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AgFoodTrade

New Issues in Agricultural, Food & Bioenergy Trade

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Working Paper 2009-10

AGFOODTRADE (*New Issues in Agricultural, Food and Bioenergy Trade*) is a Collaborative Project financed by the European Commission within its VII Research Framework. Information about the Project, the partners involved and its outputs can be found at www.agfoodtrade.eu .

Calibrating Mathematical Programming Spatial Models[1]

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1. Introduction

In the area of trade, modelers have a wide variety of tools at their disposal: spatial and non spatial partial equilibrium models, computable general equilibrium models. There is no superiority between them but rather a better adequacy or efficiency to deal with the specific issue at hand. Pros and cons of the different classes of models are addressed, among the others, in Anania (2001), Bouët (2008), Francois and Reinert (1997), and van Tongeren, van Meijl and Surry (2001). Partial equilibrium models tend to better accommodate explicit representations of complex policy instruments, allow for a more detailed representation of markets and require less restrictive assumptions. Computable general equilibrium models can deal with interdependence among sectors and income and employment effects.

In this paper we deal with spatial partial equilibrium models, e.g. with partial equilibrium models which are "naturally" able to reproduce bilateral trade flows without having to resort to the Armington assumption (Armington, 1969). These models are particularly useful when the market, or the markets, considered are relatively small with respect to the countries' overall economy and relevant trade policies include discriminatory ones, i.e. policies which discriminate by country of origin (destination) of imports (exports), such as preferential tariffs, country specific tariff rate quotas or embargos. In particular, the focus is on mathematical programming spatial partial equilibrium models.

Empirical models of international trade are subject to a common pitfall that is represented by the discrepancy between actual and optimal trade flows, that is, between realized commodity flows in a given year and the import-export patterns generated by the model solution for the same year. In fact, mathematical programming models tend to suffer from an "overspecialization" of the optimal solution with respect to observed trade flows. The main reason for this discrepancy often originates with the transaction costs per unit of commodity bilaterally traded between two countries; generally this piece of crucial information is measured with a degree of imprecision which is well above that of other parameters in the model. When this event occurs, a calibration of the trade model for the given base year allows for more effective policy simulations. Different approaches have

[1] We wish to thank Giuseppe Paletta and Francesca Vocaturo for helpful comments on an earlier draft of the paper. Financial support received by the "New Issues in Agricultural, Food and Bio-energy Trade (AGFOODTRADE)" (Small and Medium-scale Focused Research Project, Grant Agreement no. 212036) research project funded by the European Commission and by the "European Union policies, economic and trade integration processes and WTO negotiations" research project funded by the Italian Ministry of Education, University and Research (Scientific Research Programs of National Relevance 2007) is gratefully acknowledged. The views expressed in this paper are the sole responsibility of the authors and do not necessarily reflect those of the European Commission.

been used in the past to calibrate mathematical programming trade models, mostly based on including in the model additional constraints limiting the space of feasible solutions. The original calibration procedure proposed in this paper follows the approach used in Positive Mathematical Programming (PMP) (Howitt, 1995a and 1995b).

The paper is structured in two parts. The first part discusses the proposed calibration procedure with reference to a variety of mathematical programming spatial transportation and trade models; the second part provides numerical examples of the implementation of the calibration procedure proposed for models discussed in the first part.

2. Calibrating Mathematical Programming Spatial Trade Models

2.1 The Classical Transportation Model

We begin with a simple transportation model involving *J* importing and *I* exporting countries. We assume a single homogeneous commodity whose quantities consumed by the *j*-th destination, $\bar{x}_j^D$, and supplied by the *i*-th origin, $\bar{x}_i^S$, are known together with the realized trade flow, $\bar{x}_{ij}$, and the fixed accounting transaction cost per unit of commodity, tc_{ij}, transported between country pairs. In all statements, indexes range as $i = 1,...,I$ and $j = 1,...,J$.

This simple model can be stated as follows:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij} x_{ij} \tag{1}$$

		Dual variables	
subject to	$\bar{x}_j^D \le \sum_{i=1}^{I} x_{ij}$	p_j^D	(2)
	$\sum_{j=1}^{J} x_{ij} \le \bar{x}_i^S$	p_i^S	(3)

and $x_{ij} \ge 0$. The interpretation of the dual variables p_j^D and p_i^S corresponds, respectively, to commodity prices at destination and at origin.

In general, transaction costs are estimated imprecisely, often extending the same unit cost to routes for which a direct figure is not available. An initial goal of the proposed calibration procedure, therefore, is to obtain a correct marginal transaction cost by means of a dual parameter, say λ_{ij}, that is consistent with the structure of the transportation model and the knowledge of realized trade flows. Thus, the corresponding linear programming model minimizes the total transaction cost, *TTC*, subject to conventional

demand and supply constraints together with calibration constraints as in the following primal specification:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij} x_{ij} \tag{4}$$

		Dual variables	
subject to	$\bar{x}_j^D \le \sum_{i=1}^{I} x_{ij}$	p_j^D	(5)
	$\sum_{j=1}^{J} x_{ij} \le \bar{x}_i^S$	p_i^S	(6)
	$x_{ij} = \bar{x}_{ij}$	λ_{ij}	(7)

and $x_{ij} \ge 0$. λ_{ij} expresses the difference between the accounting and the effective transaction cost per unit of bilaterally traded commodity. While dual variables p_j^D and p_i^S are nonnegative by virtue of the specified direction of the associated constraints, nothing can be said *a priori* about the sign of dual variables λ_{ij} associated with calibration constraints (7). In fact, differently from the traditional PMP approach (Howitt, 1995a and 1995b), in this paper the calibrating constraints are stated as a set of equations, rather than inequalities. This means that either a reduction or an increase of the accounting – and, often, poorly measured – transaction cost is admissible. The specification of the calibration constraints admits the common event of "self-selection" that occurs when the realized trade between a given pair of countries is null. The economic justification for this occurrence is attributed to the "fact" that the marginal cost of trade is strictly greater than the associated marginal revenue.

The dual specification of the transportation model (4)-(7) is stated as the maximization of the net value added, *NVA*, of the transportation industry subject to the economic equilibrium constraints according to which its marginal cost per unit of commodity exchanged between a given pair of countries must be greater than or equal to its marginal revenue, that is

$$\max NVA = \sum_{j=1}^{J} p_j^D \bar{x}_j^D - \sum_{i=1}^{I} p_i^S \bar{x}_i^S - \sum_{i=1}^{I}\sum_{j=1}^{J} \lambda_{ij} \bar{x}_{ij} \tag{8}$$

		Dual variables	
subject to	$p_j^D \le p_i^S + (tc_{ij} + \lambda_{ij})$	x_{ij}	(9)

and $p_j^D \ge 0$, $p_i^S \ge 0$, λ_{ij} free variable. The term $(tc_{ij} + \lambda_{ij})$ constitutes the effective transaction cost per unit of commodity transported from the *i*-th to the *j*-th countries. The

supporting idea is that information about transaction costs is more difficult to obtain than information on trade flows. Hence, the utilization of all the available information – whether the accounting and, admittedly, imprecise transaction costs and the more accurate trade data – should provide a better specification of the international trade model. The level and the sign of the dual variable λ_{ij} resulted from the solution of model (4)-(7) will determine whether the accounting transaction cost tc_{ij} was originally either over- or under-estimated. The crucial realization, therefore, is that a solution of either the primal or the dual models defined above should not be regarded as a tautological statement but as a way to elicit the complete and more accurate marginal transaction costs to be used in subsequent analyses.

With knowledge of the dual variables λ_{ij} obtained from the solution of LP model (4)-(7), say λ_{ij}^*, a second phase LP model can be stated as follows:

$$\min TTC = \sum_{i=1}^{I}\sum_{j=1}^{J}(tc_{ij} + \lambda_{ij}^*)x_{ij} \tag{10}$$

Dual variables

subject to $\quad \overline{x}_j^D \le \sum_{i=1}^{I} x_{ij} \qquad p_j^D \tag{11}$

$$\sum_{j=1}^{J} x_{ij} \le \overline{x}_i^S \qquad p_i^S \tag{12}$$

with $x_{ij} \ge 0$, $i = 1,...,I$ and $j = 1,...,J$.

Classical PMP modifies a linear objective function by adding a quadratic function which accounts for additional costs inferred based on the difference between the observed realization and the solution from the uncalibrated model. The calibration procedure proposed in this paper does not alter the objective function, but only "corrects" one set of its parameters (bilateral transaction costs). The classical PMP approach assumes that costs in the uncalibrated model can be only underestimated, while the approach proposed assumes that transaction costs can be either underestimated or overestimated (λ_{ij}^* are unrestricted). Classical PMP and the calibration procedure proposed here both assume the model is well specified in all its parts but in the parameters being subject to the calibration; this means, for example, that if the model is ill-designed with respect to the representation of existing policies, these errors will be captured by the λ_{ij}^* and subsequent policy simulations will yield distorted results.

The empirical solution of model (10)-(12) should be carried out using all the available information, that includes the realized levels of activities. When the initial values of the trade flow variables are set equal to the realized level of trade flows, model (10)-(12) calibrates perfectly all its components. If initial values are set at levels different from the realized ones there is the possibility that the empirical model will detect alternative

optimal trade flow matrices (Dantzig, 1951; Koopmans, 1947; Paris, 1981). However, the optimal solution always reproduces quantities consumed and produced in each country as well as demand and supply prices; this occurs because the structure of the objective functions at the optimum and that of the constraints is identical. To illustrate this assertion it is sufficient to specify the dual of model (10)-(12), that is

$$\max NVA = \sum_{j=1}^{J} p_j^D \bar{x}_j^D - \sum_{i=1}^{I} p_i^S \bar{x}_i^S \tag{13}$$

Dual variables

subject to $\quad p_j^D \le p_i^S + (tc_{ij} + \lambda_{ij}^*) \qquad x_{ij} \tag{14}$

with $p_j^D \ge 0,\ p_i^S \ge 0$. Constraints (5), (6) and (9) in the model with calibrating constraints are identical to constraints (11), (12) and (14) in the model without calibrating constraints. Furthermore, at the optimal solution the primal and dual objective functions in the two models are equal. This establishes the equivalence of the two specifications.

A more informative discussion about the correct adjustment appearing in the objective function of the calibrating model (10)-(12) involves the Lagrangean function of model (4)-(7):

$$L = \sum_{i=1}^{I}\sum_{j=1}^{J} tc_{ij} x_{ij} + \sum_{j=1}^{J} p_j^D (\bar{x}_j^D - \sum_{i=1}^{I} x_{ij}) + \sum_{i=1}^{I} p_i^S (\sum_{j=1}^{J} x_{ij} - \bar{x}_i^S) + \sum_{i=1}^{I}\sum_{j=1}^{J} \lambda_{ij} (x_{ij} - \bar{x}_{ij}) \tag{15}$$

with derivatives

$$\frac{\partial L}{\partial x_{ij}} = tc_{ij} - p_j^D + p_i^S + \lambda_{ij} \ge 0,\ i = 1,...,I;\ j = 1,...,J, \tag{16}$$

which indicate the correct adjustment of the per-unit transaction costs in the form of $p_j^D \le (tc_{ij} + \lambda_{ij}) + p_i^S$, as given in constraints (9) and (14). Hence – because $\bar{x}_j^D$ and $\bar{x}_i^S$ are exogenously determined and the trade flows x_{ij} are the only variables – the objective function (10) expresses the desired and required parameterization for obtaining a set of multiple optimal solutions which contains the one that mimics the realized trade pattern.

The stylized nature of the above LP structures may be enriched with a more appropriate specification of an international trade model involving the paraphernalia of tariffs, subsidies, quotas, penalties, preferential trade treatments, exchange rates, etc. Hence, within reasonable parameter intervals, models (10)-(12) and (13)-(14) – augmented of the appropriate constraints – can be used to evaluate the likely effects of changes in policy interventions regarding tariffs, subsidies and other control parameters of interest.

2.2 A Samuelson-Takayama-Judge Model of International Trade with One Commodity

Analogous discussion may take place when explicit total supply and total demand functions for each country are available. In this case, the grouping between importing and exporting countries cannot be done in advance of solving the problem. Let us, therefore, define an index that covers all the regions (countries) without distinction between importers and exporters, $i, j = 1,...,R$. The known inverse demand function of the single commodity for the *j*-th country is assumed as $p_j^D = a_j - D_j x_j^D$, while the known inverse supply function for the same homogeneous commodity is assumed as $p_i^S = b_i + S_i x_i^S$. The coefficients a_j, D_j, S_i are known positive scalars. Parameter b_i is also known but may be either positive or negative. In this specification, the quantities x_j^D and x_i^S are no longer fixed, as in the previous section, and must be determined as part of the solution together with the trade flows x_{ij}. We assume the availability of information concerning realized trade flows, $\overline{x}_{ij}$, and – as a consequence – knowledge of total quantities demanded, $\overline{x}_j^D$, and supplied, $\overline{x}_i^S$, in each country.

The Samuelson-Takayama-Judge (STJ) model (Samuelson, 1952; Takayama and Judge, 1971) exhibits an objective function that maximizes a quasi-welfare function (QWF) given by the difference between the areas below the demand and above the supply functions which is netted out of total transaction costs. This specification corresponds to the maximization of the sum of consumer and producer surpluses netted out of total transaction costs.

The two elements of the QWF function – the demand and supply functions, on one side, and the total transaction costs, on the other side – may be subject to imprecise measurements. We assume that only transaction costs are measured with imprecision. In fact, this is the crucial source for the discrepancy between realized and optimal traded quantities and total quantities demanded and supplied in each country, obtained from the solution of the STJ model.[2]

When information about the realized trade pattern, $\overline{x}_{ij}$, is available, the specification of the primal model is as follows:

$$\max QWF = \sum_{j=1}^{R} (a_j - D_j x_j^D / 2) x_j^D - \sum_{i=1}^{R} (b_i + S_i x_i^S / 2) x_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} tc_{ij} x_{ij} \qquad (17)$$

Dual variables

[2] Jansson and Heckelei (2009) propose a calibration procedure for mathematical programming spatial equilibrium models based on the estimation of transportation costs and prices, assumed to be stochastic, with measurement errors independent and identically distributed with known variances.

subject to

$$x_j^D \le \sum_{i=1}^{R} x_{ij} \qquad p_j^D \tag{18}$$

$$\sum_{j=1}^{R} x_{ij} \le x_i^S \qquad p_i^S \tag{19}$$

$$x_{ij} = \overline{x}_{ij} \qquad \lambda_{ij} \tag{20}$$

and all nonnegative variables, $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0$, $(i, j = 1, 2, \ldots, R)$.

