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Piecewise deterministic Markov processes applied to fatigue crack growth modelling

Julien Chiquet\textsuperscript{a},* \ Nikolaos Limnios\textsuperscript{a}, Mohamed Eid\textsuperscript{b}

\textsuperscript{a}Université de Technologie de Compiègne, France
\textsuperscript{b}Commissariat à l'Energie Atomique, Saclay, France

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ABSTRACT

In this paper, we use a particular piecewise deterministic Markov process (PDMP) to model the evolution of a degradation mechanism that may arise in various structural components, namely, the fatigue crack growth. We first derive some probability results on the stochastic dynamics with the help of Markov renewal theory: a closed-form solution for the transition function of the PDMP is given. Then, we investigate some methods to estimate the parameters of the dynamical system, involving Bogolyubov's averaging principle and maximum likelihood estimation for the infinitesimal generator of the underlying jump Markov process. Numerical applications on a real crack data set are given.

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1. Introduction

Let us consider a structure subject to a main degradation process \((Z_t, t \in \mathbb{R}_+ )\) which increases randomly on \(\mathbb{R}^*_+ = (0, \infty)\) until it reaches a critical value \(\lambda > 0\), meaning the failure of the structure. It is assumed that \(Z_t\) is an observable process whose sample paths are obtained from experimental feedback. The reliability of such a system may be expressed by

\[ R(t) = P(Z_t < \lambda), \]

that is the probability that the process \(Z_t\) does not reach the failure boundary on the whole observation interval \([0, t]\).

The dynamical evolution of the continuous-time, real-valued process \(Z_t\) is modelled by a first order stochastic dynamical system of the following form

\[ \frac{dZ_t}{dt} = C(Z_t, X_t), \]

where \(C\) is a function with the appropriate existence and uniqueness properties. The general form of \(C\) is known from physical assumptions on the real phenomenon. The process \(Z_t\) describes the time evolution of this phenomenon whose modelling through classical differential equations is not possible due to, for instance, some random environmental effects or some inhomogeneity in material of the structure. The stochastic process \(X_t\) with state space \(E\) is thus introduced to handle with the randomness of the system.

In our study, \(X_t\) is assumed to be a jump Markov process. The use of jump processes in the framework of stochastic dynamical systems modelling seems interesting from a physical point of view. As a matter of fact, it fits well with the mathematical description of a degradation mechanism, i.e., a process observed on a continuous state space, evolving continuously in time, but whose growth

\* Corresponding author.
E-mail address: julien.chiquet@utc.fr (J. Chiquet).

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rate is likely to change through some “chocs”, even very small, occurring at discrete instants of time. Both intensities and time occurrences of the chocs are random, as described by the jump Markov process.

The coupled process \((Z_t, X_t)\) has the Markov property and no further assumption is required for \(Z_t\). The process \((Z_t, X_t)\) belongs to the class of stochastic models sometimes called \textit{piecewise deterministic Markov processes} (PDMP). PDMP are non-diffusion type stochastic models, combined with the mixture of deterministic and jump motions. Davis (1984, 1993) gave a full description of these stochastic processes as well as the basic associated stochastic calculus. Eq. (2) has also been referred in the literature as a stochastic version of the transport equations, used to model some particle systems (see, e.g., Papanicolaou, 1975; Lapeyre and Pardoux, 1998), thus \((Z_t, X_t)\) is also known as a transport process. Ezhov and Skorokhod (1969) also used similar processes under the name of Switching processes. Koroliuk and Limnios (2005) gave some asymptotic results corresponding to a large class of processes including PDMP. Note also that Jacobsen (2006) recently studied non-homogeneous PDMP and built the associated likelihood. We want to give in this paper some practical estimation methods for a particular PDMP dedicated to degradation modelling.

As a standard example of a degradation mechanism, the fatigue crack growth problem is handled in this paper to validate the models. Fig. 1 represents the extensive set of crack growth data obtained by Virkler et al. (1979) during rigorously controlled laboratory measurements. Roughly speaking, the data set consists in 68 specimens of the same material containing initially the same artificially made 9 mm-crack, that is \(Z_0 = 9\) mm. These samples were exposed to the same cyclic loading. The sample paths represent the evolution in time of the crack size for the 68 specimens, whose measures were performed until they reach the critical value \(\Delta = 49.8\) mm. One can see the large variability of the crack size despite a well controlled experiment. Thus, stochastic dynamical systems were found well adapted to the modelling of fatigue crack growth (see, e.g., Lin and Yang, 1985; Bogdanoff and Kozin, 1985; Sobczyk, 1993; Tanaka, 1999, for an introduction).

