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Abstract This paper presents theoretical advances in the application of the Stochastic Partial Differential Equation (SPDE) approach in geostatistics. We show a general approach to construct stationary models related to a wide class of SPDEs, with applications to spatio-temporal models having non-trivial properties. Within the framework of Generalized Random Fields, a criterion for existence and uniqueness of stationary solutions for a wide class of linear SPDEs is proposed and proven. Their covariance are then obtained through their associated spectral measure. We also present a result that relates the covariance in the case of a White Noise source term with that of a generic case through convolution. Using these results, we obtain a variety of SPDE-based stationary random fields. In particular, well-known results regarding the Matérn Model and models with Markovian behavior are recovered. A new relationship between the Stein model and a particular SPDE is obtained. New spatio-temporal models obtained from evolution SPDEs of arbitrary temporal derivative order are then obtained, for which properties of separability and symmetry can easily be controlled. Models with a fractional evolution in time are introduced and described, and we thereby obtain a large class of spatio-temporal models which separate regularity over space and time without separability or symmetry conditions. We also obtain results concerning stationary solutions for physically inspired models, such as solutions for the heat equation, the advection-diffusion equation, some Langevin's equations and the wave equation.

Keywords SPDE Approach, Matérn model, general random fields, spectral measure, symbol function, evolution equations, space-time geostatistics.

1 Introduction

Finding new statistical models for analyzing spatio-temporal data that appropriately capture the complex interactions between space and time observed in natural phenomenon while allowing efficient computations able to handle very large datasets is a very active field of research. The typical approach in modeling a variable that varies spatio-temporally is to consider it is a realization of a random field, i.e. a stochastic process indexed in space or in space $\times$ time. The common practice is to describe its statistical properties by its covariance function which must be non-negatively definite, thereby limiting the choice of available models and making the construction of models with realistic features intricate.

Most of the commonly used space-time covariance models are built by modifying or combining generic covariance models defined for $\mathbb{R}^d$, $d = 1, 2, \ldots$. These basic models are usually stationary and isotropic. Commonly known generic models are covariance functions of exponential, powered exponential, Matérn or Cauchy type, amongst many others (Chilès and Delfiner, 2012). Space-time separable covariance models are constructed taking a tensor product between a spatial and a temporal covariance. Separability is often an overly simplistic assumption, since it cannot capture sophisticated interaction between space and time. One of the first attempts to build nonseparable covariance functions lead to the so-called product-sum class of models which simply adds and multiplies valid covariance models in the space and time domains (De Iaco et al., 2001, 2002; Porcu et al., 2009). Even though this approach is perfectly valid from a mathematical standpoint, it is not grounded on physical considerations. In addition, product-sum models imply lower space-time correlations than separability, which is a feature rarely encountered in physical phenomenon. The Gneiting class (Gneiting, 2002) provides flexible nonseparable space-time models. Its construction is based on mixtures arguments, with no reference to physical considerations. Contrarily to the product-sum model, the Gneiting class implies higher space-time correlation than separability, in accordance to most observed phenomenon.

Another important notion characterizing some spatio-temporal covariances is that of full symmetry (Gneiting et al., 2006). In a fully symmetric model, the direction of the time evolution is ignored, obtaining equal covariance values if we look either forward or backward in time. Separable covariance functions are necessarily fully symmetric, but not vice-versa. Product sum models and the Gneiting class are fully symmetric, non separable covariance functions. Atmospheric or environmental processes are often under the influence of prevailing air or water flows which are incompatible with full symmetry. Transport effects of this type can easily be modeled with the help of a purely spatial covariance function and a possibly random velocity vector. See Benoit et al. (2018) for an example of application to precipitation fields, Ailliot et al. (2011) for an application on significant height wave fields with varying velocities. We refer to Gneiting et al. (2006) for a more detailed review on usual spatio-temporal covariance models.

The recent SPDE approach advocated in Lindgren et al. (2011) has open a new paradigm for handling large to very large ($> 10^6$) spatial datasets. This approach consists in modeling the spatial variable as arising from the solution of a particular class of stochastic partial differential equations for which the representation on a finite grid presents interesting Markov properties making it computable even for very large grids. Specifically, Lindgren et al. (2011) consider the following SPDE

$$(\kappa^2 - \Delta)^{\alpha/2} U = W, \tag{1}$$

where $\alpha > d/2$, Δ is the Laplacian, W is a white noise process and U is the unknown random field. The SPDE approach is an important paradigm shift from both theoretical and practical prospective. From a theoretical viewpoint, in contrast to the previously statistically oriented construction reviewed above, it proposes a physically grounded construction, for which the parameters carry traditional physical meanings such as diffusivity, reaction and transport, see also Whittle (1963), Dong (1990) and Kelbert et al. (2005). It allows the construction of models with interesting non-separability and non symmetry properties (Jones & Zhang, 1997; Brown et al., 2000). Non-stationarity can also easily be accounted (Fuglstad et al., 2013). Other works based on this paradigm include Vecchia (1985), Gay & Heyde (1990) and Ruiz-Medina et al. (2016). From a practical viewpoint, spatial prediction and simulation can be computed with methods brought from numerical analysis and PDE-solving methods such as finite element methods, see Lindgren et al. (2011) for details. The sparsity of the matrices involved in the computations allows extremely efficient treatment of very large datasets for which classical geostatistical methods fail due to their high computational cost, see e.g. Simpson et al. (2012) for details. Thanks to this unique combination of theoretical and practical properties, the SPDE approach has been widely used for analyzing environmental or climate datasets (Bolin & Lindgren, 2011; Cameletti et al., 2013; Huang et al., 2017) and Mena & Pfurtscheller (2017) for El Niño analysis. This has also inspired the development of other PDE-solver based methods with efficient performances for a wider class of models (Sigrist et al., 2015; Liu et al., 2016; Bolin & Kirchner, 2017). Lang & Potthoff (2011) proposes an efficient method of simulation for solutions of some SPDEs based on Fourier Analysis, taking advantage of the low computational cost of the Fast Fourier Transform.

Despite the huge potential of the SPDE approach, theoretical advances have been scarce in a spatio-temporal context. The requirement of a Markov structure for fast matrix calculations imposes some constraints. In R-INLA, which is the commonly used R package using the SPDE representation (1) of Gaussian fields, a temporal effect can be modeled as an autoregressive process or as a random walk, see Cameletti et al. (2013) for an application to particulate matters and Opitz (2017) for a recent review on R-INLA with a focus on spatio-temporal applications. Sigrist et al. (2015) shows that the solution of a stochastic advection-diffusion partial differential equation provides a flexible model for spatio-temporal processes which is computationally acceptable for large data sets. In a spatial context, actual application rely mainly on Eq. (1) with its associated Matérn covariance, and also on a restricted set of models obtained from other classes of SPDE that can be found in the aforementioned literature (Jones & Zhang, 1997; Brown et al., 2000). It is therefore of theoretical and practical interest to build, in a quite general setting, spatial and spatio-temporal covariance models that can be related to specific SPDEs, thereby offering physical interpretability and the computational efficiency brought by numerical analysis solvers. This setting should offer the possibility to handle a very general class of random fields and yet should be easy to use in order to simplify the conception, characterization and exploitation of new models. Clearly, it is expected that known results and known models will appear as special cases in this setting.

