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Abstract This paper presents an optimal control strategy allowing the maximization of the total production of a membrane filtration system over a finite time horizon. A simple mathematical model of membrane fouling is used to capture the dynamic behavior of the process which consists in the attachment of matter onto the membrane during the filtration period and the detachment of matter during the cleaning period. The control variable is the sequence of filtration/relaxation cycles over the time. Based on the maximum principle, we provide an optimal control strategy involving a singular arc and a switching curve. The proposed optimal control strategy is then compared to a classical control sequence published in the literature.

Keywords: Optimal control, membrane fouling, maximization of water production, MBR, relaxation.

1. INTRODUCTION

Membrane filtration systems are widely used as physical separation techniques in different industrial fields like water desalination, wastewater treatment, food, medicine and biotechnologies. The membrane provides a selective barrier that separates substances when a driving force is applied across the membrane.

The main disadvantages of these processes is the membrane fouling by the continuous accumulation of the filtered impurities onto the membrane surface (filter cake) and pores. Different fouling mechanisms are responsible of the flux decline at constant transmembrane pressure (TMP) or the increase of the TMP at a constant flux. Hence, the operation of the membrane filtering process requires to perform regularly cleaning actions like relaxation, aeration, backwashing and chemical cleaning to limit the membrane fouling and maintain a good water production. Usually, sequences of filtration and membrane cleaning are fixed according to the recommendations of the membrane suppliers or chosen according to the operator's experience. This leads to high operational cost and to performances (quantities of water filtered over a given period of time) that can be far from being optimal. For this reason, it is important to optimize the membrane filtration process functioning in order to maximize system performances while minimizing energy costs.

A variety of control approaches have been proposed to manage filtration processes. In practice such strategies are based on the application of a cleaning action (physical or chemical) when either the flux decline through the membrane or the TMP increase crosses predefined threshold values (Ferrero et al. (2012)). Smith et al. (1958) developed a control system that monitors the TMP evolution over time and initiates a membrane backwash when the TMP exceeds a given setpoint. Hong et al. (2008) use also the TMP as the monitoring variable but the control action was the increase or decrease of membrane aeration. Vargas et al. (2008) use the permeate flux as the controlled variable to optimize the membrane backwashing and prevent fouling. On the other hand, kowledge-based controllers find application in the control of membrane filtration process. Robles et al. (2013) have proposed an advanced control system composed of a knowledge-based controller and two classical controller (on/off and PID) to manage the aeration and backwash sequences. Finally, the permeabilty was used by Ferrero et al. (2011) as a monitoring variable in a knowledge-based control system to control membrane aeration flow.

To date, different available control systems are able to increase significantly the membrane filtration process performances. However, more enhanced optimizing control strategies are needed to cope with the dynamic operation of the purifying system and to limit membrane fouling.

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The majority of the control strategies previously cited address energy consumption. In the present work, we are interested in maximizing the water production of a membrane filtration system over a given time period T by optimizing the ratio of filtration/relaxation time-periods within a given number of filtration/cleaning sequence (the relaxation is a period during which there is no filtration: during this period, the material attached onto the membrane can then naturally detach). Assuming a simple model is available, the optimal control theory can be used to solve this problem and find an optimal control strategy that can be compared to a classical control approach.

The paper is organized as follows. First, the filtration process model considered in this work is presented. Then, the control problem is solved using the Pontryagin Maximum Principle (Pontryagin et al. (1964)). The singular arc and the switching curve are depicted and allows to obtain the optimal strategy that is illustrated with simulations. Finally, conclusions and perspectives are drawn.

2. MODEL DESCRIPTION

To describe the membrane filtration process, we consider a simple form of the model of Benyahia et al. (2013). In a previous work, it was shown that this model is very generic in the sense that it is able to capture the dynamics of a large number of models available in the literature and simple enough to be used for optimizing and control purposes, cf. Kalboussi et al. (2016). In the present work, it is assumed that the membrane fouling is only due to the particle deposition onto the membrane surface. Let m be the mass of the cake layer formed during the water filtration. One can assume that m follows a differential equation

$$\dot{m} = f_1(m) > 0. \tag{1}$$

Here, we further assume that the physical cleaning of the membrane is performed only by relaxation. During the relaxation phase, the filtration is stopped and the mass detaches from the membrane surface with the dynamics :