The paper is organized as follows: in the next section, we present the family of stochastic dynamical systems involving jump Markov processes of the general form (2) and the settings associated to the modelling of degradation mechanisms. We use Markov renewal theory to obtain a solution of the transition function of the system, with which an expression of the reliability can be given. The third section is dedicated to statistical inferences for such a system, more particularly concerning the estimation of the generator of the jump Markov process \(X_t\), whose paths cannot be directly observed. The only available data concern the process \(Z_t\), whose paths are defined on random length time intervals. Finally, numerical applications for the crack growth problem are given.

2. The model settings

Let \((Z_t, t \in \mathbb{R}_+)\) be an increasing stochastic process on \(\mathbb{R}_+^\star\), with initial condition \(Z_0 = z\). Looking toward reliability analysis, we define for the process \(Z_t\) a set of working states \(U = [z, \Delta]\), with \(0 < z < \Delta\), and a set of down states \(D = [\Delta, \infty)\). Its evolution is continuous in time with values in the real positive axis, thus \(Z_t\) necessarily passes through the point \(\Delta\) while reaching the set of down states.

---

Fig. 1. The Virkler data set for the fatigue crack growth problem.
The evolution of $Z_t$ is modeled through the following stochastic dynamical system

$$
\frac{dZ_t}{dt} = C(Z_t, X_t), \quad Z_0 = z,
$$

with the following assumptions:

**A. 1.** The process $(X_t, t \in \mathbb{R}_+)$ is an irreducible Markov process with a finite state space $E$, initial law $\pi = P(X_0 = i)$, for all $i \in E$, stationary distribution $\pi = (\pi_i)_{i \in E}$ and a matrix generator $A = (a_{ij})_{i,j \in E}$ such as

$$a_{ij} \geq 0 \quad \text{for all } i \neq j,$$

and

$$a_{ii} = -a_{i} = \sum_{k \in E, k \neq i} a_{ik}.$$

**A. 2.** The function $C : \mathbb{R}_+^+ \times E \rightarrow \mathbb{R}_+^+$ is measurable, strictly positive and of the Lipschitz type, that is, there is a function $f : E \rightarrow \mathbb{R}$ such that, for $x, y \in \mathbb{R}_+^+$ and $i \in E$,

$$|C(x, i) - C(y, i)| \leq f(i)|x - y|.$$

**A. 3.** We have $P(Z_0 = z) = 1$, with $0 < z < t$.

Because $X_t$ is Markov, it can be easily seen that the coupled process $(Z_t, X_t)$ is a Markov homogeneous process on the state space $\mathbb{R}_+^+ \times E$, provided that $Z_0$ and $X_0$ are independent. Obviously, the process $Z_t$ evokes in a deterministic way between the jumps of the process $X_t$, occurring at random discrete instants of time $S_0 = 0 < S_1 < \ldots < S_n < \ldots$. Hence, between the jumps, the evolution of a path of $Z_t$ is fully determined by a classical deterministic differential system. For instance, say that $X_t = i$ for $t < S_1$ where $S_1$ stands for the first jump time of $X_t$. Then the Cauchy problem

$$
\frac{dZ_t}{dt} = C(Z_t, i), \quad Z_0 = z,
$$

has a unique solution denoted by $\phi_{z,i}(t)$, that is $(Z_t = \phi_{z,i}(t))_{0 \leq t < S_1}$. We can thus build in a piecewise manner a unique solution $Z_t$ for system (3) on the successive jump time intervals $[S_n, S_{n+1})$ of $X_t$.

There is much to say about the Markov process $(Z_t, X_t)$. First, its generator $\mathcal{A}$ (see, e.g., Lapeyre and Pardoux, 1998) is given by

$$\mathcal{A}f(z, i) = C(z, i) \frac{\partial}{\partial z} f(z, i) + \sum_{j \in E} a_{ij} f(z, j),$$

with $i \in E, z \in \mathbb{R}_+^+$ and $f$ a measurable and differentiable function on the first argument. Unfortunately, the infinitesimal generator cannot always be exploited easily numerically. Thus, it can be more suitable to focus on an other important function of $(Z_t, X_t)$, namely the transition probability function $P(t)$:

$$P_{ij}(z, B, t) := P(Z_t \in B, X_t = j | Z_0 = z, X_0 = i).$$

It is defined for all $z \in \mathbb{R}_+^+, i,j \in E$ and $B \in \mathcal{A}$, with $\mathcal{A}$ being the Borel $\sigma$-field of $\mathbb{R}_+^+$. The transition function permits to give a simple expression of the probability distribution $\mu(t)$ of $(Z_t, X_t)$, using a conditional probability development, thus characterizing well the PDMP:

$$
\mu_{ij}(B, t) = P(Z_t \in B, X_t = j) = \sum_{i \in E} \int_{\mathbb{R}_+^+} P(Z_t \in B, X_t = j | Z_0 = y, X_0 = i) \delta_y(dy) \pi_i
$$

$$= \sum_{i \in E} \pi_i P_{ij}(z, B, t),$$

where $\delta_z$ is the Dirac distribution at point $z$.

Expression (1) of the reliability can be advantageously defined as a function of the coupled process $(Z_t, X_t)$ rather than as an expression of $Z_t$ only. Concerning $(Z_t, X_t)$, $U \times E$ is the set of working states while $D \times E$ is the set of down states. Moreover, we consider here that the system is not a reparable one. Indeed, we have

$$R(t) = P((Z_t, X_t) \in U \times E) = \sum_{j \in E} \mu_j(U, t) = \sum_{i \in E} \pi_i \mu_{ij}(z, U, t).$$
We also introduce the hitting time \( \tau \) to \( D \) for the process \( Z_t \), that is the failure time, defined by
\[
\tau = \inf\{t \geq 0 : Z_t \in D\} = \inf\{t \geq 0 : (Z_t, X_t) \in D \times E\}.
\tag{6}
\]

The distribution function \( F_{\tau} \) of \( \tau \) is directly linked to the reliability through \( F_{\tau}(t) = 1 - R(t) \).

A Markov process is a special case of a Markov renewal process, thus we can associate to \((Z_t, X_t, t \in \mathbb{R}_+\) the extended Markov renewal process \((Z_n, J_n, S_n, n \in \mathbb{N})\), where
\[
Z_n = Z_{S_n}, \quad J_n = X_{S_n}, \quad n \in \mathbb{N},
\]
with \((S_n)_{n \geq 0}\) being the jump times of \( X_t \). The Markov renewal function \( \psi(t) \), which represents at time \( t \) the mean number of times a set of \( \mathbb{R} \times E \) is visited by the process \((Z_t, X_t)\), starting from a given point \((z, i)\), is given by
\[
\psi_{ij}(z, B, t) = E_{z^i}[\text{#transitions in } B, J] = \sum_{n \geq 0} Q_{ij}^{(n)}(z, B, t),
\]
where \( Q \) is the semi-Markov kernel defined by
\[
Q_{ij}(z, B, t) = \Pr(Z_{n+1} \in B | J_{n+1} = j, S_{n+1} - S_n \leq t | Z_n = z, J_n = i).
\]
The successive \( n \)-fold convolution \( Q^{(n)} \) for \( t \geq 0 \) are defined in a Stieltjes sense, that is
\[
Q_{ij}^{(0)}(z, B, t) = 1_{[i=j]} \int_{B} \psi_{Z(t)}(t) \, dt,
\]
\[
Q_{ij}^{(1)}(z, B, t) = Q_{ij}(z, B, t), \quad t > 0,
\]
and, for \( n \geq 2 \),
\[
Q_{ij}^{(n)}(z, B, t) = \sum_{k \in \mathbb{E}} \int_{\mathbb{R}_+} \int_0^t Q_{ik}(z, dy, ds) Q_{kj}^{(n-1)}(y, B, t-s).
\]

It follows from results of Markov renewal theory (e.g., Limnios and Oprişan, 2001) the following proposition.

**Proposition 1.** The transition function \( P(t) \) is given by the following Markov renewal equation
\[
P_{ij}(z, B, t) = g_{ij}(t; z, B) + \sum_{k \in \mathbb{E}} \int_{\mathbb{R}_+} \int_0^t Q_{ik}(z, dy, ds) P_{kj}(y, B, t-s),
\tag{7}
\]
where
\[
Q_{ij}(z, B, dt) = a_{ij} e^{-a_{ij} t} \delta_{\psi_{Z(t)}(t)}(B) \, dt,
\]
and
\[
g_{ij}(z, B) = 1_{[i=j]} g_{\psi_{Z(t)}(t)}(B) \mathbb{1}_{z^i}(t).
\tag{8}
\]

Expression (8) can be computed numerically, thus with (5), the reliability \( R(t) \) of the system as well as the cumulative distribution function of the failure time \( F_{\tau} \) can also be computed. An example of a numerical computation of \( R(t) \), based upon expression (8), is detailed in Chiquet et al. (2007) for a given system with general form (3).