For this purpose, this work proposes a general framework based on the distribution theory of Schwartz (Schwartz, 1959) in which stationary models related to SPDEs can be rigorously studied. We express under which conditions a stationary covariance model corresponds to the unique stationary solution of some SPDE. When possible, the associated spectral measure is given. The well-known result by Whittle (Whittle, 1963) is easily recovered in this framework. It also includes many other known models such as stationary Markov models (Rozanov, 1977), long-range dependent random fields (Gay & Heyde, 1990; Anh et al., 1998) appearing as Matérn models without range parameter, the Stein spectral measure (Stein, 2005) and solutions to some linear evolution equations (Kelbert et al., 2005; Sigrist et al., 2015). This framework also allows us to obtain new spatio-temporal models that can present non-trivial properties such as non-separability and non-symmetry. In particular, non-symmetric evolution models with fractional evolution behavior have been obtained.

The rest of the paper is organized as follows. In section 2 we present our main result which provides a criteria for the existence and uniqueness of stationary solutions for a wide class of linear SPDEs. In this Section, we use the minimum necessary concepts to rigorously present the result. Technical details are left to Section 3 where the complete theoretical framework is presented. It is based on the concept of generalized random fields as defined in Itô (1954); Matheron (1965); Rozanov (1982), which has been relatively forgotten in the spatial statistics community. Here, the random field is no longer a function but a distribution Schwartz (1959) for which Differential calculus and Fourier analysis are well defined operations. The interest of this section is, in addition to the proof of our main result, the construction of a rigorous framework where operations on SPDEs are well-defined and relatively easy to use.

White Noise is then introduced in Section 4. We define it as a particular generalized random field playing a central role. We present an important result that relates the covariance of the solutions of the SPDEs with any source term to the covariance of the solution of the same SPDE with White Noise source term. In sections 5 and 6 we show examples of models that can be conceived within our framework, involving both known and new models. In section 5 we review some known cases in a spatial context, which involve the Matérn covariance and Markov models. In section 6 we work in a spatio-temporal context. In section 6.1 we relate the Stein model (Stein, 2005) to a particular SPDE and in section 6.2 we present a wide-class of new spatio-temporal stationary models with non-trivial properties arising as solutions to evolution equations. We show examples having special interest both in physics and statistics. A general analysis of the properties based on the spectral measures of the models is made, in particular with regards to their spatial structure. We specify the corresponding spatial SPDE when possible. We conclude in 7 with some final words.

2 Presentation of the main theoretical result

In this section we introduce our main theoretical result regarding the existence and the uniqueness of stationary solutions for a rich class of linear Stochastic Partial Differential Equations (SPDEs). All models presented later in this work, either in a spatial context in Section 5, or in a spatio-temporal one in Section 6, derive from this construction which offers a unified framework to a variety of spatial and spatio-temporal models that have been presented or revisited recently, such as Gaussian processes with Matérn covariance (Whittle, 1963; Lindgren et al., 2011) and the spatio-temporal spectral measure proposed in Stein (2005). This result is presented here in general terms. A more detailed presentation is voluntarily deferred to Section 3, where all proofs are given.

2.1 Introduction

A second order stationary real random function over $\mathbb{R}^d$ is a family of squared-integrable real random variables indexed over the euclidean space, $Z = (Z(x))_{x\in\mathbb{R}^d}$, such that its mean function $m_Z(x) = \mathbb{E}(Z(x))$ is constant and its covariance function $C_Z(x, y) = \mathbb{C}ov(Z(x), Z(y))$ depends only on the gap $x - y$. Without loss of generality, we will consider that $m_Z(x) = 0$. The stationary covariance function, $\rho_Z : \mathbb{R}^d \to \mathbb{R}$, such that $\rho_Z(x-y) = \mathbb{C}ov(Z(x), Z(y))$ must be positive-definite. By Bochner's Theorem (see for example Donoghue (1969)), it is well known that a continuous real positive-definite function is the Fourier transform of a positive, finite, even measure μ_Z, referred to as the spectral measure of Z: $\rho_Z = \mathscr{F}(\mu_Z)$, where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^d$. The covariance function ρ_Z and the spectral measure μ_Z can equivalently be used to fully characterize the covariance structure of Z.

In this work, it will be necessary to consider more general mathematical objects that allow us to deal properly with linear differential operators and Fourier transforms on random fields. We will use *Generalized Random Fields* (GeRF), which are an analogous to the generalization of functions presented in Schwartz's Distribution Theory, see for example Itô (1954) for a theory of stationary GeRFs. In this framework, the random fields have only meaning when applied to test functions in some particular functional space, and not necessarily when evaluated in points of the space. We present all the technical details in section 3. For now, we mention that the covariance structure of a stationary GeRF can be described by not necessarily finite spectral measures. To characterize those, we consider the class $\mathcal{M}^+_{SG}(\mathbb{R}^d)$ of slow-growing positive (Borel) measures on $\mathbb{R}^d$. Members of $\mathcal{M}^+_{SG}(\mathbb{R}^d)$ can have infinite total mass, but they grow at most at a polynomial rate. Specifically,

$$\mathcal{M}^+_{SG}(\mathbb{R}^d) := \left\{ \mu \text{ positive measure over } \mathbb{R}^d : \quad \int_{\mathbb{R}^d} (1+|x|^2)^{-N} d\mu(x) < \infty \quad \text{for some } N \in \mathbb{N} \right\}. \quad (2)$$

If $\mu_Z \in \mathcal{M}^+_{SG}(\mathbb{R}^d)$ is even, it can be used as a spectral measure of a real stationary GeRF Z, and its Fourier transform $\rho_Z = \mathscr{F}(\mu_Z)$ is called the *stationary covariance distribution* of Z (we will omit the adjective *stationary* when it is clear from context). This distribution is not necessarily a continuous function and thus Z is not necessarily a random function with point-wise meaning. However, when $N = 0$ in Eq. (2) we are back to the usual second order stationary random functions and to the Bochner characterization of covariance functions. From now on, every even measure in $\mathcal{M}^+_{SG}(\mathbb{R}^d)$ will be said to be a spectral measure.

The focus in this work is on a quite general class of linear stochastic equations which encompasses those considered in Whittle (1963) Lindgren et al. (2011), Sigrist et al. (2015), Bolin & Lindgren (2011), Anh et al. (1998), Kelbert et al. (2005) and Gay & Heyde (1990). Linear operators involved here are not strictly speaking *differential operators*. We refer to them as *pseudo-differential operators*. Then, following Lindgren et al. (2011), we will make a slight abuse of language and we will call SPDEs the class of stochastic equations considered in this work. This class of SPDEs is defined through operators of the form

$$\mathcal{L}_g(\cdot) := \mathscr{F}^{-1}\left(g\mathscr{F}(\cdot)\right), \quad (3)$$

where $g : \mathbb{R}^d \to \mathbb{C}$ must be a sufficiently regular and Hermitian-symmetric function, that is it must satisfy $g(x) = \overline{g(-x)}$, where $\overline{a}$ is the complex conjugate of a. Under these conditions, $\mathcal{L}_g$ is a real operator thanks to the properties of the Fourier transform. In this work, we will require that g is continuous and bounded by a polynomial, see Section 3 for a detailed exposition of all technical requirements. From now, every continuous, polynomially bounded and Hermitian-symmetric complex function g defined over $\mathbb{R}^d$ will be called a *symbol function* over $\mathbb{R}^d$, and it will be said to be the symbol function of the operator $\mathcal{L}_g$. Before presenting our main result, we first need to establish the relationship between the spectral measures of a stationary GeRF U and its transform through the operator $\mathcal{L}_g$ defined in (3). The next proposition will be proven in Section 3.

