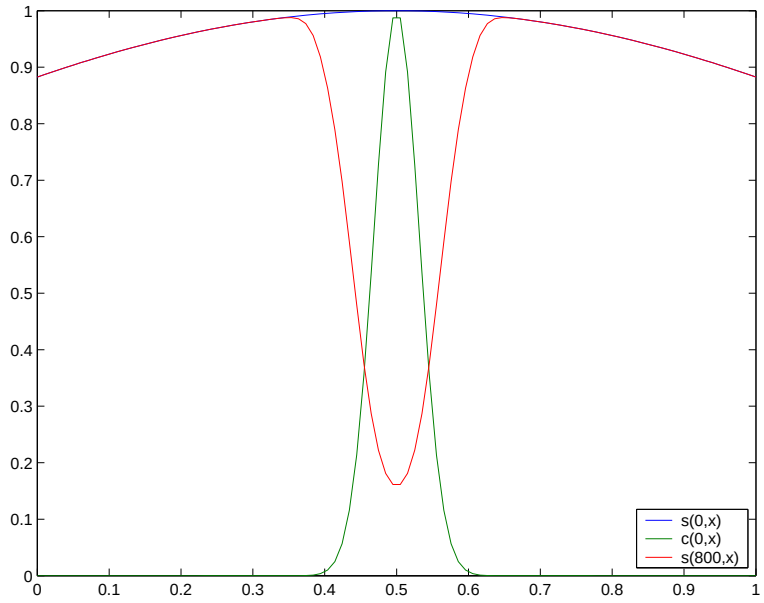

Figure 2: This figure corresponds to the local case with $\alpha = 0.0005$. The green curve corresponds to the initial distribution of bacteria, the blue curve corresponds the initial distribution of susceptible hens, and red curve corresponds to distribution of S-infectious at time t=800.

To conclude this section, we present two more figures, in which we increase the diffusion rate, and we observe that the corresponding asymptotic distribution of susceptible hens converges to the asymptotic distribution of the global case. In real hen houses, it is known that the diffusion of *Salmonella* is large. The results of Nakamura et al. [22] may illustrate the large diffusion rate. Indeed, those experiments start by inoculation of 50% of the population and observe a contamination of 87% of hens only six days later. We also refer to Hollinger [18] for further results about dispersion and persistence of *Salmonella* in hen houses.

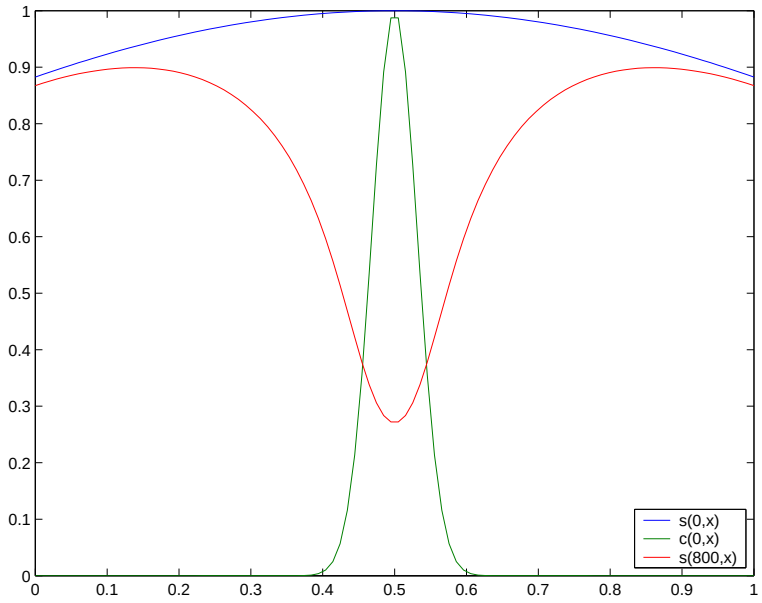

Figure 3: Local case with $\alpha = 0.01$

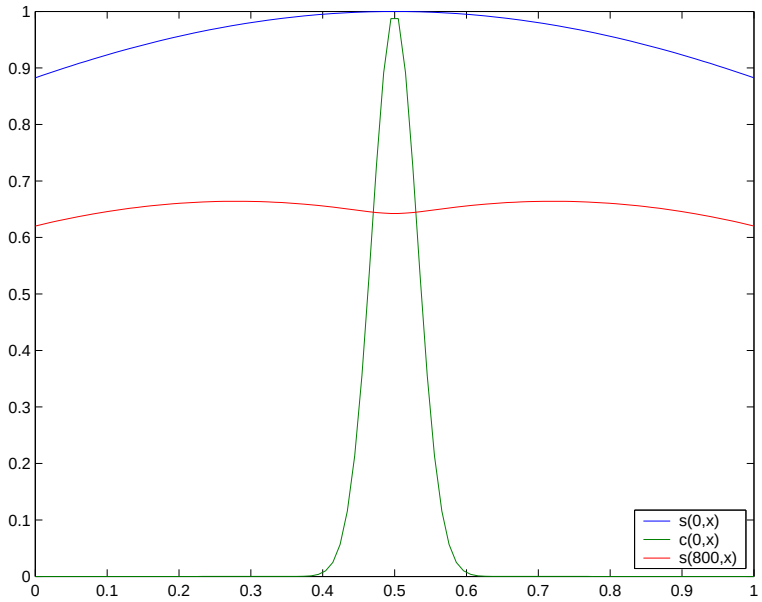

Figure 4: Local case with $\alpha = 0.1$

3. Existence and uniqueness of solutions

In this section, we investigate the existence and uniqueness of solutions of the model for both the global and local cases.

3.1. Global case

We first consider the global case, which corresponds to the following system

$$\left\{\begin{array}{lcl} \partial s(t,x)/\partial t &=& -\kappa C(t)s(t,x)+\nu r(t,x),\\ \partial i_D(t,x)/\partial t &=& \kappa C(t)s(t,x)-g i_D(t,x),\\ \partial i_S(t,x)/\partial t &=& g i_D(t,x)-\eta i_S(t,x),\\ \partial r(t,x)/\partial t &=& \eta i_S(t,x)-\nu r(t,x),\\ \partial c(t,x)/\partial t &=& \alpha^2\Delta_x c(t,x)+\beta_D i_D(t,x)+\beta_S i_S(t,x)-\lambda c(t,x). \end{array}\right. \tag{3.1}$$

From now on, we denote by

$$X = L^1(\Omega)^5, \text{ and } X_+ = L^1(\Omega)^5_+,$$

where X is endowed with the usual product norm.