As statistical inference for the dynamical system (3) is the main concern of this paper, we shall consider \( K \) independent sample paths of \( Z_t \) obtained from lab experiments. The paths are denoted by \((Z_t^k)_{k=1, \ldots, K}\) which are supposed to be the only available data for estimating the whole dynamical system. The observation procedure is following: each path \( Z_t^k \) is observed on some random time intervals \([0, \tau^k \wedge T_{\text{obs}}]\), with the \( \tau^k \)’s being \( K \) independent copies of the failure time \( \tau \) defined in (6). It means that each path of \( Z_t \) is observed until it reaches the failure threshold \( \lambda \), provided that \( \tau < T_{\text{obs}} \). Here, \( T_{\text{obs}} \) is a real positive constant representing the maximal time of observation, beyond which the lab experiment become too expensive. If we always can afford to observe the system until it reaches \( \lambda \), then we just put \( T_{\text{obs}} = \infty \), what is the case for the Virkler data set seen on Fig. 1. We point out that \( T_{\text{obs}} \) is the same for any path. We also denote by \( \tau^k \) the successive realizations of the random variables \( \tau^k \wedge T_{\text{obs}} \). Fig. 2 shows a schematic representation of the observation procedure for three paths of \( Z_t \) and the corresponding paths of \( X_t \), which are obviously defined on the same random time intervals as the corresponding \( Z_t^k \)’s.
3. Estimation

In this section we develop the scheme used for the estimation of the system, where we just have some $K$ paths of the observable degradation process $Z_t$. The estimation scheme of (3) splits in two parts:

1. First, we study the asymptotic behavior of the system using the Bogolyubov's averaging principle (see Bogolyubov and Mitropol’skii, 1961). With this technique, the random component “disappears” and we obtain a deterministic system that corresponds to the mean behavior of (3). A regression analysis is performed on the asymptotic, deterministic system so that we estimate the fixed parameters in function $C$, whose general form is assumed to be known.

2. Second, we turn back to the remaining estimation of the random part of the system, that is, of the jump Markov process $X_t$. To this purpose, we estimate the paths of $X_t$ which are not directly observed, as well as its state space. We then build the associated likelihood function, keeping in mind that these paths are defined on random time intervals. The generator of $X_t$ is estimated by maximizing an approached likelihood function.

3.1. Bogolyubov's averaging principle

An approximation of $Z_t$ is obtained by analyzing a system (3) in a series scheme (see, e.g., Koroliuk and Limnios, 2005), that is by studying the weak convergence, when $\varepsilon \to 0$, of

$$\frac{dZ_t^\varepsilon}{dt} = C(Z_t^\varepsilon, X_t^\varepsilon), \quad Z_0^\varepsilon = z,$$

where $X_t$ is assumed to be ergodic. In fact, the change of scale $t \to t/\varepsilon$ is performed for $X_t$ in order to see the behavior of the dynamical system when the random component $X_t$ just adds the information it would add after a very long time of observation of (9), since $t/\varepsilon \to \infty$ when $\varepsilon \to 0$. This so-called averaging approximation was first introduced by Bogolyubov (see Bogolyubov and Mitropol’skii, 1961) who showed that (9) converges weakly when $\varepsilon \to 0$ to the following deterministic system

$$\frac{d\bar{Z}_t}{dt} = \bar{C}(\bar{Z}_t), \quad \bar{Z}_0 = z,$$

with $\bar{Z}_t$ the limit deterministic process and $\bar{C}$ a mean function defined by

$$\bar{C}(z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T C(z, X_t) \, dt \quad a.s.$$
In the particular case where \( X_t \) is an ergodic jump Markov process with a stationary law \( \pi \), we have
\[
\bar{C}(z) = \sum_{i \in E} C(z, i) \pi_i.
\]

Through this averaging technique, we have a limit deterministic system (10) associated to stochastic differential system (3). The fixed parameters appearing in the function \( C \) are the same as the ones appearing in \( \bar{C} \) but in (10) the random part was “eliminated.” With the \( K \) sample paths \( \tilde{X}_t^k \), we can perform a classical regression analysis on (10) to estimate the fixed parameters of \( C \).

