The role of generalised p-boxes in imprecise probability models

Sébastien Destercke, Didier Dubois

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S. Destercke
Centre de coopération Internationale en Recherche Agronomique pour le Développement (CIRAD)
Ingénierie des Agropolymères et Technologies Emergentes (IATE)
sebastien.destercke@cirad.fr

D. Dubois
Institut de Recherche en Informatique de Toulouse (IRIT)
CNRS & Université de Toulouse
dubois@irit.fr

Abstract

Recently, we have introduced an uncertainty representation generalising imprecise cumulative distributions to any (pre-)ordered space as well as possibility distributions: generalised p-boxes. This representation has many attractive features, as it remains quite simple while having an interesting interpretation in terms of lower and upper confidence bounds over nested sets. However, the merits of this representation in various uncertainty processing tasks still have to be evaluated. This is the topic of this paper, in which the handling of information modelled by generalised p-boxes is studied, from the point of view of elicitation, propagation, conditioning and fusion.

Keywords. Generalized p-boxes, comonotonic clouds, fusion, conditioning, propagation.

1 Introduction

When modelling and processing uncertainty in the presence of imprecision and incompleteness, it is often desirable to use approaches whose complexity remains low rather than full-fledged generic models. The benefits of using the former is that their manipulation is often easier, implying a lower computational cost. They can also be easier to explain to non-experts, thus being useful at the elicitation and post-processing stages. The disadvantage of such simple models is that in some situations they may not be sufficient to represent the available knowledge nor to faithfully address a given problem.

Recently, we have introduced an uncertainty representation generalising imprecise cumulative distributions to any (pre-)ordered space as well as possibility distributions [5]: generalised p-boxes. We showed that this representation is quite simple, can be modeled by random sets and has strong connections with many other recent uncertainty representations such as clouds [16]. The interpretation of generalised p-boxes as collection of nested sets with associated lower and upper confidence bounds makes them promising for uncertainty elicitation. Note that general clouds, of which generalized p-boxes constitute a subfamily, are more complex, hence less attractive, in this respect [5].

However, for a given representation to be useful in uncertainty analysis, one has to study its stability across computations, and their computational complexity. Such a study has already partially been done for generalised p-boxes, whose propagation through a model and use in optimisation procedures under uncertainty have been considered previously [4, 11]. In this paper, we recall some of these previous results and complete this study by investigating other aspects of generalised p-boxes manipulation, such as conditioning or merging. When possible, we link our results with other ones originating from the frameworks of probability sets [18], belief functions [17] and possibility theory [7]. Since generalised p-boxes constitute a subfamily of Neumaier's clouds [5, 16], this study also provides some answers to questions regarding the manipulation of these clouds (in particular with respect to their merging).

Section 2 recalls basics about generalised p-boxes and their links with other uncertainty representations and frameworks. In the following sections, we explore the problems of computing probability bounds, information elicitation, propagation, conditioning and merging with generalised p-boxes. We conclude that their main practical interest lies in their simplicity for information representation and elicitation.

2 Preliminaries

Let X be a variable taking its value on a finite space $\mathcal{X}$ having N elements. First recall that two mappings f and f' from a finite indexed set $\mathcal{X} = \{x_1, \ldots, x_N\}$ to the real line $\mathbb{R}$ are said to be *comonotonic* if there is a common permutation σ of $\{1, 2, \ldots, N\}$ such that $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \cdots \geq f(x_{\sigma(N)})$ and $f'(x_{\sigma(1)}) \geq$

$f'(x_{\sigma(2)}) \geq \cdots \geq f'(x_{\sigma(N)})$. In other words, f and f' are comonotonic if and only if for any pair of elements $x, y \in \mathcal{X}$, $f(x) < f(y) \Rightarrow f'(x) \leq f'(y)$ (and $f'(x) < f'(y) \Rightarrow f(x) \leq f(y)$). Note that comonotonicity is not a transitive relation[1]. We consider here that uncertainty about X is modelled by a generalised p-box $[\underline{F}, \overline{F}]$, defined as follows:

Definition 1. *A generalised p-box $[\underline{F}, \overline{F}]$ over a finite space $\mathcal{X}$ is a pair of comonotonic mappings $\underline{F}, \overline{F}$, $\underline{F} : \mathcal{X} \to [0,1]$ and $\overline{F} : \mathcal{X} \to [0,1]$ from $\mathcal{X}$ to $[0,1]$ such that $\underline{F}$ is pointwise less than $\overline{F}$ (i.e. $\underline{F} \leq \overline{F}$) and there is at least one element x in $\mathcal{X}$ for which $\overline{F}(x) = \underline{F}(x) = 1$,*

These limit conditions ensure that $[\underline{F}, \overline{F}]$ characterizes a so-called coherent lower probability. To make notations easier, we introduce an additional element x_0 to $\mathcal{X}$, such that $\overline{F}(x_0) = \underline{F}(x_0) = 0$. As many applications involve variables taking values on the real line $\mathbb{R}$, we also consider generalised p-boxes defined on this space or on one of its product spaces. We limit ourselves to Borel sets and to discrete generalised p-boxes (i.e., when $\overline{F}, \underline{F}$ only takes a finite number of values), other situations being seldom encountered in practice. This allows us, by a proper partition, to come back to the finite space case.

A generalised p-box $[\underline{F}, \overline{F}]$ induces a particular weak order $\leq_{[\underline{F},\overline{F}]}$ between elements of $\mathcal{X}$, such that $x \leq_{[\underline{F},\overline{F}]} y$ iff $\overline{F}(x) \leq \overline{F}(y)$ and $\underline{F}(x) \leq \underline{F}(y)$. In the sequel, for sake of clarity, we assume that each distribution $\overline{F}, \underline{F}$ takes distinct values for each element $x \in \mathcal{X}$, and we consider that these elements are indexed in agreement with the ordering induced by the generalised p-box representing the uncertainty about the value of X. That is, elements $x_1, \ldots, x_N$ are indexed such that $i < j \to \overline{F}(x_i) \leq \overline{F}(x_j)$ and $\underline{F}(x_i) \leq \underline{F}(x_j)$. Given a generalised p-box $[\underline{F}, \overline{F}]$ over $\mathcal{X}$, we define $[\underline{F}, \overline{F}]$-*connected* subsets and $\sqsubseteq_{[\underline{F},\overline{F}]}$-*ordering* as follows:

Definition 2. *Given a generalised p-box $[\underline{F}, \overline{F}]$ over $\mathcal{X}$, a subset $C \subseteq \mathcal{X}$ is called $[\underline{F}, \overline{F}]$-*connected *if it can be expressed as a union of consecutive elements x_k, that is*

$$C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}.$$

Definition 3. *Let $A = \{x_i, \ldots, x_j\}, B = \{x_{i'}, \ldots, x_{j'}\} \subseteq \mathcal{X}$ be two $[\underline{F}, \overline{F}]$-connected sets. The* $\sqsubseteq_{[\underline{F},\overline{F}]}$-ordering *between these sets if defined as follows*

$$A \sqsubseteq_{[\underline{F},\overline{F}]} B \text{ if and only if } \begin{cases} i \leq i' \\ j \leq j'. \end{cases}$$

When $A \sqsubseteq_{[\underline{F},\overline{F}]} B$ and $B \not\sqsubseteq_{[\underline{F},\overline{F}]} A$, we denote it by $A \sqsubset_{[\underline{F},\overline{F}]} B$.