The dual of model (17)-(20) may be stated as follows

$$\min TCMO = \sum_{j=1}^{R} x_j^D D_j x_j^D / 2 + \sum_{i=1}^{R} x_i^S S_j x_i^S / 2 + \sum_{i=1}^{R}\sum_{j=1}^{R} \lambda_{ij} \overline{x}_{ij} + \sum_{i=1}^{R}\sum_{j=1}^{R} tc_{ij} x_{ij} \tag{21}$$

		Dual Variables	
subject to	$p_j^D \ge a_j - D_j x_j^D$	x_j^D	(22)
	$b_i + S_i x_i^S \ge p_i^S$	x_i^S	(23)
	$p_i^S + (tc_{ij} + \lambda_{ij}) \ge p_j^D$	x_{ij}	(24)

and $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0$; λ_{ij} a free variable, $(i, j = 1, 2, \ldots, R)$. The economic interpretation of the objective function is given by the minimization of the total cost of market options and of the differential total transaction costs. When interpreting a dual model it is convenient to suppose that a second economic agent – external to the primal problem – desires to "take over the enterprise" of the primal economic agent. In this case, the dual agent will have to quote prices and quantities that will reimburse the primal agent of its "potential profit" (consumer and producer surpluses) plus the differential total transaction costs. The dual constraints express the demand and supply functions as well as the condition that "marginal transaction cost" of the traded commodity between each pair of countries is greater than or equal to its "marginal revenue."

The solution of model (17)-(20) provides an estimate of the dual variables associated with the calibration constraints, λ_{ij}^*, that can be utilized for adjusting the transaction costs as in the following calibrating model:

$$\max QWF = \sum_{j=1}^{R} (a_j - D_j x_j^D / 2) x_j^D - \sum_{i=1}^{R} (b_i + S_i x_i^S / 2) x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} (tc_{ij} + \lambda_{ij}^*) x_{ij} \tag{25}$$

Dual variables

subject to $x_j^D \le \sum_{i=1}^{R} x_{ij}$ p_j^D (26)

$\sum_{j=1}^{R} x_{ij} \le x_i^S$ p_i^S (27)

and, $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0$; $(i, j = 1, 2, \ldots, R)$.

The adjustment of the per-unit transaction costs follows the same justification as presented in the previous section.

The Lagrangean function of problem (17)-(20) is:

$$L = \sum_{j=1}^{R}(a_j - D_j x_j^D / 2)x_j^D - \sum_{i=1}^{R}(b_i + S_i x_i^S / 2)x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} tc_{ij} x_{ij} + \sum_{j=1}^{R} p_j^D (\sum_{i=1}^{R} x_{ij} - x_j^D) + \sum_{i=1}^{R} p_i^S (x_i^S - \sum_{j=1}^{R} x_{ij}) + \sum_{i=1}^{R}\sum_{j=1}^{R} \lambda_{ij}(\overline{x}_{ij} - x_{ij}) \quad (28)$$

with relevant conditions:

$$\frac{\partial L}{\partial x_{ij}} = p_j^D - p_i^S - tc_{ij} - \lambda_{ij} \le 0 \text{, and } \frac{\partial L}{\partial x_{ij}} x_{ij} = 0 \ . \quad (29)$$

This implies model (25)-(27) calibrates total quantities demanded and supplied in each country. When the available information is fully exploited and the solution is searched providing observed trade flows as initial values of the trade flow variables, the model calibrates perfectly. However, in general, the calibrated model may show multiple optimal solutions, i.e. solutions where different sets of trade flows are associated to the same quantities produced and consumed in each country, the same total incurred adjusted transaction costs (calculated over all trade flows), and, as a result, the same value of the objective function. The possibility of multiple optimal solutions in terms of trade flows being associated to the unique optimal solution in terms of countries' net imports and exports does not come as a surprise because this is a common feature of this class of models (Takayama and Judge, 1971; Paris, 1983), and the proposed calibration procedure only modifies the parameters in the model without altering its structure.

Let us assume now that only information about total demand, $\overline{x}_{r'}^D$, and total supply, $\overline{x}_r^S$, is available. The STJ model assumes the following specification:

$$\max QSW = \sum_{j=1}^{R}(a_j - D_j x_j^D / 2)x_j^D - \sum_{i=1}^{R}(b_i + S_i x_i^S / 2)x_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} tc_{ij} x_{ij} \quad (30)$$

Dual
Variables

subject to $x_j^D \le \sum_{i=1}^{R} x_{ij}$ p_j^D (31)

$\sum_{j=1}^{R} x_{ij} \le x_i^S$ p_i^S (32)

$\overline{x}_j^D = x_j^D$ λ_j^D (33)

$x_i^S = \overline{x}_i^S$ λ_i^S (34)

and, $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0, (i, j = 1, 2, \ldots, R)$.

The solution of model (30)-(34) provides an estimate of the dual variables associated with the calibration constraints, λ_j^{D*} and λ_i^{S*}, that can be utilized for adjusting the trade transaction costs as in the following calibrated model:

$$\max QWF = \sum_{j=1}^{R} (a_j - D_j x_j^D / 2) x_j^D - \sum_{i=1}^{R} (b_i + S_i x_i^S / 2) x_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} (tc_{ij} + \lambda_i^{S*} + \lambda_j^{D*}) x_{ij} \quad (35)$$

Dual
Variables

subject to $x_j^D \le \sum_{i}^{R} x_{ij}$ p_j^D (36)

$\sum_{j=1}^{R} x_{ij} \le x_i^S$ p_i^S (37)

and, $x_j^D \ge 0, x_i^S \ge 0, x_{ij} \ge 0, (i, j = 1, 2, \ldots, R)$.

The solution of model (35)-(37) calibrates total demanded and supplied quantities.

In order to justify the adjustments of the transaction costs in equations (35), the Lagrangean function of model (30)-(34) comes to the rescue:

$$\begin{aligned} L = & \sum_{j=1}^{R} (a_j - D_j x_j^D / 2) x_j^D - \sum_{i=1}^{R} (b_i + S_i x_i^S / 2) x_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} tc_{ij} x_{ij} \\ & + \sum_{j=1}^{R} p_j^D (\sum_{i=1}^{R} x_{ij} - x_j^D) + \sum_{i=1}^{R} p_i^S (x_i^S - \sum_{j=1}^{R} x_{ij}) \\ & + \sum_{i=1}^{R} \lambda_i^S (\overline{x}_i^S - x_i^S) + \sum_{j=1}^{R} \lambda_j^D (\overline{x}_j^D - x_j^D) \end{aligned} \quad (38)$$

with relevant conditions

$$\frac{\partial L}{\partial x_{ij}} = -tc_{ij} + p_j^D - p_i^S - \lambda_i^S - \lambda_j^D \leq 0, \text{ or } \quad p_j^D \leq p_i^S + tc_{ij} + \lambda_i^S + \lambda_j^D , \quad (39)$$

and

$$\frac{\partial L}{\partial x_{ij}} x_{ij} = 0 \quad (40)$$

These conditions define the adjusted per-unit transaction cost ($tc_{ij} + \lambda_i^S + \lambda_j^D$) needed for the model to calibrate observed demanded and supplied quantities.[3]

2.3 A Multi-Commodity Samuelson-Takayama-Judge Model of International Trade

The extension of international trade models to multi-commodity exchanges does not require any substantial adjustment to the structure of the mathematical programming models discussed above. It requires, however, a considerably larger quantity of information that, if and when available, imposes the need of a careful management. The major shift from previous models is constituted by the specification of systems of demand and supply functions for each country. It follows that a properly defined system of demand and supply functions – for each country involved in the commodity exchange – ought to exhibit full matrices of demand and supply cross-price elasticities. This is a formidable information requirement that, when overcome, may produce adequate empirical results as well as sensible policy analyses.

We assume *K* homogeneous commodities. Each country owns a system of *K* inverse demand functions, $\mathbf{p}_j^D = \mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D$, j = 1,…, R , and an inverse system of *K* supply functions, $\mathbf{p}_j^S = \mathbf{b}_j + \mathbf{S}_j \mathbf{x}_j^S$, $j = 1,...,R$. The matrix of nominal per-unit transaction costs is defined in three dimensions as $\mathbf{T} = \left[\mathbf{tc}_{ij} \right]$, $i, j = 1,...,R$, where $\mathbf{tc}_{ij}$ is the vector of per unit transaction costs from country *i* to country *j* for the *K* commodities and $\mathbf{tc}_{rr}$ is the vector of domestic transaction costs in country *r*. We assume that information about the trade pattern for all commodities, $\bar{\mathbf{x}}_{ij}$, and, hence, total demands, $\bar{\mathbf{x}}_j^D$, and total supplies, $\bar{\mathbf{x}}_i^D$, is available for a given base year.

A special comment regards matrices $\mathbf{D}_j$ and $\mathbf{S}_j$, the matrices of cross-derivatives of the *j*-th country system of demand and supply functions. In principle, demand and production theory requires neither the symmetry nor the positive semidefinitess of such matrices. However, the statement of the STJ problem in the form of maximizing a *QWF* objective function that assumes a quadratic structure imposes the requirement that the matrices $\mathbf{D}_j$ and $\mathbf{S}_j$ be symmetric and positive semidefinite; this is quite a strong assumption, since there is no reason why $\mathbf{D}_j$ and $\mathbf{S}_j$ should satisfy these conditions. In section 2.4 this

[3] Based on (39), an alternative interpretation of the role played by the λ_i^S and λ_j^D parameters could be in terms of adjustments of the intercepts of supply and demand functions.

requirement will be relaxed and the appropriate mathematical programming problem will be specified.

2.3.1 Case 1: demand and supply functions are well measured at different market levels

We will consider two different cases. The first one is when demand and supply functions are measured at different levels, e.g. the supply function at the farm gate and the demand function at retail, and the only information in the model which is measured with imprecision are transaction costs.

Except for the dimensionality of the price, quantity and transaction cost vectors, the corresponding STJ model exhibits a structure that is similar to that of model (17)-(20):

$$\max QWF = \sum_{j=1}^{R} (\mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D / 2)' \mathbf{x}_j^D - \sum_{i=1}^{R} (\mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S / 2)' \mathbf{x}_i^S - \sum_{i=1}^{R} \sum_{j=1}^{R} \mathbf{tc}_{ij}' \mathbf{x}_{ij} \tag{41}$$

		Dual variables	
subject to	$\mathbf{x}_j^D \le \sum_{i=1}^{R} \mathbf{x}_{ij}$	$\mathbf{p}_j^D$	(42)
	$\sum_{j=1}^{R} \mathbf{x}_{ij} \le \mathbf{x}_i^S$	$\mathbf{p}_i^S$	(43)
	$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij}$	$\boldsymbol{\lambda}_{ij}$	(44)

with all nonnegative variables. The dual of model (41)-(44) is obtained in the usual fashion, by formulating the associated Lagrangean function, deriving the Karush-Kuhn-Tucker (KKT) conditions and, furthermore, by simplifying the Lagrangean function, which assumes the role of objective function in the dual problem.

$$\min TCMO = \sum_{j=1}^{R} \mathbf{x}_j'^D \mathbf{D}_j \mathbf{x}_j^D / 2 + \sum_{i=1}^{R} \mathbf{x}_i'^S \mathbf{S}_i \mathbf{x}_i^S / 2 + \sum_{i=1}^{R} \sum_{j=1}^{R} \overline{\mathbf{x}}_{ij}' \boldsymbol{\lambda}_{ij} \tag{45}$$

		Dual Variables	
subject to	$\mathbf{p}_j^D \ge \mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D$	$\mathbf{x}_j^D$	(46)
	$\mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S \ge \mathbf{p}_i^S$	$\mathbf{x}_i^S$	(47)
	$\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) \ge \mathbf{p}_j^D$	$\mathbf{x}_{ij}$	(48)

and all nonnegative variables except $\boldsymbol{\lambda}_{ij}$ which is regarded as a vector of free *K* variables. The economic interpretation of model (45)-(48) is similar to that one given for dual model (21)-(24).

The solution of model (41)-(44) provides estimates of dual variables $\boldsymbol{\lambda}_{ij}$, say $\boldsymbol{\lambda}^{*}_{ij}$, that can be used to define effective transaction costs along the line of the PMP methodology proposed above.[4] Hence, the calibrating STJ model for this more general international trade specification can be assembled as in the following structure

$$\max QSW = \sum_{j=1}^{R}(\mathbf{a}_j - \mathbf{D}_j\mathbf{x}_j^D/2)'\mathbf{x}_j^D - \sum_{i=1}^{R}(\mathbf{b}_i + \mathbf{S}_i\mathbf{x}_i^S/2)'\mathbf{x}_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^{*})'\mathbf{x}_{ij} \tag{49}$$

Dual variables

subject to $\sum_{i=1}^{R}\mathbf{x}_{ij} \geq \mathbf{x}_j^D$ $\mathbf{p}_j^D$ (50)

$\sum_{j=1}^{R}\mathbf{x}_{ij} \leq \mathbf{x}_i^S$ $\mathbf{p}_i^S$ (51)

with all nonnegative variables. The solution of model (49)-(51) will calibrate precisely the realized demanded and supplied quantities.

Extension 1: Estimation of Systems of Demand and Supply Functions

When information about the vectors of total demand quantities, $\overline{\mathbf{x}}_{jt}^D$, and supply quantities, $\overline{\mathbf{x}}_{it}^S$, is available for a number of *T* years – together with the corresponding demand prices, $\overline{\mathbf{p}}_{jt}^D$, and supply prices, $\overline{\mathbf{p}}_{it}^S$, $t = 1,...,T$, it is possible to estimate systems of demand and supply functions for each country. This estimation is performed in the same spirit of PMP; it attempts to utilize – and exploit in a logical and consistent way – all the available information.

Demand Functions

A least-squares approach is proposed for estimating the system of demand functions. In order to satisfy the integrability condition – which admits the definition of the proper

[4] Bauer and Kasnakoglu (1990) used the PMP approach to calibrate a quadratic programming model of Turkish agriculture with endogenous supply functions. Bouamra-Mechemache et al. (2002) calibrated a model similar to the one considered here by applying the classical PMP procedure (i.e. using inequality constraints to obtain the $\boldsymbol{\lambda}^{*}_{ij}$ and adding a quadratic cost function to the objective function); however, they found the calibrated solution not satisfactory and introduced further adjustments in the model.

objective function for the STJ model – the estimation is subject to the symmetry of the matrix of cross-derivatives, $\mathbf{D}_j$, as well as to its positive semidefiniteness. Hence,

$$\min \sum_{t=1}^{T} (\mathbf{u}_{jt}^{D})' \mathbf{u}_{jt}^{D} \tag{52}$$

subject to

$$\overline{\mathbf{p}}_{jt}^{D} = \mathbf{a}_j - \mathbf{D}_j \overline{\mathbf{x}}_{jt}^{D} + \mathbf{u}_{jt}^{D} \tag{53}$$

$$\mathbf{D}_j = \mathbf{L}_j \boldsymbol{\Theta}_j \mathbf{L}_j' \tag{54}$$

$$\sum_{t=1}^{T} \mathbf{u}_{jt}^{D} = \mathbf{0} \tag{55}$$

with $\Theta_{j,k,k} \geq 0$. Constraint (53) specifies the system of demand functions. Constraint (54) defines the Cholesky factorization that generates the symmetry and the positive semidefiniteness of the $\mathbf{D}_j$ matrix. The matrix $\mathbf{L}_j$ is a unit lower triangular matrix while the matrix $\boldsymbol{\Theta}_j$ is a diagonal matrix with all nonnegative elements that guarantee the positive semidefiniteness of the $\mathbf{D}_j$ matrix. Constraint (55) guarantees that all the year deviations add up to zero.