Proposition 1. *Let U be a real stationary GeRF on $\mathbb{R}^d$ with spectral measure μ_U, and let g be a symbol function over $\mathbb{R}^d$. Then, $\mathcal{L}_g U$, where $\mathcal{L}_g$ is defined in Eq. (3), is a real stationary GeRF with spectral measure $\mu_{\mathcal{L}_g U} = |g|^2 \mu_U$ and its covariance distribution is $\rho_{\mathcal{L}_g U} = \mathcal{L}_{|g|^2} \rho_U$.*

2.2 Statement of the main result

Let us consider a symbol function g over $\mathbb{R}^d$ and a stationary GeRF X, which will be called from now on the *source term*. A question that arises is to establish under which conditions on g and X the SPDE

$$\mathcal{L}_g U = X, \tag{4}$$

has a stationary solution, whether it is unique or not and, when solutions exist, whether we can characterize the associated spectral measures. Theorem 1 provides a general answer to this question in the second order sense. That is, we shall only impose that the two sides of Eq. (4) have the same (generalized) covariance, which we write

$$\mathcal{L}_g U \overset{2nd\ o.}{=} X. \tag{5}$$

This is not equivalent to require that U solves (4) strictly. Under this more restrictive requirement, the evaluations of $\mathcal{L}_g U$ and X over the same test functions (or, more simply, at the same points in the case of random functions) are almost surely equal random variables. In the language of stochastic process, this is equivalent to requiring that $\mathcal{L}_g U$ is a *modification* of X. From a direct application of Proposition 1, we get that a spectral measure μ_U of a potential stationary solution to (5) must verify

$$|g|^2 \mu_U = \mu_X. \tag{6}$$

This kind of problem is called a *division problem* in distribution theory. A criteria of existence of real stationary solutions to (5) arises from the existence of even solutions to this problem which are in $\mathcal{M}^+_{SG}(\mathbb{R}^d)$. The explicit result is now formally presented in Theorem 1, which is our main result.

Theorem 1. *Let X be a real stationary GeRF over $\mathbb{R}^d$ with spectral measure μ_X. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a symbol function, and let $\mathcal{L}_g$ be an operator as defined in (3) with symbol g. Then, there exists a real stationary GeRF solution to the equation (5) if and only if there exists $N \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}^d} \frac{d\mu_X(\xi)}{|g(\xi)|^2 (1 + |\xi|^2)^N} < \infty. \tag{7}$$

In that case, the measure

$$d\mu_U(\xi) = |g(\xi)|^{-2} d\mu_X(\xi) \tag{8}$$

is a spectral measure, and any real stationary GeRF with spectral measure μ_U solves (5). Moreover, μ_U is the unique solution in $\mathcal{M}^+_{SG}(\mathbb{R}^d)$ to (6) if and only if $|g| > 0$.

Remark 1. When $N = 0$, i.e. if $|g|^{-2}$ is integrable with respect to the measure μ_X, the measure μ_U is finite and the solution U is thus a mean-square continuous random function. This case was studied in Whittle

(1963), where it is mentioned that solutions corresponding to non finite measures μ_U still make sense in some framework, the theory of which was at that time not completely available. Our work can be seen as one possible answer to this note.

Remark 2. A Sufficient Condition for Existence and Uniqueness (SCEU), regardless of the source term X, is to require that $|g|$ is inferiorly bounded by the inverse of a strictly positive polynomial. In this case the operator $\mathcal{L}_g$ is actually invertible: $1/g$ is a symbol function and it is straightforward that $\mathcal{L}_{1/g}$ is the inverse operator of $\mathcal{L}_g$. This implies that Eq. (4) can be solved explicitly with $U = \mathcal{L}_{1/g}X$. Then, by Proposition 1, U is the unique stationary solution and its spectral measure is (8). We shall henceforth refer to this condition as the SCEU on g.

Remark 3. When the closed set $g^{-1}(\{0\}) = \{\xi \in \mathbb{R}^d \mid g(\xi) = 0\}$ is non-empty, the non-uniqueness is due to the existence of stationary solutions of the homogeneous problem

$$\mathcal{L}_g U_H = 0. \tag{9}$$

Indeed, for a spectral measure μ_{U_H} over $\mathbb{R}^d$ supported on $g^{-1}(\{0\})$, its associated stationary random field satisfies strictly Eq. (9) since $\mu_{\mathcal{L}_g U_H} = |g|^2 \mu_H = 0$. Thus, if existence is provided, the addition of any stationary solution to (5) with a non-trivial independent stationary solution to (9) is also a stationary solution to 5, which implies non-uniqueness.

3 Theoretical Framework and proof of the main result

In order to prove Theorem 1, it is necessary to lay out some theoretical background, which uses Schwartz's Distribution Theory and its application to construct GeRFs. We assume that the reader is familiar with the Schwartz class of test functions over $\mathbb{R}^d$, denoted $\mathcal{S}(\mathbb{R}^d)$, its dual space of tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$, the space of multiplicators of the Schwartz space $\mathcal{O}_M(\mathbb{R}^d)$, and the definition and properties of the Fourier Transform $\mathscr{F}$ (Schwartz, 1966). For sake of completeness, essential reminders on tempered distributions is provided in Appendix A. We suggest Itô (1954) and (Matheron, 1965, chapter 10), for a more complete introduction to GeRFs. Proposition 1 and Theorem 1 will be proven here. This Section can be skipped in a first reading by readers more interested in the spatial and spatio-temporal models of random fields implied by these results.

3.1 Slow-growing measures and pseudo-differential operators

A complex Radon measure μ over $\mathbb{R}^d$ is said to be a *slow-growing measure* if there exists $N \in \mathbb{N}$ such that the measure $(1 + |x|^2)^{-N}|\mu|$ is a finite measure, where $|\mu|$ denotes the measure of total variation of μ, see

Rudin (1987, chapter 6), or Demengel & Demengel (2000, chapter 1.A). We shall denote $\mathcal{M}_{SG}(\mathbb{R}^d)$ the set of all slow-growing complex measures over $\mathbb{R}^d$. Obviously, $\mathcal{M}_{SG}^{+}(\mathbb{R}^d) \subset \mathcal{M}_{SG}(\mathbb{R}^d)$. For a measure $\mu \in \mathcal{M}_{SG}(\mathbb{R}^d)$, the integral $\langle \mu, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x) d\mu(x)$ is well defined for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and it is straightforward that it defines a tempered distribution, thus $\mathcal{M}_{SG}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$.

