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Regulation of a Fishery: from a Local Optimal Control Problem to an "Invariant Domain" Approach

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Abstract. This study aims at regulating a fishery by reducing its catch and effort variations around a reference equilibrium point. A simple bioeconomic continuous time model is considered and the control variable acts on the rate of change of effort. Two approaches have been implemented: the first one is a local optimal approach that sets the problem in the classical linear-quadratic framework; it is extended to an "invariant domain" approach, whose robustness constitutes a major advantage.

1. Introduction. The optimal control theory has often been applied to fisheries management, for the calculation of bioeconomic optima, generally in order to maximize the (discounted) revenue. C.W Clark's [1976, 1985] books are a reference in this particular field. Mostly, the optimal solutions obtained are bang-bang controls. Optimization in fisheries has also been reviewed by J.W. Horwood [1993], also with P. Whittle [1986a, 1986b], who apply their results on several discrete time fisheries models. They obtain locally-optimal linear feedback controls.

However, it can be more interesting to maintain selected variables (catch, effort, stock size, etc.) at given levels. Rather than seeking for an optimum bioeconomic point, the fishery could be regulated: variations of these variables around a reference equilibrium point could be reduced, which would also have a stabilization effect. Another possible application of optimal control consists then in minimizing these fluctuations. Horwood et al. [1990], [1991] have proposed this original approach : they aim at minimizing the catch and effort variations thanks to control term on the catch. In these papers, a suboptimal control is obtained.

In this article we retain the same criterion, but we apply it to a global (no age classes) continuous time model, using the rate of change of fishing effort as control variable – and not the catch. Our goal is to reduce the catch and effort variations, which thereby guarantees the fishermen relatively stable revenues and labor. It can thus be considered as a simple economic criterion.

Two approaches to this simple bioeconomic regulation are implemented. They require quite different resolution methods. The first one consists in local optimal control, which sets the problem in the classical framework of linear-quadratic optimization and Riccati's theorem, as in the papers of Horwood and Whittle [1986a], [1986b]. The "invariant domain" approach, detailed in S. Touzeau's PhD [1997], tries to keep the system within preset effort and catch boundaries thanks to an appropriate control. The philosophy of this study is very similar to the viability theory developed by Aubin

[1991] and his team, however methods and resolution differ. The invariant domain methodology is described in more detail in this paper. The extension of the regulation process to this nonoptimal control approach aims at increasing the robustness of the regulation.

Furthermore, for greater likelihood, the behavior of the fishermen is taken into account. We assume it is governed by profit constraints, as in Schaefer's model [1991]; very simple price and cost functions are therefore introduced in the model.

In Section 2, we first present the fishery model and its properties. Then in Section 3, we concentrate on the regulation process; a fishery example is chosen to illustrate both methods: the local optimization of Subsection 3.1 and the invariant domain approach described in Subsection 3.2. Finally Section 4 compares the two approaches; some conclusions are drawn and perspectives for future work are exposed.

2. Modeling. We have retained a very general model of fishery, that is similar for the two following regulation approaches. It is introduced in the subsection below, along with the hypotheses it implies. In the following subsection the mathematical properties of the model are presented.

2.1. Presentation. We consider a simple fishery dynamic system constituted of one fleet and one stock. The fleet is represented by its fishing effort E, whereas the biomass (or total abundance) X describes the state of the stock. The links between these two components are shown in Figure 1. The fishing effort generated by the fleet is applied to the stock and produces a certain catch Y. The fleet is also submitted to an exogenous control U.

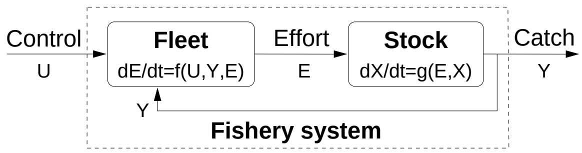

Figure 1. Representation of the fishery system.

The fleet dynamics are considered to be driven by a profit constraint: as long as the revenue is higher than the harvesting cost, the effort increases. Simple yield and cost functions are chosen. The cost is proportional to the effort and assuming a fixed fish price, the yield is proportional to the catch. Proportional costs take into account the fuel and labor costs for instance, but do not include long term investments such as fishing vessels.

No age structure is introduced on the stock. It is represented by Schaefer' model, assuming logistic growth for the unexploited stock, minus the catch that is considered proportional to the effort and the total biomass.

A further general modeling assumption has been made: we use a continuous time model and the control is introduced as an additional term on in the fleet dynamics. The resulting mathematical model follows:

$$
\left\{
\begin{aligned}
\dot{X} &= rX(1 - X/k) - qEX \\
\dot{E} &= c\,(\alpha Y - \beta E) + U \\
Y &= qEX
\end{aligned}
\right.
\tag{1}
$$

with r as the growing factor, k the carrying capacity, q the catchability, α the price of a catch unit,

β the mean cost of an effort unit and c the conversion parameter. Parameter c is a simple conversion factor, because the profit $\alpha Y - \beta E$ is expressed in currency unit, whereas the effort is not.