We assume that $\Omega \subset \mathbb{R}^2$ is an open subset of class $C^{2+\gamma}$ for some $\gamma \in (0,1)$. Consider $\widehat{B} : C^2(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ the linear operator defined by

$$\widehat{B}\phi = \alpha^2\Delta\phi, \forall \phi \in D(\widehat{B}) = \left\{\phi \in C^2(\overline{\Omega}) : \eta(x).\bigtriangledown\phi(x) = 0, \forall x \in \partial\Omega\right\},$$

where $\eta(x)$ denotes the outer normal unit vector at the boundary point $x \in \partial\Omega$.

By combining the approach used by Pazy [27, Section 7.4 p.218-219], and Theorem 2.3 p.33 in the book of Wu [33] (which is due to Mora [21, Theorem 2.4]), we deduce that $\widehat{B}$ is closable in $L^1(\Omega)$, and $\alpha^2\Delta$ its closure is the infinitesimal generator of an analytic semigroup $\{T_{\alpha^2\Delta}(t)\}_{t\geq 0}$ on $L^1(\Omega)$, and $(0,+\infty) \subset \rho(\alpha^2\Delta)$ the resolvent set of $\alpha^2\Delta$. Moreover, from the maximum principle for the Laplacian operator with Neumann boundary conditions in $L^2(\Omega)$, we also deduce that

$$\left(\delta I - \alpha^2\Delta\right)^{-1} L^1_+(\Omega) \subset L^1_+(\Omega), \forall \delta > 0.$$

Therefore, $\alpha^2\Delta$ is the infinitesimal generator of a positive analytic semigroup.

We consider $A : D(A) \subset X \to X$ the linear operator defined by

$$A\begin{pmatrix} s\\ i_D\\ i_S\\ r\\ c \end{pmatrix} = \begin{pmatrix} \nu r\\ -g i_D\\ g i_D - \eta i_S\\ \eta i_S - \nu r\\ \alpha^2\Delta c + \beta_D i_D + \beta_S i_S - \lambda c \end{pmatrix},$$

with

$$D(A) = L^1(\Omega)^4 \times D(\alpha^2\Delta).$$

Then, by using classical bounded perturbation technics (see for example Pazy [27] Chapter 3), we deduce that A generates $\{T_A(t)\}_{t\geq 0}$ a strongly continuous semigroup of positive and bounded linear operator on X. We define $F : X \to X$ by

$$F\begin{pmatrix} s \\ i_D \\ i_S \\ r \\ c \end{pmatrix} = \begin{pmatrix} -\kappa s C(t) \\ \kappa s C(t) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

is a Lipschitz continuous function on the bounded sets of X.

The system (3.1) can be rewritten as the abstract semi-linear Cauchy problem

$$\frac{dU(t)x}{dt} = AU(t)x + F(U(t)x), t \geq 0,\ U(0)x = x. \tag{3.2}$$

We now recall that a family of maps $\{U(t)\}_{t\geq 0}$, on a metric space (M, d) is called a **continuous semiflow** if the three following assertions are satisfied:

i) $U(0) = Id$;

ii) $U(t + s) = U(t) \circ U(s), \forall t, s \geq 0$;

iii) The map $(t, x) \longrightarrow U(t)x$ is continuous map from $\mathbb{R}_+ \times M$ into M.

Then we have the following result about existence and uniqueness for system (3.2).

Theorem 1. *There exists a continuous semiflow $\{U(t)\}_{t\geq 0}$ on X_+ such that for each $x \in X_+$, $U(.)x \in C([0, +\infty), X_+)$ is the unique solution of*

$$U(t)x = T_A(t)x + \int_0^t T_A(t - s)F(U(s)x)ds, \forall t \geq 0. \tag{3.3}$$

Moreover, if we consider

$$K(t) = \int_\Omega k(t, x)dx, \forall k = s, i_D, i_S, r, c,$$

then $S(t)$, $I_D(t)$, $I_S(t)$, $R(t)$, $C(t)$, satisfy the following system of ordinary differential equations:

$$\begin{cases} dS(t)/dt &= -\kappa C(t)S(t) + \nu R(t), \\ dI_D(t)/dt &= \kappa C(t)S(t) - gI_D(t), \\ dI_S(t)/dt &= gI_D(t) - \eta I_S(t), \\ dR(t)/dt &= \eta I_S(t) - \nu R(t), \\ dC(t)/dt &= \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t). \end{cases} \tag{3.4}$$

Proof. The existence of a unique positive maximal solution for each positive initial value follows from the fact that A generates a positive semigroup on X, F is Lipschitz on bounded sets of X, and the following property for each $M > 0$, there exists $\delta > 0$, such that

$$(F + \delta I)(x) \geq 0, \forall x \in B(0, M) \cap X_+.$$

So for each $x \in X_+$, we know that (3.2) has a unique maximal solution $U(t)x \in C([0, T(x)), X_+)$ with the following property

$$T(x) < +\infty \Rightarrow \lim_{t \to T(x)^-} \|U(t)x\| = +\infty. \tag{3.5}$$

We refer to Cazenave and Haraux [5] for further precisions. By integration on Ω of each component of (3.1) we obtain the system (3.4), and we obtain

$$\frac{d}{dt}\left[S(t) + I_D(t) + I_S(t) + R(t)\right] = 0,$$

so

$$0 \leq K(t) \leq \overline{N}, \forall t \in [0, T(x)), \forall K = S, I_D, I_S, R.$$

Moreover,

$$dC(t)/dt \leq (\beta_D + \beta_S)\bar{N} - \lambda C(t) \Rightarrow C(t) \leq \hat{C}(t); \forall t \in [0, T(x)),$$

where $\hat{C}(t)$is the unique solution of

$$d\hat{C}(t)/dt = (\beta_D + \beta_S)\bar{N} - \lambda\hat{C}(t), \forall t \geq 0, \text{ with } \hat{C}(0) = C(0).$$

Finally, if $T(x) < +\infty$, we have

$$\sup\{\|U(t)x\|, t \in [0, T(x))\} < +\infty,$$

a contradiction with (3.5). So $T(x) = +\infty$. □

3.2. Local case

The local case corresponds to the following system

$$\begin{cases} \partial s(t,x)/\partial t &= -\kappa c(t,x)s(t,x) + \nu r(t;x), \\ \partial i_D(t,x)/\partial t &= \kappa c(t,x)s(t,x) - g i_D(t,x), \\ \partial i_S(t,x)/\partial t &= g i_D(t,x) - \eta i_S(t,x), \\ \partial r(t,x)/\partial t &= \eta i_S(t,x) - \nu r(t;x), \\ \partial c(t,x)/\partial t &= \alpha^2 \Delta c(t,x) + \beta_D i_D(t,x) + \beta_S i_S(t,x) - \lambda c(t,x). \end{cases} \tag{3.6}$$