$$\dot{m} = -f_2(m) \leq 0. \tag{2}$$

The water flow rate that passes through the membrane is modeled by a function which only depends on the accumulated mass onto the membrane surface

$$g : m \mapsto g(m) > 0. \tag{3}$$

3. THE OPTIMAL CONTROL PROBLEM

3.1 Statement of the problem

The considered system is operated by alternating two functioning modes: filtration and relaxation. For this reason, we consider a control u that only takes values 1 during filtration period and 0 during membrane relaxation period. Then, the dynamics of the fouling layer formed by the attachment of a mass m onto the membrane surface can be written as follows:

$$\dot{m} = uf_1(m) - (1-u)f_2(m), \quad m(0) = m_0. \tag{4}$$

As mentioned in the introduction section, the aim of this work is to determine the optimal switching between the two functioning modes that maximize the water production of the membrane filtration process during a time interval $[0, T]$. Then, the objective function of the optimal control problem can be expressed as:

$$J_T(m_0, u(\cdot)) = \int_0^T u(t)g(m(t))dt. \tag{5}$$

Given an initial mass $m_0 > 0$ attached onto the membrane, the objective is to determine an optimal strategy $u(\cdot)$ that takes values 0 or 1 for maximizing $J_T(m_0, u(\cdot))$.

Remark 1. The fact that an admissible control takes values within $\{0, 1\}$ (i.e. within a non-convex set) is not usual in optimal control. This point will be discussed in the next section.

We now consider the following hypotheses on the model.

Hypothesis 2. The functions f_1, f_2 and g are of class C^1 over $\mathbf{R}_+$ and satisfy the following assumptions:

- $f_1(m) > 0$ and $g(m) > 0$ for any $m \geq 0$
- $f_2(0) = 0$ and $f_2(m) > 0$ for $m > 0$
- f_1 and g are decreasing
- f_2 is increasing

One can straightforwardly check the following property

Lemma 3. Under Hypothesis 2, the domain $\{m > 0\}$ is positively invariant whatever the control $u(\cdot)$ is.

3.2 Application of Pontryagin's Principle

Now, we use the Pontryagin Maximum Principle Pontryagin et al. (1964) in order to determine necessary conditions on optimal trajectories. We introduce the Hamiltonian H associated to the controlled system (equation 4) and the cost functional (equation 5):

$$H(m, \lambda, u) = -\lambda f_2(m) + u[g(m) + \lambda(f_1(m) + f_2(m))], \tag{6}$$

where λ is called the adjoint variable and satisfies the adjoint equation $\dot{\lambda} = -\frac{\partial H}{\partial m}$, that is:

$$\dot{\lambda} = \lambda f_2'(m) - u^\star[g'(m) + \lambda(f_1'(m) + f_2'(m))]. \tag{7}$$

together with the terminal condition $\lambda(T) = 0$.

Furthermore, the maximum principle states that an optimal control $u^\star$ maximizes almost everywhere the Hamiltonian with respect to u, that is:

$$u^\star = \left| \begin{array}{l} 1 \text{ when } \phi(m, \lambda) > 0 \\ 0 \text{ when } \phi(m, \lambda) < 0 \end{array} \right.$$

where ϕ is the switching function

$$\phi(m, \lambda) = g(m) + \lambda(f_1(m) + f_2(m)). \tag{8}$$

The switching function determines when the control must be switched between the two functioning modes (i.e. filtration and relaxation).

Proposition 4. According to the Hypothesis 2 the adjoint variable satisfies $\lambda(t) < 0$ for any time $t \in [0, T[$. Moreover, for any initial condition m_0 there exists $\bar{t} < T$ such that $u(t) = 1$ is optimal for $t \in [\bar{t}, T]$.

Proof. As λ has to vanish, the adjoint equation straightforwardly implies that $\dot{\lambda}(t_0) \geq 0$ at every time $t_0 \in [0,T]$ such that $\lambda(t_0) = 0$. Suppose now that there exists a time $t < T$ such that $\lambda(t) > 0$ and let $t_0 > t$ be the first time where λ is vanishing (t_0 exists from the transversality condition). We necessarily have $\dot{\lambda}(t_0) \leq 0$ as $\lambda > 0$ in a left neighborhood of t_0. Hence, we obtain a contradiction. It follows that one has $\lambda \leq 0$ over $[0,T]$.