By this mean, we obtain the mean deterministic process, that is \( \bar{Z}_t \), associated to the dynamical system. The only unknowns that remain are the initial law \( \pi \), the generator \( A \) and the state space \( E \) of the random process \( X_t \). The stationary law \( \pi \) will be obtained from \( A \) through
\[
\pi A = 0, \quad \sum_{i \in E} \pi_i = 1.
\]

3.2. Estimating the paths and the state space of the jump Markov process

Prior to any estimation of \( A \) or \( \pi \), some representations of the paths \( \{X_t^k\}_{k=1,...,K} \) are needed. To this purpose, it is assumed that there exists a function \( G \) from expression (3) giving \( X_t \) as a function of \( Z_t \) and its derivative denoted by \( \dot{Z}_t = dZ_t/dt \). Such a function does not always exist. Indeed, when the stochastic process \( X_t \) is a linear additive or a multiplicative term in the function \( C \), we may easily find the corresponding function \( G \). This is the case for the so-called Paris–Erdogan model, which will be used to describe the crack growth mechanism of the Virkler data. Hence, we may obtain a first estimation for the \( X_t^k \)'s with the following relationship:
\[
\bar{X}_t^k = G(Z_t^k, \tilde{Z}_t^k),
\]
where the derivative of \( Z_t \) can be estimated through
\[
\dot{Z}_t^k = \frac{Z_{t+\Delta t}^k - Z_t^k}{\Delta t},
\]
with \( \Delta t \) being the time discretization step of the data set.

Due to the way the \( \bar{X}_t^k \) are computed, that is through expression (11), we obtain some noisy paths taking their values in \( \mathbb{R} \), which values may be quite nearby. Our model requires a finite state space for the underlying jump Markov process \( X_t \) with some piecewise shape paths, thus it is appropriate to “regroup” the values which are very close from each other to an unique state. By these means, we build a state space which is an approximation of the real state space. Two cases may occur: first, if the state space \( E \) of the process \( X_t \) is assumed to be known, each point of the \( \bar{X}_t^k \)'s is rounded to the closest element of \( SN \), the so-called embedded Markov chain of the Markov process \( X_t \). The application of one of this clustering methods leads to \( K \) approximated paths \( \bar{X}_t^k \), still defined on the random time interval \([0, t^k \wedge T_{\text{obs}}]\). The \( \bar{X}_t^k \) are then used for further estimations linked to the jump Markov process \( X_t \), namely for estimating \( \pi \) and \( A \).

3.3. Maximum likelihood method

Concerning the initial law of the jump Markov process \( X_t \), it is assumed that \( X_0 \) is independent of the generator and of the censoring time \( \tau \). We thus have the classical empirical estimator (see, e.g., Sadek and Limnios, 2005)
\[
\tilde{Z}_t^k = \frac{1}{R} \sum_{k=1}^K \left( X_t^k, t \wedge \tau^k \right).
\]

It turns out that the estimation problem of the infinitesimal generator \( A \) requires further attention. In order to build the likelihood function for \( A \) on the \( \bar{X}_t^k \)'s, we need to give an expression of the density function of a path of \( X_t \) defined on a random time interval \([0, \tau]\). To this purpose, we first remind fundamental results of statistical inference for jump Markov processes, obtained by Billingsley (1961) and Albert (1962), who built the density function on the space of sample paths defined on the same fixed time interval \([0, \tau]\) with \( \theta \) a real positive constant. The following notation is required in the sequel:

- \( \{S_n, n \in \mathbb{N}\} \), the jump times of the process \( X_t \), with \( S_0 = 0 \);
- \( W_n = S_{n+1} - S_n \), for \( n \in \mathbb{N} \), the sojourn times in the successively visited states;
- \( \{j_n, n \in \mathbb{N}\} \), the so-called embedded Markov chain of the Markov process \( X_t \), that is \( j_n = X_{S_n} \), for \( n \in \mathbb{N} \);
- \( N(\theta) \), the number of jumps in \([0, \theta]\), that is the largest integer for which \( S_{N(\theta)} \leq \theta \);

Let us now build the likelihood function based upon the density paths are not necessarily observed until the random failure time estimated from Section 3.2, defined on $[0, T_{\text{obs}}]$, that is, $V_i(\theta) = \int_0^\theta \delta_{\{X_t = i\}} \, ds$.