We define

$$D_\delta = \left\{(s, i_D, i_S, r) \in L^1(\Omega)^4 : s, i_D, i_S, r \geq 0 \text{ and } s + i_D + i_S + r \leq \delta 1_\Omega\right\},$$

and

$$\hat{D}_\delta = \left\{\varphi \in L^1(\Omega, \mathbb{R}) : 0 \leq \varphi \leq \delta 1_\Omega\right\}.$$

The system (3.6) is now considered as the following abstract Cauchy problem on $D_{\delta_1} \times \hat{D}_{\delta_2}$

$$\frac{du}{dt} = Au(t) + H\left(u(t)\right), t \geq 0,\ u(0) = x \in D_{\delta_1} \times \hat{D}_{\delta_2}, \tag{3.7}$$

where

$$H\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} G\begin{pmatrix} v \\ c \end{pmatrix} \\ L(v) \end{pmatrix},$$

with $G : D_{\delta_1} \times \hat{D}_{\delta_2} \to L^1(\Omega, \mathbb{R})^4$ and $L : L^1(\Omega, \mathbb{R})^4 \to L^1(\Omega, \mathbb{R})$ are defined by

$$G\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} -\kappa sc \\ \kappa sc \\ 0 \\ 0 \end{pmatrix},\ L\left(v\right) = \beta_D i_D + \beta_S i_S$$

whenever $v = (s, i_D, i_S, r) \in D_{\delta_1}$, and $c \in \hat{D}_{\delta_2}$.

We also note that by using the matrix operator formalism,we have

$$A = \begin{pmatrix} B & 0 \\ 0 & \alpha^2\Delta \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & \nu Id \\ 0 & -gId & 0 & 0 \\ 0 & gId & -\eta Id & 0 \\ 0 & 0 & \eta & -\nu Id \end{pmatrix}.$$

By choosing some suitable $\delta_1 > 0$ and $\delta_2 > 0$, we can prove the existence and the uniqueness of the solution of (3.7) on $D_{\delta_1} \times \hat{D}_{\delta_2}$.

Theorem 2. *Let be $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_2 > (\beta_D + \beta_S)\,\delta_1/\lambda$. Then there exists a continuous semiflow $\{U(t)\}_{t\geq 0}$ on $D_{\delta_1} \times \hat{D}_{\delta_2}$ such that $\forall x \in D_{\delta_1} \times \hat{D}_{\delta_2}, U(.)x$ is the unique mild solution of (3.7) in $C([0, +\infty), D_{\delta_1} \times \hat{D}_{\delta_2})$.*

Proof. First, it is clear that since H is Lipschitz continuous on $D_{\delta_1} \times \hat{D}_{\delta_2}$, the Cauchy problem has at most one solution on $D_{\delta_1} \times \hat{D}_{\delta_2}$. In the following we demonstrate that such a solution exits.

Let $(v_0, c_0) \in D_{\delta_1} \times \hat{D}_{\delta_2}$ be fixed. We consider the following fixed problem

$$\begin{cases} v = \Psi_1(C), \\ C = \Psi_2(v), \end{cases}$$

where $\Psi_1(c)(t)$ is the unique solution of

$$v(t) = e^{Bt}v_0 + \int_0^t e^{B(t-s)}F\left(v(s), c(s)\right) ds,$$

and

$$\Psi_2(v)(t) := T_{\alpha^2\Delta-\lambda I}(t)c_0 + \int_0^t T_{\alpha^2\Delta-\lambda I}(t-s)L(v(s))ds,$$

where $T_{\alpha^2\Delta-\lambda I}(t) = e^{\lambda t}T_{\alpha^2\Delta}(t), \forall t \geq 0$, is the C_0-semigroup generated by $\alpha^2\Delta - \lambda I$.

We have $T_{\alpha^2\Delta}(t)1_\Omega = 1_\Omega, \forall t \geq 0$. Assuming that $v \in D_{\delta_1}$, then we have

$$\Psi_2(v)(t) = T_{\alpha^2\Delta-\lambda I}(t)c_0 + \int_0^t T_{\alpha^2\Delta-\lambda I}(t-s)\left[\beta_D i_D(s) + \beta_S i_S(s)\right] ds,$$

so

$$0 \leq H_2(v)(t) \leq \max(\delta_2, (\beta_D + \beta_S)\delta_1/\lambda)1_\Omega.$$

Consequently, for each $\delta_2 \geq (\beta_D + \beta_S)\delta_1/\lambda$, we have

$$\Psi_2(C\left([0,\tau], D_{\delta_1}\right)) \subset C\left([0,\tau], \hat{D}_{\delta_2}\right), \forall \tau \geq 0.$$

Next, we remark that $e^{(B-\delta I)t} \geq 0, \forall t \geq 0$, $\forall \delta \geq 0$, and if we set $v(t) = \Psi_1(c)(t)$ for some $c \in C\left([0,\tau], \hat{D}_{\delta_2}\right)$, we have for each $\delta \geq 0$,

$$v(t) = e^{(B-\delta I)t}v_0 + \int_0^t e^{(B-\delta I)(t-s)}\left[G\left(v(s), c(s)\right) + \delta v(s)\right] ds,$$

since $c \in C\left([0,\tau], \hat{D}_{\delta_2}\right)$, it follows that

$$v(t) \geq 0, \forall t \geq 0. \tag{3.8}$$

Moreover

$$\frac{dv(t)}{dt} = Bv(t) + G(v(t), c(t)), \forall t \geq 0,$$

and if we set $v(t) = (s(t), i_D(t), i_S(t), r(t))$, then we deduce that

$$\frac{d\left(s(t) + i_D(t) + i_S(t) + r(t)\right)}{dt} = 0, \forall t \geq 0. \tag{3.9}$$

Now by combining (3.8) and (3.9), we deduce that

$$\Psi_1(C\left([0,\tau], \hat{D}_{\delta_2}\right)) \subset C\left([0,\tau], D_{\delta_1}\right), \forall \tau \geq 0.$$

Finally by using Gronwall's lemma, we deduce that there exist $K_1 > 0$ and $K_2 > 0$ such that for each $i = 1, 2$, Ψ_i is τK_i-Lipschitz continuous. So for $\tau > 0$ small enough, $\Psi_1 \circ \Psi_2$ is a contraction strict independently of the initial value (v_0, c_0). The result follows. □

4. Asymptotic behavior in the global case

In this section we investigate the asymptotic behavior of the model in the global case. Therefore, we consider the system of ordinary differential equations

$$\left\{\begin{array}{lcl} dS(t)/dt &=& -\kappa C(t)S(t) + \nu R(t), \\ dI_D(t)/dt &=& \kappa C(t)S(t) - gI_D(t), \\ dI_S(t)/dt &=& gI_D(t) - \eta I_S(t), \\ dR(t)/dt &=& \eta I_S(t) - \nu R(t), \\ dC(t)/dt &=& \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t), \end{array}\right. \tag{4.1}$$

with initial values

$$S(0) = S_0, I_D(0) = I_{D0}, I_S(0) = I_{S0}, R(0) = R_0, \text{ and } C(0) = C_0,$$

with $S_0 \geq 0, I_{D0} \geq 0, I_{S0} \geq 0, R_0 \geq 0$, and $C_0 \geq 0$.