This model was introduced very early by Schaefer [1991], except for the control term U.

The control is applied to the rate of change of effort. Sometimes in the literature, the catch appears as an input to the system. This choice proceeds certainly from the fact that the TAC (Total Allowable Catch) is an limitation on the catch. But when the biomass is low and/or the stock's accessibility decreases, it is difficult to control, i.e., apply a certain catch; whereas applying a certain effort seems more logical. However in our case, the fishing effort E follows its own dynamics, governed by profit, and is therefore not a suitable control. Even if the profit constraint was omitted, it would still be preferable to apply the control to the rate of change of effort. The effort as control variable might have to follow steep changes or even a "bang-bang" trajectory (possibly the solution of optimal control, especially when the Hamiltonian is linear in the control); applied to a fishery where the effort represents the number of boats, it could mean building three boats, then destroying seven the following year, etc., which is not realistic. Controlling with the rate of change of effort is smoother; in the previous example it would indicate whether to build or destroy boats, and with which speed.

2.2. Properties.

Positivity of the variables. This model doesn't make any sense for negative values of biomass and effort. With no control, the structure of the model however ensures that if these variables initially are nonnegative, they will remain nonnegative in the course of time. The control variable U however can be negative, so we may have to provide further constraints.

Equilibria. There are two trivial equilibria for this model, when no harvesting takes place: the first one corresponds to a depleted stock and the second one corresponds to a stock at the carrying capacity level K, i.e., a nonexploited stock.

The existence of a positive and nontrivial equilibrium is necessary to have a sustainable fishery. The following constraint should therefore be respected:

$$X_e = \frac{\beta}{q\alpha} < k \tag{2}$$

It is a profitability constraint: price α should be high enough compared to the costs (proportional to β) in order to allow the fishermen to make a profit from this stock in the long term.

We will assume that this inequality is verified. Table 1 summarizes the three equilibrium points hence determined.

Table 1. Equilibria of the system (1), when the profit constraint (2) is verified ($U = 0$).

X_e	E_e	
0	0	$\rightarrow$ depleted stock
k	0	$\rightarrow$ virgin stock
$\beta/(q\alpha)$	$(r/q)(1-\beta/(kq\alpha)))$	$\rightarrow$ exploited stock

Normalization. This work is not applied to any particular stock. Therefore, in order to simplify the notations and without any loss of generality, we normalize the model, i.e., we set that:

$$r = k = q = c = 1 \tag{3}$$

The following change of variables and parameters transforms the parametrized system of equation (1) into the normalized system (noted with subscript$_1$):

$$\begin{cases} t = \dfrac{t_1}{r} & X = kX_1 & E = \dfrac{r}{q}\,E_1 \\ U = \dfrac{r^2}{q}\,U_1 & \alpha = \dfrac{r}{cqk}\,\alpha_1 & \beta = \dfrac{r}{c}\,\beta_1 \end{cases}$$

This proves the normalization (3) keeps the generality of the model. The normalized system will mainly be used when applying our results to an example, in order to reduce the numbers of parameters to be set. In the following sections, when dealing with the normalized system, subscript$_1$ will be omitted.

3. Regulation. The aim of the regulation is to keep the system described in the previous section around a reference equilibrium point by reducing the effort and catch variations. A very simple "economic" interpretation of this criterion is to provide rather stable revenues and labor level to the fleet. The means available to achieve this goal lie in the control term applied to the rate of change of effort.

Two approaches are presented to this regulation problem: a local optimal control resolution, quite classical, and an "invariant domain" approach that is described in details in Section 3.2.

3.1. Local optimal control. This regulation approach, compared to a more classical optimization framework, presents the following advantages. From a given and satisfying state, what is the best way to react towards a sudden increase of the fishing effort or a decrease of the stock? With the criterion selected here, the solution is not: "maximize the catch" but it is a combination of: "maintain a preset effort level" and "allow the fishermen a certain catch". Moreover, as it is a local problem, the linearization of the model is licit and allows a more complex initial model. Combined with a quadratic criterion, we can obtain a local optimal solution quite easily thanks to Riccati's theorem. Tendencies can then be extracted from these results.

So our aim is to minimize the local effort and catch variations around an equilibrium point on a finite time horizon T. The following subsection describes the mathematical problem and its resolution, then results are produced on an example.

3.1.1. Resolution of the optimization problem. The first step consists in linearizing the system around the nontrivial equilibrium (X_e, E_e); this simplification is quite justified in the case a local problem. We note with lowercase the local variables. For example: $x = X - X_e$. The linear system corresponding to equation (1) follows:

$$\begin{cases} \dot{x}(t) = -rX_e\,x(t)/k - qX_e\,e(t) \\ \dot{e}(t) = c\alpha qE_e\,x(t) + c\,(\alpha qX_e - \beta)e(t) + u(t) = c\alpha qE_e\,x(t) + u(t) \\ y(t) = qE_e\,x(t) + qX_e\,e(t) \end{cases} \tag{4}$$

We choose a quadratic criterion, so the problem is situated in the very classical framework Riccati's theorem. The expression of the optimization problem follows.