Let $g : \mathbb{R}^d \to \mathbb{C}$ be a polynomially bounded continuous function. The multiplication of g with a slow-growing measure $\mu \in \mathcal{M}_{SG}(\mathbb{R}^d)$, noted $g\mu$, is defined as the measure that, for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, applies

$$\langle g\mu, \varphi \rangle = \langle \mu, g\varphi \rangle = \int_{\mathbb{R}^d} \varphi(x) g(x) d\mu(x) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{10}$$

It is thus straightforward that $g\mu \in \mathcal{M}_{SG}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. Note that we could have defined equivalently the measure $g\mu$ by defining $(g\mu)(A) := \int_A g(x) d\mu(x)$ for every bounded borel set $A \subset \mathbb{R}^d$.

As a consequence, pseudo-differential operators of the form $\mathcal{L}_g = \mathscr{F}^{-1}(g\mathscr{F}(\cdot))$, as defined in Eq. (3), with g being a symbol function, are well defined within our framework whenever the Fourier transform of the argument is a slow-growing measure. The domain of definition of $\mathcal{L}_g$ is thus the space of all tempered distributions such that its Fourier transform is a slow-growing measure:

$$D(\mathcal{L}_g) = \{T \in \mathcal{S}'(\mathbb{R}^d) \mid \mathscr{F}(T) \in \mathcal{M}_{SG}(\mathbb{R}^d)\}. \tag{11}$$

This class of operators includes for example linear combinations of differential operators which correspond to g being an Hermitian-symmetric polynomial. Some fractional-differential operators are also included by taking g to be a suitable continuous functions. A more comprehensive list of specific examples will be worked out in Sections (5) and (6)).

3.2 Generalized Random Fields

A real L^2-tempered random distribution Z, referred to as real *Generalized Random Field* (GeRF) from now on, is a real and continuous linear application from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$, for some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will write $\langle Z, \varphi \rangle := Z(\varphi)$ to emphasize that Z acts as a continuous linear functional. All linear operators that are well defined for tempered distributions can be used without restrictions on GeRFs, since they are defined through actions on test functions. In particular, differentiation and Fourier transforms are admissible operations on GeRFs (see Appendix A for their definitions in the deterministic case).

If Z is a real GeRF, there exists a real mean distribution $m_Z \in \mathcal{S}'(\mathbb{R}^d)$ and a real covariance distribution $C_Z \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying $\mathbb{E}(\langle Z, \varphi \rangle) = \langle m_Z, \varphi \rangle$ and $\mathbb{C}ov(\langle Z, \varphi \rangle, \langle Z, \phi \rangle) = \langle C_Z, \varphi \otimes \overline{\phi} \rangle$ respectively for all $\varphi, \phi \in \mathcal{S}(\mathbb{R}^d)$. Without loss of generality, we will assume $m_Z = 0$. The covariance distribution must be a positive-definite kernel, i.e. it must verify $\langle C_Z, \varphi \otimes \overline{\varphi} \rangle \geqslant 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where $\otimes$ denotes the tensor product, with $(\varphi \otimes \phi)(x,y) = \varphi(x)\phi(y)$ and $\varphi, \phi \in \mathcal{S}(\mathbb{R}^d)$. The existence of this covariance distribution, which does not follow obviously from our assumptions, can be guaranteed by Schwartz's Kernel Theorem applied to the space $\mathcal{S}'(\mathbb{R}^d)$. See Schwartz (1959) or Trèves (1967), Theorem 51.6 and its corollary.

A real GeRF Z is *second order stationary* (from now on, more simply *stationary*) if there exists a real and even distribution $\rho_Z \in \mathcal{S}'(\mathbb{R}^d)$ such that $\langle C_Z, \varphi \otimes \overline{\phi} \rangle = \langle \rho_Z, \varphi * \check{\overline{\phi}} \rangle$, where $*$ denotes the convolution product and $\check{}$ denotes the reflection operator, with $\check{\phi}(x) = \phi(-x)$. The distribution ρ_Z is the *stationary covariance distribution* of Z, or more simply *covariance distribution* if stationarity is clear from the context. It must be positive-definite, i.e., it must verify $\langle \rho_Z, \varphi * \check{\overline{\varphi}} \rangle \geqslant 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. A generalization of Bochner's Theorem, known as Bochner-Schwartz Theorem (see for example Donoghue (1969), chapter 42), states that any positive-definite and even distribution ρ_Z is the Fourier transform of a positive and even slow-growing measure: $\rho_Z = \mathscr{F}(\mu_Z)$, known as the *spectral measure* of the associated real GeRF. Since both μ_Z and ρ_Z are even distributions, we will use extensively the following fact: $\rho_Z = \mathscr{F}(\mu_Z) = \mathscr{F}^{-1}(\mu_Z)$.

3.3 Slow-growing Orthogonal Random Measures

A (not necessarily real) GeRF Z is said to be a *slow-growing random measure* if its covariance distribution C_Z is a slow-growing measure, i.e. if $C_Z \in \mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R}^d)$. Similarly to slow-growing measures, slow-growing random measures can be multiplied by (deterministic) polynomially bounded continuous functions, thereby defining a new slow-growing random measure with covariance distribution in $\mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R}^d)$. Specifically we have the following Proposition which is proven in Appendix B:

Proposition 2. *Let Z be a slow-growing random measure with covariance $C_Z \in \mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R}^d)$ and let $g : \mathbb{R}^d \to \mathbb{C}$ be a polynomially bounded continuous function. Let us define the multiplication gZ as a GeRF determined by $\langle gZ, \varphi \rangle = \langle Z, g\varphi \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then the multiplication gZ is well defined as a GeRF and it is a slow-growing random measure with $C_{gZ} = (g \otimes \overline{g})C_Z \in \mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R}^d)$.*

A particular class of slow-growing random measures is the class of *slow-growing orthogonal random measures*, characterized by covariances of the form

$$\langle C_Z, \varphi \otimes \overline{\phi} \rangle = \int_{\mathbb{R}^d} \varphi(x)\overline{\phi}(x) d\nu_Z(x), \quad \varphi, \phi \in \mathcal{S}(\mathbb{R}^d), \tag{12}$$

with $\nu_Z \in \mathcal{M}^+_{SG}(\mathbb{R}^d)$. This form is obtained when the covariance measure C_Z is supported over the hyper-plane $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid x = y\}$. The measure ν_Z is called *the weight* of Z. An important characteristic of this class is that these random measures take non-correlated values when evaluated over test functions that are orthogonal with respect to the weight measure, in particular when they have disjoint supports. From Proposition 2 we get directly the following corollary.

Corollary 1. *Let Z be an orthogonal random measure with weight ν_Z and let g be a symbol function. Then, gZ is an orthogonal random measure with weight $|g|^2\nu_Z$.*

Slow-growing orthogonal random measures are in close connection with stationary GeRFs. A well known result, which is also easy to prove within the framework of GeRFs, (see Itô (1954) or Matheron (1965), chapter 10), is that the Fourier transform of a real stationary GeRF with spectral measure μ_Z is a Hermitian-symmetric complex slow-growing Orthogonal random measure with weight $(2\pi)^{d/2}\mu_Z$. Grounded on this result, the Fourier transform of a stationary GeRF can be seen as a slow-growing measure. Operators of the form (3), defined trough a symbol g, can therefore be applied without restrictions. Having laid out these theoretical foundations, we are now able to establish Proposition 1.