In the sequel, we will make the following assumption.

Assumption 4.1: *We assume that* $\kappa > 0$, $g > 0$, $\eta > 0$, $\lambda > 0$, $\beta_D + \beta_S > 0$, *and*

$$\overline{N} := S_0 + I_{D0} + I_{S0} + R_0 > 0.$$

From here on, we focus on the asymptotic behavior of system (4.1). We consider separately the cases $\nu = 0$ and $\nu > 0$ which correspond to a perfect (respectively a partial) immunization of hens after one infection.

4.1. Case $\nu = 0$

The case $\nu = 0$ can be considered as an extended SI model without input flux for susceptible hens. Hence, the idea is to extend the method used by Hethcote [14, 15, 16] to describe the asymptotic behavior of the model.

From now on, we set

$$a := \frac{\eta g \lambda}{\kappa(\eta\beta_D + g\beta_S)}.$$

Theorem 3. *Let Assumption 4.1 be satisfied. Assume in addition that* $\nu = 0$. *Then*

$$(I_D(t), I_S(t), C(t)) \to (0, 0, 0) \text{ as } t \to +\infty,$$

and we have the following alternative:

(i) If $I_{D0} + I_{S0} + C_0 = 0$ *or* $S_0 = 0$, *then*

$$(S(t), R(t)) \to \left(S_0, \overline{N} - S_0\right) \text{ as } t \to +\infty.$$

(ii) If $S_0 > 0$ and $I_{D0} + I_{S0} + C_0 > 0$, then

$$(S(t), R(t)) \to \left(S_\infty, \overline{N} - S_\infty\right) \text{ as } t \to +\infty,$$

where S_∞ is the unique root in $(0, a]$ of the equation

$$\sigma - S_\infty + a\ln(S_\infty) = 0, \tag{4.2}$$

where

$$\sigma = S_0 + I_{D0} + a\left[\frac{\beta_S}{\eta}\frac{\kappa}{\lambda}I_{S0} + \frac{\kappa}{\lambda}C_0 - \ln(S_0)\right].$$

Proof. The proof of (i) is trivial. We now prove (ii). Consider the following system

$$\begin{cases} dS(t)/dt &= -\kappa S(t)C(t), & S(0) = S_0 > 0, \\ dI_D(t)/dt &= \kappa S(t)C(t) - gI_D(t), & I_D(0) = I_{D0} \geq 0, \\ dI_S(t)/dt &= gI_D(t) - \eta I_S(t), & I_S(0) = I_{S0} \geq 0, \\ dC(t)/dt &= \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t), & C(0) = C_0 \geq 0. \end{cases} \tag{4.3}$$

From the first equation of (4.3), we deduce that $t \to S(t)$ is non-increasing. Hence,

$$\lim_{t \longrightarrow +\infty} S(t) = S_\infty \geq 0.$$

We can split the last equation of (4.3), into a system

$$\begin{cases} dC_D(t)/dt = \beta_D I_D(t) - \lambda C_D(t), & C_D(0) = C_{D0} \geq 0, \\ dC_S(t)/dt = \beta_S I_S(t) - \lambda C_S(t), & C_S(0) = C_{S0} \geq 0, \end{cases}$$

with

$$C_{D0} + C_{S0} = C_0,$$

and we obtain

$$C(t) = C_D(t) + C_S(t), \forall t \geq 0.$$

We set

$$A_{I_S} = \frac{\kappa\beta_S}{\lambda\eta}a, \text{ and } A_D = A_S = \frac{a\kappa}{\lambda}.$$

We will use the following change of variable

$$\begin{cases} \hat{C}_D(t) &= A_D C_D(t), & \hat{C}_D(0) = A_D C_D(0) = A_D C_{D0}, \\ \hat{C}_S(t) &= A_S C_S(t), & \hat{C}_S(0) = A_S C_S(0) = A_S C_{S0}, \\ \hat{I}_S(t) &= A_{I_S} I(t), & \hat{I}_S(0) = A_{I_S} I(0) = A_{I_S} I_{S0}. \end{cases}$$

Then from (4.3), we derive the following system of ordinary differential equations

$$\begin{cases} dS(t)/dt &= -\kappa S(t)[C_D(t) + C_S(t)], & S(0) = S_0 > 0, \\ dI_D(t)/dt &= \kappa S(t)[C_D(t) + C_S(t)] - gI_D(t), & I_D(0) = I_{D0} \geq 0, \\ d\hat{I}_S(t)/dt &= A_{I_S} gI_D(t) - \eta\hat{I}_S(t), & \hat{I}_S(0) = A_{I_S} I_{S0} \geq 0, \\ d\hat{C}_D(t)/dt &= A_D\beta_D I_D(t) - \lambda\hat{C}_D(t), & \hat{C}_D(0) = A_D C_{D0} \geq 0, \\ d\hat{C}_S(t)/dt &= \frac{A_S\beta_S}{A_{I_S}}\hat{I}_S(t) - \lambda\hat{C}_S(t), & \hat{C}_S(0) = A_S C_{S0} \geq 0. \end{cases} \tag{4.4}$$

Let be $V(t)$ defined by

$$V(t) = S(t) - a\ln(S(t)) + I_D(t) + \hat{I}_S(t) + \hat{C}_D(t) + \hat{C}_S(t),$$

then we obtain

$$dV(t)/dt = 0.$$

Consequently

$$V(t) = \sigma, \forall t \geq 0, \tag{4.7}$$

where

$$\sigma = S_0 - a\ln(S_0) + I_{D0} + A_{I_S} I_{S0} + A_D C_{D0} + A_S C_{S0},$$

so

$$\sigma = V(0) = S_0 + I_{D0} + a\left[\frac{\kappa\beta_S}{\lambda\eta} I_{S0} + \frac{\kappa}{\lambda}\left(C_{D0} + C_{S0}\right) - \ln(S_0)\right],$$

and since $(C_{D0} + C_{S0}) = C_0$, we obtain

$$\sigma = S_0 + I_{D0} + a\frac{\kappa}{\lambda}\left[\frac{\beta_S}{\eta} I_{S0} + C_0 - \frac{\lambda}{\kappa}\ln(S_0)\right].$$