Hence, the history $\mathcal{H}_\theta$ of the process $X_t$ on the fixed time interval $[0, \theta]$ can be represented as an ordered sequence: $\mathcal{H}_\theta = ((J_0, W_0), \ldots, (W_{N(\theta)-1}, W_{N(\theta)-1}), (U_{N(\theta)}, U_\theta))$.

where $U_\theta = \theta - S_{N(\theta)}$. It means that the path starts at state $J_0$ at time zero, remains in $J_0$ for $W_0$ units of time, makes a jump to $J_1$, and so on, until it makes the final jump to $J_{N(\theta)}$ where it stays for $U_{\theta}$ units of time, that is until the censoring time $\theta$.

The probability density function $f_A$ associated to a sample path defined on a fixed time interval $[0, \theta]$, was given by Albert (1962). We put $h_\theta = \mathcal{H}_\theta(\omega)$ a realization of the history of $X_t$ associated to a given sample path. It can be shown that:

$$f_A(h_\theta) = \pi(x_0) \prod_{i \neq j} \frac{n_{ij}(\theta)}{n_{ij}} e^{-a_{ij}v_i(\theta)}.$$  

The lower case letters $n_{ij}(\theta), v_i(\theta)$ have been used to write down the realizations associated to the corresponding random variables.

These results can be slightly generalized to get an expression of the density for a path defined on a random time interval $[0, \tau]$, with $\tau$ the random censoring variable. Let us put $\mathcal{H}_\tau$ the history of $X_t$ observed on the random interval $[0, \tau]$. A realization $h_\theta = \mathcal{H}_\theta(\omega)$ of the history associated to a sample path of $X_t$ censored by $\tau$ has the same structure as in the fixed time case. Yet the density $f_A(\cdot)$ of a randomly censored path $h_\theta$ is written as a function of the conditional density $f_A(\cdot | \tau)$ and of the density function $f_\tau$ of $\tau$ through

$$f_A(h_\theta) = f_A(\cdot | \tau = \theta) f_\tau(\theta).$$  

(14)

Let us now build the likelihood function based upon the density $f_A(\cdot | \tau)$, yet under the observation scheme of Fig. 2, that is when the paths are not necessarily observed until the random failure time $\tau$, but until $\tau < T_{\text{obs}}$, that is still random. We have $K$ paths of $X_t^k$ estimated from Section 3.2, defined on $[0, t_k]$, with $t_k$ being a realization of $\tau^k < T_{\text{obs}}$. We associate $K$ ordered sequences $h^k_{\theta}$ to the estimated paths $X^k_{t_k}$'s.

Let us also introduce the following function, associated to the $k$th path:

$$s_\tau^k(t_k) = \left\{ \begin{array}{ll}
  f_\tau(t_k) & \text{if } t_k \leq T_{\text{obs}} \\
  \mathbb{P}(\tau > T_{\text{obs}}) & \text{if } t_k > T_{\text{obs}}
\end{array} \right.$$  

By expression (14) of $f_A(\cdot | \tau)$ and using the function $s_\tau$, it turns out that the corresponding log-likelihood $\mathcal{L}_A$ for the generator $A$ is

$$\mathcal{L}_A(\tau) = \sum_{k=1}^{K} \log f_A(\{h^k_{\theta} | \tau^k = t_k\}) + \sum_{k=1}^{K} \log s_\tau^k(t_k).$$  

(15)

The second term in the right hand of (15) can be written

$$\sum_{k=1}^{K} \log s_\tau^k(t_k) = \sum_{k: t_k < T_{\text{obs}}} \log f_\tau(t_k) + \sum_{k: t_k > T_{\text{obs}}} \log \mathbb{P}(\tau > t_k),$$  

and if we denote by $L$ the number of paths censored by $T_{\text{obs}}$, with $L \leq K$, we have for these $L$ paths $\mathbb{P}(\tau > t_k) = 1 - F_\tau(T_{\text{obs}})$. Hence, the log-likelihood (15) becomes

$$\mathcal{L}_A(\tau) = \sum_{k=1}^{K} \log f_A(\{h^k_{\theta} | \tau^k = t_k\}) + \sum_{k: t_k < T_{\text{obs}}} \log f_\tau(t_k) + L \log(1 - F_\tau(T_{\text{obs}))}.$$  

(16)