3.4 Proofs of our main result

3.4.1 Proof of Proposition 1

Let g be a symbol function and let $\mathcal{L}_g$ its associated operator. Let U be a real stationary GeRF with spectral measure μ_U and covariance distribution ρ_U. We know that $\mathscr{F}(U)$ is a Hermitian-symmetric complex slow-growing orthogonal random measure with weight $(2\pi)^{d/2}\mu_U$. Thus, by the corollary of Proposition 2, its multiplication by g is well defined and is also a slow-growing orthogonal random measure with weight $(2\pi)^{d/2}|g|^2\mu_U \in \mathcal{M}_{SG}^+(\mathbb{R}^d)$. Moreover, it is Hermitian-symmetric since g is a symbol function.

Hence, the inverse Fourier transform of $g\mathscr{F}(U)$, which is equal to $\mathcal{L}_gU$, is a real stationary GeRF with spectral measure $|g|^2\mu_U$. The expression of the covariance of $\mathcal{L}_gU$ is obtained immediately from $\rho_{\mathcal{L}_gU} = \mathscr{F}^{-1}\left(|g|^2\mu_U\right) = \mathscr{F}^{-1}\left(|g|^2\mathscr{F}(\rho_U)\right) = \mathcal{L}_{|g|^2}\rho_U$. ■

3.4.2 Proof of Theorem 1

Let X be a real stationary GeRF over $\mathbb{R}^d$ with spectral measure μ_X. Let g be a symbol function over $\mathbb{R}^d$ and let $\mathcal{L}_g$ be its associated operator. We start by proving the existence criterion. Let us prove the necessity. Suppose there exists a real stationary GeRF, say U, satisfying (5). By Proposition 1, this implies that $|g|^2\mu_U = \mu_X$, and in particular we have that $\mu_X(g^{-1}(\{0\})) = 0$. As $\mu_U \in \mathcal{M}_{SG}^+(\mathbb{R}^d)$, we can take $N \in \mathbb{N}$ such that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-N}d\mu_U(\xi) < \infty$. We have thus that

$$\int_{\mathbb{R}^d} \frac{d\mu_X(\xi)}{(1+|\xi|^2)^N|g(\xi)|^2} = \int_{\{g\neq 0\}} \frac{|g(\xi)|^2}{(1+|\xi|^2)^N}\frac{d\mu_U(\xi)}{|g(\xi)|^2} = \int_{\{g\neq 0\}} \frac{d\mu_U(\xi)}{(1+|\xi|^2)^N} \leqslant \int_{\mathbb{R}^d} \frac{d\mu_U(\xi)}{(1+|\xi|^2)^N} < \infty. \tag{13}$$

Let us prove the sufficiency. The condition (7) implies in particular that the function $|g|^{-2}$ is locally integrable with respect to μ_X. We can therefore define the Radon measure $\mu_U(A) := \int_A |g(\xi)|^{-2}d\mu_X(\xi)$, for any bounded Borel set $A \subset \mathbb{R}^d$. By (7) and by the fact that both μ_X and $|g|^2$ are even, we see in addition that $\mu_U \in \mathcal{M}^+_{SG}(\mathbb{R}^d)$ and that it is even, thus that μ_U is a spectral measure. Condition (7) implies also that $\mu_X(g^{-1}(\{0\})) = 0$. It is therefore straightforward that $|g|^2\mu_U = \mu_X$. Thus, any real stationary GeRF with spectral measure μ_U satisfies (5).

Let us now prove the uniqueness criterion. For the necessity, suppose that g does have zeros. Let us consider μ_H, a tempered positive measure supported in the closed manifold $g^{-1}(\{0\})$. For instance, we can take any point $\xi_0 \in \mathbb{R}^d$ such that $g(\xi_0) = 0$ and use $\mu_H = \delta_{\xi_0} + \delta_{-\xi_0}$, which is a spectral measure. Hence, we have that $|g|^2\mu_H = 0$. Thus, μ_H can be added to any solution μ_U of (6) and we will still get $|g|^2(\mu_U + \mu_H) = \mu_X$. We conclude that the solution is not unique. For the sufficiency, suppose $|g| > 0$ and that there are two different spectral measures μ_1 and μ_2 satisfying (8). Then, the signed measure $\mu = \mu_1 - \mu_2$ satisfies $|g|^2\mu = 0$, and thus for any continuous function with compact support φ we have $\langle |g|^2\mu, \varphi\rangle = 0$. As $|g|$ is continuous and strictly positive, $|g|^{-2}\varphi$ is also continuous with compact support, and we can argue that for all φ continuous with compact support,

$$\langle \mu, \varphi\rangle = \langle \mu, |g|^2|g|^{-2}\varphi\rangle = \langle |g|^2\mu, |g|^{-2}\varphi\rangle = 0. \tag{14}$$

We conclude that $\mu = 0$ necessarily, and so $\mu_1 = \mu_2$ and the solution is unique. ■

4 White Noise as a fundamental case: a convolution theorem

Let introduce the *White Noise*, denoted W, defined as a real GeRF whose covariance distribution over $\mathbb{R}^d \times \mathbb{R}^d$ is

$$\langle C_W, \varphi \otimes \overline{\phi}\rangle = \int_{\mathbb{R}^d} \varphi(x)\overline{\phi}(x)dx, \quad \varphi, \phi \in \mathcal{S}(\mathbb{R}^d). \tag{15}$$

W is stationary with covariance distribution $\rho_W = \delta$, where $\delta \in \mathcal{S}'(\mathbb{R}^d)$ is the Dirac measure in 0. Its spectral measure is then proportional to the Lebesgue measure, $d\mu_W(x) = (2\pi)^{-d/2}dx$. W is also a particular case of an orthogonal random measure whose weight is the Lebesgue measure. Since μ_W is not a finite measure, W is not a random function and it can only be defined as a GeRF or as a random measure. We will see that SPDEs with White Noise as source term correspond to a *fundamental case* that can be used to obtain the covariance of solutions with more general source terms.

Let us consider the SPDE

$$\mathcal{L}_g U \overset{2nd\ o.}{=} W. \tag{16}$$

Theorem (1) allows us to conclude that there are stationary solutions of (16) if and only if the density $|g|^{-2}(\xi)d\xi$ defines a measure in $\mathcal{M}^+_{SG}(\mathbb{R}^d)$. We suppose this holds and we note $d\mu_U^W(\xi) = (2\pi)^{-\frac{d}{2}}|g|^{-2}(\xi)d\xi$, and $\rho_U^W = \mathscr{F}(\mu_U^W)$. According to Proposition 1, Eq. (16) implies that

$$\mathcal{L}_{|g|^2}\rho_U^W = \rho_W = \delta. \tag{17}$$

Hence, the covariance ρ_U^W can be seen as a *Green's Function* of the operator $\mathcal{L}_{|g|^2}$. It turns out that in order to find the covariance of a solution to (5) with an arbitrary source term X, we have to study the convolvability between ρ_U^W and ρ_X. If convolvability is verified, we get $\rho_U = \rho_U^W * \rho_X$. Theorem 2 provides a sufficient criteria regarding the applicability of this procedure regardless of the source term X.