The interpretation of the term $\mathbf{u}_{jt}^{D}$ deserves a special comment. Within the context of a calibrating PMP approach, and under the assumption that only information for a very limited number of years is available, it is convenient to interpret this term as a yearly deviation from the average system of demand functions rather than as either an "error" or a "disturbance term." In other words, the yearly realization of the demand prices in the *r*-th region would deviate from the average prices by the amount $\mathbf{u}_{rt}^{D}$. Knowledge of this deviation, therefore, is crucial for assuring the calibration of the model over every region and every year.

An analogous approach may be used to estimate the system of supply functions.

Extension 2: A Multi-Year STJ Model of International Trade

With the estimation of the demand and supply systems, a PMP model may be specified over *T* years along the lines presented in equations (52)-(55). Thus, assuming that information about the trade flows in each year, $\overline{\mathbf{x}}_{ijt}$, is available:

$$\begin{aligned} \max QWF = & \sum_{t=1}^{T} \sum_{j=1}^{R} (\hat{\mathbf{a}}_j - \hat{\mathbf{D}}_j \mathbf{x}_{jt}^{D} / 2 + \hat{\mathbf{u}}_{jt}^{D})' \mathbf{x}_{jt}^{D} \\ & - \sum_{t=1}^{T} \sum_{i=1}^{R} (\hat{\mathbf{b}}_i + \hat{\mathbf{S}}_i \mathbf{x}_{it}^{S} / 2 + \hat{\mathbf{u}}_{it}^{S})' \mathbf{x}_{it}^{S} - \sum_{t=1}^{T} \sum_{i=1}^{R} \sum_{j=1}^{R} \mathbf{tc}_{ijt}' \mathbf{x}_{ijt} \end{aligned} \tag{56}$$

Dual variables

subject to $\sum_{i=1}^{R} \mathbf{x}_{ijt} \geq \mathbf{x}_{jt}^{D}$ $\mathbf{p}_{jt}^{D}$ (57)

$\sum_{j=1}^{R} \mathbf{x}_{ijt} \leq \mathbf{x}_{it}^{S}$ $\mathbf{p}_{it}^{S}$ (58)

$\mathbf{x}_{ijt} = \overline{\mathbf{x}}_{ijt}$ λ_{ijt} (59)

with all nonnegative variables, but λ_{ijt} which is unrestricted. This first phase model provides the essential estimates of the dual variables λ_{ijt}, say λ^{*}_{ijt}. Therefore, the calibrating PMP model can be specified as follows

$$\begin{aligned} \max QWF = & \sum_{t=1}^{T}\sum_{j=1}^{R} (\hat{\mathbf{a}}_j - \hat{\mathbf{D}}_j \mathbf{x}_{jt}^{D} / 2 + \hat{\mathbf{u}}_{jt}^{D})' \mathbf{x}_{jt}^{D} \\ & - \sum_{t=1}^{T}\sum_{i=1}^{R} (\hat{\mathbf{b}}_i + \hat{\mathbf{S}}_i \mathbf{x}_{it}^{S} / 2 + \hat{\mathbf{u}}_{it}^{S})' \mathbf{x}_{it}^{S} - \sum_{t=1}^{T}\sum_{i=1}^{R}\sum_{j=1}^{R} (\mathbf{tc}_{ijt} + \lambda_{ijt}^{*})' \mathbf{x}_{ijt} \end{aligned} \quad (60)$$

Dual variables

subject to $\sum_{i=1}^{R} \mathbf{x}_{ijt} \geq \mathbf{x}_{jt}^{D}$ $\mathbf{p}_{jt}^{D}$ (61)

$\sum_{j=1}^{R} \mathbf{x}_{ijt} \leq \mathbf{x}_{it}^{S}$ $\mathbf{p}_{it}^{S}$ (62)

with all nonnegative variables. The above model calibrates the quantity demanded and supplied by each country.

2.3.2 Case 2: demand and supply functions are measured with imprecision at the same market level

We consider a second case where demand and supply functions are measured at the same market level - the retail one - and transaction costs as well as demand and supply functions are measured with imprecision. This means that $\mathbf{tc}_{rr}$, the vector of domestic transaction costs in country *r*, is the null vector and $\mathbf{p}_r^D = \mathbf{p}_r^S$, for all *r* =1, 2, …, *R* .

Essentially this is the case where the calibration procedure, not only makes the model reproduce observed trade patterns, but, at the same time, adjusts parameters of the

demand and supply functions to make these and observed trade flows, $\overline{\mathbf{x}}_{ij}$, consistent with the condition $\mathbf{p}_r^D = \mathbf{p}_r^S$.

In this case in phase I, a least-squares approach is proposed to estimate simultaneously the adjustments of transaction costs and those of demand and supply functions needed for the model to calibrate observed trade patterns and comply with the condition that supply prices equal demand prices. In order to satisfy the conditions needed for the definition of the proper objective function for the STJ model, the estimation takes into account the need to assure the symmetry and positive semidefiniteness of the adjusted matrices of cross-derivatives.

The objective function is composed of the Sum of Squared Residuals of the intercepts and slopes of the demand and supply functions, plus a primal-dual component that represents the combination of the dual objective function of the problem minus the primal objective function. At the optimum this primal-dual portion of the objective function should achieve the value of zero. The constraints combine primal and dual constraints.

Using the familiar notation the model can be specified as follows

$$\min LS = \sum_{j=1}^{R} \mathbf{u}_j'\mathbf{u}_j / 2 + \sum_{i=1}^{R} \mathbf{v}_i'\mathbf{v}_i / 2 + \sum_{j=1}^{R} trace(\mathbf{W}'_j\mathbf{W}_j) / 2 + \sum_{i=1}^{R} trace(\mathbf{Y}'_i\mathbf{Y}_i) / 2 +$$
$$\{\sum_{j=1}^{R}\sum_{i=1}^{R} \overline{\mathbf{x}}_{ij}\boldsymbol{\lambda}_{ij} - [\sum_{j=1}^{R}((\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\ \mathbf{x}_j^D)'\mathbf{x}_j^D - \sum_{i=1}^{R}((\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\ \mathbf{x}_i^S)'\mathbf{x}_i^S \qquad (63)$$
$$- \sum_{i=1}^{R}\sum_{j=1}^{R} \mathbf{tc}'_{ij}\mathbf{x}_{ij}]\}$$

subject to

$$\sum_{i=1}^{R} \mathbf{x}_{ij} \geq \mathbf{x}_j^D \qquad (64)$$

$$\sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S \qquad (65)$$

$$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij} \qquad (66)$$

$$\mathbf{W}_j = \mathbf{L}_j\boldsymbol{\Theta}_j\mathbf{L}'_j \qquad (67)$$

$$\mathbf{Y}_i = \mathbf{M}_i\boldsymbol{\Phi}_i\mathbf{M}_i' \qquad (68)$$

$$\mathbf{p}_j^D \geq (\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\mathbf{x}_j^D \qquad (69)$$

$$(\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\mathbf{x}_i^S \geq \mathbf{p}_i^S \qquad (70)$$

$$\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) \geq \mathbf{p}_j^D \qquad (71)$$

$$\mathbf{p}_j^S = \mathbf{p}_j^D \qquad (72)$$

with $\Theta_{j,k,k} \geq 0$, $\Phi_{i,k,k} \geq 0$. The matrices $\mathbf{L}_j$ and $\mathbf{M}_i$ are unit lower triangular matrices, while matrices $\boldsymbol{\Theta}_j$ and $\boldsymbol{\Phi}_i$ are diagonal matrices with all nonnegative elements

Constraints (67) and (68) define the Cholesky factorization that generates the symmetry and positive semidefiniteness of the $\mathbf{W}_j$ and $\mathbf{Y}_i$ matrices, a sufficient, although not necessary, condition for the symmetry and semidefiniteness of the matrices of the adjusted slopes, ($\mathbf{D}_j + \mathbf{W}_j$) and ($\mathbf{S}_i + \mathbf{Y}_i$), in the systems of demand and supply functions.

The phase II calibrating model takes on the familiar maximization structure:

$$\max QWF = \sum_{j=1}^{R}((\mathbf{a}_j + \hat{\mathbf{u}}_j) - (\mathbf{D}_j + \hat{\mathbf{W}}_j)\mathbf{x}_j^D / 2)'\mathbf{x}_j^D - \sum_{i=1}^{R}((\mathbf{b}_i + \hat{\mathbf{v}}_i) + (\mathbf{S}_i + \hat{\mathbf{Y}}_i)\mathbf{x}_i^S / 2)'\mathbf{x}_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \hat{\boldsymbol{\lambda}}_{ij})'\mathbf{x}_{ij} \tag{73}$$

subject to

$$\sum_{i=1}^{R}\mathbf{x}_{ij} \geq \mathbf{x}_j^D \tag{74}$$

$$\sum_{j=1}^{R}\mathbf{x}_{ij} \leq \mathbf{x}_i^S \tag{75}$$

where $\hat{\mathbf{u}}_j, \hat{\mathbf{v}}_i, \hat{\mathbf{W}}_j, \hat{\mathbf{Y}}_i$ and $\hat{\boldsymbol{\lambda}}_{ij}$ are the least-squares estimates obtained in phase I of the corresponding parameters.

This model calibrates produced and consumed quantities and yields in each region demand prices equal to supply prices. As usual, prices correspond to the dual variables of the primal constraints.

2.4 The Equilibrium Problem

When the $\mathbf{D}_j$ and the $\mathbf{S}_j$ matrices are not symmetric, the system of demand and supply functions cannot be integrated and no suitable objective function exists for the STJ model. In this case, the appropriate mathematical programming structure is given by the Equilibrium Problem.

Definition
Let us consider the demand (*Dem*) and supply (*Sup*) of a commodity with quantity (*Q*), price (*P*) and marginal cost (*MC*). Then, the Equilibrium Problem is jointly defined by the following two sets of relations:

$$\begin{cases} \text{Primal: } P \geq 0, & Dem \leq Sup, & (Sup - Dem)P = 0 & (76) \\ \text{Dual: } Dem,\ Sup \geq 0 & MC \leq P, & (MC - P)Q = 0 & (77) \end{cases}$$

Hence, the calibrated Equilibrium Problem with a system of demand and supply functions whose matrices $\mathbf{D}_j$ and $\mathbf{S}_i$ are not symmetric is specified as follows:

Primal relations: $\mathbf{p}_j^D \geq \mathbf{0}$, $\mathbf{x}_j^D \leq \sum_{i=1}^{R} \mathbf{x}_{ij}$, $(\sum_{i=1}^{R} \mathbf{x}_{ij} - \mathbf{x}_j^D)'\mathbf{p}_j^D = \mathbf{0}$ (78)

$\mathbf{p}_i^S \geq \mathbf{0}$, $\sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S$, $(\mathbf{x}_i^S - \sum_{j=1}^{R} \mathbf{x}_{ij})'\mathbf{p}_i^S = \mathbf{0}$ (79)

$\boldsymbol{\lambda}_{ij}$ free, $\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij}$, $(\overline{\mathbf{x}}_{ij} - \mathbf{x}_{ij})'\boldsymbol{\lambda}_{ij} = \mathbf{0}$ (80)

Dual relations: $\mathbf{x}_j^D \geq \mathbf{0}$, $\mathbf{a}_j - \mathbf{D}_j\mathbf{x}_j^D \leq \mathbf{p}_j^D$, $(\mathbf{p}_j^D - \mathbf{a}_j + \mathbf{D}_j\mathbf{x}_j^D)'\mathbf{x}_j^D = \mathbf{0}$ (81)

$\mathbf{x}_i^S \geq \mathbf{0}$, $\mathbf{p}_i^S \leq \mathbf{b}_i + \mathbf{S}_i\mathbf{x}_i^S$, $(\mathbf{b}_i + \mathbf{S}_i\mathbf{x}_i^S - \mathbf{p}_i^S)'\mathbf{x}_i^S = \mathbf{0}$ (82)

$\mathbf{x}_{ij} \geq \mathbf{0}$, $\mathbf{p}_j^D \leq \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij})$, $[\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) - \mathbf{p}_j^D]'\mathbf{x}_{ij} = \mathbf{0}$. (83)

The fact that the $\mathbf{D}_j$ and $\mathbf{S}_i$ matrices are not symmetric causes neither theoretical nor computational difficulties since the system of demand and supply functions appears directly into the dual constraints (81) and (82) without the need for passing through the integral of the system – that does not exist, in this case – and the corresponding (not existent) primal objective function.

2.4.1 Case 1: demand and supply functions are well measured at different market levels

As we did for the STJ model in section 2.3, for the Equilibrium Problem too we consider two different cases. Again, the first one is when demand and supply functions are measured at different levels and the only information in the model which is measured with imprecision are transaction costs.

The solution of Equilibrium Problem (78)-(83) can be obtained by introducing primal and dual slack variables into the structural constraints and exploiting the complementary slackness conditions – that add up to zero – in the form of an auxiliary objective function that should be minimized, since each term is nonnegative. Thus, using nonnegative slack variables $\mathbf{z}_{jP1}, \mathbf{z}_{iP2}, \mathbf{z}_{jD1}, \mathbf{z}_{iD2}, \mathbf{z}_{ijD3}$, (where the subscript of $\mathbf{z}_{jP1}, \mathbf{z}_{iP2}$ stands for primal constraints 1 and 2 and the subscript of $\mathbf{z}_{jD1}, \mathbf{z}_{iD2}, \mathbf{z}_{ijD3}$ stands for dual constraints 1, 2 and 3) the solution of the calibrated Equilibrium Problem can be obtained by solving the following Phase I specification:

$$\min\{\sum_{ij}[\mathbf{z}'_{jP1}\mathbf{p}_j^D + \mathbf{z}'_{iP2}\mathbf{p}_i^S + \mathbf{z}'_{jD1}\mathbf{x}_j^D + \mathbf{z}'_{iD2}\mathbf{x}_i^S + \mathbf{z}'_{ijD3}\mathbf{x}_{ij}]\} \quad (84)$$

		Dual variables	
subject to	$\mathbf{x}_j^D + \mathbf{z}_{jP1} = \sum_{i=1}^{R} \mathbf{x}_{ij}$,	$\mathbf{p}_j^D \geq \mathbf{0}$	(85)

$$\sum_{i=1}^{R} \mathbf{x}_{ij} + \mathbf{z}_{iP2} = \mathbf{x}_i^S, \qquad \mathbf{p}_i^S \geq \mathbf{0} \tag{86}$$

$$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij}, \qquad \boldsymbol{\lambda}_{ij} \text{ free} \tag{87}$$

$$\mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D + \mathbf{z}_{jD1} = \mathbf{p}_j^D, \qquad \mathbf{x}_j^D \geq \mathbf{0} \tag{88}$$

$$\mathbf{p}_i^S + \mathbf{z}_{iD2} = \mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S, \qquad \mathbf{x}_i^S \geq \mathbf{0} \tag{89}$$

$$\mathbf{p}_j^D + \mathbf{z}_{ijD3} = \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}), \qquad \mathbf{x}_{ij} \geq \mathbf{0}. \tag{90}$$

With all nonnegative variables, but $\boldsymbol{\lambda}_{ij}$ which are unrestricted. One advantage of this mathematical programming specification is that the optimal value of the objective function is known and it is equal to zero.