[1] Otherwise all mappings would be comonotonic, since all mappings are comonotonic with the constant mapping.

2.1 Link with convex sets of probability

Convex sets of probabilities constitute one of the most general uncertainty model available nowadays. Their use has been popularised by Walley [18] and studied by numerous authors (see Miranda [13] for a recent review). In this paper, we will restrict ourselves to sets of probabilities $\mathcal{P}_{\underline{P}}$ induced by lower probabilities. Given a probability set $\mathcal{P}$, its lower probability $\underline{P}$ on an event $A \subseteq \mathcal{X}$ is defined as $\underline{P}(A) = \inf_{P \in \mathcal{P}} P(A)$. Upper probability can be defined similarly (i.e., $\overline{P}(A) = \sup_{P \in \mathcal{P}} P(A)$) and the two measures are dual, in the sense that, for any event $A \subseteq \mathcal{X}$, $\underline{P}(A) = 1 - \overline{P}(A^c)$, where A^c is the complement of A. Then $\mathcal{P}_{\underline{P}} = \{P \geq \underline{P}\}$. The lower probability $\underline{P}$ is called coherent if $\underline{P}(A) = \inf\{P(A), P \in \mathcal{P}_{\underline{P}}\}, \forall A$. The probability set $\mathcal{P}_{\underline{P}}$ is then called a *credal set.*

A generalised p-box $[\underline{F}, \overline{F}]$ induces a particular credal set $\mathcal{P}_{[\underline{F},\overline{F}]}$ such that

$$\mathcal{P}_{[\underline{F},\overline{F}]} = \{P \in \mathbb{P}_{\mathcal{X}} | \underline{F}(x_i) \leq P(\{x_1, \ldots, x_i\}) \leq \overline{F}(x_i)\}$$

with $\mathbb{P}_{\mathcal{X}}$ the set of all probability measures over $\mathcal{X}$. When $\mathcal{X}$ is the real line ($\mathcal{X} = \mathbb{R}$) and when sets A_i are of the type $(-\infty, x_i]$ with $x_i < x_j$ when $i \leq j$, we retrieve classical p-boxes [10].

2.2 Link with random sets

Formally, a random set [2] is a mapping from a probability space to the power set of another space. In the discrete case [17], a random set can also be constructed as a mass assignment $m : \wp(\mathcal{X}) \to [0,1]$ s.t. $\sum_{E \in \wp(\mathcal{X})} m(E) = 1$. In this case, subsets E having a strictly positive mass are called focal sets. We denote the set of focal sets by $\mathcal{F}$, and a random set by $(m, \mathcal{F})$. From a random set, we can define two uncertainty measures, respectively the belief and plausibility functions, which reads, for all $A \subseteq X$:

$$Bel(A) = \sum_{E, E \subseteq A} m(E); \; Pl(A) = \sum_{E, E \cap A \neq \emptyset} m(E).$$

The belief function quantifies our credibility in event A, by summing all the masses that **surely** support A, while the plausibility function measures the maximal confidence that can be given to event A, by summing all masses that **could** support A. They are dual measures, in the sense that for all events A, we have $Bel(A) = 1 - Pl(A^c)$. The belief function can be interpreted as a lower probability, and in this case induces a credal set $\mathcal{P}_{(m,\mathcal{F})} = \{P \in \mathbb{P}_{\mathcal{X}} | P \geq Bel\}$, and $Bel(A) = \underline{P}(A)$, $Pl(A) = \overline{P}(A)$ for any event $A \subseteq \mathcal{X}$.

A generalised p-box $[\underline{F}, \overline{F}]$ also induces a particular random set [5]. This random set can be built by the following procedure: consider a threshold $\theta \in [0,1]$.

When $\underline{F}(x_{i+1}) > \theta \geq \underline{F}(x_i)$ and $\overline{F}(x_{j+1}) > \theta \geq \overline{F}(x_j)$, then, the corresponding focal set is $A_{i+1} \setminus A_j$, with weight

$$m(A_{i+1} \setminus A_j) = \min(\underline{F}(x_{i+1}), \overline{F}(x_{j+1})) - \max(\underline{F}(x_i), \overline{F}(x_j)). \quad (1)$$

This allows to apply results concerning random sets to generalised p-boxes. Let us note $(m, \mathcal{F})_{[\underline{F},\overline{F}]}$ the random set induced by a generalised p-box $[\underline{F}, \overline{F}]$. The focal sets of $(m, \mathcal{F})_{[\underline{F},\overline{F}]}$ have very specific features, which can be summarised as follows:

$[\underline{F}, \overline{F}]$**-connectedness**: If $A \in \mathcal{F}_{[\underline{F},\overline{F}]}$, then A is $[\underline{F}, \overline{F}]$-connected.

$[\underline{F}, \overline{F}]$**-ordered**: focal sets are completely ordered with respect to ordering $\sqsubseteq_{[\underline{F},\overline{F}]}$, i.e., for any two distinct sets $A, B \in \mathcal{F}_{[\underline{F},\overline{F}]}$, either $A \sqsubset_{[\underline{F},\overline{F}]} B$ or $B \sqsubset_{[\underline{F},\overline{F}]} A$.

2.3 Link with possibility distributions and clouds

A possibility distribution [7] is a mapping $\pi : \mathcal{X} \to [0, 1]$ from a space $\mathcal{X}$ to the unit interval such that $\pi(x) = 1$ for at least one element x in $\mathcal{X}$. Formally, a possibility distribution is equivalent to the membership function of a fuzzy set. From this possibility distribution are defined two uncertainty measures, respectively the possibility and necessity functions, which reads, for all $A \subset X$:

$$\Pi(A) = \sup_{x \in A} \pi(x); \; N(A) = 1 - \Pi(A^c).$$

Given a possibility distribution π and a degree $\alpha \in [0, 1]$, the strong and regular α-cuts are subsets respectively defined as the sets $E_{\overline{\alpha}} = \{x \in \mathcal{X} | \pi(x) > \alpha\}$ and $E_\alpha = \{x \in \mathcal{X} | \pi(x) \geq \alpha\}$. These α-cuts are nested, since if $\alpha > \beta$, then $E_\alpha \subseteq E_\beta$. In the finite case, a possibility distribution takes at most N values. Let us denote these N values by $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N = 1$. We denote the set of probabilities $\mathcal{P}_\pi = \{P \in \mathbb{P}_\mathcal{X} | P \geq N\}$ associated to a possibility distribution π by $\mathcal{P}_\pi$.