The above expression (16) is quite general. Let us here treat the case where the failure time $\tau$ and the process $X_t$ are independent, which is obviously not true when dealing with the dynamical system (3), but can be considered as a first approximation. If so, the
structure of the likelihood is almost the same as in Albert (1962). The conditional density \( f_{A,t} \) turns to \( f_A \) and the log-likelihood \( \mathcal{L}_A \) to be maximized is just

\[
\mathcal{L}_A(K) = \sum_{k=1}^{K} \log f_A(h^k) = \sum_{k=1}^{K} \log \prod_{\substack{ij \in E \ni \neq j}} a_{ij}^{n_{ij}^k(t_k)} e^{-a_{ij}v_i^k(t_k)},
\]

where \( n_{ij}^k \) and \( v_i^k \) are, respectively, the number of transitions from \( i \) to \( j \) and the time spent in \( i \) for the \( k \)th observed history \( h^k \). If we denote \( n_{ij}(K) \) and \( v_i(K) \) the corresponding variables for all of the \( K \) paths, that is

\[
n_{ij}(K) = \sum_{k=1}^{K} n_{ij}^k(t_k), \quad v_i(K) = \sum_{k=1}^{K} v_i^k(t_k),
\]

then we have

\[
\mathcal{L}_A(K) = \sum_{\substack{ij \in E \ni \neq j}} n_{ij}(K) \log a_{ij} - a_{ij}v_i(K).
\]

For an element \( a_{ij} \) with \( i \neq j \), we get

\[
\hat{a}_{ij}(K) = \frac{n_{ij}(K)}{v_i(K)}.
\]

This is a straightforward generalization of the maximum likelihood estimator obtained in Billingsley (1961) and Albert (1962), computed here on several sample paths with random time lengths. In Chiquet and Limnios (2006), an estimator (17) was proved to be consistent with an asymptotic normality property, even when defined on randomly censored sample paths, in the independent case.

In the following numerical application, we use the estimator for the generator of the underlying jump Markov process of the dynamical system, in the framework of fatigue crack growth analysis. This is obviously an approximation because we consider the failure time and the process \( X_t \) independent, which is not the case as seen in expression (6) of the failure time. Although good numerical results are obtained with this approximation, it could be significantly improved by introducing the dependency in the estimation procedure.

4. Application to crack growth modelling

To handle with the fatigue crack growth problem, some empirical laws have been suggested in fracture mechanics to describe the mean behavior of a crack. One of the most versatile one is the law given by Paris and Erdogan (1963), describing deterministically the evolution of the crack size function \( z_t \) by

\[
\frac{dz_t}{dt} = C(\Delta K(z_t))^n,
\]

where \( C \) and \( n \) are some material constants and \( \Delta K(z) = Y(z)\Delta \sqrt{S} \) is the so-called stress intensity factor range. \( \Delta S \) is the stress range of the load applied on the specimen, while \( Y(z) \) is a structure geometry function that is approximately equal to 1 when considering large structure. With this assumption, Eq. (18) turns to the much simpler form

\[
\frac{dz_t}{dt} = a(z_t)^b,
\]

where \( a = C(\Delta S)^n \pi^{n/2} \) and \( b = n/2 \). We suggest to randomized (19) with a right-hand side multiplicative factor given by a jump Markov process \( X_t \), thus we model the phenomenon by the following stochastic dynamical system

\[
\frac{dZ_t}{dt} = a(Z_t)^b \times X_t, \quad Z_0 = z,
\]

with \( z \) being the initial crack size.

Applying the Bogolyubov’s averaging principle described in Section 3.1, we have

\[
\frac{dZ^\varepsilon_t}{dt} = a(Z_t^\varepsilon)^b \times X_t^\varepsilon, \quad Z^\varepsilon_0 = z,
\]

where \( \varepsilon > 0 \), a series parameter. The asymptotic behavior, when \( \varepsilon \to 0 \), gives

\[
\frac{dZ_t}{dt} = a_0(Z_t)^b, \quad Z_0 = z,
\]

where \( a_0 = a_0 \sum_{i \in E} \xi_i \). The following stochastic dynamical system is equivalent to (20)

\[
\frac{dZ_t}{dt} = a_0 (Z_t)^b \times \nu(X_t), \quad Z_0 = z.
\]

where \( \nu(X_t) \) is a jump Markov process with the same generator as \( X_t \), but a state space \( E' \) that is linked to the state space \( E \) of \( X_t \) through the linear mapping \( \nu \) defined by \( \nu(x) = \nu' \sum_{i \in E} \xi_i, x \in E \). We consider system (22) rather than (20) so that we “normalize” the process \( X_t \) to \( \nu(X_t) \) by dividing by the state mean \( \sum_{i \in E} \xi_i \). The term \( a_0 \) absorbs the compensatory term for the normalization. Hence, practically, the stationary law \( \pi \) does not need to be computed. Moreover, the system is more suitable to regression analysis.