So, we conclude that given an initial condition $m_0 > 0$, there exists a time $\bar{t} > 0$ such that one should trigger a filtration cycle on the time interval $[\bar{t},T]$.

Now, the question that remains is what values take the optimal control for the other t-values in the control interval $[0,\bar{t}]$ and if any singular arc exist (when ϕ is vanishes over a time interval). Moreover, $\bar{t}$ depends on the initial condition, and from an applicative point of view, it is important to know a locus (in the (t,m)-plane) where the optimal control u switches to its final value 1.

4. OPTIMALITY RESULT

4.1 Existence of a singular arc

To solve the optimal control problem, we suppose that the following assumptions hold :

Hypothesis 5. The admissible control variable is a measurable time function that can take any values within [0,1].

Remark 6. When dealing with control functions taking values within a non-convex set (in our setting it is equal to $\{0,1\}$), optimal control theory does not guarantee the existence of an optimal control. So we have enlarged the set of controls to ensure the existence of an optimal control. Nevetheless, it is well known that a control that takes values different from 0 and 1 can be approximated by a sequence of "chattering" controls (see for instance Zelikin (2013)), as we shall consider in Section 5.

Now, we study the existence of singular arcs in our context. Recall that singular arcs occur in many optimal control problems. Roughly speaking, a singular arc is a time interval where the control is not bang-bang i.e. the maximization condition in the Hamiltonian does not provide any information on the optimal control. This situation typically appears when the system is linear w.r.t. the control. We refer for instance to Boscain et al. (2005) for a thorough study of this notion.

In our setting, the computation of the singular arc is as follows. Let us define a function $\psi : \mathbf{R}_+ \to \mathbf{R}$ by:

$$\begin{aligned}\psi(m) = g(m)(f_1'(m)f_2(m) - f_1(m)f_2'(m))\\ -g'(m)f_2(m)(f_1(m)+f_2(m)).\end{aligned} \tag{9}$$

We consider the following hypothesis:

Hypothesis 7. The function ψ admits an unique positive root $\bar{m}$ and is such that $\psi(m)(m-\bar{m}) > 0$ for any positive $m \neq \bar{m}$. .

Remark 8. This assumption can be verified numerically using the explicit expression of f_1, f_2 and g in the model.

Proposition 9. There exists a unique singular arc which is characterized by $m = \bar{m}$.

Proof. Let us write the derivative of the switching function represented by the equation (8). For simplicity,we drop the m dependency of functions f_1, f_2 and g. Also,we implicitly write ϕ instead of $\phi(m,\lambda)$:

$$\begin{aligned}\dot{\phi} &= [g' + \lambda(f_1' + f_2')].[uf_1 - (1-u)f_2]\\ &\quad +[\lambda f_2' - u(g' + \lambda(f_1' + f_2'))].(f_1 + f_2)\\ &= -g'f_2 + \lambda(f_2'f_1 - f_1'f_2)\\ &= -g'f_2 - g\frac{(f_2'f_1 - f_1'f_2)}{f_1+f_2} + \phi\frac{f_2'f_1 - f_1'f_2}{f_1+f_2}.\end{aligned} \tag{10}$$

or equivalently

$$\dot{\phi} = \frac{\psi}{f_1+f_2} + \phi\frac{f_2'f_1 - f_1'f_2}{f_1+f_2} \tag{11}$$

As a singular arc has to fulfill $\phi = 0$ and $\dot{\phi} = 0$, equation (11) and Hypothesis 7 give $\psi = 0$. Then, the only possibility for having a singular arc on a time interval $[t_1,t_2]$ is to have $m(t) = \bar{m}$ for any $t \in [t_1,t_2]$.

By a direct computation using the state equation, the singular control $\bar{u}$ (i.e. the constant control value for which $m(t) = \bar{m}$) is given by:

$$\bar{u} = \frac{f_2(\bar{m})}{f_1(\bar{m}) + f_2(\bar{m})}. \tag{12}$$

Notice that under Hypothesis 2 the value $\bar{u}$ belongs to $[0,1]$.