By using (4.7), we deduce that

$$-a\ln(S(t)) \leq \sigma, \forall t \geq 0,$$

so

$$S(t) \geq e^{-\frac{\sigma}{a}} > 0, \forall t \geq 0.$$

We deduce that

$$\lim_{t\to+\infty} S(t) = S_\infty > 0,$$

and

$$\hat{I}_S(t) + I_D(t) + \hat{C}_D(t) + \hat{C}_S(t) \longrightarrow \sigma - S_\infty + a\ln(S_\infty), t \longrightarrow +\infty.$$

In order to show $\hat{I}_S(t) + I_D(t) + \hat{C}_D(t) + \hat{C}_S(t) \longrightarrow 0$, we consider the system (4.4) on ω-limit set. Then we have

$$\begin{cases} dS^*(t)/dt &= -\kappa S^*(t)[C_D^*(t) + C_S^*(t)], & S^*(0) = S_\infty > 0, \\ dI_D^*(t)/dt &= \kappa S^*(t)[C_D^*(t) + C_S^*(t)] - gI_D^*(t), & I_D^*(0) = I_{D0}^* \geq 0, \\ d\hat{I}_S^*(t)/dt &= A_{I_S} g I_D^*(t) - \eta \hat{I}_S^*(t), & \hat{I}_S^*(0) = I_{S0}^* \geq 0, \\ d\hat{C}_D^*(t)/dt &= A_D \beta_D I_D^*(t) - \lambda \hat{C}_D^*(t), & \hat{C}_D^*(0) = \hat{C}_{D0}^* \geq 0, \\ d\hat{C}_D^*(t)/dt &= \frac{A_S\beta_S}{A_{I_S}} \hat{I}_S^*(t) - \lambda \hat{C}_S^*(t), & \hat{C}_S^*(0) = \hat{C}_{S0}^* \geq 0. \end{cases}$$

Invariance of ω-limit set implies $S^*(t) = S_\infty, \forall t \geq 0,$ and $dS^*(t)/dt = 0$, so

$$-\kappa S_\infty [C_D^*(t) + C_S^*(t)] = 0.$$

But since $\kappa > 0$, and $S_\infty > 0$ then $\hat{C}_D^*(t) = \hat{C}_S^*(t) = 0$. So $\hat{I}_S^*(t)$ and $I_D^*(t)$ satisfy the system

$$\begin{cases} dI_D^*(t)/dt = -gI_D^*(t), \\ d\hat{I}_S^*(t)/dt = A_{I_S} gI_D^*(t) - \eta \hat{I}_S^*(t), \end{cases}$$

and by invariance of the omega-limit set, we deduce that

$$\hat{I}_S^*(t) = I_D^*(t) = 0, \forall t \geq 0.$$

So we obtain

$$\hat{I}_S(t) + I_D(t) + \hat{C}_D(t) + \hat{C}_S(t) \longrightarrow 0, t \longrightarrow +\infty,$$

and we deduce that S_∞ is a root of

$$\sigma - S_\infty + a \ln(S_\infty) = 0.$$

It remains to show that $S_\infty < a$. Assume that $S_\infty > a$. Then we have

$$\begin{cases} dI_D(t)/dt \geq \kappa S_\infty C(t) - gI_D(t), \\ dI_S(t)/dt = gI_D(t) - \eta I_S(t), \\ dC(t)/dt = \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t), \end{cases}$$

so by using classical arguments coming from monotone dynamical system theory, we deduce that

$$I_D(t) \geq I_D^*(t), I_S(t) \geq I_S^*(t), \text{ and } C(t) \geq C^*(t), \forall t \geq 0,$$

where $(I_D^*(t), I_S^*(t), C^*(t))$ is the solution of

$$\begin{cases} dI_D^*(t)/dt = \kappa S_\infty C^*(t) - gI_D^*(t), \\ dI_S^*(t)/dt = gI_D^*(t) - \eta I_S^*(t), \\ dC^*(t)/dt = \beta_D I_D^*(t) + \beta_S I_S^*(t) - \lambda C^*(t), \end{cases}$$

with initial value

$$I_D^*(0) = I_{D0}, I_S^*(0) = I_{S0}, \text{ and } C^*(0) = C_0.$$

Now since by assumption $\beta_D + \beta_S > 0$, and since $I_{D0} + I_{S0} + C_0 > 0$, we deduce

$$I_D^*(t) > 0, I_S^*(t) > 0, C^*(t) > 0, \forall t > 0.$$

Now by using the same change of variable as above we obtain

$$\frac{d}{dt}\left(I_D^*(t) + \hat{I}_S^*(t) + \hat{C}_D^*(t) + \hat{C}_S^*(t)\right) = \kappa\left[S_\infty - a\right]\left[C_D^*(t) + C_S^*(t)\right].$$

Now, if $S_\infty > a$, then we have

$$\frac{d}{dt}\left(I_D(t) + \hat{I}_S(t) + \hat{C}_D(t) + \hat{C}_S(t)\right) > 0, \forall t > 0.$$

So we obtain a contradiction with the fact that $\left(I_D(t) + \hat{I}_S(t) + \hat{C}_D(t) + \hat{C}_S(t)\right)$ converges to 0 as t goes to $+\infty$, and it follows that $S_\infty \leq a$. Finally from the fact that $\overline{N} = S(t) + I_D(t) + I_S(t) + R(t), \forall t \geq 0$, we obtain

$$\lim_{t \longrightarrow +\infty} R(t) = R_\infty = \overline{N} - S_\infty.$$

□

As an immediate consequence of Theorem 3, we have the following result for the system (3.1).

Corollary 4. *Let Assumption 4.1 be satisfied, assume that $\nu = 0$, and that*

$$s_0 + i_{D0} + i_{D0} + r_0 = n.$$

Then $(i_D(t), i_S(t), c(t)) \to (0,0,0)$ as $t \to +\infty$, in $L^1(\Omega)^3$, and we have the following alternative for system (3.1):

(i) If $i_{D0} + i_{S0} + c_0 = 0$ or $s_0 = 0$, then

$$(s(t), r(t)) \to (s_0, n - s_0), \text{ as } t \to +\infty \text{ in } L^1(\Omega)^2.$$

(ii) If $i_{D0} + i_{S0} + c_0 > 0$ and $s_0 > 0$, then

$$(s(t), r(t)) \to \left(S_\infty \frac{s_0(.)}{\int_\Omega s_0(y)dy}, n - S_\infty \frac{s_0(.)}{\int_\Omega s_0(y)dy}\right), \text{ as } t \to +\infty \text{ in } L^1(\Omega)^2,$$

where S_∞ is the solution of (4.2).