Possibility distributions can also be interpreted as particular random sets: they are equivalent to random sets whose focal elements are nested. A belief function (resp. a plausibility function) is a necessity measure (resp a possibility measure) if and only if it derives from a mass function with nested focal sets. Such a random set is called consonant by Shafer [17]. Given a possibility distribution π, the corresponding random set will have the following focal sets E_i with masses $m(E_i)$, $i = 1, \ldots, N$:

$$\begin{cases} E_i = \{x \in X | \pi(x) \geq \alpha_i\} = E_{\alpha_i} \\ m(E_i) = \alpha_i - \alpha_{i-1}. \end{cases} \quad (2)$$

Uncertainty modelled by a generalised p-box $[\underline{F}, \overline{F}]$ can also be modelled by a pair of possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ such that, for $i = 1, \ldots, N$,

$$\pi_{\overline{F}}(x_i) = \overline{F}(x_i), \quad (3)$$
$$\pi_{\underline{F}}(x_i) = 1 - \underline{F}(x_{i-1}), \quad (4)$$

in the sense that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\underline{F}}} \cap \mathcal{P}_{\pi_{\overline{F}}}$. The random sets with mass assignments $m_{\pi_{\overline{F}}}$ and $m_{\pi_{\underline{F}}}$ modeling the uncertainty of distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ are such that, for $i = 0, \ldots, N-1$,

$$m_{\pi_{\overline{F}}}(A_i^c) = \overline{F}(x_{i+1}) - \overline{F}(x_i)$$
$$m_{\pi_{\underline{F}}}(A_{i+1}) = \underline{F}(x_{i+1}) - \underline{F}(x_i).$$

If we denote the M distinct values taken by $\overline{F}, \underline{F}$ by $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_M = 1$, then the following random set, defined for $j = 1, \ldots, M$ as

$$\begin{cases} E_j = \{x_i \in X | (\pi_{\overline{F}}(x_i) \geq \gamma_j) \wedge (1 - \pi_{\underline{F}}(x_i) < \gamma_j)\}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases} \quad (5)$$

is the same as the random set given by Eq. (1).

Due to their links with possibility distributions, generalised p-boxes also have strong connections with clouds, an uncertainty representation introduced by Neumaier [16]. A cloud $[\pi, \delta]$ is a pair of distributions δ, π from $\mathcal{X}$ to $[0, 1]$ such that $\delta \leq \pi$, $\pi(x) = 1$ for at least one $x \in \mathcal{X}$ and $\delta(y) = 0$ for at least one element $y \in \mathcal{X}$. A cloud $[\pi, \delta]$ induces a set of probabilities $\mathcal{P}_{[\pi,\delta]} = \{P \in \mathbb{P}_X | P(\delta_\alpha) \leq 1 - \alpha \leq P(\pi_{\overline{\alpha}})\}$, with $\delta_\alpha = \{x | \delta(x) \geq \alpha\}$ and $\pi_{\overline{\alpha}} = \{x | \pi(x) > \alpha\}$. It can be shown that clouds whose distributions δ, π are comonotonic are equivalent to generalised p-boxes [5], in the sense that they model exactly the same family of probability sets. A so-called comonotonic cloud $[\pi, \delta]$ models the same uncertainty as the generalised p-box $[\underline{F}, \overline{F}]$ for which $\pi_{\overline{F}} = \pi$ and $\pi_{\underline{F}} = 1 - \delta$, and conversely. That is, for any cloud $[\pi, \delta]$, we have $\mathcal{P}_{[\pi,\delta]} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$, with $\pi, 1 - \delta$ possibility distributions. Using the fact that clouds $[\pi, \delta]$ and $[1 - \delta, 1 - \pi]$ represent the same uncertainty, in the sense that $\mathcal{P}_{[\pi,\delta]} = \mathcal{P}_{[1-\delta,1-\pi]}$, it is immediate that a generalised p-box $[\underline{F}, \overline{F}]$ represents the same uncertainty as the generalised p-box $[\underline{F}_*, \overline{F}_*]$, where, for $i = 1, \ldots, N$

$$\underline{F}_*(x_i) = 1 - \overline{F}(x_{i-1}) \text{ and } \overline{F}_*(x_i) = 1 - \underline{F}(x_{i-1})$$

with the ordering $\leq_{[\underline{F}_*,\overline{F}_*]}$ being the reverse of $\leq_{[\underline{F},\overline{F}]}$.

3 Computing probability bounds

Given a generalised p-box $[\underline{F}, \overline{F}]$, computing lower and upper probabilities over any event $A \subseteq \mathcal{X}$ is rather easy. First, we consider events forming $[\underline{F},\overline{F}]$-*connected* sets $C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$ where x_i, x_j are respectively the two elements of C with least and greatest index with respect to ordering $\leq_{[\underline{F},\overline{F}]}$. The lower probability of such a set is clearly obtained as [5]

$$\underline{P}(C) = \max\{0, \underline{F}(x_j) - \overline{F}(x_{i-1})\}.$$

Now the focal sets induce, via their intersections, a partition of $\mathcal{X}$. Any subset $E \in \mathcal{X}$ in the Boolean sub-algebra $\mathcal{H}$ induced by this partition is made of a disjoint union of $[\underline{F},\overline{F}]$-*connected* sets $C_k : E = C_1 \cup \ldots \cup C_M$. Then [5]:

$$\underline{P}_{[\underline{F},\overline{F}]}(E) = \sum_{k=1}^{M} \underline{P}_{[\underline{F},\overline{F}]}(C_k).$$

Now we can compute the lower and upper probabilities of any event $A \subseteq \mathcal{X}$. Namely, let A_* be the lower approximation of A in $\mathcal{H}$ (i.e. the maximal subset $A_* \subseteq A$ in $\mathcal{H}$). It can be proved [5, 14] that $\underline{P}(A) = \underline{P}(A_*)$, hence, if $A_* = C_1 \cup \ldots \cup C_M$ and $C_i = \{x_{\underline{i}}, x_{\underline{i}+1}, \ldots, x_{\overline{i}}\}$, that

$$\underline{P}(A) = \sum_{i=1}^{M} \max\{0, \underline{F}(x_{\overline{i}}) - \overline{F}(x_{\underline{i}-1})\}.$$

Upper probabilities are easily retrieved by duality. In particular, if $C = \{x_k \in \mathcal{X} | 0 < i \leq k \leq j \leq N\}$ is a $[\underline{F},\overline{F}]$-*connected* subset, then

$$\overline{P}(C) = \overline{F}(x_j) - \underline{F}(x_{i-1}). \qquad (6)$$

Note that these bounds always coincide with the lower envelope of $\mathcal{P}_{[\underline{F},\overline{F}]}$, contrary to other conservative bounds using the relations between possibility distributions and ordinary p-boxes [1] or clouds and possibility distributions [16] in their respective computations.

4 Elicitation of generalised p-boxes

To shorten notations, we adopt, from now on, the following notation: for $i = 1, \ldots, N$, let $\alpha_i := \overline{F}(x_i)$ and $\beta_i := \underline{F}(x_i)$ be the lower and upper probability bounds of sets $\{x_1, \ldots, x_i\}$, themselves denoted by A_i. A generalised p-box can then be described as a set of N probabilistic constraints on nested sets

$$i = 1, \ldots, N, \quad \alpha_i \leq P(A_i) \leq \beta_i.$$

Hence, generalised p-boxes can be elicited by asking an expert to provide upper and lower uncertainty bounds over a finite set of nested sets or intervals. There are many situations where asking information

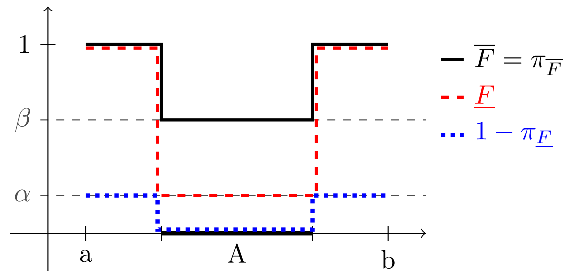

Figure 1: Illustration of $[\underline{F},\overline{F}]$ and associated cloud $[\pi_{\overline{F}}, 1 - \pi_{\underline{F}}]$ of Example 1

under this form is more natural than asking for a set of (imprecise) quantiles, as would be done for ordinary p-boxes. A typical situation is when a parameter or physical quantity θ can be assumed to have an unknown but constant value: in such cases, it sounds natural to ask for confidence bounds around a best estimate $\widehat{\theta}$. Other situations where generalised p-boxes may prove interesting is: (1) when working with categorical variables for which a natural ordering does exist and; (2) when $\theta \in \mathbb{R}^n$ and when sets A_i are convex nested regions of $\mathbb{R}^n$, in which case generalised p-boxes may fit in, while ordinary p-boxes does not.