Theorem 2. *Let X be a real stationary GeRF over $\mathbb{R}^d$ with covariance distribution ρ_X. Let g be a symbol function over $\mathbb{R}^d$ such that $\frac{1}{g}$ is smooth with polynomially bounded derivatives of all orders. Then, there exists a unique stationary solution to (5) and its covariance distribution is given by*

$$\rho_U = \rho_U^W * \rho_X, \tag{18}$$

where ρ_U^W is the covariance of the unique stationary solution to (16).

Proof: Since $1/g$ is smooth and polynomially bounded, the SCEU holds, and there exists a unique stationary solution to equation (5). The spectral measure of the solution is given by $\mu_U = |g|^{-2}\mu_X$, μ_X being the spectral measure of X. The regularity and boundedness conditions for $1/g$ and its derivatives imply that both $\frac{1}{g}$ and $|g|^{-2}$ are in the space $\mathcal{O}_M(\mathbb{R}^d)$ of multiplicators of the Schwartz space (see Appendix A). Since the expression $|g|^{-2}\mu_X$ is the multiplication between $|g|^{-2} \in \mathcal{O}_M(\mathbb{R}^d)$ and $\mu_X \in \mathcal{S}'(\mathbb{R}^d)$, the exchange formula for the Fourier transform can be applied. We thus obtain

$$\rho_U = \mathscr{F}(\mu_U) = \mathscr{F}(|g|^{-2}\mu_X) = \mathscr{F}((2\pi)^{-\frac{d}{2}}|g|^{-2}) * \mathscr{F}(\mu_X) = \mathscr{F}(\mu_U^W) * \mathscr{F}(\mu_X) = \rho_U^W * \rho_X. \quad \blacksquare \tag{19}$$

Although the condition on g required in Theorem 2 may seem restrictive, it turns out that it is satisfied by most models studied in the statistical literature on spatio-temporal random fields. For example, the Matérn model, the Stein model and Markov models satisfy these conditions, as will be detailed in Sections 5.1, 5.3 and 6.1. A more general analysis could be done by studying the convolvability between ρ_U^W and ρ_X in the more general framework of the $\mathcal{S}'$–convolution, see e.g. Dierolf & Voigt (1978).

Theorem 2 shows that solutions of SPDEs with White Noise source term is the starting point of more general solutions, when the source term can be any stationary GeRF. In the next Sections, devoted to spatial and spatio-temporal models we shall always start our analysis by considering White Noise source term.

5 Applications to spatial models

5.1 Matérn Model

As a first example, we start with a well-known and increasingly popular model, namely the Matérn model. The relationship between the Matérn Model and the SPDE over $\mathbb{R}^d$,

$$(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = W, \tag{20}$$

with $\kappa > 0$, $\alpha \in \mathbb{R}$ and where Δ denotes the Laplace operator, established a long time ago (Whittle, 1963) and recently revisited in Lindgren et al. (2011), can easily be obtained from Theorem 1.

The operator $(\kappa^2 - \Delta)^{\frac{\alpha}{2}}$ is of the form (3) with symbol function $g(\xi) = (\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}}$, satisfying the SCEU defined in Remark 2. This allows us to conclude that there exists a unique stationary solution to (20), with spectral measure

$$d\mu_U^W(\xi) = \frac{d\xi}{(2\pi)^{d/2}(\kappa^2 + |\xi|^2)^{\alpha}}. \tag{21}$$

If $\alpha > d/2$ the measure (21) is finite, and thus its associated random field is a mean-square continuous random function, with stationary Matérn covariance function

$$\rho_U^W(h) = \frac{1}{(2\pi)^{d/2} 2^{\alpha-1} \kappa^{2\alpha-d} \Gamma(\alpha)} (\kappa|h|)^{\alpha-d/2} K_{\alpha-d/2}(\kappa|h|), \tag{22}$$

where Γ is the Gamma function and $K_{\alpha-d/2}$ is the modified Bessel function of the second kind of order $\alpha - d/2$. When $\alpha \leqslant d/2$, we still obtain a unique stationary solution, but it is only defined in a distributional sense. We refer to this covariance as the *generalized Matérn covariance* .

Since g also satisfies the conditions in Theorem (2), we get that for any real stationary GeRF X the SPDE

$$(\kappa^2 - \Delta)^{\alpha/2} U = X \tag{23}$$

has a unique stationary solution whose covariance is the convolution between ρ_X and the Matérn covariance.

5.2 Matérn Model without range parameter

The condition $\kappa > 0$ in the Matérn SPDE defined in Eq. (20) can be relaxed. Setting $\kappa = 0$, we obtain a fractional Laplacian operator $(-\Delta)^{\alpha/2}$ with symbol function $g(\xi) = |\xi|^{\alpha}$ for $\alpha > 0$. Let us consider the SPDE

$$(-\Delta)^{\alpha/2} U = W, \tag{24}$$

which corresponds to the limit case of a Matérn model as $\kappa \to 0$. In Theorem 1, the existence condition (7) requires that there exists $N \in \mathbb{N}$ such that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-N}|\xi|^{-2\alpha}d\xi < \infty$. Because of the singularity at the origin, this is only possible if $\alpha < d/2$. In this case, the spectral measure of a stationary solution to the equation (24) is

$$d\mu_U(\xi) = \frac{1}{(2\pi)^{d/2}}\frac{d\xi}{|\xi|^{2\alpha}}. \tag{25}$$

The associated covariance distribution is its Fourier transform, which is the locally integrable function (see Donoghue (1969), chapter 32)

$$\rho_U(h) = \frac{1}{\pi^{d/2}}\frac{\Gamma(\frac{d}{2}-\alpha)}{\Gamma(\alpha)}\frac{1}{|h|^{d-2\alpha}}. \tag{26}$$

Note that the function ρ_U in (26) is not defined at $h = 0$. It is not continuous, but it is still positive-definite in distributional sense. The associated random field must be interpreted as a GeRF and not as a continuous random function. This is an example of the kind of covariance structures we obtain when working with non-finite spectral measures. Such models have a long-range dependence behavior. They have been studied in Anh et al. (1998) and in Gay & Heyde (1990), in which Eq. (24) is specified with a slightly different definition of the operator $(-\Delta)^{\frac{\alpha}{2}}$.