Problem 1. *Find the control $u(t)$ on the time interval $[0, T]$ minimizing:*

$$J(u) = \frac{1}{2} \int_0^T [\, p_1\, e(t)^2 + p_2\, y(t)^2 + p_3\, u(t)^2 \,]\, dt \tag{5}$$

where the p_i are positive weighting parameters, normalized so that their sum equals to 1.

Rather than bounding the control, it is integrated in the criterion. It is simpler than introducing a constraint on u.

The mathematical resolution of the problem, according to Riccati's theorem, is exposed in the appendix. It gives an optimal feedback solution (cf., for example, Kwakernaak and Sivan [1972]) that has the following form:

$$u^*(t) = F_1(t)\, x^*(t) + F_2(t)\, e^*(t) \tag{6}$$

where F_1 and F_2 only depend on the model's parameters and time. The formal expression F_1 and F_2 requires the resolution of a set of ordinary differential equations. So numerical resolution is performed. In the following subsection an example is presented. The methodology has also been applied on several real stocks (cf. Touzeau [1997]).

3.1.2. Application to an example. To visualize the optimal control obtained from the regulation problem, we take a numerical example. The normalized stock model is chosen, according to equation (3), as well as two values for the price and cost parameters (α and β) verifying the profitability constraint (2). They determine a nontrivial equilibrium point (X_e, E_e) (cf. Table 1). Then the initial condition of the system needs to be set, creating the perturbation that needs to be regulated. We apply a 50% decrease of the biomass and a more or less equivalent increase of the fishing effort. The last step is to choose the weighting parameters of the criterion (5), as well as the optimization horizon. The latter is set to 10 (unit of time depends on the parameterization). The weight p_3 on the control term is set rather high, in order to limit the rate of change of effort; catch and effort terms are given the same importance in the criterion. The parameters chosen fitting this description are summarized in Table 2.

Table 2. Local optimal regulation parameters.

Fishery system parameters		
Normalized stock	$r = k = q = 1$	
Profit	$\alpha = 0.8$	$\beta = 0.24$
$\Rightarrow$ Equilibrium	$X_e = 0.3$	$E_e = 0.7$
Minimization parameters		
Initial condition	$X_0 = 0.15$	$E_0 = 1$
Criterion's weights	$p_1 = p_2 = 0.25$	$p_3 = 0.5$
Time horizon	$T = 10$	

The resulting locally optimal curves are presented in Figure 2. The biomass, effort, catch and control trajectories are displayed. The optimal regulation consists here first in rapidly reducing the effort and then slowly increasing it. The state of the system at the end of the regulation period is very close to equilibrium, which is rather satisfying.

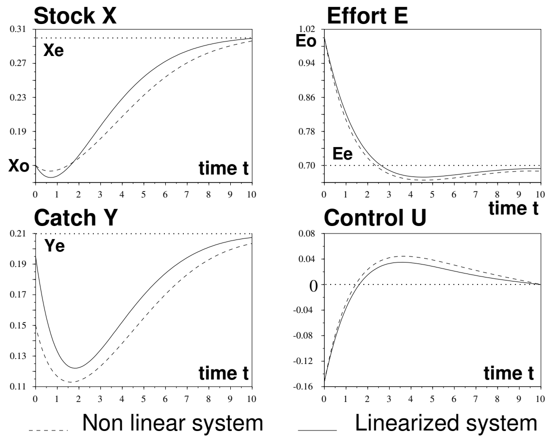

Figure 2. Local optimal regulation; comparison with the suboptimal trajectories.

Figure 2 also allows us to appraise the linear approximation. So the same feedback law, optimal for the linearized system (4), is fed into the non linear system (1) (only suboptimal in the non linear case). Comparison of both regulated trajectories of the various variables shows that the approximation is rather good. It can be noted that the catch curve differ appreciably, especially in the beginning. Both initial points are quite apart here, because the linear assumption neglects the second term component of the catch.

3.2. "Invariant domain" approach. This approach is merely inspired from the previous study. But, instead of looking for an optimal control reducing the effort and catch variations, we try here to regulate the system for it to stay "close" to the equilibrium. "Close" means here that the catch and effort variations around the equilibrium are bounded.

The aim here is hence to keep the system within preset effort and catch boundaries, defining a domain around a given equilibrium point. More precisely, we would like to investigate if, starting from a point within this domain, it is possible to find an admissible control in order to stay in it as time goes by. This fits in the viability studies (cf. Doyen and Saint-Pierre [1997] for example). Given viability constraints, this type of problem consists in determining the initial condition set beside which, whatever the admissible control may be, the system won't verify the constraints.

In the present case, the viability constraints would be: "stay within catch and effort boundaries". However, if our philosophy is the same, the resolution methods differ. In our applied example, we have chosen an essentially graphic resolution, very simple in this case.