Example 1. *Given a parameter $\theta \in [a, b]$, with $[a, b] \subseteq \mathbb{R}$, an expert provides an interval A as a best guess about the value of θ, together with upper and lower confidence estimates whether θ is in A. This answer (which can be given, for example, as a level on a symbolic scale) is translated into confidence bounds α, β such that $\alpha \leq P(A) \leq \beta$. Define $\mathcal{X}$ as $\{A, [a, b] \setminus A\}$. This information can be translated into a generalised p-box taking values $\overline{F}(x) = \underline{F}(x) = 1$ if $x \in [a, b] \setminus A (= x_2)$ and $\overline{F}(x) = \beta, \underline{F}(x) = \alpha$ if $x \in A (= x_1)$. Note that this is a generalisation of the so-called simple support function [17], where an upper confidence bound (β) is given in addition to a lower one. Figures 1 and 2 provides a graphical illustration of this simple generalised p-box, while its induced random set is such that*

$$m(A) = \alpha; \; m([a, b]) = \beta - \alpha; \; m([a, b] \setminus A) = 1 - \beta.$$

Note that, from a purely practical viewpoint, the cloud $\pi_{\underline{F}}, 1 - \pi_{\overline{F}}$ on figure 2 looks more self-explanatory, at least graphically.

The next example is more complex, illustrating how p-boxes can help in uncertainty elicitation.

Example 2. *Consider an expert having to assess a pH value in a certain situation. His best guess is $pH \in [4.5, 5.5]$. He is not very certain about these bounds and only judges them fairly plausible. He provides another wider interval $[4, 6]$ about which he feels*

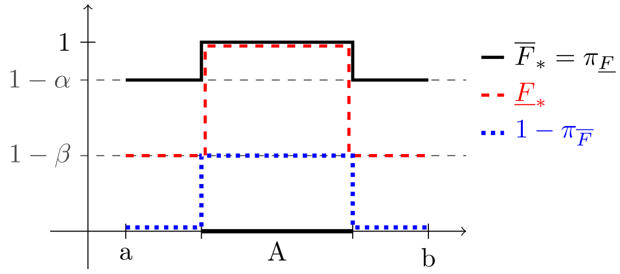

Figure 2: Illustration of , $\underline{F}_*, \overline{F}_*$ and associated cloud $[\pi_{\underline{F}}, 1-\pi_{\overline{F}}]$ of Example 1

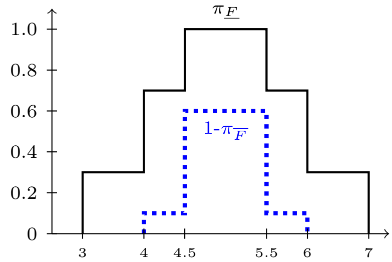

Figure 3: $\pi_{\underline{F}}, 1-\pi_{\overline{F}}$ of generalised p-box of Example 2

more confident. He is however absolutely sure that pH values outside $[3,7]$ *are impossible. His opinion can be modelled as follows:*

- $0.3 \leq P(pH \in [4.5, 5.5]) \leq 0.6$,
- $0.7 \leq P(pH \in [4, 6]) \leq 0.9$,
- $1 \leq P(pH \in [3, 7]) \leq 1$.

The resulting distributions $\pi_{\underline{F}}, 1-\pi_{\overline{F}}$ *of this generalised p-box are pictured in Figure 3.*

5 Propagating generalised p-boxes

Let f be a function of variable X such that $f(X) = Y$, with Y a variable taking values on a space $\mathcal{Y}$. Recall that X can be any pre-ordered space (e.g., the discretization of a multi-dimensional continuous space). First recall that the propagation of a random set $(m, \mathcal{F})$, and of its induced set of probabilities $\mathcal{P}_{(m,\mathcal{F})}$, comes down to computing, for every focal set $A \in \mathcal{F}$, the image $f(A)$ and to assigning the same mass to this set as to A in the original random set $(m, \mathcal{F})$. In a previous paper [4], we studied how to propagate a generalised p-box $[\underline{F}, \overline{F}]$ on $\mathcal{X}$, defined by the constraints $\alpha_i \leq P(A_i) \leq \beta_i$ for $i = 1, \ldots, N$, through the model f. We compared three different methods:

- by computing the image of each focal set of $(m, \mathcal{F})_{[\underline{F},\overline{F}]}$, ending up with the random set denoted by $(m, \mathcal{F})_{f((m,\mathcal{F}))}$ and such that to any threshold $\theta \in [0,1]$ corresponds the focal set
$$\left.\begin{array}{l}\alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j\end{array}\right\} \begin{array}{l} m(f(A_{i+1} \setminus A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j);\end{array}$$
- by considering constraints $\alpha_i \leq P(f(A_i)) \leq \beta_i$ on the probabilities of images of sets A_i. Sets $f(A_i)$ being still nested, these constraints again correspond to a generalized p-box, inducing the random set denoted by $(m, \mathcal{F})_{f([\underline{F},\overline{F}])}$ and such that to any threshold $\theta \in [0,1]$ corresponds the focal set
$$\left.\begin{array}{l}\alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j\end{array}\right\} \begin{array}{l} m(f(A_{i+1}) \setminus f(A_j)) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j).\end{array}$$
Note that $f(A_{i+1}) \setminus f(A_j) \subseteq f(A_{i+1} \setminus A_j)$ the former possibly being empty ;
- by separately propagating the focal sets of each possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$, ending up with two propagated random sets $(m, \mathcal{F})_{f(\pi_{\underline{F}})}$ and $(m, \mathcal{F})_{f(\pi_{\overline{F}})}$ which respectively have, for $i = 0, \ldots, N-1$, mass assignments $m(f(A_i^c)) = \beta_{i+1} - \beta_i$ and $m(f(A_{i+1})) = \alpha_{i+1} - \alpha_i$. Rearranging them as in the original generalised p-box, we end up with the random set denoted by $(m, \mathcal{F})_{f(\pi_{\underline{F}}, \pi_{\overline{F}})}$ and such that to any threshold $\theta \in [0,1]$ corresponds the focal set
$$\left.\begin{array}{l}\alpha_{i+1} > \theta \geq \alpha_i \\ \beta_{j+1} > \theta \geq \beta_j\end{array}\right\} \begin{array}{l} m(f(A_{i+1}) \setminus f(A_j^c)^c) = \\ \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j).\end{array}$$
Here, $f(A_{i+1} \setminus A_j) \subseteq f(A_{i+1}) \setminus f(A_j^c)^c)$.