We remark that the symbol function $g(\xi) = |\xi|^\alpha$ has a zero at the origin. Hence, the SCEU conditions do not hold. The stationary solution associated to the covariance (26) is not the unique possible solution. To describe all possible stationary solutions, we follow Remark 3 and we consider spectral measures which are supported at the origin, i.e., which are proportionals to the Dirac measure $\mu_{U_H} = a\delta$, with $a > 0$. The associated covariance distributions are then constant positive functions, and thus the associated GeRF are *random constants*, that is, $U_H(x) = A$, for all $x \in \mathbb{R}^d$, with A being a centered random variable with variance $(2\pi)^{-\frac{d}{2}}a < \infty$. In other words, *the only stationary solutions to the homogeneous equation* $(-\Delta)^{\alpha/2}U_H = 0$ *are random constants.*

5.3 Isotropic Markov Models

Let $p : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly positive polynomial over $\mathbb{R}^+$. We consider the SPDE over $\mathbb{R}^d$

$$\sqrt{p(-\Delta)}U = W, \tag{27}$$

where the operator $\sqrt{p(-\Delta)}$ is of the form (3) with symbol function $g(\xi) = \sqrt{p(|\xi|^2)}$. The SCEU holds, and thus the SPDE (27) has a unique stationary solution, and its spectral measure if of the form

$$d\mu_U^W(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\frac{d\xi}{p(|\xi|^2)}. \tag{28}$$

This is a measure whose density is the inverse of a strictly positive and isotropic polynomial. Rozanov's Theorem (Rozanov, 1977) allows us to conclude that this model is an isotropic stationary Markov Random Field (MRF). According to Rozanov's Theory, a MRF is, broadly speaking, a GeRF such that for every domain of $\mathbb{R}^d$, evaluations of the random field in the interior of the domain are independent upon evaluations in the interior of the complement of the domain, conditionally to the behavior of the random field in a neighborhood of the boundary of the domain. By *evaluations*, we mean the action of the GeRF to test functions whose supports are in the interior of the corresponding set. Rozanov's theorem states that every stationary MRF has a spectral measure whose density is the inverse of a strictly positive polynomial. Thus, in the case of isotropic models, MRFs satisfy equation (27). See Rozanov (1982) for a complete theory of MRFs which also uses the theory of GeRFs.

Note that g satisfies the conditions of Theorem 2. Hence, for any real stationary GeRF X there exists a unique stationary solution to the SPDE $\sqrt{p(-\Delta)}U = X$, whose covariance is the convolution between ρ_X and the covariance of the MRF solution to Eq. (27).

6 Applications to spatio-temporal models

We now present stationary spatio-temporal models which can be obtained and described within our framework. From now on, d will always denote the spatial dimension, and we will explicitly write $\mathbb{R}^d \times \mathbb{R}$ referring to the spatio-temporal domain. We will denote $\mathscr{F}$, $\mathscr{F}_S$ and $\mathscr{F}_T$, respectively the spatio-temporal, spatial and temporal Fourier transforms. We will use the variables $(\xi, \omega) \in \mathbb{R}^d \times \mathbb{R}$ for the frequency space-time domain (that is, after applying a spatio-temporal Fourier transform). When working with stationary covariance functions or distributions, the spatial gap will always be denoted by $h \in \mathbb{R}^d$ and the temporal gap by $u \in \mathbb{R}$. The function g will always denote a *spatial* symbol function. Thus, $\mathcal{L}_g$ denotes the operator $\mathscr{F}_S^{-1}(g\mathscr{F}_S(\cdot))$, which will be applied to stationary GeRFs over $\mathbb{R}^d \times \mathbb{R}$. We denote g_R and g_I the real and imaginary parts of g respectively. As g is Hermitian-symmetric, g_R is even and g_I is odd.

We recall the important concepts of *separability* and *symmetry* of a spatio-temporal stationary model. A stationary GeRF Z over $\mathbb{R}^d \times \mathbb{R}$ is said to be *separable* if its covariance $\rho_Z \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R})$ can be expressed as the tensor product of a spatial covariance and a temporal covariance, $\rho_Z = \rho_{Z_S} \otimes \rho_{Z_T}$, with $\rho_{Z_S} \in \mathcal{S}'(\mathbb{R}^d)$ and $\rho_{Z_T} \in \mathcal{S}'(\mathbb{R})$, obtaining $\rho_Z(h,u) = \rho_{Z_S}(h)\rho_{Z_T}(u)$ in the case with functional meaning. This is equivalent to require the spatio-temporal spectral measure to be the tensor product of a spatial and a temporal measure, $d\mu_Z(\xi,\omega) = d\mu_{Z_S}(\xi)d\mu_{Z_T}(\omega)$. If Z is separable, we write $Z = Z_S \otimes Z_T$, Z_S and Z_T *representing* the corresponding spatial and temporal GeRFs with covariance ρ_{Z_S} and ρ_{Z_T} respectively. A stationary GeRF Z over $\mathbb{R}^d \times \mathbb{R}$ is said to be *symmetric* if its covariance satisfies $\rho_Z(h,u) = \rho_Z(h,-u) = \rho_Z(-h,u) = \rho_Z(-h,-u)$ in the case with functional meaning, with its corresponding generalization in the case of distributions. If Z is a stationary GeRF with spectral measure $d\mu_Z(\xi,\omega) = f(\xi,\omega)d\xi d\omega$, a necessary and sufficient condition for Z to be non-symmetric is that the density f does not depend on the variable ω only through its absolute value $|\omega|$.

The spatial behavior of a stationary random function over $\mathbb{R}^d \times \mathbb{R}$, say Z, with covariance function ρ_Z is studied by fixing the time component at any particular time $t \in \mathbb{R}$, obtaining the spatial random function $Z_S = (Z(x,t))_{x\in\mathbb{R}^d}$. Because of time stationarity, Z_S has the same spatial covariance function for any chosen t, with $\rho_{Z_S}(h) = \rho_Z(h,0)$. We refer to Z_S as a *spatial trace* of Z. This can be generalized to any stationary GeRF Z such that its spectral measure satisfies $\mu_Z(A\times\mathbb{R}) < \infty$ for every bounded Borel set $A \subset \mathbb{R}^d$. In such a case, the covariance distribution ρ_Z has a continuous point-wise meaning in time, and thus $\rho_{Z_S} := \rho_Z(\cdot, 0)$ is a spatial covariance distribution. Any spatial GeRF Z_S with ρ_{Z_S} as covariance distribution is said to be a spatial trace of Z. We can equivalently describe the spectral measure of a spatial trace, with $\mu_{Z_S}(A) = (2\pi)^{-\frac{1}{2}}\mu_Z(A\times\mathbb{R})$ for every bounded Borel set $A \subset \mathbb{R}^d$.

6.1 Stein Model

Rather than solving a SPDE and analyzing the covariance structure of the solution, we start in this example from the spectral measure over $\mathbb{R}^d \times \mathbb{R}$ proposed in Stein (2005)

$$d\mu_U(\xi,\omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}}\frac{d\xi d\omega}{(a(s^2+\omega^2)^\beta + b(\kappa^2+|\xi|^2)^\alpha)^\nu}, \tag{29}$$

with $a, b > 0$, $s^2+\kappa^2 > 0$, and $\alpha, \beta, \nu \in \mathbb{R}$. It is always a well-defined spectral measure since its density is the inverse of a positive and polynomially bounded continuous function. Using the spatio-temporal White Noise, $d\mu_W(\xi,\omega) = (2\pi)^{-\frac{d+1}{2}}d\xi d\omega$, we consider the spatio-temporal symbol function

$$(\xi,\omega) \mapsto (a(s^2+\omega^2)^\beta + b(\kappa^2+|\xi|^2)^\alpha)^{\nu/2} \tag{30}$$

which satisfies the SCEU as long as α and β are not simultaneously equal to 0, which we exclude from now on. The corresponding SPDE is

$$\left(a\left(s^2 - \frac{\partial^2}{\partial t^2}\right)^\beta + b\left(\kappa^2 - \Delta\right)^\alpha\right)^{\nu/2} U = W, \tag{31}$$

where W is a White Noise on $\mathbb{R}^d \times \mathbb{R}$. As a consequence of Theorem 1, there exists a unique stationary solution to (31) and its spectral measure is (29). When α, β and ν are positive, and if $\frac{1}{\beta\nu}+\frac{d}{\alpha\nu} < 2$ holds, Stein (2005) shows that the measure (29) is finite and that its associated random field is a mean-square continuous random function. The interesting property of the Stein model is that, without being a separable model, the spatial and temporal smoothness of the paths of the random function can be controlled separately thanks to the parameters α and β. Except for some particular values for the parameters, there is no closed-form expression of the covariance.