3.2.1. Presentation of the domain approach. To introduce the invariant domain approach, we first need to define precisely what invariant means in this case. The two following definitions clarify this term.

Definition 1. A domain is said to be *invariant (or viable) without control* if, from whatever initial point in the domain, the system remains within the domain limits along time, the control being kept to zero.

Definition 2. A domain is said to be *invariant (or viable) with control* if, from whatever initial point in the domain, there is an admissible (bounded) control $U(t)$ allowing the system to stay within the domain limits.

In this approach, we add a constraint on the control: it is bounded ($U_{min} \leqslant U \leqslant U_{max}$), so as to avoid too steep variations of the effort. Generally the minimum rate of change of effort will be negative, the maximum positive, so that the fishing effort may decrease and increase.

The main steps of the invariant domain research are roughly described in this paragraph, but the method will be demonstrated on a numerical example of the fishery system (1). Prior to any further analysis, an initial domain D needs to be defined. So we fix the effort and catch values ($E_{min}, E_{max}, Y_{min}, Y_{max}$) within which the fishery should evolve. Then the following analysis should be performed:

- If the domain D is invariant without control, i.e. if there is an ingoing field at the borders of the domain, no need to pursue the analysis: doing nothing, $U = 0$, will ensure that the fishery state remains in the domain.
- If not, a subdomain of the initial domain D, invariant with control, should be determined. This is achieved by border control: whenever possible, the control at the domain's limits is set so as to get an ingoing field.
  - If at all borders of the domain D, an admissible control can be found that produces an ingoing field, the domain is invariant with control.
  - If there are points at the border where all admissible controls fail to obtain an ingoing field, the domain is reduced consequently. This last step will be more clearly illustrated in the application.

For the applied problem in the next subsection, a mainly graphic resolution has been implemented. To ease the analysis of the field direction at the borders of domain D, a change of variables is performed (we use the normalized system). In the phase plane (E, Y), system (1) becomes:

$$\left\{\begin{aligned} \dot{E} &= \alpha Y - \beta E + U \\ \dot{Y} &= \dot{E} X + \dot{X} E \\ &= \left[(\alpha - 1)Y + (1-\beta)E - E^2 + U\right] Y/E \end{aligned}\right. \tag{7}$$

The isoclines of this new system without control, $U = 0$, are defined in the following way:

- $\dot{E} = 0 \quad \Leftrightarrow \quad Y = \dfrac{\beta}{\alpha} E$
- $\dot{Y} = 0 \quad \Leftrightarrow \quad Y = 0 \quad \text{or} \quad \begin{cases} Y = \frac{E}{\alpha - 1}(E + \beta - 1) & \text{if } \alpha \neq 1 \\ E = 1 - \beta & \text{if } \alpha = 1 \end{cases}$

They will allow us to determine the direction of the field without control at the borders of the domain D.

3.2.2. Application to an example. The same fishery system as the one used in the local optimal regulation and described in Table 2 is chosen for the application of the invariant domain approach. The domain boundaries are chosen so as to enclose the nontrivial equilibrium point (cf. Table 1). The upper and lower limit on the control variable, i.e., the rate of change of effort, are respectively positive and negative so they can produce an increasing or decreasing trend on the effort; the limits are chosen relatively small compared to the effort level at equilibrium. These values are presented in Table 3.

Table 3. Parameters associated to the invariant domain approach.

Fishery system parameters		
cf. table 2		
Regulation parameters		
Limits of domain D	$E_{min} = 0.6$	$E_{max} = 1$
	$Y_{min} = 0.15$	$Y_{max} = 0.35$
Admissible control	$U_{min} = -0.1$	$U_{max} = 0.1$

The determination of the invariant subdomain of domain D can be easily visualized on graphics in the phase plane (E, Y), corresponding to system (7). The resolution therefore follows the series of figures.

Figure 3. This first graph shows the initial domain D, the isoclines, the nontrivial equilibrium point **Eq**, as well as the field without control. They are represented in the initial phase plane (X, E), corresponding to equations (1), and in the working phase plane (E, Y), corresponding to (7).

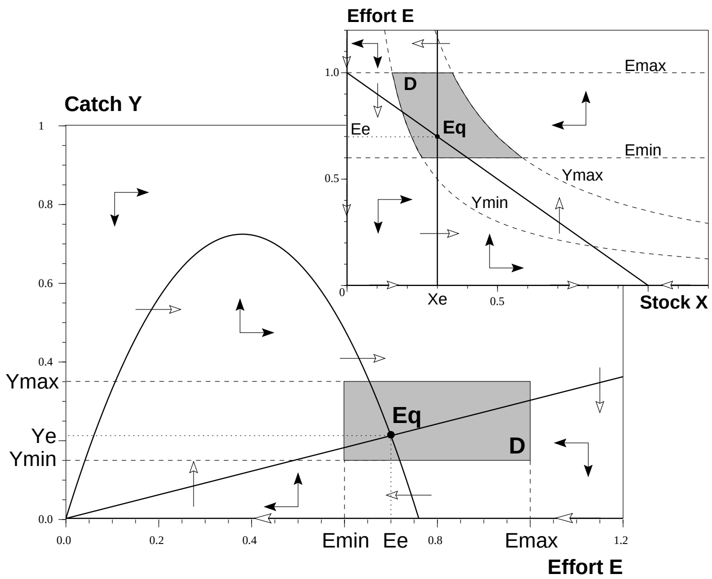

Figure 3. Initial domain, isoclines and field without control in both phase planes.