4.2 Switching curve

We will now show that the optimal synthesis exhibits a switching curve in the plane (t,m) from the control value 0 to the control value 1. This curve then allows to determine the instant $\bar{t}$ where the control switches.

First, we consider the following assumption. Let $\bar{m}_T \in \mathbf{R}$ and $\bar{T} \in \mathbf{R}$ be defined by:

$$\bar{m}_T = g^{-1}\left(\frac{g(\bar{m})f_2(\bar{m})}{f_1(\bar{m}) + f_2(\bar{m})}\right), \quad \bar{T} = T - \int_{\bar{m}}^{\bar{m}_T} \frac{dm}{f_1(m)}.$$

Proposition 10. Assume $\bar{T} > 0$. There exists a continuous function $\zeta : [\bar{T},T) \to [\bar{m},+\infty)$ such that:

- ζ is increasing and satisfies $\zeta(\bar{T}) = \bar{m}$.
- An optimal trajectory $m(\cdot)$ starting at some point (t_0,m_0) with $t_0 \geq \bar{T}$ and $m_0 \geq \zeta(t_0)$ satisfies $u = 0$ until $m(\cdot)$ intersects ζ at a time $\bar{t} \in [t_0,T)$, and then we have $u = 1$ over $[\bar{t},T]$.

Notice that the proof of the optimality results is in line with the preprint Kalboussi et al. (2016 bis) by the same authors which is devoted to the study of an analogous optimal control problem. For lack of place, we omit the proof.

Remark 11. The end point of the switching curve is $(\bar{T},\bar{m})$. Therefore $\bar{T}$ is the exact time from which it is optimal to leave (with the control $u = 1$) the singular arc $m = \bar{m}$.

4.3 Optimal synthesis

The following proposition is the main result of the paper and provides an optimal feedback control of the problem.

Proposition 12. Let $\xi : [0,T] \to [\bar{m},+\infty)$ be the function

$$\xi(t) = \begin{vmatrix} \bar{m} & \text{if } t \in [0, \bar{T}], \\ \zeta(t) & \text{if } t \in [\bar{T}, T]. \end{vmatrix}$$

where $\zeta(\cdot)$ and $\bar{T}$ are given by Proposition 10. The optimal feedback control strategy maximizing the cost functional (5) is given by:

$$u^{\star}(t,m) = \begin{vmatrix} 0 & \text{if } m > \xi(t), \\ \bar{u} & \text{if } m = \bar{m} \quad \text{and} \quad t \leq \bar{T}, \\ 1 & \text{if } m < \xi(t). \end{vmatrix}$$

The proof is described in detail in Kalboussi et al. (2016 bis). To summarize, according to the position of $m(t)$ with respect to $\bar{m}$, the membrane filtration process operates as follows:

- *First case*: the singular arc is reached over $[0, \bar{T}]$. Then, the process operates in filtration ($u = 1$) or in relaxation ($u = 0$) until it reaches the singular arc and stay at the steady state ($m = \bar{m}$) by applying the constant control ($u = \bar{u}$) until the switching time $\bar{T}$ is reached. After the time $\bar{T}$, it is optimal to filtrate ($u = 1$) until the terminal time T.
- *Second case*: the singular arc cannot be reached before the time $\bar{T}$, either because the optimal trajectory stays above $\bar{m}$ with $u = 0$ and reaches first the switching curve, either because the optimal trajectory stays below the concatenation of the singular arc and the switching curve with the control $u = 1$ which is optimal up to the final time.

5. PRACTICAL IMPLEMENTATION

In practice, applying the singular control $\bar{u}$ has no physical meaning and only $u = 1$ or $u = 0$ can be applied on a real process. The problem we address in this section is thus to find an appropriate filtration/cleaning sequence given $\bar{u}$. In other words, we want to approximate the singular control $\bar{u}$ by successive switchings of only 0 or 1 (known in the literature as chattering controls, see Zelikin (2013)) such that $m(t)$ remains close to $\bar{m}$, in the spirit of former works in biotechnologies such as in Betancur et al. (2013). Logically, as a first strategy, we can consider that $\bar{u}$ is the percentage of filtration time in an operating cycle, such that:

$$\begin{cases} \bar{u} &= \dfrac{T_f}{T_p} \\ T_p &= \dfrac{T_{SA}}{N} = T_f + T_r \end{cases} \tag{13}$$

with T_f and T_r the filtration and the relaxation times, respectively, over an operating cycle, T_p the duration of the operating cycle, T_{SA} the total time for which it is optimal to applied the constant control $\bar{u}$ and N the number of cycles fixed by the user over the time-period T_{SA}.