Once again, the crucial task of the Equilibrium Problem that is represented by relations (84)-(90) is to obtain consistent estimates of the dual variables $\boldsymbol{\lambda}_{ij}$ associated to the calibrating constraint (87), say $\boldsymbol{\lambda}_{ij}^*$. With such estimates, a calibrating Equilibrium Problem can be stated as the following Phase II specification:

$$\min\{\textstyle\sum_{ij} \mathbf{z}'_{jP1}\mathbf{p}_j^D + \mathbf{z}'_{iP2}\mathbf{p}_i^S + \mathbf{z}'_{jD1}\mathbf{x}_j^D + \mathbf{z}'_{iD2}\mathbf{x}_i^S + \mathbf{z}'_{ijD3}\mathbf{x}_{ij}\} \tag{91}$$

		Dual variables	
subject to,	$\mathbf{x}_j^D + \mathbf{z}_{jP1} = \sum_{i=1}^{R} \mathbf{x}_{ij}$,	$\mathbf{p}_j^D \geq \mathbf{0}$	(92)
	$\sum_{j=1}^{R} \mathbf{x}_{ij} + \mathbf{z}_{iP2} = \mathbf{x}_i^S$,	$\mathbf{p}_i^S \geq \mathbf{0}$	(93)
	$\mathbf{a}_j - \mathbf{D}_j \mathbf{x}_j^D + \mathbf{z}_{jD1} = \mathbf{p}_j^D$,	$\mathbf{x}_j^D \geq \mathbf{0}$	(94)
	$\mathbf{p}_i^S + \mathbf{z}_{iD2} = \mathbf{b}_i + \mathbf{S}_i \mathbf{x}_i^S$,	$\mathbf{x}_i^S \geq \mathbf{0}$	(95)
	$\mathbf{p}_j^D + \mathbf{z}_{ijD3} = \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^*)$,	$\mathbf{x}_{ij} \geq \mathbf{0}$.	(96)

This calibrating model can now be used to estimate the response to changes in specific policy measures.

2.4.2 Case 2: demand and supply functions are measured with imprecision at the same market level

The second case we consider is when demand and supply functions are measured at the same market level and they are inconsistent with observed trade flows, $\mathbf{x}_{ij}$, and with the

condition that $\mathbf{p}_r^D = \mathbf{p}_r^S$, for all r =1, 2, …, R. The assumption is that when this happens demand and supply functions, as well as transaction costs, are measured with imprecision.

In phase I the same least-squares approach proposed for the analogous situation for the STJ modeling framework is used. Adjustments of transaction costs and of demand and supply functions needed for the model to calibrate observed trade patterns and produce the equality between supply and demand prices in each country are jointly estimated. However, in contrast with the STJ modeling framework, in this case we do not need to impose the symmetry of the adjusted matrices of cross-derivatives; only their positive semidefiniteness is needed.

The phase I model can be specified as follows

$$\begin{aligned}\min LS = &\sum_{j=1}^{R} \mathbf{u}_j' \mathbf{u}_j / 2 + \sum_{i=1}^{R} \mathbf{v}_i' \mathbf{v}_i / 2 + \sum_{j=1}^{R} trace(\mathbf{W}'_j \mathbf{W}_j) / 2 + \sum_{i=1}^{R} trace(\mathbf{Y}'_i \mathbf{Y}_i) / 2 + \\ &\{\sum_{j=1}^{R}\sum_{i=1}^{R} \overline{\mathbf{x}}_{ij} \boldsymbol{\lambda}_{ij} - [\sum_{j=1}^{R} ((\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\ \mathbf{x}_j^D)' \mathbf{x}_j^D - \\ &\sum_{i=1}^{R} ((\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\ \mathbf{x}_i^S)' \mathbf{x}_i^S - \sum_{i=1}^{R}\sum_{j=1}^{R} \mathbf{tc}_{ij}' \mathbf{x}_{ij}]\}\end{aligned} \tag{97}$$

subject to

$$\sum_{i=1}^{R} \mathbf{x}_{ij} \geq \mathbf{x}_j^D \tag{98}$$

$$\sum_{j=1}^{R} \mathbf{x}_{ij} \leq \mathbf{x}_i^S \tag{99}$$

$$\mathbf{x}_{ij} = \overline{\mathbf{x}}_{ij} \tag{100}$$

$$\mathbf{W}_j = \mathbf{G}_j \mathbf{H}_j \mathbf{T}_j' \tag{101}$$

$$\mathbf{G}_j \mathbf{G}_j' = \mathbf{I} \tag{102}$$

$$\mathbf{T}_j \mathbf{T}_j' = \mathbf{I} \tag{103}$$

$$\mathbf{Y}_i = \mathbf{M}_i \mathbf{K}_i \mathbf{N}_i' \tag{104}$$

$$\mathbf{M}_i \mathbf{M}_i' = \mathbf{I} \tag{105}$$

$$\mathbf{N}_i \mathbf{N}_i' = \mathbf{I} \tag{106}$$

$$\mathbf{p}_j^D \geq (\mathbf{a}_j + \mathbf{u}_j) - (\mathbf{D}_j + \mathbf{W}_j)\mathbf{x}_j^D \tag{107}$$

$$(\mathbf{b}_i + \mathbf{v}_i) + (\mathbf{S}_i + \mathbf{Y}_i)\mathbf{x}_i^S \geq \mathbf{p}_i^S \tag{108}$$

$$\mathbf{p}_i^S + (\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}) \geq \mathbf{p}_j^D \tag{109}$$

$$\mathbf{p}_j^S = \mathbf{p}_j^D \tag{110}$$

with $\mathbf{H}_j$ and $\mathbf{K}_i$ diagonal matrices and $h_{j,k,k} \geq 0$, $k_{i,k,k} \geq 0$. Constraints (101)-(106) give Singular Value Decompositions of matrices $\mathbf{W}_j$ and $\mathbf{Y}_i$, which assure their positive semidefiniteness without imposing symmetry. This is a sufficient, although not necessary, condition for the matrices of the adjusted slopes of the demand and supply functions, ($\mathbf{D}_j + \mathbf{W}_j$) and ($\mathbf{S}_i + \mathbf{Y}_i$), to be positive semidefinite.

The phase II calibrated model includes the estimates of the adjustment coefficients obtained in phase I ($\hat{\mathbf{u}}_j$, $\hat{\mathbf{v}}_i$, $\hat{\mathbf{W}}_j$, $\hat{\mathbf{Y}}_i$, and $\hat{\boldsymbol{\lambda}}_{ij}$) in the minimization structure of the Equilibrium Problem proposed above for case 1:

$$\min\{\sum_{ij} \mathbf{z}'_{jP1}\mathbf{p}_j^D + \mathbf{z}'_{iP2}\mathbf{p}_i^S + \mathbf{z}'_{jD1}\mathbf{x}_j^D + \mathbf{z}'_{iD2}\mathbf{x}_i^S + \mathbf{z}'_{ijD3}\mathbf{x}_{ij}\} \tag{111}$$

		Dual variables	
subject to,	$\mathbf{x}_j^D + \mathbf{z}_{jP1} = \sum_{i=1}^{R} \mathbf{x}_{ij}$,	$\mathbf{p}_j^D \geq \mathbf{0}$	(112)
	$\sum_{j=1}^{R} \mathbf{x}_{ij} + \mathbf{z}_{iP2} = \mathbf{x}_i^S$,	$\mathbf{p}_i^S \geq \mathbf{0}$	(113)
	$(\mathbf{a}_j + \hat{\mathbf{u}}_j) - (\mathbf{D}_j + \hat{\mathbf{W}}_j)\mathbf{x}_j^D + \mathbf{z}_{jD1} = \mathbf{p}_j^D$,	$\mathbf{x}_j^D \geq \mathbf{0}$	(114)
	$\mathbf{p}_i^S + \mathbf{z}_{iD2} = (\mathbf{b}_i + \hat{\mathbf{v}}_i) + (\mathbf{S}_i + \hat{\mathbf{Y}}_i)\mathbf{x}_i^S$,	$\mathbf{x}_i^S \geq \mathbf{0}$	(115)
	$\mathbf{p}_j^D + \mathbf{z}_{ijD3} = \mathbf{p}_i^S + (\mathbf{tc}_{ij} + \hat{\boldsymbol{\lambda}}_{ij})$,	$\mathbf{x}_{ij} \geq \mathbf{0}$.	(116)

3. Numerical Examples and Empirical Implementation

A series of numerical examples of increasing complexity will illustrate the application of the PMP methodology to mathematical programming spatial trade models. The list of models developed is given as follows:

1. four exporting countries and four distinct importing countries of a single commodity;
2. four countries that are potentially export or import traders of a single commodity;
3. four countries that are potentially export or import traders of three commodities, diagonal demand and supply matrices;
4. four countries that are potentially export or import traders of three commodities, full, symmetric positive semidefinite demand and supply slope matrices;
5. four countries that are potentially export or import traders of three commodities, full, symmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured at the same market level;

6. four countries that are potentially export or import traders of three commodities, full, asymmetric positive semidefinite demand and supply slope matrices;
7. four countries that are potentially export or import traders of three commodities, full, asymmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured at the same market level.

The matrix of transaction costs may be regarded as the array of effective marginal transaction costs between trading countries with the following structure

$$\mathbf{TC} = \left[tc_{ij} + \lambda_{ij}^{*} \right] \tag{117}$$

where tc_{ij} is the accounting transaction cost generally measured rather imprecisely, and λ_{ij}^{*} is the differential between the effective and the accounting marginal transaction cost implied by the observed trade flows. As discussed above, in this paper, and contrary to the traditional PMP literature, the calibrating constraints are stated as a set of equalities, rather than inequalities, with the consequence that the sign of λ_{ij}^{*} is *a priori* undetermined. This choice is based on the consideration that, if the accounting transaction costs are measured incorrectly, than they may be either over or under estimated. Thus, the value and sign of the estimated λ_{ij}^{*} will determine the effective marginal transaction costs that will produce a calibrated solution of the quantities produced and consumed in each country.

In general, a meaningful effective transaction cost will be nonnegative. However, when trade policies are not explicitly modeled, effective transaction costs will include their effects; when export subsidies are larger than the sum of the other transaction costs, the overall effective transaction cost will be negative.

Example 1: Four exporting countries and four distinct importing countries of a single commodity

The inverse demand and supply functions of the two sets of distinct countries are given below.

Exporting countries are given by $I = A, B, U, E$; importing countries by $J = DA, DB, DU, DE$.

Inverse demand functions:

$$\mathbf{a} = \begin{matrix} \text{DA} \\ \text{DB} \\ \text{DU} \\ \text{DE} \end{matrix} \begin{bmatrix} 30.0 \\ 22.0 \\ 25.0 \\ 29.0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0.55 & & & \\ & 0.37 & & \\ & & 0.42 & \\ & & & 0.49 \end{bmatrix};$$

inverse supply functions:

$$\mathbf{b} = \begin{matrix} A \\ B \\ U \\ E \end{matrix}\begin{bmatrix} 0.4 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 1.4 & & & \\ & 2.4 & & \\ & & 1.9 & \\ & & & 0.6 \end{bmatrix}.$$

Matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{matrix} \\ A \\ B \\ U \\ E \end{matrix}\begin{matrix} \begin{matrix} DA & DB & DU & DE \end{matrix} \\ \begin{bmatrix} 1.2 & 1.5 & 1.0 & 0.1 \\ 1.0 & 1.0 & 0.4 & 0.5 \\ 2.0 & 0.5 & 1.5 & 2.1 \\ 3.0 & 1.2 & 2.0 & 1.0 \end{bmatrix} \end{matrix}.$$

The optimal solution obtained without calibrating the model is as shown below:

optimal trade flow matrix:

$$\mathbf{X}^{*} = \begin{matrix} \\ A \\ B \\ U \\ E \end{matrix}\begin{matrix} \begin{matrix} DA & DB & DU & DE \end{matrix} \\ \begin{bmatrix} 13.394 & & & \\ 3.318 & & 4.662 & \\ 0.832 & 8.511 & & \\ & & 7.836 & 20.916 \end{bmatrix} \end{matrix}$$

total supply quantities:

$$\mathbf{x}^{S^*} = \underset{A}{[13.394,} \ \underset{B}{7.980,} \ \underset{U}{9.343,} \ \underset{E}{28.752]}$$

total demand quantities:

$$\mathbf{x}^{D^*} = \underset{A}{[17.543,} \ \underset{B}{8.511,} \ \underset{U}{12.497,} \ \underset{E}{20.916]}$$

corresponding supply prices:

$$\mathbf{p}^{S^*} = \underset{A}{[19.151,} \ \underset{B}{19.351,} \ \underset{U}{18.351,} \ \underset{E}{17.751]}$$

and corresponding demand prices:

$$\mathbf{p}^{D^*} = \underset{DA}{[20.351,} \ \underset{DB}{18.851,} \ \underset{DU}{19.751,} \ \underset{DE}{18.751]}.$$

Let's now consider the matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 11.500 & & 2.000 & \\ 2.500 & & 3.500 & \\ 2.000 & 7.000 & & \\ & & 6.000 & 22.500 \end{bmatrix} \end{array}$$

and the corresponding value of realized produced and consumed quantities in the eight countries considered:

$$\bar{\mathbf{x}}^{S} = \underset{A}{[13.500,} \ \underset{B}{6.000,} \ \underset{U}{9.000,} \ \underset{E}{28.500]}$$

$$\bar{\mathbf{x}}^{D} = \underset{DA}{[16.000,} \ \underset{DB}{7.000,} \ \underset{DU}{11.500,} \ \underset{DE}{22.500]}$$

When the calibrating constraints are included in the model, the matrix of dual variables, $\mathbf{\Lambda}$, associated with these constraints is:

$$\mathbf{\Lambda}^{*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 0.700 & & -0.130 & \\ 5.600 & 3.810 & 5.170 & 2.875 \\ 1.500 & 1.210 & 0.970 & \\ 0.600 & 0.610 & 0.570 & -0.625 \end{bmatrix} \end{array} .$$

Some of its elements are negative. However, all elements of the matrix of total effective transaction costs, $\mathbf{TC} + \mathbf{\Lambda}^{*}$, are positive:

$$\mathbf{TC} + \mathbf{\Lambda}^{*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 1.900 & 1.500 & 0.870 & 0.100 \\ 6.600 & 4.810 & 5.570 & 3.375 \\ 3.500 & 1.710 & 2.470 & 2.100 \\ 3.600 & 1.810 & 2.570 & 0.375 \end{bmatrix} \end{array} .$$

The optimal solution obtained using the PMP approach, i.e. after replacing the original transaction costs with $\mathbf{TC} + \mathbf{\Lambda}^{*}$ is as shown below:

total supply quantities:

$$\mathbf{x}^{S*} = \underset{A}{[13.500,} \ \underset{B}{6.000,} \ \underset{U}{9.000,} \ \underset{E}{28.500]}$$

total demand quantities:

$$\mathbf{x}^{D^*} = \underset{DA}{[16.000,} \ \underset{DB}{7.000,} \ \underset{DU}{11.500,} \ \underset{DE}{22.500]}$$

with the corresponding supply prices:

$$\mathbf{p}^{S^*} = \underset{A}{[19.300,} \ \underset{B}{14.600,} \ \underset{U}{17.700,} \ \underset{E}{17.600]}$$

and demand prices:

$$\mathbf{p}^{D^*} = \underset{DA}{[21.200,} \ \underset{DB}{19.410,} \ \underset{DU}{20.170,} \ \underset{DE}{17.975]}.$$

The model calibrates exactly each country's total production and consumption. However, multiple sets of optimal trade flows are associated to this calibration. Three examples of matrices of optimal trade flows associated to the same optimal solution (quantities consumed and produced in each country) are provided below. These optimal sets of trade flows have been obtained by providing the solver with different starting points for its search of the optimal solution.

Matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 11.500 & & 2.000 & \\ 2.500 & & 3.500 & \\ 2.000 & 7.000 & & \\ & & 6.000 & 22.500 \end{bmatrix} \\ \begin{bmatrix} 16.000 & 7.000 & 11.500 & 22.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 13.500 \\ 6.000 \\ 9.000 \\ 28.500 \end{bmatrix} \\ \\ \end{array}$$

This optimal solution calibrates realized trade flows; it has been obtained by using the latter as initial values in the optimization procedure.

Matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 2.000 & & 11.500 & \\ 6.000 & & & \\ 8.000 & 1.000 & & \\ & 6.000 & & 22.500 \end{bmatrix} \\ \begin{bmatrix} 16.000 & 7.000 & 11.500 & 22.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 13.500 \\ 6.000 \\ 9.000 \\ 28.500 \end{bmatrix} \\ \\ \end{array}$$

Matrix of trade flows 3:

$$\mathbf{X}_3^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} DA & DB & DU & DE \end{array} \\ \begin{bmatrix} 11.000 & & 2.500 & \\ & & & 6.000 \\ & & 9.000 & \\ 5.000 & 7.000 & & 16.500 \end{bmatrix} \\ \begin{bmatrix} 16.000 & 7.000 & 11.500 & 22.500 \end{bmatrix} \end{array} \begin{bmatrix} 13.500 \\ 6.000 \\ 9.000 \\ 28.500 \end{bmatrix}$$

The value of total transaction costs (e.g. $\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^*)\mathbf{x}_{ij}$) is the same in all three cases and is equal to 102.412.

Example 2: Four countries that are potentially export or import traders of a single commodity

Four countries, $R = A, B, U, E$, can potentially either export or import a single homogeneous commodity. Each country supplies and demands that commodity. The required information is as follows:

inverse demand functions:

$$\mathbf{a} = \begin{array}{c} A \\ B \\ U \\ E \end{array} \begin{bmatrix} 35.0 \\ 59.0 \\ 36.0 \\ 38.0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1.2 & & & \\ & 1.4 & & \\ & & 1.1 & \\ & & & 0.9 \end{bmatrix}.$$

inverse supply functions:

$$\mathbf{b} = \begin{array}{c} A \\ B \\ U \\ E \end{array} \begin{bmatrix} 0.4 \\ 0.2 \\ 0.6 \\ 0.5 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 1.4 & & & \\ & 2.4 & & \\ & & 1.9 & \\ & & & 0.6 \end{bmatrix}.$$

matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 0.10 & 4.50 & 7.50 & 9.00 \\ 4.50 & 0.10 & 7.50 & 12.00 \\ 7.50 & 7.50 & 0.10 & 7.50 \\ 9.00 & 12.00 & 7.50 & 0.10 \end{bmatrix} \end{array}.$$

Without calibrating constraints – that is without using the PMP approach – the optimal solution is as shown below:

optimal trade flow matrix:

$$\mathbf{X}^{*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.179 & 7.596 & & \\ & 11.702 & & \\ & & 11.767 & \\ & 2.570 & & 23.905 \end{bmatrix} \end{array};$$

total supply quantities:

$$\mathbf{x}^{S^{*}} = \underset{A}{[16.775,} \ \underset{B}{11.702,} \ \underset{U}{11.767,} \ \underset{E}{26.475]};$$

total demand quantities:

$$\mathbf{x}^{D^{*}} = \underset{A}{[9.179,} \ \underset{B}{21.868,} \ \underset{U}{11.767,} \ \underset{E}{23.905]};$$

corresponding supply prices:

$$\mathbf{p}^{S^{*}} = \underset{A}{[23.885,} \ \underset{B}{28.285,} \ \underset{U}{22.957,} \ \underset{E}{16.385]} \quad ;$$

and corresponding demand prices:

$$\mathbf{p}^{D^{*}} = \underset{A}{[23.985,} \ \underset{B}{28.385,} \ \underset{U}{23.057,} \ \underset{E}{16.485]} \ .$$

Let's now consider the matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.000 & 6.000 & & \\ & 9.000 & 1.000 & \\ & 1.000 & 8.500 & 0.500 \\ 1.000 & 3.000 & & 21.000 \end{bmatrix} \end{array},$$

and the corresponding values of realized produced and consumed quantities in the four countries considered:

$$\bar{\mathbf{x}}^{S} = \underset{A}{[15.000,} \ \underset{B}{10.000,} \ \underset{U}{10.000,} \ \underset{E}{25.000]} \ ,$$

$$\bar{\mathbf{x}}^{D} = \underset{A}{[10.000,} \ \underset{B}{19.000,} \ \underset{U}{9.500,} \ \underset{E}{21.500]} \ .$$

The matrix of dual variables, $\mathbf{\Lambda}$, associated with the calibrating constraints is:

$$\mathbf{\Lambda}^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 1.500 & 6.500 & & \\ & 8.100 & -6.150 & \\ & 5.300 & 5.850 & -8.450 \\ -1.500 & 4.900 & 2.550 & 3.050 \end{bmatrix} \end{array},$$

And the matrix of total effective transaction costs, $\mathbf{TC} + \mathbf{\Lambda}^*$:

$$\mathbf{TC} + \mathbf{\Lambda}^* = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 1.600 & 11.000 & 7.500 & 9.000 \\ 4.500 & 8.200 & 1.350 & 12.000 \\ 7.500 & 12.800 & 5.950 & -0.950 \\ 7.500 & 16.900 & 10.050 & 3.150 \end{bmatrix} \end{array}.$$

With the calibrating constraints – that is using the PMP approach – the optimal solution is as shown below:

total supply quantities:

$$\mathbf{x}^{S*} = \underset{A}{[15.000,} \ \underset{B}{10.000,} \ \underset{U}{10.000,} \ \underset{E}{25.000]};$$

total demand quantities:

$$\mathbf{x}^{D*} = \underset{A}{[10.000,} \ \underset{B}{19.000,} \ \underset{U}{9.500,} \ \underset{E}{21.500]};$$

supply prices:

$$\mathbf{p}^{S*} = \underset{A}{[21.400,} \ \underset{B}{24.200,} \ \underset{U}{19.600,} \ \underset{E}{15.500]}; \text{ and}$$

demand prices:

$$\mathbf{p}^{D*} = \underset{A}{[23.000,} \ \underset{B}{32.400,} \ \underset{U}{25.550,} \ \underset{E}{18.650]}.$$

As previously, the model calibrates exactly each country's total production and consumption. Multiple sets of optimal trade flows are associated to this calibration. When realized trade flows are used as initial values in the optimization procedure the optimal solution calibrates them as well (matrix 1 below).

Three examples of matrices of optimal trade flows are provided:

Matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 9.000 & 6.000 & & \\ & 9.000 & 1.000 & \\ & 1.000 & 8.500 & 0.500 \\ 1.000 & 3.000 & & 21.000 \end{bmatrix} \\ \begin{bmatrix} 10.000 & 19.000 & 9.500 & 21.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 15.000 \\ 10.000 \\ 10.000 \\ 25.000 \end{bmatrix} \\ \\ \end{array}$$

Matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 10.000 & 5.000 & & \\ & 10.000 & & \\ & 0.500 & 9.500 & \\ & 3.500 & & 21.500 \end{bmatrix} \\ \begin{bmatrix} 10.000 & 19.000 & 9.500 & 21.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 15.000 \\ 10.000 \\ 10.000 \\ 25.000 \end{bmatrix} \\ \\ \end{array}$$

Matrix of trade flows 3:

$$\mathbf{X}_3^* = \begin{array}{c} \\ A \\ B \\ U \\ E \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} A & B & U & E \end{array} \\ \begin{bmatrix} 10.000 & 5.000 & & \\ & 10.000 & & \\ & & & 10.000 \\ & 4.000 & 9.500 & 11.500 \end{bmatrix} \\ \begin{bmatrix} 10.000 & 19.000 & 9.500 & 21.500 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 15.000 \\ 10.000 \\ 10.500 \\ 25.000 \end{bmatrix} \\ \\ \end{array}$$

The value of total transaction costs, $\sum_{i=1}^{R}\sum_{j=1}^{R}(\mathbf{tc}_{ij} + \boldsymbol{\lambda}_{ij}^{*})\mathbf{x}_{ij}$, is the same in all three cases and equal to 342.800.

Example 3: Four countries that are potentially export or import traders of three commodities, diagonal demand and supply matrices

Being the matrices of the demand and supply slopes diagonal, it is assumed that no linkages exist across commodities either in production or consumption. This means that solving this problem is analogous to solving the three individual commodity models individually.

Countries: $R = A, B, U, E$. Commodities: $M = 1, 2, 3$.

The input data:

the matrix of inverse demand intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.000 & 25.000 & 20.000 \\ 22.000 & 18.000 & 15.000 \\ 25.000 & 10.000 & 18.000 \\ 28.000 & 20.000 & 19.000 \end{bmatrix} \end{array};$$

the matrix of inverse demand slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & & \\ & 2.1 & \\ & & 0.7 \\ 0.8 & & \\ & 1.6 & \\ & & 2.6 \\ 0.8 & & \\ & 0.9 & \\ & & 1.7 \\ 1.1 & & \\ & 0.8 & \\ & & 0.9 \end{bmatrix} \end{array};$$

the matrix of inverse supply intercepts:

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array};$$

the matrix of inverse supply slopes:

$$\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & & \\ & 2.1 & \\ & & 1.7 \\ 2.4 & & \\ & 1.6 & \\ & & 1.8 \\ 1.9 & & \\ & 2.8 & \\ & & 2.1 \\ 0.6 & & \\ & 1.1 & \\ & & 0.5 \end{bmatrix} \end{array} ; \text{ and}$$

the matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.5 & 0.5 & 0.5 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.5 & 0.5 & 0.5 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} .$$

The optimal solution obtained without calibrating the model is as shown below:

optimal trade flow matrix:

$$\mathbf{X}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5.543 & 4.492 & 5.487 \\ 1.020 & & \\ 3.553 & & \\ & & 2.744 \\ 6.401 & 5.594 & 2.105 \\ & 2.635 & 0.038 \\ 8.244 & 0.519 & 4.690 \\ 6.904 & & 5.264 \\ 14.034 & 11.105 & 12.192 \end{bmatrix} \end{array};$$

total supply quantities:

$$\mathbf{x}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.116 & 4.492 & 5.487 \\ 6.401 & 5.594 & 4.849 \\ 8.244 & 3.155 & 4.727 \\ 20.938 & 11.105 & 17.455 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{x}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 12.448 & 7.127 & 13.532 \\ 7.421 & 5.594 & 2.105 \\ 11.796 & 0.519 & 4.690 \\ 14.034 & 11.105 & 12.192 \end{bmatrix} \end{array};$$

corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.563 & 9.533 & 10.028 \\ 15.563 & 8.550 & 9.028 \\ 15.063 & 9.033 & 9.528 \\ 12.063 & 10.616 & 7.528 \end{bmatrix} \end{array}; \text{ and}$$

corresponding demand prices:

$$\mathbf{p}^{D^*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.063 & 10.033 & 10.528 \\ 16.063 & 9.050 & 9.528 \\ 15.563 & 9.533 & 10.028 \\ 12.563 & 11.116 & 8.028 \end{bmatrix} \end{array}.$$

Let's now consider the following matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5.000 & 4.000 & 6.000 \\ 1.000 & & \\ 3.000 & & \\ 1.000 & & 2.000 \\ 5.000 & 5.000 & 2.000 \\ & 1.000 & \\ & 2.000 & \\ 7.000 & & 2.500 \\ & 1.500 & \\ 6.000 & & 4.500 \\ 1.000 & & 0.500 \\ 12.000 & 8.000 & 10.500 \end{bmatrix} \end{array},$$

and the corresponding value of realized produced and consumed quantities in the four countries considered:

$$\bar{\mathbf{x}}^S = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 9.000 & 4.000 & 6.000 \\ 6.000 & 6.000 & 4.000 \\ 7.000 & 3.500 & 2.500 \\ 19.000 & 8.000 & 15.500 \end{bmatrix} \end{array}$$

$$\bar{\mathbf{x}}^D = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 12.000 & 6.000 & 12.500 \\ 7.000 & 5.000 & 2.500 \\ 10.000 & & 2.500 \\ 12.000 & 10.500 & 10.500 \end{bmatrix} \end{array}.$$