Proof. i) is trivial. To prove ii) it is sufficient to remark that

$$s(t,x) = \exp\left(-\int_0^t \kappa C(s)ds\right) s_0(x),$$

thus

$$\frac{s(t,x)}{\int_\Omega s(t,y)dy} = \frac{s_0(x)}{\int_\Omega s_0(y)dy},$$

so

$$\lim_{t \to +\infty} s(t,x) = \lim_{t \to +\infty} \int_\Omega s(t,x)dx \frac{s_0(x)}{\int_\Omega s_0(y)dy} = S_\infty \frac{s_0(x)}{\int_\Omega s_0(y)dy}.$$

The last part of the corollary follows from the fact that

$$\frac{d}{dt}\left(s(t,x) + i_D(t,x) + i_S(t,x) + r(t,x)\right) = 0.$$

□

4.2. Case $\nu > 0$

In this subsection we consider system (4.1) when $\nu > 0$. We study the stability of equilibriums (*i.e.* disease free and endemic equilibriums), and the uniform persistence of the disease. We also provide a sufficient condition for the global asymptotic stability of the endemic equilibrium.

From now on, we denote by

$$R_0 = \bar{N}\frac{\kappa}{\lambda}\left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right),$$

the basic reproductive number.

Equilibriums: The disease free equilibrium of system (4.1) is

$$\overline{X}_f = \left(\overline{N}, 0, 0, 0, 0\right).$$

If $R_0 \leq 1$, system (4.1) has no other equilibrium. If $R_0 > 1$, the system (4.1) has a unique endemic equilibrium given by

$$\overline{X}_e = \left(\frac{\bar{N}}{R_0}, \frac{\bar{N}\kappa}{gR_0}\bar{C}, \frac{\bar{N}\kappa}{\eta R_0}\bar{C}, \frac{\bar{N}\kappa}{\nu R_0}\bar{C}, \bar{C}\right),$$

where

$$\bar{C} = \frac{(R_0 - 1)}{\kappa\left(\frac{1}{g} + \frac{1}{\eta} + \frac{1}{\nu}\right)}.$$

Uniform Persistence and Extinction: To study the uniform persistence of *D*-infectious hens, we will use the following auxiliary lemma.

Lemma 5. *Let Assumption 4.1 be satisfied, and consider the matrix*

$$B = \begin{pmatrix} -g & 0 & \theta \\ g & -\eta & 0 \\ \beta_D & \beta_S & -\lambda \end{pmatrix}, \text{ with } \theta > 0.$$

We set

$$\rho := \frac{\theta}{\lambda}\left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right).$$

Then there exist $(a, b, c) \in (0, +\infty)^+$ *and* $s(B) \in \mathbb{R}$*, such that*

$$(a, b, c)\, B = s(B)\, (a, b, c).$$

Moreover we have the following alternative:

$$\textit{1) } \rho > 1 \Leftrightarrow s(B) > 0;\ \textit{2) } \rho = 1 \Leftrightarrow s(B) = 0;\ \textit{3) } \rho < 1 \Leftrightarrow s(B) < 0.$$

Proof. We remark that $\delta I + B$ is an irreducible matrix for all $\delta > 0$ large enough (see example Horn and Johnson [17] or Minc [20] for a precise definition of irreducibility). It follows that there exist some vector $(a, b, c) \in (0, +\infty)^3$ and $s(B) \in \mathbb{R}$ such that

$$(a, b, c)\, B = s(B)\, (a, b, c)\,,$$

where $s(B)$ is the largest real eigenvalue of B. The characteristic polynomial of B is

$$P(\gamma) = -\,(\gamma + g)\,(\gamma + \eta)\,(\gamma + \lambda) + \theta\,(\beta_S g + \beta_D\,(\gamma + \eta))\,.$$

We remark that

$$P(0) = \theta\,(\beta_S g + \beta_D \eta) - g\eta\lambda.$$

Since $P(\gamma) \to -\infty$ as $\gamma \to +\infty$, and $s(B)$ is the largest real eigenvalue of B, it is clear that

$$P(0) > 0 \Rightarrow s(B) > 0,$$

or equivalently

$$\frac{\theta}{\lambda}\left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right) > 1 \Rightarrow s(B) > 0.$$

We have

$$P'(\gamma) = -\left[(\gamma + \eta)\,(\gamma + \lambda) + (\gamma + g)\,(\gamma + \lambda) + (\gamma + g)\,(\gamma + \eta)\right] + \theta\beta_D,$$

$$P''(\gamma) = -2\left[(\gamma + \lambda) + (\gamma + g) + (\gamma + \eta)\right].$$

We note that $P''(0) = -2\,(g + \eta + \lambda) < 0$, so $P''(\gamma) < 0, \forall \gamma > 0$, and $P'(\gamma)$ is decreasing on $[0, +\infty)$. If $P(0) \leq 0$ ($\Leftrightarrow \theta\,(\beta_S g + \beta_D \eta) \leq g\eta\lambda$.) we have

$$P'(0) = -\left[\eta\lambda + g\lambda + g\eta\right] + \theta\beta_D,$$

since $\theta\beta_D\eta \leq g\eta\lambda$, we deduce that $\theta\beta_D - g\lambda \leq 0$, so $P'(0) < 0$. The result follows. □

From now on, we set

$$M = \left\{(S, I_D, I_S, R, C) \in \mathbb{R}^5_+ : S + I_D + I_S + R = \overline{N}\right\},$$

and

$$M_0 := \left\{(S, I_D, I_S, R, C) \in \mathbb{R}^5_+ : S + I_D + I_S + R = \overline{N},\ \text{and } I_D > 0\right\}.$$

We denote by $\{\Pi(t)\}_{t\geq 0}$ the continuous semiflow generated by system (4.1) on M. Then it is clear that M_0 is positively invariant by $\{\Pi(t)\}_{t\geq 0}$. We set

$$\partial M_0 = M \setminus M_0.$$

The following theorem concerns the strong uniform persistence of *D*-infected hens for system (4.1), or equivalently the strong uniform persistence of $\{\Pi(t)\}_{t\geq 0}$ with respect to $(\partial M_0, M_0)$.

Theorem 6. ***(Uniform persistence)***

Let Assumption 4.1 be satisfied. Assume in addition that $\nu > 0$, and $R_0 > 1$. Then there is strong uniform persistence of D-infected hens for system (4.1). That is to say that there exists $\varepsilon_D > 0$, such that for each non-negative initial value of system (4.1) in M_0, we have

$$\liminf_{t\to+\infty} I_D(t) \geq \varepsilon_D.$$

Moreover for each $K = S$, I_S, R, or C, we also have

$$\liminf_{t\to+\infty} K(t) \geq \varepsilon_K,$$

where $\varepsilon_K > 0$ is an appropriated constant, which is also independent of the initial value in M_0.