6. NUMERICAL RESULTS AND DISCUSSION

We consider the following functions

$$f_1(m) = \frac{b}{e+m}, \; f_2(m) = am, \; g(m) = \frac{1}{e+m}$$

where a, b and e are positive parameters. One can check that Hypothesis 2 is fulfilled. A straightforward computation of the function ψ gives

$$\psi(m) = \frac{d}{e+m}\left(-\frac{bam}{(e+m)^2} - \frac{ba}{e+m}\right) + \frac{dam}{(e+m)^2}\left(\frac{b}{e+m} + am\right) = \frac{da\left(am^2 - b\right)}{(e+m)^2}$$

and allows to check that Hypothesis 7 is fulfilled with

$$\bar{m} = \sqrt{\frac{b}{a}}.$$

Figure 1 shows the general synthesis of the optimal control with a model where $a = b = e = 1$ and for a time horizon of 10 hours.

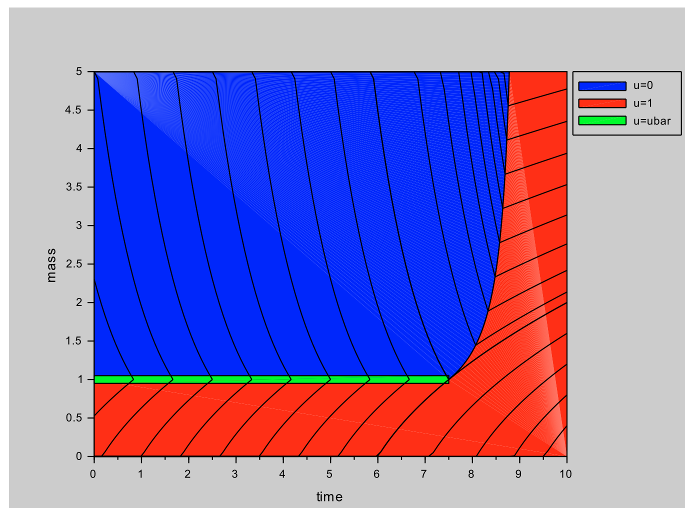

Figure 1. Synthesis for $a = b = e = 1$ and $T = 10$ hours; in red, $u = 1$, in blue $u = 0$, the horizontal line which separates them corresponds to $\bar{u}$ while the curve connected to this line and which separates the red and blue regions on the right of the figure is the "switching curve"

In Figure 2, we plotted an example of the control sequences propopsed in Section 5 for a case where $\bar{u} = 0.75$ and $N = 4$ over 10 time units.

In this section we also compare the optimal control solution with the operating strategy published in the work of Benyahia et al. (2013), hereafter denoted as the "classical control strategy". In addition, we investigate how much the theoretical optimal strategy degraded is when the number of cycles operated on the singular arc is modified (constrained by practical considerations). All simulations were performed with MATLAB using the same model, parameter values and initial conditions. This model has been validated using experimental data (cf. Benyahia et al. (2013)). The model parameters are summarized in table 1.

Table 1. Model parameters

Parameter	a	b	e	d
Value	25	2.75e+03	20	1800

First, what about the "classical control results"? Whatever the initial conditions, the classical control strategy consists

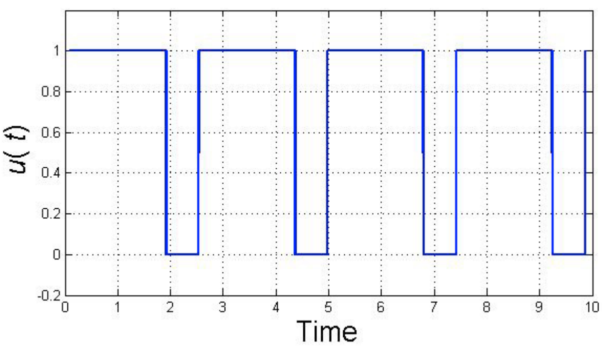

Figure 2. An example of a control sequence to approximate a constant control when $\bar{u} = 0.75$ and $N = 4$ over a period of time of 10 units