When the calibrating constraints are included in the model, the matrix of dual variables Λ, associated with these constraints is:

$$\Lambda^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2.100 & 3.400 & -0.150 \\ 1.900 & & \\ 3.000 & 0.500 & 1.850 \\ & 0.100 & \\ -0.500 & 1.700 & 2.250 \\ 1.300 & 0.300 & 0.500 \\ 0.200 & & 4.050 \\ & -1.600 & \\ 1.900 & 1.400 & 5.400 \\ 1.500 & & 1.450 \\ 3.800 & -0.500 & 8.400 \\ & -2.100 & 1.000 \\ 1.700 & 2.200 & 1.700 \\ 1.500 & & -2.050 \\ 2.400 & & 3.500 \\ 3.400 & 3.900 & 2.500 \end{bmatrix} \end{array} .$$

Many elements of Λ^* are negative. However, the elements of the matrix of effective transaction costs, $\mathbf{TC} + \mathbf{\Lambda}^*$, are all not negative:

$$\mathbf{TC}+\boldsymbol{\Lambda}^{*}=\begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 2.600 & 3.900 & 0.350 \\ 3.400 & 1.500 & 1.500 \\ 4.000 & 1.500 & 2.850 \\ 3.000 & 3.100 & 3.000 \\ 1.000 & 3.200 & 3.750 \\ 1.800 & 0.800 & 1.000 \\ 2.400 & 2.200 & 6.250 \\ 4.000 & 2.400 & 4.000 \\ 2.900 & 2.400 & 6.400 \\ 3.700 & 2.200 & 3.650 \\ 4.300 & 0.000 & 8.900 \\ 3.700 & 1.600 & 4.700 \\ 4.700 & 5.200 & 4.700 \\ 5.500 & 4.000 & 1.950 \\ 6.100 & 3.700 & 7.200 \\ 3.900 & 4.400 & 3.000 \end{array}\right] \end{array}.$$

The optimal solution obtained using the PMP approach, i.e. after replacing the original transaction costs with $\mathbf{TC}+\boldsymbol{\Lambda}^{*}$ is as shown below:

total supply quantities:

$$\mathbf{x}^{S*}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 9.000 & 4.000 & 6.000 \\ 6.000 & 6.000 & 4.000 \\ 7.000 & 3.500 & 2.500 \\ 19.000 & 8.000 & 15.500 \end{array}\right] \end{array};$$

total demand quantities:

$$\mathbf{x}^{D*}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 12.000 & 6.000 & 12.500 \\ 7.000 & 5.000 & 2.500 \\ 10.000 & & 2.500 \\ 12.000 & 10.500 & 10.500 \end{array}\right] \end{array};$$

with the corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.000 & 8.500 & 10.900 \\ 14.600 & 9.200 & 7.500 \\ 12.700 & 10.000 & 4.850 \\ 10.900 & 7.200 & 6.550 \end{bmatrix} \end{array},$$

and demand prices

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.600 & 12.400 & 11.250 \\ 16.400 & 10.000 & 8.500 \\ 17.000 & 10.000 & 13.750 \\ 14.800 & 11.600 & 9.550 \end{bmatrix} \end{array}.$$

As previously, the model calibrates exactly each country's total production and consumption. Again multiple sets of optimal trade flows are associated to this calibration. Three examples of matrices of optimal trade flows associated to the same optimal solution are provided below.

Matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 5.000 & 4.000 & 6.000 \\ 1.000 & & \\ 3.000 & & \\ 1.000 & & 2.000 \\ 5.000 & 5.000 & 2.000 \\ & 1.000 & \\ & 2.000 & \\ 7.000 & & 2.500 \\ & 1.500 & \\ 6.000 & & 4.500 \\ 1.000 & & 0.500 \\ 12.000 & 8.000 & 10.500 \end{bmatrix} \end{array}$$

The optimal solution above calibrates realized trade flows; it has been obtained by using the latter as initial values in the optimization procedure.

Matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.U \\ B.E \\ U.B \\ U.E \\ E.A \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 9.000 & & 3.500 \\ & 4.000 & \\ & & 2.500 \\ 3.000 & & 1.500 \\ & 1.000 & 2.500 \\ 3.000 & & \\ & 5.000 & \\ 7.000 & & \\ & 3.500 & 2.500 \\ & 6.000 & 7.500 \\ 7.000 & & \\ 12.000 & 2.000 & 8.000 \end{bmatrix} \end{array}$$

Matrix of trade flows 3:

$$\mathbf{X}_3^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & & 6.000 \\ 6.000 & 4.000 & \\ 3.000 & & \\ 6.000 & & 1.500 \\ & 1.000 & 2.500 \\ & 5.000 & \\ & 3.500 & \\ 7.000 & & 2.500 \\ 6.000 & 2.500 & 5.000 \\ 1.000 & & \\ 12.000 & 5.500 & 10.500 \end{bmatrix} \end{array}$$

It can be easily verified that in all three cases the model calibrates exactly on total demands and supplies. Total transaction costs are the same in all three cases and equal to 300.875.

Example 4: Four countries that are potentially export or import traders of three commodities, full, symmetric positive semidefinite demand and supply slope matrices

With full matrices of demand and supply slopes, this example constitutes a serious test of the PMP methodology. The input data are as follows:

the matrix of inverse demand intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array};$$

the matrix of inverse demand slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.3 & -0.2 \\ 0.3 & 2.1 & 0.1 \\ -0.2 & 0.1 & 0.7 \\ 0.8 & -0.2 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.2 & 0.4 & 2.6 \\ 0.8 & 0.3 & 0.4 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & -0.1 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.2 & 0.9 \end{bmatrix} \end{array};$$

the matrix of inverse supply intercepts:

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array};$$

the matrix of inverse supply slopes:

$$\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.4 & 2.1 & 0.2 \\ 0.3 & 0.2 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.5 & 1.6 & 0.3 \\ 0.2 & 0.3 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.5 & 0.4 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.2 & 0.5 & 0.5 \end{bmatrix} \end{array} \quad ; \text{ and}$$

the matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.5 & 0.5 & 0.5 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.5 & 0.5 & 0.5 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} .$$

The optimal solution obtained without calibrating the model is as shown below:

optimal trade flow matrix:

$$\mathbf{X}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.520 & 2.764 & 4.398 \\ 3.478 & 2.787 & \\ 3.866 & & \\ & & 5.083 \\ 5.321 & 3.442 & \\ & 2.453 & 3.140 \\ 7.481 & & 0.538 \\ 9.662 & & 0.332 \\ 12.128 & 10.407 & 3.037 \end{bmatrix} \end{array};$$

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.843 & 5.552 & 4.398 \\ 5.321 & 3.442 & 5.083 \\ 7.481 & 2.453 & 3.678 \\ 21.790 & 10.407 & 3.370 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.182 & 5.218 & 12.954 \\ 8.798 & 6.229 & \\ 11.347 & & 0.538 \\ 12.128 & 10.407 & 3.037 \end{bmatrix} \end{array};$$

corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.707 & 8.293 & 12.547 \\ 15.707 & 9.293 & 11.547 \\ 15.207 & 7.793 & 12.047 \\ 12.207 & 9.354 & 10.047 \end{bmatrix} \end{array}; \text{ and}$$

corresponding demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.207 & 8.793 & 13.047 \\ 16.207 & 9.793 & 10.749 \\ 15.707 & 6.650 & 12.547 \\ 12.707 & 9.854 & 10.547 \end{bmatrix} \end{array}.$$

Let's now consider the matrix of realized trade flows:

$$\overline{\mathbf{X}} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.000 & 2.500 & 4.500 \\ 2.500 & 2.000 & \\ 4.000 & & \\ & 0.500 & 4.000 \\ 2.500 & 3.500 & \\ 0.500 & & \\ 1.000 & 1.500 & 2.000 \\ 6.000 & 0.500 & \\ & & 1.000 \\ 7.000 & & 0.500 \\ 8.500 & 10.000 & 3.500 \end{bmatrix} \end{array}$$

and the corresponding value of realized produced and consumed quantities in the four countries considered:

$$\overline{\mathbf{x}}^{S} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 9.500 & 4.500 & 4.500 \\ 3.000 & 4.000 & 4.000 \\ 7.000 & 2.000 & 3.000 \\ 15.500 & 10.000 & 4.000 \end{bmatrix} \end{array} \quad \text{and}$$

$$\overline{\mathbf{x}}^{D} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 4.500 & 11.000 \\ 5.000 & 5.500 & \\ 10.000 & 0.500 & \\ 9.000 & 10.000 & 4.500 \end{bmatrix} \end{array}$$

The matrix of dual variables, $\mathbf{\Lambda}^{*}$, has positive and negative elements:

$$\boldsymbol{\Lambda}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.900 & 4.000 & 1.450 \\ 4.350 & 2.050 & \\ 2.600 & & 0.950 \\ & 0.550 & \\ 5.950 & 0.950 & 3.250 \\ 8.400 & 1.000 & 2.000 \\ 4.450 & & 2.550 \\ 1.550 & & \\ 2.650 & 3.850 & 2.850 \\ 2.900 & 1.700 & \\ 2.350 & -0.250 & 3.350 \\ & 0.200 & -3.650 \\ 6.050 & & 2.150 \\ 6.500 & & \\ 4.550 & & 1.450 \\ 6.650 & -0.150 & 0.850 \end{bmatrix} \end{array} .$$

The elements of the matrix of effective transaction costs, on the contrary, are all positive:

$$\mathbf{TC} + \boldsymbol{\Lambda}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4.400 & 4.500 & 1.950 \\ 5.850 & 3.550 & 1.500 \\ 3.600 & 1.000 & 1.950 \\ 3.000 & 3.550 & 3.000 \\ 7.450 & 2.450 & 4.750 \\ 8.900 & 1.500 & 2.500 \\ 6.650 & 2.200 & 4.750 \\ 5.550 & 4.000 & 4.000 \\ 3.650 & 4.850 & 3.850 \\ 5.100 & 3.900 & 2.200 \\ 2.850 & 0.250 & 3.850 \\ 3.700 & 3.900 & 0.050 \\ 9.050 & 3.000 & 5.150 \\ 10.500 & 4.000 & 4.000 \\ 8.250 & 3.700 & 5.150 \\ 7.150 & 0.350 & 1.350 \end{bmatrix} \end{array} .$$

With the calibrating constraints – that is using the PMP approach – the optimal solution is as shown below:

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 9.500 & 4.500 & 4.500 \\ 3.000 & 4.000 & 4.000 \\ 7.000 & 2.000 & 3.000 \\ 15.500 & 10.000 & 4.000 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 4.500 & 11.000 \\ 5.000 & 5.500 & \\ 10.000 & 0.500 & \\ 9.000 & 10.000 & 4.500 \end{bmatrix} \end{array};$$

supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.250 & 6.650 & 12.100 \\ 10.200 & 8.700 & 9.300 \\ 14.000 & 6.300 & 10.200 \\ 8.600 & 9.850 & 8.900 \end{bmatrix} \end{array}; \text{ and}$$

demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 17.650 & 11.150 & 14.050 \\ 19.100 & 10.200 & 11.800 \\ 16.850 & 6.550 & 14.050 \\ 15.750 & 10.200 & 10.250 \end{bmatrix} \end{array}$$

The model calibrates exactly each country's production and consumption of the three commodities. Three examples of optimal trade flows associated to the same optimal solution are provided below. The first optimal solution calibrated realized trade flows.

Matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.U \\ U.E \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.000 & 2.500 & 4.500 \\ 2.500 & 2.000 & \\ 4.000 & & \\ & 0.500 & 4.000 \\ 2.500 & 3.500 & \\ 0.500 & & \\ 1.000 & 1.500 & 2.000 \\ 6.000 & 0.500 & \\ & & 1.000 \\ 7.000 & & 0.500 \\ 8.500 & 10.000 & 3.500 \end{bmatrix} \end{array} .$$

Matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & & 4.500 \\ & 4.500 & \\ 9.500 & & \\ 3.000 & 4.000 & 4.000 \\ 6.500 & 0.500 & \\ & 1.000 & \\ 0.500 & 0.500 & \\ & & 3.000 \\ 1.500 & & 2.500 \\ 5.000 & & \\ 9.000 & 10.000 & 1.500 \end{bmatrix} \end{array} .$$

Matrix of trade flows 3:

$$\mathbf{X}_3^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4.400 & 4.500 & 4.500 \\ 5.000 & & \\ 0.100 & & \\ & & 4.000 \\ & 4.000 & \\ 3.000 & & \\ 6.600 & & \\ & 1.500 & \\ 0.400 & 0.500 & \\ & & 3.000 \\ & & 2.500 \\ 9.500 & & \\ 6.000 & 10.000 & 1.500 \end{bmatrix} \end{array} .$$

It can be easily verified that in the three cases the model calibrates exactly each country's total demand and supply. The value of total transaction costs is the same in all three cases and equal to 290.675.

Example 5: Four countries that are potentially export or import traders of three commodities, full, symmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured at the same market level

Here the model is calibrated in order to reproduce trade patterns as well as to adjust intercepts and the slopes of demand and supply functions so that demand prices are equal to supply prices in each region. Except for the transaction costs, which have been modified to make all the domestic ones equal to zero, input data are the same as in example 4:

the matrix of inverse demand intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} ;$$

the matrix of inverse demand slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.3 & -0.2 \\ 0.3 & 2.1 & 0.1 \\ -0.2 & 0.1 & 0.7 \\ 0.8 & -0.2 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.2 & 0.4 & 2.6 \\ 0.8 & 0.3 & 0.4 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & -0.1 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.2 & 0.9 \end{bmatrix} \end{array} ;$$

the matrix of inverse supply intercepts:

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array} ;$$

the matrix of inverse supply slopes:

$$\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.4 & 2.1 & 0.2 \\ 0.3 & 0.2 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.5 & 1.6 & 0.3 \\ 0.2 & 0.3 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.5 & 0.4 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.2 & 0.5 & 0.5 \end{bmatrix} \end{array} ; \text{ and}$$

the matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0 & 0 & 0 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0 & 0 & 0 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The optimal solution obtained without calibrating the model is as shown below:

optimal trade flow matrix:

$$\mathbf{X}^* = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.071 & 5.456 & 4.710 \\ & 0.110 & \\ & & 5.076 \\ 5.418 & 3.732 & \\ & & 2.826 \\ & 2.399 & \\ 7.787 & & 0.834 \\ 2.126 & & 0.177 \\ 3.535 & & \\ 3.314 & & \\ 12.609 & 10.583 & 3.289 \end{bmatrix} \end{array};$$

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.071 & 5.567 & 4.710 \\ 5.418 & 3.732 & 5.076 \\ 7.787 & 2.399 & 3.661 \\ 21.584 & 10.583 & 3.466 \end{bmatrix} \end{array}$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.196 & 5.456 & 12.790 \\ 8.954 & 6.242 & \\ 11.101 & & 0.834 \\ 12.609 & 10.583 & 3.289 \end{bmatrix} \end{array}$$

corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.085 & 8.304 & 13.141 \\ 16.085 & 9.804 & 11.641 \\ 15.785 & 7.604 & 12.141 \\ 12.085 & 9.615 & 10.141 \end{bmatrix} \end{array} \text{; and}$$

corresponding demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.085 & 8.304 & 13.141 \\ 16.085 & 9.804 & 10.712 \\ 15.785 & 6.753 & 12.141 \\ 12.085 & 9.615 & 10.141 \end{bmatrix} \end{array}$$

Let’s now consider the matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 1.000 & & 3.000 \\ 3.000 & 2.000 & \\ 0.500 & & 2.000 \\ & 2.000 & \\ 6.000 & 0.500 & 0.500 \\ 2.000 & & \\ 3.000 & & \\ 2.000 & 0.500 & \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array}$$

The optimal solution obtained when imposing that demand prices must be equal to supply prices and other calibrating constraints is as shown below:

$$\hat{\mathbf{\Lambda}} = \begin{array}{c} \\ A.B \\ A.U \\ A.E \\ B.A \\ B.U \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.B \\ E.U \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -2.341 & 0.107 & -2.854 \\ -0.465 & -0.195 & -0.410 \\ -4.733 & 0.467 & -3.469 \\ -0.659 & -3.107 & -0.146 \\ -0.824 & -3.002 & -0.256 \\ -4.892 & -2.140 & -3.115 \\ -1.535 & -1.805 & -1.590 \\ -3.576 & -1.398 & -4.144 \\ -5.969 & -1.038 & -4.759 \\ -1.267 & -6.467 & -2.531 \\ -3.108 & -5.860 & -4.885 \\ -1.431 & -6.362 & -2.641 \end{bmatrix} \end{array}$$

the matrix of adjustment to transaction costs, $\hat{\mathbf{\Lambda}}$, has positive and negative elements and all $\hat{\mathbf{\Lambda}}_{i,i,k}$ are zero, as expected.