Figure 4. Some parts of the border admit an outgoing field without control. They are limited by the intersection point between the border and the corresponding isocline, e.g. border $Y = Y_{min}$ and isocline $\dot{Y} = 0$ at point (E_{lim}, Y_{min}); the field changes from outgoing to ingoing (or vice versa) when going beyond this point, because the sign of the relevant field component, e.g. $\dot{E}$, changes. The invariant subdomain without control can be deducted from this considerations: it is delimited by the trajectories of system (7) without control, $U = 0$, reaching the domain border at these threshold

points. This first result is illustrated in Figure 4, that displays the relevant field component at the border and the invariant subdomain without control, domain D minus the nonviable grey areas.

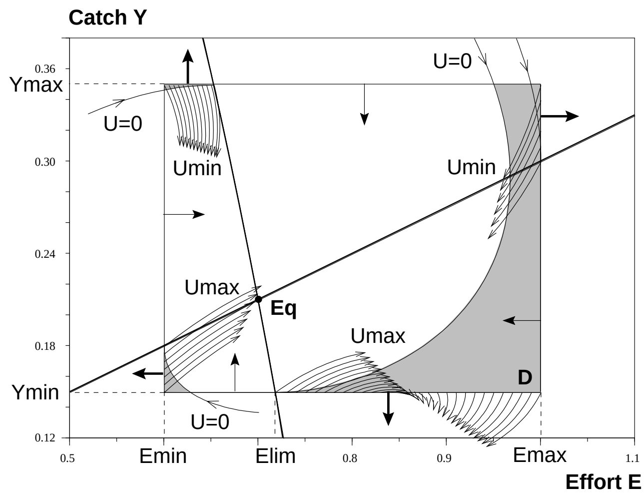

FIGURE 4. Invariant domain without control and most favorable control at the border.

In the parts of the border where the field is going out, we then have to set the control so as to possibly get an ingoing field.

Border $\mathbf{E} = \mathbf{E_{min}}$ The $\dot{E}$ field component should be nonnegative. Now:

$$\dot{E} = \alpha Y - \beta E_{min} + U$$

The most favorable case is to have control U as high as possible, that is to say U_{max}. The field then becomes positive if the following inequality holds:

$$Y \geqslant \frac{\beta E_{min} - U_{max}}{\alpha} = 0.055$$

This condition being always verified in domain D, it is possible to find an admissible control so that the system won't come out of the domain through the E_{min} border.

Border $\mathbf{E} = \mathbf{E_{max}}$ In a way similar to the previous case, we show that U_{min} is the most favorable control and that the corresponding inequality is then always verified. So the nonviable zone without control associated with the E_{max} border becomes invariant with control.

Border $\mathbf{Y} = \mathbf{Y_{max}}$ $\dot{Y}$ should be nonpositive. Now:

$$\dot{Y} = [(\alpha - 1)Y_{max} + (1 - \beta)E - E^2 + U]\, Y_{max}/E$$

The most favorable case is to choose U_{min} as control. The condition for the field to be ingoing is:

$$-E^2 + (1 - \beta)E + (\alpha - 1)Y_{max} + U_{min} \leqslant 0$$

The associated polynomial doesn't have any real root, so this inequality is always verified. The nonviable zone without control associated with this border becomes invariant with control U_{min}.

Border $\mathbf{Y} = \mathbf{Y_{min}}$ In a way similar to the previous case, we show that U_{max} is the most favorable control. The condition for the field to be ingoing is:

$$-E^2 + (1-\beta)E + (\alpha - 1)Y_{min} + U_{max} \geqslant 0$$

The roots of the associated polynomial are $E_1 \simeq 0.843$ and -0.083, so the equality is verified on the Y_{min} border if: $E \leqslant E_1$. For efforts greater than E_1 in the neighborhood of this border, there is a nonviable zone with control.

Reducing the effort when its level is high, or on the contrary increasing it for small values in order to keep the system in domain D is a logical result. It is less obvious as far as the catch is concerned. Decreasing the fishing effort has two opposite consequences: a priori it reduces the catch, but it also makes the biomass increase, which can finally lead to an rise of the catch in a more or less long term. It depends on the state of the stock, which is clearly demonstrated on the Y_{min} border.

Figure 5. Thanks to the determination of the most favorable control at the border, the a priori invariant subdomain of Figure 4 has been increased. Only one part of the Y_{min} border, for effort levels higher than E_1, remains nonviable: whatever admissible control is applied there, the catch will fall below its minimum value. As a first attempt, we may delimit this zone with the trajectory without control reaching the border at the (E_1, Y_{min}) point, as shown in Figure 5.