If we respectively denote the probability sets, induced by the three propagated random sets, by $\mathcal{P}_{f((m,\mathcal{F}))}, \mathcal{P}_{f(\pi_{\underline{F}},\pi_{\overline{F}})}$, and $\mathcal{P}_{f([\underline{F},\overline{F}])}$, we have the following inclusion relations:

$$\mathcal{P}_{f([\underline{F},\overline{F}])} \subseteq \mathcal{P}_{f((m,\mathcal{F}))} \subseteq \mathcal{P}_{f(\pi_{\underline{F}},\pi_{\overline{F}})},$$

with the inclusions being usually strict. The above relations turn into equalities when f is an injective function, however restricting oneself to such functions is very limiting. When f is not injective, only the second set $\mathcal{P}_{f((m,\mathcal{F}))}$ provides the correct result.

6 Conditioning with generalised p-boxes

Since the lower probability $\underline{P}_{[\underline{F},\overline{F}]}$ induced by a generalised p-box is also a belief function, there are two main ways of conditioning $\underline{P}_{[\underline{F},\overline{F}]}$ when uncertainty on X is modelled by a generalised p-box $[\underline{F}, \overline{F}]$: the first is Dempster's rule of conditioning, while the second is Walley's rule of conditioning. Both extend classical Bayes conditioning, but correspond to different interpretations of conditioning [8]. In this section, we study whether the conditional uncertainty measures obtained by both conditionings can still be modelled by generalised p-boxes.

6.1 Dempster conditioning

Given a random set $(m, \mathcal{F})$ and a conditioning event $B = \{x_{b_1}, \dots, x_{b_M}\}$, we denote the conditional (plausibility and belief) measures obtained by Dempster conditioning [2]by $\overline{P}_{[B]}, \underline{P}_{[B]}$. These conditional measures, which are still belief and plausibility measures, can be obtained by computing, for each event $A \subseteq \mathcal{X}$

$$\overline{P}_{[B]}(A) = \frac{\overline{P}(A \cap B)}{\overline{P}(B)},$$

where $\overline{P}$ is the plausibility measure of $(m, \mathcal{F})$. Since $\overline{P}_{[B]}$ is a plausibility function, it has positive mass assignment $m_{[B]}$, which can also be built from the initial distribution m, by transferring it to subsets of B, computing for every subset $A \in \mathcal{X}$,

$$m_{[B]}(A) = \begin{cases} \dfrac{\sum_{C \subseteq \{\mathcal{X} \setminus B\}} m(A \cup C)}{1 - \sum_{A \subseteq B^c} m(A)}, & \text{if } A \subseteq B \\ 0, & \text{otherwise.} \end{cases}$$

This means that the mass $m(A)$ is transferred to $A \cap B$ if $A \cap B \neq \emptyset$, and that the masses given to non-empty sets are then normalised (so that $\sum_{A \subseteq \mathcal{X}} m_{[B]}(A) = 1$). Now, the question is to know whether the upper and lower measures $\overline{P}_{[B]}, \underline{P}_{[B]}$ are still induced by a generalised p-box? The answer is yes, as the following proposition indicates.

Proposition 1. *Let $\underline{P}_{[\underline{F},\overline{F}]}$ be the lower probability induced by a generalised p-box, and B a conditioning event, then the lower measure $\underline{P}_{[B]}$ obtained by Dempster's conditioning stems from a generalised p-box $[\underline{F}, \overline{F}]_{[B]}$ defined on $\mathcal{X} \cap B$ and yielding the restriction of weak order $\leq_{[\underline{F},\overline{F}]}$ of the original p-box to elements $x \in B$.*

Proof. Since $\underline{P}_{[B]}$ is still induced by a random set, it suffices to shows that $m_{[B]}$ remains a mass assignment induced by a generalised p-box, that is that focal sets of $m_{[B]}$ are $[\underline{F}, \overline{F}]$-connected and $[\underline{F}, \overline{F}]$-ordered on B with pre-ordering $\leq_{[\underline{F},\overline{F}]}$.

First, as we consider the weak ordering $\leq_{[\underline{F},\overline{F}]}$ restricted to elements of B, and as any focal set $A = \{x_i, \dots, x_j\}$ is transformed after conditioning to the focal set $A \cap B$, thus retaining all elements in A and B, $A \cap B$ is still $[\underline{F}, \overline{F}]$-connected if the (pre)-ordering is restricted to elements of B.

We have then to show that two distinct focal sets A, A' remain $[\underline{F}, \overline{F}]$-ordered after conditioning on B. Assume $A = \{x_i, \dots, x_j\} \sqsubset_{[\underline{F},\overline{F}]} A' = \{x_k, \dots, x_l\}$, meaning that $i \leq k$ and $j \leq l$, with at least one of the two inequalities strict. Let us consider an element $x_{b_i} \in B$ and the sets $A \setminus x_{b_i}, A' \setminus x_{b_i}$. If $x_{b_i} \in A \cap A'$, then $k \leq b_i \leq j$, and $A \setminus x_{b_i} \sqsubset_{[\underline{F},\overline{F}]} A' \setminus x_{b_i}$. If $x_{b_i} \in A \setminus A'$, then $i \leq b_i < k$, and either $A \setminus x_{b_i} = A' \setminus x_{b_i}$ or $A \setminus x_{b_i} \sqsubset_{[\underline{F},\overline{F}]} A' \setminus x_{b_i}$, as A, A' are $[\underline{F}, \overline{F}]$-connected, thus we have $A \setminus x_{b_i} \sqsubseteq_{[\underline{F},\overline{F}]} A' \setminus x_{b_i}$. As we can do it repeatedly for each element $x \in B$, this finishes the proof. □

The above proposition indicates that all the information contained in conditional measures $\overline{P}_{[B]}, \underline{P}_{[B]}$ is captured by a generalised p-box. If $B = \{x_{b_1}, \dots, x_{b_M}\}$, with elements indexed accordingly to $\leq_{[\underline{F},\overline{F}]}$, and if we let $B_i = \{x_{b_1}, \dots, x_{b_i}\}$, then it is sufficient to compute $\overline{P}_{[B]}(B_i), \underline{P}_{[B]}(B_i)$ for $i = 1, \dots, M$ and to consider the induced generalised p-box $[\underline{F}, \overline{F}]_{[B]}$ to model all the conditional information. Let us consider the case (which is the most likely to happen in practice) of conditioning on a $[\underline{F}, \overline{F}]$-connected set $B = \{x_{b_i} | b_1 \leq b_i \leq b_M\}$, then the conditioned generalised p-box is easy to compute, as we have, for $i = 1, \dots, M$ (Using Eq. (6))

$$\begin{aligned}\overline{P}_{[B]}(B_i) &= \frac{\overline{P}(B_i \cap B)}{\overline{P}(B)} = \frac{\overline{P}(\{x_{b_1}, \dots, x_{b_i}\})}{\overline{P}(\{x_{b_1}, \dots, x_{b_M}\})} \\ &= \frac{\overline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \overline{F}_{[B]}(x_{b_i}), \\ \underline{P}_{[B]}(B_i) &= \underline{P}_{[B]}(B_i) = 1 - \overline{P}_{[B]}(B_i^c) = 1 - \frac{\overline{P}(B_i^c \cap B)}{\overline{P}(B)} \\ &= 1 - \frac{\overline{P}(\{x_{b_i+1}, \dots, x_{b_M}\})}{\overline{P}(\{x_{b_1}, \dots, x_{b_M}\})} \\ &= \frac{\underline{F}(x_{b_i}) - \underline{F}(x_{b_1 - 1})}{\overline{F}(x_{b_M}) - \underline{F}(x_{b_1 - 1})} = \underline{F}_{[B]}(x_{b_i}).\end{aligned}$$