As a matter of fact, the parameters \( a_0 \) and \( b \) can be estimated by a least squares method on the data set, by taking the logarithm on both sides of (21). Hence,

\[
\ln Z_t = \ln a_0 + b \ln Z_t + \ln \nu(X_t).
\]

The available data, that is the \( Z_t \)'s, can be represented as a \( N \)-sample composed with all of the data points from the crack growth curves:

\[
((t_i, Z_{t_i}) : i = 1, \ldots, N).
\]

This \( N \)-sample can be transformed in \( ((\ln Z_{t_i}, \ln \hat{Z}_{t_i})) \), with \( \hat{Z}_{t_i} \) being obtained through (12). Introducing the notation \( x_i = \ln Z_{t_i}, y_i = \ln \hat{Z}_{t_i} \) and \( e_i = \ln \nu(X_{t_i}) \), we have the following classical regression problem:

\[
y_i = \ln a_0 + b x_i + e_i, \quad i = 1, \ldots, N.
\]

When \( \varepsilon \to 0 \), that is in the series scheme of the Bogolyubov’s principle described in Section 3.1, system (22) tends to (21). It also means that, on average, \( \nu(X_t) = 1 \). The term “average” must be understood here in the series scheme sense of the Bogolyubov’s principle. Thus, the \( \ln \nu(X_{t_i}) \)'s are “close to 0” in the regression equation (23) and correspond to the so-called residuals \( e_i \).

Minimizing \( \Sigma e_i^2 \), we get the estimators associated to the well-known least squares method:

\[
\hat{a}_0 = \exp(\bar{y} - b \bar{x}), \quad \hat{b} = \frac{\Sigma x_i y_i - N \bar{x} \bar{y}}{\Sigma x_i^2 - N \bar{x}^2},
\]

with

\[
\bar{x} = \frac{1}{N} \Sigma x_i, \quad \bar{y} = \frac{1}{N} \Sigma y_i.
\]

Some paths \( (\nu(X_{t_i}))^k, k = 1, \ldots, K \), are then obtained explicitly by “reversing” the dynamical system, that is through the function \( G \) given by (11). In the case at hand, we have

\[
(\nu(X_{t_i}))^k = \frac{1}{a_0} (Z_{t_i})^{-b} \times Z_{t_i}^k.
\]

We then apply the K-means clustering method, initially suggested by MacQueen (1967) (see, e.g., Jain and Dubes, 1988, for implementation) to reduce the state space of \( \nu(X_t) \), thus obtaining some \( (\nu(X_t))^k \)'s suitable for any further estimation of the jump Markov process. For the Virkler data, the histogram of a 18-state space obtained with the K-means method is represented on Fig. 3. The values of the estimated state space of \( \nu(X_t) \) are then obtained explicitly by “reversing” the dynamical system, that is through the function \( G \) given by (11). In the case at hand, we have

\[
\hat{F}_t(x) = \hat{b}(Z_t \leq x) = \frac{1}{M} \Sigma_{m=1}^M 1_{[y_m \leq x]}, \quad x \in [z, A],
\]

with \( (y_m)_{m=1, \ldots, M} \) being \( M \) sample paths simulated with the stochastic dynamical system (22). Estimator (24) can be compared with the empirical cumulative distribution function, that is, computed on the Virkler data set from Fig. 1. Results are represented at Fig. 4 for various values of \( t \).

In the same way, we may estimate the reliability defined here by (1) through

$$\hat{R}(t) = \frac{1}{M} \sum_{m=1}^{M} \mathbb{I}_{[I_m < \tau]}.$$  \hfill (25)

Results are compared with the data points on Fig. 5.

5. Concluding remarks and perspectives

Concerning the generator $A$ of the jump Markov process $X_t$, the estimator used in the previous numerical application is an approximation, since we have used the maximum likelihood estimator for $K$ sample paths censored by a random variable independent of the process. We pointed out that it was not the case in the framework of dynamical system developed in the present paper, as long as the random censoring $\tau$ is also the failure time of the system. In Section 3, we gave the general form of the log-likelihood to be maximized. One can see that the density function of $\tau$ appears within (16), and an important issue is to give an formal expression of $f_\tau$ if we want to include it in the maximization process. This could be achieved using Proposition 1 and consecutive results, what the authors are working on.

Fig. 5. Reliability estimator compared with the data points.

References