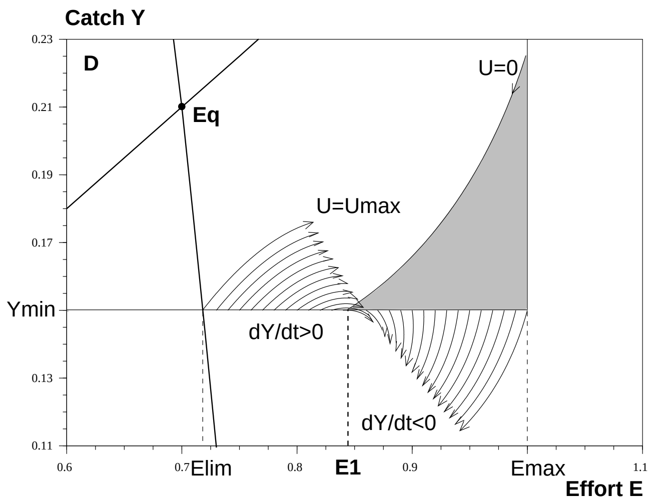

Figure 5. A priori nonviable zone with control.

This zone corresponds to a highly exploited stock with a low biomass. It is therefore not surprising the catch cannot instantly go up. Increasing the harvesting in this region doesn't seem very wise, however it is a punctual strategy: it is all that can be done when the minimum catch has been reached, but it is not always sufficient to stay within the domain.

Figure 6. The most favorable control in Figure 5 tends to drive the system towards the nonviable zone. To make sure that the fishery will move away from this area, the following conditions should be simultaneously verified:

$$\dot{Y} \geqslant 0 \quad \text{and} \quad \dot{E} \leqslant 0 \qquad \text{for } Y = Y_{min}$$

In the previous paragraph, we have shown that for $E \leqslant E_1$, there is an admissible control such that $\dot{Y} \geqslant 0$. Furthermore, effort E and catch Y_{min} being set, $\dot{Y}$ is an increasing function of U. For the second condition to be verified, $\dot{E} \leqslant 0$, the control needs to be taken "as small as possible". Therefore we should choose a border control such that $\dot{Y} = 0$, that is:

$$U = E^2 + (\beta - 1)E + (1 - \alpha)Y_{min}$$

Replacing U by this value in equations (7), the assumption $\dot{E} \leqslant 0$ becomes equivalent to:

$$E^2 - E + Y_{min} \leqslant 0$$

The roots of the associated polynomials being 0.18 and $E_2 = 0.5 + \sqrt{0.1} \simeq 0.816 < E_1$, this condition is verified at the border for $E \leqslant E_2$. So finally, the invariant subdomain of D should only include points of the Y_{min} border for which the effort is lower than E_2. This new effort limit is represented in Figure 6.

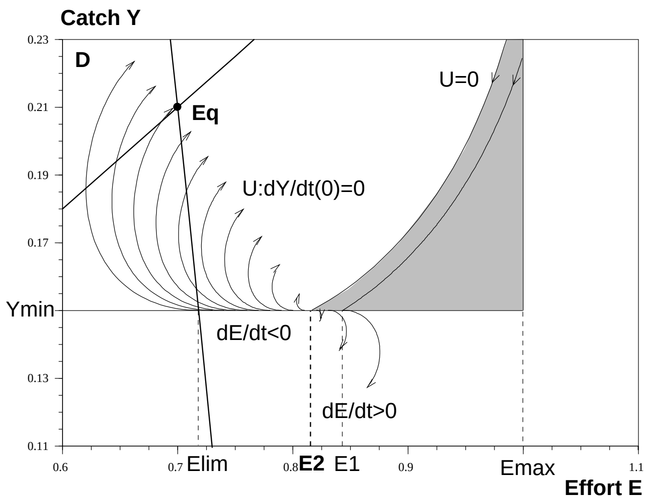

Figure 6. Effort limit for the invariant subdomain with control.

E_2 is a limit that probably defines a smaller invariant domain with control that is actually required. But our approach is not optimal: we prefer cutting more of the initial domain and having simpler strategy to keep the system within the subdomain, than determining what is the maximum invariant subdomain.

Figure 7. This figure represents the trajectories leading to the limit point (E_2, Y_{min}), for different admissible values of a constant control. They all admit the same tangent at this point, because for all control U:

$$\dot{E}(E_2, Y_{min}) = \dot{Y}(E_2, Y_{min})\, Y_{min}/E_2$$

The figure shows that the trajectory delimiting the greatest invariant subdomain is obtained for the control U_{min}.

This result is quite intuitive: when the stock level is low, it seems natural to try to diminish the effort as fast as possible to help the stock recover, otherwise it would rapidly bring the catch down.