6.2 Walley's conditioning

Let us now study Walley's conditioning [18]. Given a set of probabilities $\mathcal{P}$, its associated lower and upper probabilities $\underline{P}, \overline{P}$ and a conditioning event B for which $\underline{P}(B) > 0$,[2] we denote the (dual) measures obtained after applying Walley's conditioning by $\underline{P}_{|B}$ and $\overline{P}_{|B}$. For any event $A \subseteq \mathcal{X}$, $\underline{P}_{|B}(A)$ is

$$\underline{P}_{|B}(A) = \inf_{P \in \mathcal{P}} \frac{P(A \cap B)}{P(B)}.$$

[2] We avoid dealing with the case where there are $P \in \mathcal{P}$ such that $P(B) = 0$, which requires more caution (See Miranda [13] for an introduction)

	x_1	x_2	x_3	x_4
$\overline{F}$	0.3	0.5	0.9	1
$\underline{F}$	0.1	0.4	0.7	1

Table 1: Generalised p-box $[\underline{F}, \overline{F}]$ of Example 3

When lower probabilities are belief functions, $\underline{P}_{|B}(A)$ can be computed by the following formula:

$$\underline{P}_{|B}(A) = \frac{\underline{P}(A \cap B)}{\underline{P}(A \cap B) + \overline{P}(A^c \cap B)}.$$

We can then ask ourselves the same question as for Dempster's conditioning: can the information of $\underline{P}_{|B}$, which is known to still be a belief function [12], be totally captured by a generalised p-box? The next example shows that this is not the case.

Example 3. *Consider the space $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and the p-box $[\underline{F}, \overline{F}]$ summarized in Table 1. Consider now the conditioning event $B = \{x_1, x_2, x_4\}$. Computing the conditional measure $\underline{P}_{|B}$ for $\{x_1\}$, $\{x_4\}$ and $\{x_1, x_4\}$, we get*

$$\underline{P}_{|B}(\{x_1\}) = 1/8; \underline{P}_{|B}(\{x_4\}) = 1/6; \underline{P}_{|B}(\{x_1, x_4\}) = 2/6.$$

Were $\underline{P}_{|B}$ induced by a generalised p-box, it would satisfy $\underline{P}_{|B}(\{x_1, x_4\}) = \underline{P}_{|B}(\{x_1\}) + \underline{P}_{|B}(\{x_4\})$ (after Section 3), since $\{x_1\}$ and $\{x_4\}$ are disjoint $[\underline{F}, \overline{F}]$-connected sets. It is not the case here, hence $\underline{P}_{|B}$ cannot be modelled by a generalised p-box

This example shows that generalised p-box models are not preserved under Walley's conditioning. However, lower conditional probabilities remain easy to compute. Also, conditional probabilities $\underline{P}_{|B}, \overline{P}_{|B}$ should not be further used in iterated procedures (contrary to $\underline{P}_{[B]}, \overline{P}_{[B]}$). Indeed this type of conditioning is tailored to question-answering of statistical knowledge modelled by credal sets, on the basis of singular information B [8]. If additional singular information C comes up, one has to compute $\underline{P}_{|B\cap C}, \overline{P}_{|B\cap C}$ directly from $\underline{P}, \overline{P}$, therefore the non-preservation of generalised p-boxes under this kind of conditioning is not really an issue.

7 Merging generalised p-boxes

In this section, we assume that S different generalised p-boxes $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$ are available to model our uncertainty about X. They can be provided by different experts, sensors, or any other source of information. In such cases, it is desirable to provide rules to merge uncertain information, possibly taking into account source dependencies. In the following, we say that generalised p-boxes $[\underline{F}, \overline{F}]_1, \dots, [\underline{F}, \overline{F}]_S$ form a comonotonic set if $\underline{F}_i, \overline{F}_i$, $i = 1, \dots, S$, are all comonotonic (i.e. all orderings $\leq_{[\underline{F}, \overline{F}]_i}$ are the same).

7.1 Idempotent merging rules

When dependencies between sources are not well known, it is usual to use merging rules satisfying the property of idempotence, as this ensures that the merging of two identical information items $[\underline{F}, \overline{F}]_1, [\underline{F}, \overline{F}]_2$ will result in the same representation (thus not adding unwarranted information). Given the strong connections between generalised p-boxes, p-boxes and possibility distributions, it appears natural to define idempotent merging rules as follows:

Conjunction: we define the conjunctively merging $[\underline{F}, \overline{F}]_\cap$ of generalised p-boxes, for any $x \in \mathcal{X}$ as the following pair of mappings

$$\underline{F}_\cap(x) = \max_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cap(x) = \min_{i=1,S} \overline{F}_i(x). \quad (7)$$

We say that the conjunction is empty when $\underline{F}_\cap(x) > \overline{F}_\cap(x)$ for at least one $x \in \mathcal{X}$

Disjunction: we define the conjunctively merging $[\underline{F}, \overline{F}]_\cup$ of generalised p-boxes as the pair of mappings $\underline{F}_\cap, \overline{F}_\cap$ such that, for any $x \in \mathcal{X}$

$$\underline{F}_\cup(x) = \min_{i=1,S} \underline{F}_i(x) \text{ and } \overline{F}_\cup(x) = \max_{i=1,S} \overline{F}_i(x). \quad (8)$$

Convex combination: Let $\lambda_1, \dots, \lambda_S$ be non negative weights summing up to one ($\sum_{i=1}^S \lambda_i = 1$) and associated to sources. We then define the arithmetic weighted mean $[\underline{F}, \overline{F}]_\Sigma$ as the pair of mappings $\underline{F}_\Sigma, \overline{F}_\Sigma$ such that, for any $x \in \mathcal{X}$

$$\underline{F}_\Sigma(x) = \sum_{i=1}^S \lambda_i \underline{F}_i(x) \text{ and } \overline{F}_\Sigma(x) = \sum_{i=1}^S \lambda_i \overline{F}_i(x). \quad (9)$$

One can check that, when generalised p-boxes are restricted to ordinary p-boxes, idempotent fusion rules proposed by Ferson's et al. [10] are retrieved. The merging results $[\underline{F}, \overline{F}]_\cup$, $[\underline{F}, \overline{F}]_\Sigma$ and $[\underline{F}, \overline{F}]_\cap$ are not guaranteed to be generalised p-boxes (as comonotonicity can be lost), but they can still be interpreted as clouds (thus offering a possible answer as how to merge clouds [16]). However, when generalised p-boxes form a comonotonic set, the fact that the maximum, minimum and mean operators are non-decreasing in their arguments ensures that the result will still be a generalised p-box with the same induced ordering.

It is also useful to notice that the possibility distribution pairs induced by $[\underline{F}, \overline{F}]_\cup$, $[\underline{F}, \overline{F}]_\cap$ and $[\underline{F}, \overline{F}]_\Sigma$ are such that

- $\pi_{\underline{F}_\cup} = \max_{i=1,S} \pi_{\underline{F}_i}$ and $\pi_{\overline{F}_\cup} = \max_{i=1,S} \pi_{\overline{F}_i}$,
- $\pi_{\underline{F}_\cap} = \min_{i=1,S} \pi_{\underline{F}_i}$ and $\pi_{\overline{F}_\cap} = \min_{i=1,S} \pi_{\overline{F}_i}$,
- $\pi_{\underline{F}_\Sigma} = \sum_{i=1,S} \lambda_i \pi_{\underline{F}_i}$ and $\pi_{\overline{F}_\Sigma} = \sum_{i=1,S} \lambda_i \pi_{\overline{F}_i}$.