Proof. To prove this theorem, we use the fact that weak implies strong uniform persistence. We refer to [11, 28, 29, 19] for precise definitions and results. Assume that $\{\Pi(t)\}_{t\geq 0}$ is not weakly uniformly persistence with respect to $(M_0, \partial M_0)$. Then for each $\varepsilon > 0$, we can find $x_0 = (S_0, I_{D0}, I_{S0}, R_0, C_0) \in M_0$ such that

$$\limsup_{t\to+\infty} I_D(t) < \varepsilon.$$

So $\varepsilon_1 > 0$ be fixed such that

$$\left[\bar{N} - \varepsilon_1\right] \frac{\kappa}{\lambda} \left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right) > 1.$$

Let $\varepsilon_0 > 0$ be fixed such that

$$\varepsilon_0 \left(1 + \frac{g}{\eta} + \frac{g}{\nu}\right) \leq \frac{\varepsilon_1}{2}.$$

By construction we can find an initial value such that

$$I_D(t) \leq \varepsilon_0, \forall t \geq 0.$$

This implies

$$\limsup_{t\to+\infty} I_S(t) \leq \frac{g\varepsilon_0}{\eta}, \text{ and } \limsup_{t\to+\infty} R(t) \leq \frac{g\varepsilon_0}{\nu}.$$

So there exists $t_1 > 0$, such that

$$I_D(t) + I_S(t) + R(t) \leq \varepsilon_1, \forall t \geq t_1.$$

Without loss of generality, we can assume that $t_1 = 0$ (by replacing the initial value x_0 by $\Pi(t_1)x_0$). Now, since

$$S(t) + I_D(t) + I_S(t) + R(t) = \overline{N}, \forall t \geq 0,$$

we can reduce the system (4.1) to the following system of ordinary differential equations

$$\begin{cases} dI_D(t)/dt = \kappa C(t)\left[\overline{N} - (I_D(t) + I_S(t) + R(t))\right] - gI_D(t), \\ dI_S(t)/dt = gI_D(t) - \eta I_S(t), \\ dR(t)/dt = \eta I_S(t) - \nu R(t), \\ dC(t)/dt = \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t). \end{cases} \tag{4.8}$$

By construction we have

$$\begin{cases} dI_D(t)/dt \geq \kappa C(t)\left[\overline{N} - \varepsilon_1\right] - gI_D(t), \\ dI_S(t)/dt = gI_D(t) - \eta I_S(t), \\ dC(t)/dt = \beta_D I_D(t) + \beta_S I_S(t) - \lambda C(t). \end{cases}$$

So, by using monotonicity arguments, we have

$$I_D(t) \geq I_D^*(t), I_S(t) \geq I_S^*(t), C(t) \geq C^*(t), \forall t \geq 0,$$

where $(I_D^*(t), I_S^*(t), C^*(t))$ is the solution of

$$\begin{cases} dI_D^*(t)/dt = \kappa\left[\overline{N} - \varepsilon_1\right] C^*(t) - gI_D^*(t), \\ dI_S^*(t)/dt = gI_D^*(t) - \eta I_S^*(t), \\ dC^*(t)/dt = \beta_D I_D^*(t) + \beta_S I_S^*(t) - \lambda C^*(t), \end{cases} \tag{4.9}$$

with

$$I_D^*(0) = I_{D0}, I_S^*(0) = I_{S0}, \text{ and } C^*(0) = C_0.$$

The system (4.9) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} I_D^*(t) \\ I_S^*(t) \\ C^*(t) \end{pmatrix} = B\begin{pmatrix} I_D^*(t) \\ I_S^*(t) \\ C^*(t) \end{pmatrix},$$

where B is the matrix defined in Lemma 5 with $\theta = \kappa\left[\overline{N} - \varepsilon_1\right]$. So, by using Lemma 5, we deduce that

$$\frac{d}{dt}(a,b,c)\begin{pmatrix} I_D^*(t) \\ I_S^*(t) \\ C^*(t) \end{pmatrix} = S(B)(a,b,c)\begin{pmatrix} I_D^*(t) \\ I_S^*(t) \\ C^*(t) \end{pmatrix},$$

and since $\left[\bar{N} - \varepsilon_1\right]\frac{\kappa}{\lambda}\left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right) > 1$, Lemma 5 also implies that $S(B) > 0$. So

$$(a,b,c)\begin{pmatrix} I_D^*(t) \\ I_S^*(t) \\ C^*(t) \end{pmatrix} \to +\infty, \text{ as } t \to +\infty,$$

and we obtain a contradiction with the fact that $I_D(t)$, $I_S(t)$, and $C(t)$ are bounded. □

Theorem 7. ***(Extinction of the disease)***
Let Assumption 4.1 be satisfied. Assume in addition that $\nu > 0$, and $R_0 \leq 1$. Then the disease free equilibrium is global asymptotically stable for system (4.1).

Proof. We have

$$\begin{aligned} dI_D(t)/dt &= \kappa C(t)\left[\overline{N} - (I_D(t) + I_S(t) + R(t))\right] - gI_D(t), \\ &\leq \kappa C(t)\overline{N} - gI_D(t). \end{aligned}$$

So

$$\frac{d}{dt}\begin{pmatrix} I_D(t) \\ I_S(t) \\ C(t) \end{pmatrix} \leq B \begin{pmatrix} I_D(t) \\ I_S(t) \\ C(t) \end{pmatrix},$$

where B is the matrix defined in Lemma 5 with $\theta = \kappa\overline{N}$. By using Lemma 5 the result follows. □

Stability of endemic equilibrium

Theorem 8. ***(Local stability of the endemic equilibrium)***
Let Assumption 4.1 be satisfied. Assume in addition that $\nu > 0$, and $R_0 > 1$. Then the endemic equilibrium of system (4.1) is locally asymptotically stable.

Proof. To prove the local stability of the endemic equilibrium, we compute the linearized equation of system (4.8) at the endemic equilibrium. We obtain the following linear system

$$Y\prime = DY,$$

where $Y = (I_D(t), I_S(t), R(t), C(t))$ and

$$D = \begin{bmatrix} -\kappa\bar{C} - g & -\kappa\bar{C} & -\kappa\bar{C} & \kappa\bar{S} \\ g & -\eta & 0 & 0 \\ 0 & \eta & -\nu & 0 \\ \beta_D & \beta_S & 0 & -\lambda \end{bmatrix},$$

where $\overline{X}_e = \left(\bar{S}, \bar{I}_D, \bar{I}_S, \bar{R}, \bar{C}\right)$ is the endemic equilibrium.