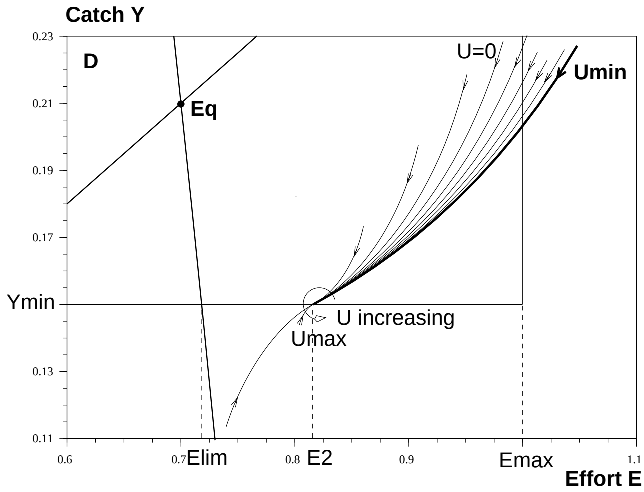

Figure 7. Limiting trajectory for the invariant subdomain with control.

Figure 8. This figure represents the final invariant subdomain with control Δ. Compared to the initial domain D, it excludes the nonviable zone corresponding to a high fishing pressure on a stock of low biomass.

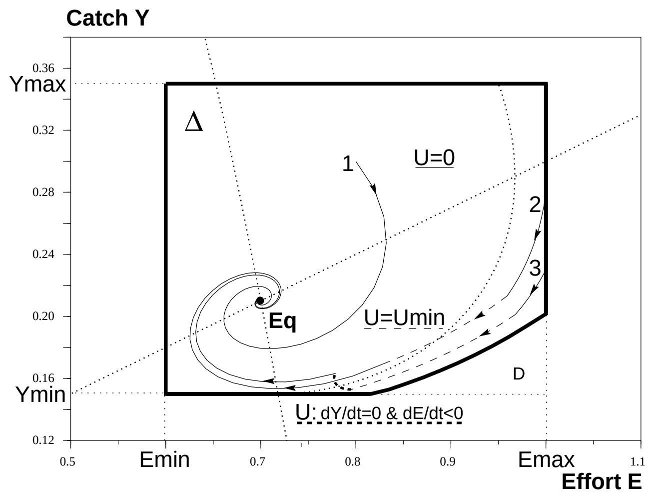

Figure 8. Invariant subdomain with control (Δ); examples of trajectories with control.

Some trajectories with control are also displayed. Trajectory 1 does not need any control. Trajectory 2 requires a quick decrease of the fishing effort as it approaches the nonviable zone: when the effort is high and the catch starts to decrease noticeably (an obvious sign of a diminution of the stock), the former should intuitively be reduced, and the faster the decrease the better. This strategy is also applied to trajectory 3, but in spite of this measure, the catch comes close to its minimum value. So the effort should be increased (border control, cf. Figure 6), in order to increase the catch and move away from the new Y_{min} border.

Note. What happens outside of the domain is not described in this paper. In the viability theory, it is partly considered by the means of a return time (cf. J.-P. Aubin [1991]). But the philosophy of this approach is to determine whether or not it is possible to remain within certain boundaries. Outside, the criterion is not respected and the system is therefore not studied.

4. Discussion. The aim of this study was to control a fishery so that the catch and effort variables remain close to given equilibrium values. This management criterion tends to stabilize the revenues, costs and labor of the fishery. Two approaches to regulate the fairly simple bioeconomic fishery system implemented have been considered. The first one defines local optimal trajectories, minimizing the catch and effort variations around the equilibrium. They depend on the initial state of the system, as well as on some more technical parameters for the minimization. The second is a domain approach: given catch and effort boundaries, it determines an invariant subdomain with control. Provided the fishery's initial state is within the subdomain, it will evolve within these boundaries, if an appropriate control (not unique) is applied when the system comes near to the border of the subdomain. Figure 9 displays results of both regulation approaches for the particular fishery system described in Tables 2 and 3. The invariant subdomain with control and local optimal trajectories (feedback law (6) introduced in the non linear system (7)) are represented in the phase plane (E, Y).

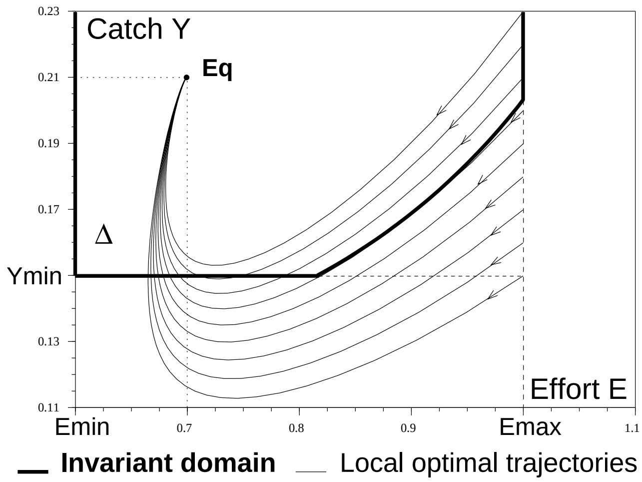

Figure 9. Comparison of the two approaches: local optimal trajectories and limit of the invariant subdomain with control.