The proposed idempotent merging rules are therefore equivalent to applying the classical idempotent rules of possibility theory twice (those rules are retrieved when p-boxes reduce to single possibility distributions).

With regard to sets of probabilities, these merging rules can be used as approximations of exact computations. Let $[\pi_{\overline{F}_\cup}, 1-\pi_{\underline{F}_\cup}]$, $[\pi_{\overline{F}_\cap}, 1-\pi_{\underline{F}_\cap}]$ denote the clouds resulting from disjunctions, conjunctions of generalised p-boxes $[\underline{F},\overline{F}]_1,\ldots,[\underline{F},\overline{F}]_S$, and $\mathcal{P}_{[\underline{F},\overline{F}]_\cup}, \mathcal{P}_{[\underline{F},\overline{F}]_\cap}$ their induced sets of probabilities (possibly empty). The following proposition holds:

Proposition 2. *Let $\mathcal{P}_{[\underline{F},\overline{F}]_1},\ldots,\mathcal{P}_{[\underline{F},\overline{F}]_S}$ be the sets of probabilities induced by $[\underline{F},\overline{F}]_1,\ldots,[\underline{F},\overline{F}]_S$. Then, the following inclusions hold*

$$\mathcal{P}_{[\underline{F},\overline{F}]_\cap} \subseteq \bigcap_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i},$$
$$\mathcal{P}_{[\underline{F},\overline{F}]_\cup} \supseteq \bigcup_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i},$$

the first inclusion turning into an equality when generalised p-boxes form a comonotonic set.

Proof. First recall that, if π_1,π_2 are two possibility distributions, $\min\{\pi_1,\pi_2\}$, $(\max\{\pi_1,\pi_2\})$ their minimum (maximum) and $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_{\min_{12}}$ $(\mathcal{P}_{\max_{12}})$ their induced sets of probabilities, then $\mathcal{P}_{\min_{12}} \subseteq \mathcal{P}_1 \cap \mathcal{P}_2$ $(\mathcal{P}_1 \cup \mathcal{P}_2 \subseteq \mathcal{P}_{\max_{12}})$.

Using the relation between clouds and sets of probabilities, we have, for conjunction:

$$\mathcal{P}_{[\underline{F},\overline{F}]_\cap} = \mathcal{P}_{\pi_{\overline{F}_\cap}} \cap \mathcal{P}_{\pi_{\underline{F}_\cap}},$$

and since $\pi_{\underline{F}_\cap} = \min_{i=1,S} \pi_{\underline{F}_i}$ and $\pi_{\overline{F}_\cap} = \min_{i=1,S} \pi_{\overline{F}_i}$, we have

$$\bigcap_{i=1}^{S} \left(\mathcal{P}_{\pi_{\overline{F}_i}} \cap \mathcal{P}_{\pi_{\underline{F}_i}}\right) \supseteq \left(\mathcal{P}_{\min_{i=1,S} \pi_{\overline{F}_i}} \cap \mathcal{P}_{\min_{i=1,S} \pi_{\underline{F}_i}}\right)$$

and since $\mathcal{P}_{\pi_{\overline{F}_i}} \cap \mathcal{P}_{\pi_{\underline{F}_i}} = \mathcal{P}_{[\underline{F},\overline{F}]_i}$, this shows the inclusion relation for the conjunction. If we consider now the case where generalised p-boxes form a comonotonic set, then it means that all constraints bear on the same events A_i, $i=1,\ldots,N$, and are of the kind $\alpha_{i,j} \leq P(A_i) \leq \beta_{i,j}$, where $\alpha_{i,j},\beta_{i,j}$ are the lower and upper bounds of p-box $[\underline{F},\overline{F}]_j$ for the set A_i. Thus, the intersection $\cap_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i}$ is induced by the set of following constraints:

$$\max_{i=1,S} \alpha_i \leq P(A_i) \leq \min_{i=1,S} \beta_i,$$

and these constraints exactly describe the generalised p-box $[\underline{F},\overline{F}]_\cap$. So, $\mathcal{P}_{[\underline{F},\overline{F}]_\cap} = \bigcap_{i=1}^{S} \mathcal{P}_{[\underline{F},\overline{F}]_i}$.

To see the inclusion relation for disjunction, it is sufficient to note that $\cup_{i=1}^{S}(\mathcal{P}_{\pi_{\underline{F}_i}} \cap \mathcal{P}_{\pi_{\overline{F}_i}}) \subseteq (\cup_{i=1}^{S}\mathcal{P}_{\pi_{\underline{F}_i}}) \cap (\cup_{i=1}^{S}\mathcal{P}_{\pi_{\overline{F}_i}})$, for any $i=1,\ldots,S$. The first probability set is sometimes not convex even in the comonotonic case. □

In particular, Proposition 2 indicates that the conjunction of sets of probabilities induced by ordinary p-boxes or of sets of comonotonic possibility distributions is induced by the result of the proposed merging rule. The conjunctive and disjunctive merging rules can also be interpreted in terms of random sets, as the next proposition indicates. It shows that merging rules can be associated to a random set merging applying a commensuration process [9], with a hypothesis of level-wise merging (i.e. correlation between the sources).

Proposition 3. *Consider the set $\{\gamma_1,\ldots,\gamma_M\} = \bigcup_{i=1}^{S} \{\overline{F}_i(x), \underline{F}_i(x) | x \in \mathcal{X}\}$ of distinct values taken by the generalised p-box $[\underline{F},\overline{F}]_i$, $i=1,\ldots,S$, and indexed such that $0=\gamma_0<\gamma_1<\ldots<\gamma_M=1$. Assume $[\underline{F},\overline{F}]_\cap$ and $[\underline{F},\overline{F}]_\cup$ are generalised p-boxes, then they respectively induce the random sets $(m,\mathcal{F})_\cap$,$(m,\mathcal{F})_\cup$ having, for $j=1,\ldots,M$, the following focal sets:*

$$m_\cap(\cap_{i=1}^{S} E_{i,j}) = \gamma_j - \gamma_{j-1} \tag{10}$$

and

$$m_\cup(\cup_{i=1}^{S} E_{i,j}) = \gamma_j - \gamma_{j-1}, \tag{11}$$

with $E_{i,j} = \{x \in \mathcal{X} | (\pi_{\overline{F}_i}(x) \geq \gamma_j) \wedge (1-\pi_{\underline{F}_i}(x) < \gamma_j)\}$ the set obtained by Eq. (5) for $[\underline{F},\overline{F}]_i$.