The characteristic polynomial of this matrix D is given by

$$p(\theta) = (\eta + \theta)(\nu + \theta)\, p_1(\theta),$$

where

$$p_1(\theta) = \theta^2 + \left(\kappa\bar{C} + 2g + \lambda\right)\theta + \lambda(2g + \kappa\bar{C}) - \kappa\bar{S}\beta_D.$$

Clearly $-\eta$ and $-\nu$ are two negative roots of $p(\theta)$. Moreover

$$p_1(\theta) > 0, \forall\theta \geq 0 \Leftrightarrow \lambda(2g + \kappa\bar{C}) - \kappa\bar{S}\beta_D > 0.$$

But

$$\bar{C} = \frac{(R_0 - 1)}{\kappa\left(\frac{1}{g} + \frac{1}{\eta} + \frac{1}{\nu}\right)}, \text{ and } \bar{S} = \frac{\bar{N}}{R_0},$$

so

$$\lambda(2g + \kappa\bar{C}) - \kappa\bar{S}\beta_D > 0 \Leftrightarrow 2g + \kappa\bar{C} > \frac{\kappa\bar{N}}{\lambda R_0}\beta_D,$$

and by using the formula for $R_0 = \bar{N}\frac{\kappa}{\lambda}\left(\frac{\beta_D}{g} + \frac{\beta_S}{\eta}\right)$, we obtain

$$\lambda(2g + \kappa\bar{C}) - \kappa\bar{S}\beta_D > 0 \Leftrightarrow 1 + \frac{\kappa\bar{C}}{2g} > \frac{1}{2}\left(\frac{\frac{\beta_D}{g}}{\frac{\beta_D}{g} + \frac{\beta_S}{\eta}}\right).$$

It is clear that the last inequality is always true, consequently $\overline{X}_e$ is locally asymptotically stable. □

Theorem 9. ***(Global stability of the endemic equilibrium)***
Let assumptions of Theorem 8 be satisfied. Assume in addition that

$$\left(\frac{g}{\eta} + \frac{g}{\nu}\right) < 1.$$

Then the endemic equilibrium is globally asymptotically stable for system (4.1) restricted to M_0.

Proof. To study global stability of endemic equilibrium, we first consider $\omega(x_0)$ the omega-limit set of a point $x_0 \in M_0$ for $\{\Pi(t)\}_{t\geq 0}$ the continuous semiflow generated by system (4.1). We consider a complete orbit of system (4.1) in $\omega(x_0)$ passing through $x_1 \in \omega(x_0)$ at $t = 0$. From Theorem 6, we know that

$$I_{D-} := \inf\{I_D : (S, I_D, I_S, R, C) \in \omega(x_0)\} > 0.$$

If we consider a complete orbit of (4.1) in $\omega(x_0)$ we have

$$I_D(t) \geq I_{D-}, \forall t \in \mathbb{R},$$

and that

$$I_S(t) \geq I_{S-} := \frac{g}{\eta}I_{D-}, R(t) \geq R_- := \frac{g}{\nu}I_{D-}, \forall t \in \mathbb{R}.$$

We also have

$$0 \leq S(t) = \overline{N} - (I_S(t) + R(t) + I_D(t)) \leq \overline{N} - (I_{S-} + R_-) - I_D(t), \forall t \in \mathbb{R}. \tag{4.10}$$

So we deduce that for each $t_0 \in \mathbb{R}$,

$$I_D(t) \leq I_{D*}(t), I_S(t) \leq I_{S*}(t), C(t) \leq C_*(t), \forall t \geq t_0,$$

where $(I_{D*}(t), I_{S*}(t), C_*(t))$ is the solution of the monotone system

$$\left\{\begin{array}{lcl} dI_{D*}(t)/dt & = & \kappa C_*(t)\left[\overline{N} - (I_{S-} + R_-) - I_{D*}(t)\right]^+ - gI_{D*}(t), \\ dI_{S*}(t)/dt & = & gI_{D*}(t) - \eta I_{S*}(t), \\ dC_*(t)/dt & = & \beta_D I_{D*}(t) + \beta_S I_{S*}(t) - \lambda C_*(t), \end{array}\right. \tag{4.11}$$

where $x^+ = \max(x, 0)$, and with the initial value

$$I_{D*}(t_0) = I_D(t_0),\ I_{S*}(t_0) = I_S(t_0), C_*(t_0) = C(t_0).$$

Note that the above system is monotone, because if we denote by $G : \mathbb{R}^3 \to \mathbb{R}^3$ the second member of (4.11), then for each $M > 0$, there exists $\lambda = \lambda(M) > 0$, such that $G + \lambda I$ is monotone increasing on $\mathbb{R}^3_+ \cap \overline{B}_{\mathbb{R}^3}(0, M)$ (where $\overline{B}_{\mathbb{R}^3}(0, M) = \{x \in \mathbb{R}^3 : \|x\| \leq M\}$).

The system (4.11) admits a unique positive equilibrium $\left(\overline{I}_{D*}, \overline{I}_{S*}, \overline{C}_*\right)$ which is given by

$$\overline{I}_{S*} = \frac{g}{\eta}\overline{I}_{D*}, \overline{C}_* = \frac{\left(\beta_D + \beta_S \frac{g}{\eta}\right)}{\lambda}\overline{I}_{D*},$$

$$\overline{I}_{D*} = \overline{N}\left(1 - \frac{1}{R_0}\right) - \left(\frac{g}{\eta} + \frac{g}{\nu}\right) I_{D-}.$$

Furthermore, by using upper and lower solutions, it follows that this equilibrium is globally asymptotically stable for system (4.11) restricted to $(0, +\infty)^3$. So in particular, this positive equilibrium attracts the compact subsets of $(0, +\infty)^3$. From this, we deduce that

$$I_D(t) \geq I_{D-}, \forall t \in \mathbb{R} \Rightarrow I_D(t) \leq h(I_{D-}), \forall t \in \mathbb{R},$$

where

$$h(y) = \overline{N}\left(1 - \frac{1}{R_0}\right) - \left(\frac{g}{\eta} + \frac{g}{\nu}\right) y.$$

By symmetry, we also deduce that

$$I_D(t) \leq I_{D+}, \forall t \in \mathbb{R} \Rightarrow I_D(t) \geq h(I_{D+}), \forall t \in \mathbb{R}.$$

Finally, since by assumption $\left(\frac{g}{\eta} + \frac{g}{\nu}\right) < 1$, h is a contraction, and the result follows. □

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