The local optimal trajectories do converge towards the equilibrium, but they do not stay in the invariant subdomain with control. The optimization criterion is a quadratic sum of the variations, hence we might prefer the domain approach that reduces the distance to the equilibrium in phase plane (E, Y).

Furthermore, the robustness of the domain approach compared to trajectories constitutes the major advantage of this approach. The domain approach is more robust with regard to the model uncertainties (r,k,etc.), for the aim is not to head towards an equilibrium but in a domain. Even if knowledge of the stock dynamics and the profitability of the harvest is approximative, we could maximize all uncertainties in a precautionary approach (worst case) and still apply this method. An estimate of the catch and effort is necessary; however, the application is easier in the case of a fishery involving few landing ports and gears.

The main perspective of this study is to apply the invariant domain approach on real stocks for management purposes. It may require to develop the stock description: stages or age classes could be introduced or multispecific features. This would increase the dimension of the system, so the graphic resolution presented in Section 3.2 could not be implemented.

For a more realistic application of this invariant domain technique, we may change the control variable. In the present model, it is an authoritative component on the rate of change of effort, which otherwise is driven by profit constraints. It represents an exterior decider decreeing a rule on the effort variation: should it decrease and at which speed. However, such existing rules are mainly the TACs (Total Allowable Catches) and quotas in the ICES regions (North Atlantic, Baltic Sea, etc.), which are adapted every year. In the Mediterranean technical measures on the effort are implemented, such as the time at sea, the mesh size, etc. However these are historical rules that remain constant along time. For Mediterranean fisheries, the subsidies (e.g. on the fuel price) could be a good control variable. It would act on the rate of change of effort, but would require a more developed economic model, linking the subsidies to the effort changes.

Appendix

Resolution of the optimal regulation problem. This Appendix is a complement for Section 3.1, presenting optimal control results with quadratic cost applied to the linear fishery system (4). It sets the problem in the classical framework of Riccati's theorem (cf. Kwakernaak and Sivan [1972] or d'Andréa Novel and Cohen de Lara [1994]).

Given the following dynamic linear system, controlled and continuous:

$$\dot{x} = Ax + Bu$$

with:

$$x \in \mathbf{R}^n, \quad u \in \mathbf{R}^p$$

and assuming the following matrices are known:

Q : symmetric positive matrix of size $n \times n$
R : symmetric definite positive matrix of size $p \times p$
S : matrix of size $n \times p$
L : symmetric positive matrix of size $n \times n$

Theorem 1 (Linear quadratic optimization – finite horizon). *Let $K_0 = K(0)$ be the final value of the solution of the following retrograde differential equation, called Riccati's equation:*

$$\begin{gathered} \dot{K} + A'K + KA - (KB + S)R^{-1}(B'K + S') + Q = 0 \\ K(T) = L. \end{gathered} \tag{9}$$

Let us moreover consider the following quadratic criterion on a finite horizon for $T > 0$:

$$J(u) = \frac{1}{2}\int_0^T \left[x(t)'Q\,x(t) + u(t)'R\,u(t) + x(t)'S\,u(t) + u(t)'S'x(t)\right]\,dt + x(T)'L\,x(T). \tag{10}$$

Then, noting $x_0 = x(0)$, the minimum of criterion (10) is:

$$\min_u J(u) = x_0'\,K_0\,x_0.$$

Furthermore, this minimum is reached for the following feedback control:

$$u^*(t) = -R^{-1}(B'K + S')\,x^*(t) \tag{11}$$

where $x^*(t)$ is the solution of:

$$\begin{aligned} \dot{x}^* &= Ax^* + Bu^* = [A - BR^{-1}(B'K + S')]\,x^* \\ x_0^* &= x_0. \end{aligned}$$

To apply this theorem to the fishery system (4) with criterion (5), we first need to identify the above-mentioned matrices, replacing vector x by (x, e), with $X_e = \beta/(q\alpha)$ and $E_e = (r/q)(1 - \beta/(kq\alpha))$:

$$A = \begin{pmatrix} -(r/k)X_e & -qX_e \\ c\alpha q E_e & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad Q = \begin{pmatrix} p_2\, q^2 E_e^2 & p_2\, q^2 E_e X_e \\ p_2\, q^2 E_e X_e & p_1 + p_2\, q^2 X_e^2 \end{pmatrix}$$

$$R = (p_3) \qquad S = 0 \qquad L = 0$$

If the weighting coefficient p_3 is positive (p_1 and p_2 being nonnegative), matrices Q, R, S, L verify the hypotheses above; this constraint means that the rate of change of effort should be included in the criterion, which is not restrictive because we do want to limit the control variations. Then Problem 1. leads to the resolution of Riccati's equation 9, where K is a 2×2 matrix. There is no simple formal solution, but a numerical resolution on an example is feasible.

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