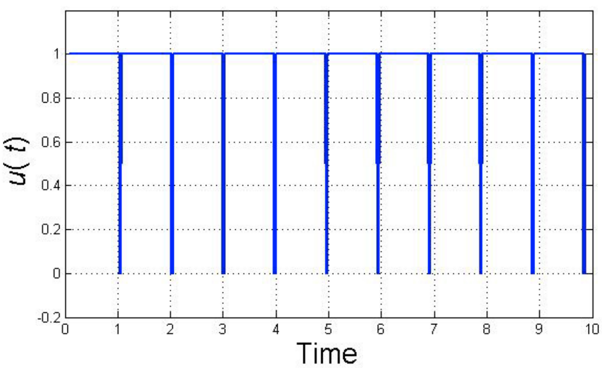

Figure 3. The "classical control sequence"

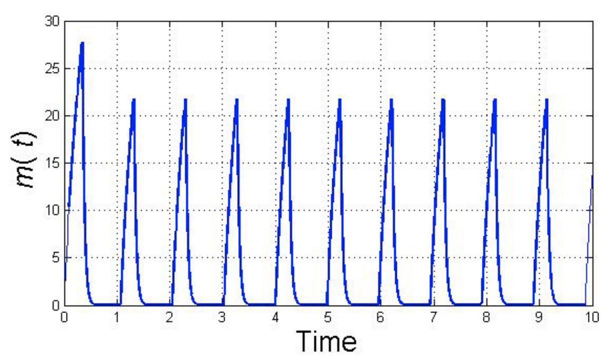

Figure 4. The dynamic of $m(t)$ when the "classical control" is applied

in starting at $t = 0$ by $u = 1$ (filtration) for 2 hours and then switching to $u = 0$ (relaxation) for 0.08 hours and continue until T is reached. The corresponding control sequence over $T = 10$ h is represented in Figure 3.

The corresponding dynamic of $m(t)$ is plotted in Figure 4 while the total water produced over T is then about 250 litters.

Now, what about the optimal strategy? For the parameter values reported in Table 1 and for a prediction horizon of 10 hours, one may easily check that $\bar{m} = 10.5$ g and $\bar{T} = 9.8$ time units. Figure 5 shows the optimal control $u(t)$. In this simulation, as $m_0 = 10^{-3}$ g is lower than $\bar{m}$, the optimal strategy is to filtrate ($u = 1$) until $m(t)$ reaches $\bar{m}$. Then the singular control $\bar{u} = 0.74$ is applied until $t = \bar{T} = 9.8$ h, before finally switching back to $u = 1$ until $t = T$ is reached.

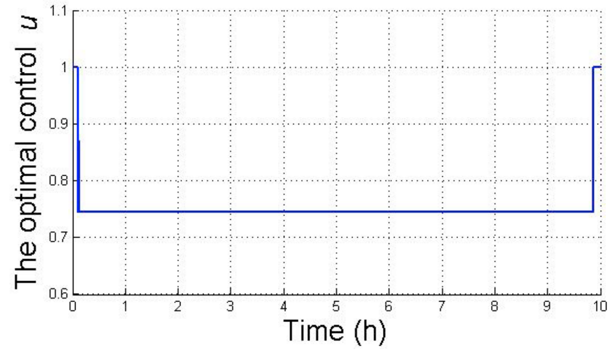

Figure 5. The optimal control sequence

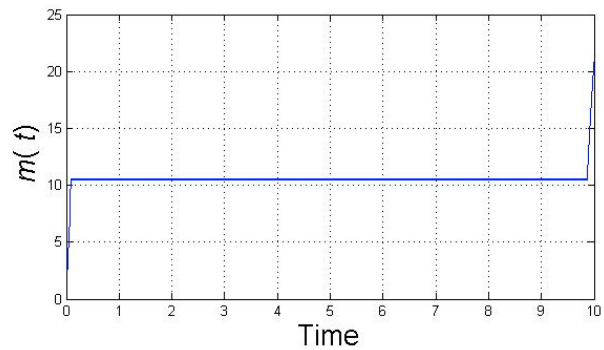

Figure 6. The dynamic of $m(t)$ when the theoretical optimal control is applied

The corresponding dynamic of $m(t)$ is plotted in Figure 6 while the total water produced over T is then about 450 litters.