Proof. Again, we provide only the proof for $[\underline{F},\overline{F}]_\cup$. If we consider $[\underline{F},\overline{F}]_\cup$ and the induced pair of possibility distributions $\pi_{\underline{F}_\cup},\pi_{\overline{F}_\cup}$, the induced random $(m,\mathcal{F})$ have, for $j=1,\ldots,M$, masses $m(E_j)=\gamma_j-\gamma_{j-1}$ assigned to focal sets such that

$$\begin{aligned}
E_j &= \{x | \pi_{\overline{F}_\cup}(x) \geq \gamma_j \wedge (1-\pi_{\underline{F}_\cup}(x) < \gamma_j)\} \\
&= \left\{x | \max_{i=1,S} \pi_{\overline{F}_i}(x) \geq \gamma_j \wedge (1-\max_{i=1,S} \pi_{\underline{F}_i}(x) < \gamma_j)\right\} \\
&= \left\{x | \max_{i=1,S} \pi_{\overline{F}_i}(x) \geq \gamma_j\right\} \cap \left\{x | \max_{i=1,S} \pi_{\underline{F}_i} \geq 1-\gamma_j\right\} \\
&= \cup_{i=1,S} \left\{x | \pi_{\overline{F}_i}(x) \geq \gamma_j\right\} \cap \cup_{i=1,S} \left\{x | \pi_{\underline{F}_i} \geq 1-\gamma_j\right\} \\
&= \bigcup_{i=1,S} \left\{x | \pi_{\overline{F}_i}(x) \geq \gamma_j \wedge \pi_{\underline{F}_i} \geq 1-\gamma_j\right\} = \bigcup_{i=1,S} (E_{i,j}).
\end{aligned}$$

This ends the proof. The fourth equality following from known relation between possibilistic disjunction with maximum rule and random sets combination (namely, that the maximum of a set of possibility distributions boils down to computing level-wise unions of their α-cuts [9]). $\square$

Note that when $[\underline{F}, \overline{F}]_\cap$ and $[\underline{F}, \overline{F}]_\cup$ are only clouds, the random sets in the Proposition are only inner approximations [5]. Finding out a relation between $[\underline{F}, \overline{F}]_\Sigma$ and the convex mixture of sets of probabilities (i.e. $\mathcal{P}_\Sigma = \{\sum_{i=1}^S \lambda_i P_i | P_i \in \mathcal{P}_{[\underline{F},\overline{F}]_i}\}$) or of random sets (the two procedure inducing the same set of probabilities) looks harder, except when generalised p-boxes form a comonotonic set, in which case $[\underline{F}, \overline{F}]_\Sigma$ can be seen as an approximation of the result that is exact on sets A_i (due to the fact that $\underline{P}_\sigma(A_i) = \sum_{i=1}^S \lambda_i \underline{P}_{[\underline{F},\overline{F}]_i}(A_i) = \sum_{i=1}^S \lambda_i \overline{F}(x_i)$).

8 Other merging rules

In cases where the independence of sources or some dependence structures between them can be assumed, the property of idempotence can be dropped, and it is desirable to use merging rules reflecting the known (in)dependence structure. We are not aware of merging rules exploiting such information in settings emphasising the use of probability sets, but such rules do exist in the settings of possibility theory and of random sets. Exploiting the links between generalised p-boxes and possibility distributions, we can therefore propose an extension of the idempotent merging rules proposed in Section 7.1, such that conjunctive and disjunctive rules respectively become

$$\underline{F}_\top(x) = \bot_{i=1,S} \underline{F}_i(x);\ \overline{F}_\top(x) = \top_{i=1,S} \overline{F}_i(x). \quad (12)$$

$$\underline{F}_\bot(x) = \top_{i=1,S} \underline{F}_i(x);\ \overline{F}_\bot(x) = \bot_{i=1,S} \overline{F}_i(x). \quad (13)$$

with $\top$ a triangular norm and $\bot$ its dual triangular conorm, possibly restricted to associative copulas [15][3] if a probabilistic interpretation is to be preserved. A t-norm is a function $\top : [0,1]^2 \to [0,1]$ that is associative, commutative, non-decreasing in each variable and $\top(x,1) = x$, $\top(x,0) = 0$. The dual t-conorm of a t-norm is such that $\bot(x,y) = 1 - \top(1-x, 1-y)$ for any $(x,y) \in [0,1]^2$. For instance, if all sources can be judged independent, it makes sense to use the product t-norm and its associated t-conorm $\bot(x,y) = x + y - x \cdot y$. Note that this rule is still equivalent to a pair-wise application of the t-norm to possibility distributions $\pi_{\overline{F}_i}, \pi_{\underline{F}_i}$, and that inclusions in Proposition 2 remain valid and are, in this case, always strict.

[3] t-norms $\top$ satisfying $\top(c,d) - \top(c,b) - \top(a,d) + \top(a,b) \geq 0$ for any $(a,b,c,d) \in [0,1]^4$ such that $a \leq c$, $b \leq d$

	A	A^c	$\mathcal{X}$
B	$A \cap B$	$A^c \cap B$	B
B^c	$A \cap B^c$	$A^c \cap B^c$	B^c
$\mathcal{X}$	A	B	$\mathcal{X}$

Table 2: Dempster's rule allocation for Example 4.

As generalised p-boxes constitute particular instances of random sets, it is also possible to merge their induced random sets by families of rules used in this setting [3]. For example, one can apply unnormalised Dempster's rule if sources can be judged independent. Given two random sets with mass assignments m_1, m_2 on $\mathcal{X}$, the random set with mass assignment m_{12} resulting from unnormalised Dempster's rule is such that, for any $A \subseteq \mathcal{X}$,

$$m_{12}(A) = \sum_{\substack{B \cap C = A \\ B, C \subseteq \mathcal{X}}} m_1(B) \cdot m_2(C).$$

The disjunctive rule is obtained by replacing $\cap$ with $\cup$ in the formula. As for possibility distributions [9], applying this rule to random sets induced by a set of generalised p-boxes does not, in general, result in a random set induced by a generalised p-box as the next example indicates.

Example 4. *Let us consider two generalised p-boxes as in Example 1, such that the first source provide bounds α_1, β_1 on set A and the second source provides bounds α_2, β_2 for a distinct set B, such that $B \cap A \neq \{A, B, \emptyset\}$. Table 2 summarises which sets receive a positive mass for the conjunctive allocation.*

Since $A \cap B$, $A \cap B^c$, $A^c \cap B^c$, $A \cap B^c$ are disjoint focal sets strictly included in $\mathcal{X}$, the result is not a generalised p-box, since there are no weak order on elements of $\mathcal{X}$ such that all focal sets are connected *and* ordered. *The same argument holds for the disjunctive counterpart of Dempster's rule.*

9 Summary and Conclusions

This paper suggests that generalised p-boxes are not very stable uncertainty representations, in the sense that most information processing tasks (e.g. propagation, conditioning), once applied to generalised p-boxes, result in representations that are no longer generalised p-boxes. However, even in such situations, using these representations can alleviate the computational burden (e.g., by using quick approximation). There are also specific processing tasks (i.e. propagation through injective functions, dempsterian conditioning, merging of comonotonic sets of generalised p-boxes) where the final result is still a generalised p-box.

Consequently, processing information solely by the means of generalised p-boxes appears of poor interest when one want to make exact computations, as their expressive power remains limited (they can be, however, useful to provide quick approximations). It thus appears that the main interest of generalised p-boxes lies in the elicitation and post-processing stages. Indeed, assigning lower and upper confidence bounds to a set of nested sets is a quite natural way to characterise and to represent information tainted with uncertainty. Recent works on comonotonic clouds [11] also show that generalised p-boxes (which have an expressive power equivalent to comonotonic clouds, as they can model the same sets of probabilities) are convenient for modelling uncertainty in high-dimensional spaces and facilitate optimisation tasks (exploiting the convexity of confidence regions).

Concerning future works, there are still a number of practical results concerning ordinary p-boxes and possibility distributions whose extensions to generalised p-boxes need to be explored. Among these results are fuzzy [6] and probabilistic [19] arithmetic, respectively allowing easy propagation of fuzzy sets and ordinary p-boxes under different (in)dependence assumptions.

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