The matrix of effective transaction costs now contains both positive and negative elements:

$$\mathbf{TC} + \hat{\mathbf{\Lambda}} = \begin{array}{c} \\ A.B \\ A.U \\ A.E \\ B.A \\ B.U \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.B \\ E.U \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -0.841 & 1.607 & -1.354 \\ 0.535 & 0.805 & 0.590 \\ -1.733 & 3.467 & -0.469 \\ 0.841 & -1.607 & 1.354 \\ 1.376 & -0.802 & 1.944 \\ -0.892 & 1.860 & 0.885 \\ -0.535 & -0.805 & -0.590 \\ -1.376 & 0.802 & -1.944 \\ -2.269 & 2.662 & -1.059 \\ 1.733 & -3.467 & 0.469 \\ 0.892 & -1.860 & -0.885 \\ 2.269 & -2.662 & 1.059 \end{bmatrix} \end{array}$$

The deviations from supply and demand intercepts are given by:

$$\hat{\mathbf{V}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -0.522 & 0.174 & 0.028 \\ 0.054 & 0.116 & 0.104 \\ 0.029 & -0.043 & 0.098 \\ 0.006 & 0.005 & 0.014 \end{bmatrix} \end{array} \text{ and;}$$

$$\hat{\mathbf{U}} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.522 & -0.174 & -0.028 \\ -0.054 & -0.116 & -0.104 \\ -0.029 & 0.043 & -0.098 \\ -0.006 & -0.005 & -0.014 \end{bmatrix} \end{array}$$

The deviations from supply and demand slopes are given by:

$$\hat{\mathbf{Y}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.001 & -0.025 & -0.012 \\ -0.025 & 0.611 & 0.302 \\ -0.012 & 0.302 & 0.149 \\ 0.253 & 0.255 & 0.282 \\ 0.255 & 0.257 & 0.284 \\ 0.282 & 0.284 & 0.314 \\ 0.267 & -0.012 & 0.286 \\ -0.012 & 5.2E\text{-}4 & -0.013 \\ 0.286 & -0.013 & 0.306 \\ 0.124 & 0.078 & 0.100 \\ 0.078 & 0.049 & 0.063 \\ 0.100 & 0.063 & 0.081 \end{bmatrix} \end{array} \text{ and;}$$

$$\hat{\mathbf{W}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.019 & -0.111 & -0.110 \\ -0.111 & 0.638 & 0.638 \\ -0.110 & 0.638 & 0.637 \\ 0.398 & 0.433 & 0.218 \\ 0.433 & 0.471 & 0.237 \\ 0.218 & 0.237 & 0.119 \\ 0.352 & -0.074 & 0.265 \\ -0.074 & 0.016 & -0.056 \\ 0.265 & -0.056 & 0.200 \\ 0.072 & 0.060 & 0.066 \\ 0.060 & 0.051 & 0.056 \\ 0.066 & 0.056 & 0.062 \end{bmatrix} \end{array}$$

In phase II, when the estimates of the adjustments are included in the model and calibrating constraints omitted, the optimal solution is as shown below:

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 4.000 & 2.000 & 3.000 \\ 6.500 & 2.500 & 2.500 \\ 18.000 & 9.500 & 2.000 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.500 & 8.000 \\ 6.000 & 4.000 & \\ 8.000 & 1.000 & 0.500 \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array};$$

supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.665 & 6.594 & 11.197 \\ 13.824 & 8.201 & 9.844 \\ 15.200 & 7.399 & 11.787 \\ 12.931 & 10.061 & 10.728 \end{bmatrix} \end{array}; \text{ and}$$

demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.665 & 6.594 & 11.197 \\ 13.824 & 8.201 & 9.844 \\ 15.200 & 7.399 & 11.787 \\ 12.931 & 10.061 & 10.728 \end{bmatrix} \end{array}$$

The model calibrates exactly each country's production and consumption of the three commodities and in each country demand prices equal supply prices.

Three examples of optimal trade flows associated to this optimal solution are provided below; the first one calibrates realized trade flows.

Matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 3.500 & 3.000 \\ 1.000 & & 3.000 \\ 3.000 & 2.000 & \\ 0.500 & & 2.000 \\ & 2.000 & \\ 6.000 & 0.500 & 0.500 \\ 2.000 & & \\ 3.000 & & \\ 2.000 & 0.500 & \\ 11.000 & 9.000 & 2.000 \end{bmatrix} \end{array}.$$

Matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A.A \\ A.B \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & & 3.000 \\ 0.010 & & \\ 10.990 & 3.500 & \\ & & 0.500 \\ & 1.451 & \\ 3.990 & & 0.500 \\ 0.010 & 0.549 & 2.000 \\ & & 2.500 \\ 5.990 & 2.500 & \\ 0.510 & & \\ 14.500 & 3.500 & 2.000 \\ & 0.049 & \\ 3.500 & 1.000 & \\ & 4.951 & \end{bmatrix} \end{array}.$$

Matrix of trade flows 3:

$$\mathbf{X}_3^* = \begin{array}{c} \\ A.A \\ A.B \\ A.E \\ B.A \\ B.B \\ B.U \\ U.A \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & & 1.000 \\ & 3.089 & \\ 11.000 & 0.411 & 2.000 \\ & 0.089 & 3.000 \\ 4.000 & 0.911 & \\ & 1.000 & \\ 6.500 & & 2.500 \\ & 2.500 & \\ 8.000 & 3.411 & 1.500 \\ 2.000 & & \\ 8.000 & & 0.500 \\ & 6.089 & \end{bmatrix} \end{array} .$$

It can be easily verified that in the three cases the model calibrates exactly on total demanded and supplied quantities in each country. The value of total transaction costs is the same in all three cases and equal to 14.406.

Example 6: Four countries that are potentially export or import traders of three commodities, full, asymmetric positive semidefinite demand and supply slope matrices

In general, systems of demand and supply functions do not exhibit symmetric matrices of first own and cross-derivatives (slopes). When three or more commodities are involved, these systems cannot be integrated into a meaningful STJ objective function. The solution of such trade models relies upon the specification and solution of an Equilibrium Problem, as illustrated in section 2.4 above.

The following numerical example exhibits asymmetric matrices of demand and supply slopes. The data are as follows:

the matrix of inverse demand intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} ;$$

the matrix of inverse demand slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.2 & -0.2 \\ 0.3 & 2.1 & 0.2 \\ -0.1 & 0.1 & 0.7 \\ 0.8 & -0.1 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.3 & 0.3 & 2.6 \\ 0.8 & 0.2 & 0.5 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & 0.0 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.0 & 0.8 & 0.2 \\ 0.4 & 0.3 & 0.9 \end{bmatrix} \end{array};$$

the matrix of inverse supply intercepts:

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array};$$

the matrix of inverse supply slopes:

$$\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.2 & 2.1 & 0.2 \\ 0.2 & 0.3 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.7 & 1.6 & 0.3 \\ 0.1 & 0.5 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.6 & 0.5 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.3 & 0.3 & 0.5 \end{bmatrix} \end{array};$$

the matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 0.5 & 0.5 & 0.5 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.5 & 0.5 & 0.5 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.5 & 0.5 & 0.5 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.5 & 0.5 & 0.5 \end{array}\right] \end{array}.$$

The optimal solution obtained in “phase 0”, without calibrating the model, is shown below:

optimal trade flows matrix:

$$\mathbf{X}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3.910 & 1.740 & 4.637 \\ 2.834 & 2.887 & \\ 3.684 & & \\ & & 4.835 \\ 5.356 & 3.037 & \\ & 2.704 & 2.037 \\ 7.618 & & 0.837 \\ 9.809 & & 0.450 \\ 12.909 & 12.124 & 0.158 \end{array}\right] \end{array};$$

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.429 & 4.627 & 4.637 \\ 5.356 & 3.037 & 4.835 \\ 7.618 & 2.704 & 2.873 \\ 22.718 & 12.124 & 0.608 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.719 & 4.445 & 11.958 \\ 8.190 & 5.924 & \\ 11.302 & & 0.837 \\ 12.909 & 12.124 & 0.158 \end{bmatrix} \end{array};$$

corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.540 & 8.659 & 12.057 \\ 15.540 & 9.659 & 11.057 \\ 15.040 & 8.159 & 11.557 \\ 12.040 & 9.769 & 9.557 \end{bmatrix} \end{array};$$

corresponding demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 15.040 & 9.159 & 12.557 \\ 16.040 & 10.159 & 10.766 \\ 15.540 & 8.659 & 12.057 \\ 12.540 & 10.269 & 10.057 \end{bmatrix} \end{array}.$$

Let’s now consider the matrix of realized trade flows:

$$\bar{\bar{\mathbf{X}}} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3.000 & 2.000 & 3.000 \\ 2.500 & 2.500 & \\ 2.000 & & \\ 0.500 & & 4.000 \\ 5.000 & 2.000 & \\ 1.000 & 1.000 & 1.000 \\ 6.000 & & \\ 10.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array},$$

and the corresponding values of realized produced and consumed quantities of the three products in the four countries considered:

$$\bar{\mathbf{x}}^{S} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 4.500 & 3.000 \\ 5.500 & 2.000 & 4.000 \\ 7.000 & 1.000 & 1.000 \\ 22.000 & 10.000 & \end{bmatrix} \end{array}$$

and

$$\bar{\mathbf{x}}^{D} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.000 & 8.000 \\ 7.500 & 4.500 & \\ 8.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array}.$$

When the calibrating constraints (phase I) are included in the model, the matrix of dual variables $\mathbf{\Lambda}^{*}$, associated with these constraints is:

$$\boldsymbol{\Lambda}^{*} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3.100 & 3.600 & 6.400 \\ 4.950 & 2.150 & 1.250 \\ 7.600 & -2.050 & 5.150 \\ 0.800 & 0.350 & -0.450 \\ -3.100 & 3.400 & 5.000 \\ 0.750 & 3.950 & 1.850 \\ 1.200 & -2.450 & 3.550 \\ -5.400 & 0.150 & -1.850 \\ -0.500 & 9.050 & 8.150 \\ 1.150 & 7.400 & 2.800 \\ 5.000 & 4.400 & 7.900 \\ -3.000 & 5.600 & 1.100 \\ -1.100 & 2.550 & 12.550 \\ 0.750 & 1.100 & 7.400 \\ 3.200 & -3.300 & 11.100 \\ 1.600 & 4.300 & 10.700 \end{array}\right] \end{array} .$$

Many of its elements are negative. The same is true for the matrix of effective transaction costs $\mathbf{TC} + \boldsymbol{\Lambda}^{*}$:

$$\mathbf{TC} + \mathbf{\Lambda}^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3.600 & 4.100 & 6.900 \\ 6.450 & 3.650 & 2.750 \\ 8.600 & -1.050 & 6.150 \\ 3.800 & 3.350 & 2.550 \\ -1.600 & 4.900 & 6.500 \\ 1.250 & 4.450 & 2.350 \\ 3.400 & -0.250 & 5.750 \\ -1.400 & 4.150 & 2.150 \\ 0.500 & 10.050 & 9.150 \\ 3.350 & 9.600 & 5.000 \\ 5.500 & 4.900 & 8.400 \\ 0.700 & 9.300 & 4.800 \\ 1.900 & 5.550 & 15.550 \\ 4.750 & 5.100 & 11.400 \\ 6.900 & 0.400 & 14.800 \\ 2.100 & 4.800 & 11.200 \end{array}\right] \end{array}$$

The phase I equilibrium matrix of trade flows

$$\mathbf{X}^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 3.000 & 2.000 & 3.000 \\ 2.500 & 2.500 & \\ 2.000 & & \\ 0.500 & & 4.000 \\ 5.000 & 2.000 & \\ 1.000 & 1.000 & 1.000 \\ 6.000 & & \\ 10.000 & & \\ 12.000 & 10.000 & \end{array}\right] \end{array}$$

equals the matrix of realized trade flows.