Now, we assess the degradation of the optimal strategy when we consider practical considerations in the sense the theoretical optimal control cannot be applied in practice. In other terms, as underlined in the previous section, the optimal control u, when considered on the singular arc, must be replaced by a sequence of filtration/relaxation periods. Let us investigate how the control performances change with respect to the number of cycles N which are applied during the period of time in which the singular control $\bar{u}$ should be applied. In Figure 7, we plotted the final value of the criterion as a function of N (*i.e.* the total water production over T when the number of cycles N is varied between 4 and 100). The criterion value is significantly improved by increasing the number of cycle N. For large N, J tends towards 450 litters which corresponds to the total water production when the system operates according to the theoretical optimal solution determined by the maximum principle. The main point to be highlighted here is that even with a very limited number of cycles N (actually less than 10, cf. Figure 7), the performances are already better - increased by about 20% - than those obtained using the "classical control strategy".

In Figure 8, we represent for each value of N the mean value of the mass accumulated onto the membrane surface

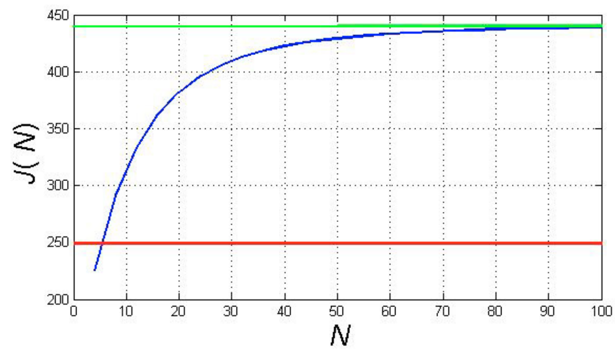

Figure 7. Optimal control and classical control water production; in red, the total water production of the "classical control"; in green, the total water production in the optimal case

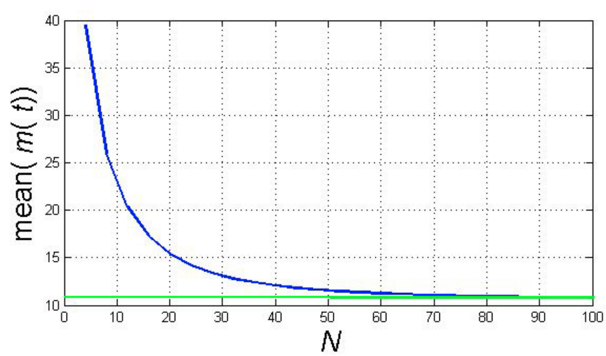

Figure 8. The mean mass accumulated in the membrane surface; in green we have plotted $\bar{m}$

over T_{SA}; when N increases the mean accumulated mass onto the membrane surface tends towards $\bar{m}$. In other terms, closer m to $\bar{m}$ on the singular arc, better the performances.

7. CONCLUSIONS AND PERSPECTIVES

In this work, the application of the Pontryagin Maximum Principle for the synthesis of the optimal control of a switched system showed interesting results for maximizing the treated volume of water in a filtration system. Our approach consisted in studying first the optimal control problem from a mathematical point of view and then to adapt the optimal solution to the process practical constraints. The determined optimal strategy improves the water production of the membrane filtration process compared to a classical operating strategy proposed in the literature. The main advantage of the optimal control approach proposed here is that it has been synthesized for a very large class of models, essentially defined by qualitative properties of functions f_1, f_2 and g. The main drawback of the proposed approach is that the variable used for deciding what is the control to be applied at a given instant ($m(t)$) is not measured in practice. Thus, perspectives of this work include not only extensions to more realistic models and control problems (including the case of backwash for membrane cleaning instead of only considering relaxation) but also the synthesis of observers to be able to estimate the decision variable $m(t)$ from realistic measurements.

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