The Phase II equilibrium matrix of supply quantities:

$$\mathbf{x}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 4.500 & 3.000 \\ 5.500 & 2.000 & 4.000 \\ 7.000 & 1.000 & 1.000 \\ 22.000 & 10.000 & \end{bmatrix} \end{array};$$

equilibrium matrix of demand quantities:

$$\mathbf{x}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.500 & 3.000 & 8.000 \\ 7.500 & 4.500 & \\ 8.000 & & \\ 12.000 & 10.000 & \end{bmatrix} \end{array};$$

equilibrium matrix of supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 8.650 & 8.650 \\ 15.200 & 7.850 & 9.050 \\ 13.100 & 2.700 & 6.400 \\ 11.700 & 7.200 & 8.400 \end{bmatrix} \end{array};$$

equilibrium matrix of demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.600 & 12.750 & 15.550 \\ 16.450 & 12.300 & 11.400 \\ 18.600 & 7.600 & 14.800 \\ 13.800 & 12.000 & 11.200 \end{bmatrix} \end{array}.$$

As it was the case with examples 1-5, the equilibrium model too calibrates exactly each country's production and consumption of the three commodities and exhibits multiple optimal solutions; three examples of optimal sets of trade flows associated to that same optimal solution are provided below:

matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ U.A \\ U.U \\ E.A \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 4.500 & & 3.000 \\ 2.000 & 2.500 & \\ 1.000 & & \\ & 2.000 & \\ & & 4.000 \\ 5.500 & 2.000 & \\ & 1.000 & 1.000 \\ 7.000 & & \\ 10.000 & 2.000 & \\ 12.000 & 8.000 & \end{bmatrix} \end{array};$$

matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.500 & 2.000 & 3.000 \\ & 2.500 & \\ 0.500 & & 4.000 \\ & 2.000 & \\ 5.000 & & \\ & & 1.000 \\ 7.000 & 1.000 & \\ 6.500 & 1.000 & \\ 7.500 & & \\ 8.000 & & \\ & 9.000 & \end{bmatrix} \end{array};$$

matrix of trade flows 3

$$\mathbf{X}_3^* = \begin{array}{c} \\ A.A \\ A.B \\ A.E \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.E \\ E.A \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.500 & & 3.000 \\ 0.500 & 2.500 & \\ 6.500 & 2.000 & \\ & & 4.000 \\ & 2.000 & \\ 5.500 & & \\ & & 1.000 \\ 7.000 & & \\ & 1.000 & \\ 14.000 & 3.000 & \\ 8.000 & & \\ & 7.000 & \end{bmatrix} \end{array} .$$

The value of total transaction costs is the same in all three cases and equal to 267.400.

Example 7: Four countries that are potentially export or import traders of three commodities, full, asymmetric positive semidefinite demand and supply slope matrices, demand and supply functions are measured at the same market level

As in example 5, the model is calibrated in order to reproduce trade patterns as well as to adjust intercepts and slopes of demand and supply functions in order for demand prices to equal supply prices in each region. Except for the transaction costs, input data are the same as in example 6. The data are as follows:

the matrix of inverse demand intercepts:

$$\mathbf{A} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 30.0 & 25.0 & 20.0 \\ 22.0 & 18.0 & 15.0 \\ 25.0 & 10.0 & 18.0 \\ 28.0 & 20.0 & 19.0 \end{bmatrix} \end{array} ;$$

the matrix of inverse demand slopes:

$$\mathbf{D} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.2 & 0.2 & -0.2 \\ 0.3 & 2.1 & 0.2 \\ -0.1 & 0.1 & 0.7 \\ 0.8 & -0.1 & 0.2 \\ -0.2 & 1.6 & 0.4 \\ 0.3 & 0.3 & 2.6 \\ 0.8 & 0.2 & 0.5 \\ 0.3 & 0.9 & -0.1 \\ 0.4 & 0.0 & 1.7 \\ 1.1 & 0.1 & 0.3 \\ 0.0 & 0.8 & 0.2 \\ 0.4 & 0.3 & 0.9 \end{bmatrix} \end{array};$$

the matrix of inverse supply intercepts:

$$\mathbf{B} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.7 \\ 0.2 & -0.4 & 0.3 \\ -0.6 & 0.2 & -0.4 \\ -0.5 & -1.6 & -1.2 \end{bmatrix} \end{array};$$

the matrix of inverse supply slopes:

$$\mathbf{S} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -0.4 & 0.3 \\ -0.2 & 2.1 & 0.2 \\ 0.2 & 0.3 & 1.7 \\ 2.4 & 0.5 & 0.2 \\ 0.7 & 1.6 & 0.3 \\ 0.1 & 0.5 & 1.8 \\ 1.9 & -0.1 & 0.5 \\ -0.1 & 2.8 & 0.4 \\ 0.6 & 0.5 & 2.1 \\ 0.6 & -0.1 & 0.2 \\ -0.1 & 1.1 & 0.5 \\ 0.3 & 0.3 & 0.5 \end{bmatrix} \end{array};$$

the matrix of accounting transaction costs:

$$\mathbf{TC} = \begin{array}{c} \\ A.A \\ A.B \\ AU \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 1.5 & 1.5 & 1.5 \\ 1.0 & 1.0 & 1.0 \\ 3.0 & 3.0 & 3.0 \\ 1.5 & 1.5 & 1.5 \\ 0.0 & 0.0 & 0.0 \\ 2.2 & 2.2 & 2.2 \\ 4.0 & 4.0 & 4.0 \\ 1.0 & 1.0 & 1.0 \\ 2.2 & 2.2 & 2.2 \\ 0.0 & 0.0 & 0.0 \\ 3.7 & 3.7 & 3.7 \\ 3.0 & 3.0 & 3.0 \\ 4.0 & 4.0 & 4.0 \\ 3.7 & 3.7 & 3.7 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \end{array}.$$

The optimal solution obtained without calibrating the model is shown below:

optimal trade flows matrix:

$$\mathbf{X}^* = \begin{array}{c} \\ A.A \\ B.A \\ B.B \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.640 & 4.653 & 4.943 \\ & & 4.794 \\ 5.461 & 3.305 & \\ & & 1.673 \\ & 2.646 & \\ 7.930 & & 1.159 \\ 3.125 & & 0.423 \\ 2.886 & & \\ 3.074 & & \\ 13.405 & 12.311 & 0.352 \end{bmatrix} \end{array};$$

total supply quantities:

$$\mathbf{X}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.640 & 4.653 & 4.943 \\ 5.461 & 3.305 & 4.794 \\ 7.930 & 2.646 & 2.832 \\ 22.490 & 12.311 & 0.775 \end{bmatrix} \end{array};$$

total demand quantities:

$$\mathbf{X}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13.765 & 4.653 & 11.833 \\ 8.347 & 5.951 & \\ 11.003 & & 1.159 \\ 13.405 & 12.311 & 0.352 \end{bmatrix} \end{array};$$

corresponding supply prices:

$$\mathbf{p}^{S*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.918 & 8.732 & 12.628 \\ 15.918 & 10.148 & 11.128 \\ 15.618 & 7.948 & 11.628 \\ 11.918 & 10.081 & 9.628 \end{bmatrix} \end{array};$$

corresponding demand prices:

$$\mathbf{p}^{D*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14.918 & 8.732 & 12.628 \\ 15.918 & 10.148 & 11.128 \\ 15.618 & 7.948 & 11.628 \\ 11.918 & 10.081 & 9.628 \end{bmatrix} \end{array}.$$

Let’s now consider the matrix of realized trade flows:

$$\bar{\mathbf{X}} = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & \\ 0.500 & & 3.500 \\ 5.000 & 2.000 & \\ & 0.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 0.500 & \\ 1.500 & & \\ 11.000 & 11.000 & \end{bmatrix} \end{array},$$

The optimal solution obtained when imposing that demand prices must be equal to supply prices and other calibrating constraints is as shown below:

the matrix of adjustment to transaction costs:

$$\hat{\mathbf{\Lambda}} = \begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -3.2E\text{-}16 & -2.3E\text{-}16 & -7.7E\text{-}16 \\ -2.354 & 2.569 & -2.442 \\ -1.557 & -0.224 & -0.785 \\ -3.810 & 3.123 & -1.868 \\ -0.646 & -2.569 & 0.942 \\ 2.6E\text{-}18 & 7.3E\text{-}16 & -2.0E\text{-}16 \\ -0.703 & -2.793 & 1.657 \\ -2.956 & -3.446 & 0.574 \\ 1.557 & 0.224 & -0.215 \\ 0.703 & 0.593 & -1.657 \\ -8.4E\text{-}16 & -4.1E\text{-}16 & 5.8E\text{-}16 \\ -2.253 & 3.347 & -1.083 \\ 0.810 & -3.123 & -1.132 \\ -1.044 & -4.554 & -0.574 \\ -1.447 & -3.347 & 1.083 \\ -4.8E\text{-}16 & -7.4E\text{-}16 & -4.7E\text{-}16 \end{bmatrix} \end{array}.$$

the matrix of effective transaction costs $\mathbf{TC}+\hat{\mathbf{\Lambda}}$:

$$\mathbf{TC}+\hat{\mathbf{\Lambda}}=\begin{array}{c} \\ A.A \\ A.B \\ A.U \\ A.E \\ B.A \\ B.B \\ B.U \\ B.E \\ U.A \\ U.B \\ U.U \\ U.E \\ E.A \\ E.B \\ E.U \\ E.E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \text{-3.2E-16} & \text{-2.3E-16} & \text{-7.7E-16} \\ \text{-0.854} & 4.069 & \text{-0.942} \\ \text{-0.557} & 0.776 & 0.215 \\ \text{-0.810} & 6.123 & 1.132 \\ 0.854 & \text{-1.069} & 2.442 \\ 2.6\text{E-18} & 7.3\text{E-16} & \text{-2.0E-16} \\ 1.497 & \text{-0.593} & 3.857 \\ 1.044 & 0.554 & 4.574 \\ 2.557 & 1.224 & 0.785 \\ 2.903 & 2.793 & 0.543 \\ \text{-8.4E-16} & \text{-4.1E-16} & \text{-5.8E-16} \\ 1.447 & 7.047 & 2.617 \\ 3.810 & \text{-0.123} & 1.868 \\ 2.956 & \text{-0.554} & 3.426 \\ 2.253 & 0.353 & 4.783 \\ \text{-4.8E-16} & \text{-7.4E-16} & \text{-4.7E-16} \end{bmatrix} \end{array}.$$

Deviations of the supply and demand intercepts are given by:

$$\hat{\mathrm{V}}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7.6\text{E-4} & 0.031 & 0.010 \\ 0.002 & 0.028 & 0.019 \\ 0.053 & 0.014 & 0.010 \\ 0.009 & 0.001 & 0.005 \end{bmatrix} \end{array} \text{ and;}$$

$$\hat{\mathrm{U}}=\begin{array}{c} \\ A \\ B \\ U \\ E \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \text{-7.6E-4} & \text{-0.031} & \text{-0.010} \\ \text{-0.002} & \text{-0.028} & \text{-0.019} \\ \text{-0.053} & \text{-0.014} & \text{-0.010} \\ \text{-0.009} & \text{-0.001} & \text{-0.005} \end{bmatrix} \end{array}$$

Deviations of supply (**Y**) and demand (**W**) slopes are given by:

$$\hat{\mathbf{Y}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.008 & 0.002 & 0.003 \\ 0.342 & 0.062 & 0.125 \\ 0.115 & 0.021 & 0.042 \\ 0.012 & 0.005 & 0.007 \\ 0.154 & 0.070 & 0.098 \\ 0.104 & 0.047 & 0.066 \\ 0.318 & 0.106 & 0.159 \\ 0.084 & 0.028 & 0.042 \\ 0.059 & 0.020 & 0.030 \\ 0.151 & 0.102 & 0.009 \\ 0.024 & 0.016 & 0.001 \\ 0.083 & 0.056 & 0.005 \end{bmatrix} \end{array} \text{ and;}$$

$$\hat{\mathbf{W}} = \begin{array}{c} \\ A.1 \\ A.2 \\ A.3 \\ B.1 \\ B.2 \\ B.3 \\ U.1 \\ U.2 \\ U.3 \\ E.1 \\ E.2 \\ E.3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.010 & 0.002 & 0.007 \\ 0.389 & 0.062 & 0.296 \\ 0.130 & 0.021 & 0.099 \\ 0.018 & 0.009 & 1.1\text{E-}10 \\ 0.237 & 0.126 & 1.5\text{E-}9 \\ 0.160 & 0.085 & 1.0\text{E-}9 \\ 0.398 & 3.8\text{E-}9 & 0.106 \\ 0.104 & 1.0\text{E-}9 & 0.028 \\ 0.074 & 7.1\text{E-}10 & 0.020 \\ 0.097 & 0.102 & 1.8\text{E-}9 \\ 0.015 & 0.016 & -5.7\text{E-}11 \\ 0.054 & 0.056 & 1.0\text{E-}9 \end{bmatrix} \end{array}$$

The phase I equilibrium matrix of trade flows

$$\mathbf{X^*} = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & \\ 0.500 & & 3.500 \\ 5.000 & 2.000 & \\ & 0.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 0.500 & \\ 1.500 & & \\ 11.000 & 11.000 & \end{bmatrix} \end{array}$$

equals the matrix of realized trade flows.

The Phase II equilibrium matrix of supply quantities:

$$\mathbf{x}^{S^*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 2.000 & 4.000 \\ 5.500 & 2.500 & 3.500 \\ 6.000 & 2.000 & 3.000 \\ 17.000 & 11.500 & 1.000 \end{bmatrix} \end{array};$$

equilibrium matrix of demand quantities:

$$\mathbf{x}^{D^*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 12.500 & 2.000 & 9.500 \\ 8.500 & 4.500 & \\ 7.500 & & 2.000 \\ 11.000 & 11.500 & \end{bmatrix} \end{array};$$

equilibrium matrix of supply prices:

$$\mathbf{p}^{S^*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 16.308 & 7.321 & 11.780 \\ 15.454 & 9.889 & 9.338 \\ 14.752 & 7.097 & 10.994 \\ 12.498 & 10.443 & 9.912 \end{bmatrix} \end{array};$$

equilibrium matrix of demand prices:

$$\mathbf{p}^{D^*} = \begin{array}{c} \\ A \\ B \\ U \\ E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 16.308 & 7.321 & 11.780 \\ 15.454 & 9.889 & 9.338 \\ 14.752 & 7.097 & 10.994 \\ 12.498 & 10.443 & 9.912 \end{bmatrix} \end{array} .$$

This model too calibrates exactly each country's production and consumption of the three commodities. The model exhibits multiple optimal solutions; three examples of optimal sets of trade flows associated to that same optimal solution are provided below:

matrix of trade flows 1:

$$\mathbf{X}_1^* = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & \\ 0.500 & & 3.500 \\ 5.000 & 2.000 & \\ & 0.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 0.500 & \\ 1.500 & & \\ 11.000 & 11.000 & \end{bmatrix} \end{array} ;$$

matrix of trade flows 2:

$$\mathbf{X}_2^* = \begin{array}{c} \\ A.A \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 11.000 & 2.000 & 4.000 \\ & & 3.500 \\ 5.500 & 2.500 & \\ & 0.500 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 1.500 & & 1.000 \\ 3.000 & & \\ 1.500 & & \\ 11.000 & 11.500 & \end{bmatrix} \end{array};$$

matrix of trade flows 3

$$\mathbf{X}_3^* = \begin{array}{c} \\ A.A \\ A.B \\ B.A \\ B.B \\ B.E \\ U.A \\ U.B \\ U.U \\ E.A \\ E.B \\ E.U \\ E.E \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 10.000 & 2.000 & 4.000 \\ 1.000 & & \\ 0.500 & & 3.500 \\ 5.000 & 2.476 & \\ & 0.024 & \\ & & 1.000 \\ & 2.000 & \\ 6.000 & & 2.000 \\ 2.000 & & 1.000 \\ 2.500 & 0.024 & \\ 1.500 & & \\ 11.000 & 11.476 & \end{bmatrix} \end{array}.$$

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