When $\kappa, s, a, b > 0$ and α, β, ν are not null, the symbol function (30) satisfies conditions in Theorem 2. Hence, if the source term X is any stationary GeRF, $\left(a\left(s^2 - \frac{\partial^2}{\partial t^2}\right)^{\beta} + b\left(\kappa^2 - \Delta\right)^{\alpha}\right)^{\nu/2} U = X$ has a unique stationary solution whose covariance is the convolution between ρ_X and the covariance of the (generalized) Stein model.

6.2 Models derived from Evolution Equations

In this section we study models associated to the following class of SPDEs over $\mathbb{R}^d \times \mathbb{R}$:

$$\frac{\partial^{\beta} U}{\partial t^{\beta}} + \mathcal{L}_g U = X \tag{32}$$

where X is a stationary spatio-temporal GeRF, and $\beta > 0$. For this class of SPDEs, we study in details several examples with both physical and statistical interest. They involve diffusion, Langevin-type equation, heat equation and wave propagation phenomena.

We present sufficient conditions for existence and uniqueness of a stationary solution. We describe the spectral measure of the covariance distribution when the source term is a spatio-temporal White Noise. When we can, we also specify the spatial SPDE to be satisfied by the spatial trace of the model when it exists (i.e. when $\beta > \frac{1}{2}$), and in this case we describe its spatial spectral measure.

For some specific values of β, namely when $\beta \in \{1, 2\}$, we provide a more detailed analysis. We specify the cases where the solutions can be conceived as random functions. We also detail the cases where X is a separable model with a White Noise structure in time and any other structure in space, which we write $X = X_S \otimes W_T$. We also specify when Theorem 2 can be applied for an arbitrary X. Finally, our framework allows us to provide a well adapted definition of the operator $\frac{\partial^{\beta}}{\partial t^{\beta}}$ for non integer values of β.

6.2.1 First Order Evolution Models

First order evolution models correspond to $\beta = 1$ in (32). The SPDE is then determined by the symbol function over $\mathbb{R}^d \times \mathbb{R}$:

$$(\xi, \omega) \mapsto i\omega + g(\xi). \tag{33}$$

The null-set of this function is $\mathcal{N}_1 = \{(\xi, \omega) \in \mathbb{R}^d \times \mathbb{R} \ : \ g_R(\xi) = 0 \text{ and } g_I(\xi) = -\omega\}$. Hence, a sufficient condition to have a unique stationary solution is that g_R satisfies the SCEU, for in this case the function (33) also satisfies the SCEU. Let us first consider the case where the source term is a spatio-temporal White Noise

$$\frac{\partial U}{\partial t} + \mathcal{L}_g U = W. \tag{34}$$

Following (8), the spectral measure of the unique stationary solution to (34) is

$$d\mu_U^W(\xi, \omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{(\omega + g_I(\xi))^2 + g_R^2(\xi)}, \tag{35}$$

leading, after temporal Fourier transform to its covariance

$$\rho_U^W(h, u) = \mathscr{F}_S \left(\xi \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{e^{iug_I(\xi) - |u||g_R(\xi)|}}{2|g_R(\xi)|} \right)(h). \tag{36}$$

For ease of reading, we have used a functional notation for the variables (h, u) in (36) even though this covariance is not necessarily a function. In general, it is a distribution, the spatial Fourier transform being interpreted in a distributional sense. However, when $|g_R|^{-1}$ is integrable, the equation (36) does define a positive-definite continuous covariance function, and in that case the solution is a stationary, mean square continuous random function.

This spatio-temporal model is in general neither separable nor symmetrical. Symmetry is obtained if and only if $g_I = 0$. The spatial structure is controlled by g_R, as can be seen by evaluating (36) in $u = 0$:

$$\rho_U^W(h, 0) = \mathscr{F}_S \left(\xi \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{2|g_R(\xi)|} \right)(h). \tag{37}$$

It is thus clear that $(2\pi)^{-\frac{d}{2}} \frac{d\xi}{2|g_R(\xi)|}$ is the spectral measure of the spatial covariance.

Let us now consider the more general case $X = X_S \otimes W_T$. In this case, the covariance of the solution is

$$\rho_U(h, u) = \mathscr{F}_S \left(\xi \mapsto \frac{e^{iug_I(\xi) - |u||g_R(\xi)|}}{2|g_R(\xi)|} d\mu_{X_S}(\xi) \right)(h). \tag{38}$$

Characteristics of separability and symmetry are then similar to those obtained with White Noise as source term. Evaluating at $u = 0$ we get the covariance describing the spatial behavior, with associated spectral measure $d\mu_{U_S}(\xi) = \dfrac{d\mu_{X_S}(\xi)}{2|g_R(\xi)|}$. Observing that this measure is of the form (8), it follows that U_S satisfies the SPDE over $\mathbb{R}^d$:

$$\sqrt{2}\mathcal{L}_{\sqrt{|g_R|}} U_S \overset{2nd\ o.}{=} X_S. \tag{39}$$

In the completely general case, when X is any spatio-temporal stationary GeRF, Theorem 2 is of application under the condition that g_R, g_I and $\frac{1}{g_R}$ are in $\mathcal{O}_M(\mathbb{R}^d)$, since in this case the reciprocal of the spatio-temporal symbol function (33) is in $\mathcal{O}_M(\mathbb{R}^d \times \mathbb{R})$. Under these conditions on g, the only stationary solution to the SPDE (32) with $\beta = 1$ has a covariance of the form $\rho_U = \rho_U^W * \rho_X$, with ρ_U^W given by (36).

Example 6.1. First order evolution of Matérn model. We call *First Order Evolution of Matérn Model* the unique stationary solution to the equation over $\mathbb{R}^d \times \mathbb{R}$

$$\frac{\partial U}{\partial t} + a(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = W, \tag{40}$$

Where $a > 0, \kappa > 0$, and $\alpha \in \mathbb{R}$. This is a first order evolution model with spatial symbol function $g(\xi) = a(\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}}$. The associated spectral measure is

$$d\mu_U(\xi, \omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{\omega^2 + a^2(\kappa^2 + |\xi|^2)^{\alpha}}. \tag{41}$$

Following equation (36), and using expressions of the Fourier transform of radial functions (see for example Donoghue (1969), chapter 41), we obtain the spatio-temporal covariance

$$\rho_U(h, u) = \frac{1}{(2\pi)^{\frac{d}{2}} |h|^{\frac{d-2}{2}}} \int_0^\infty J_{\frac{d-2}{2}}(|h| r) \frac{e^{-a(\kappa^2 + r^2)^{\frac{\alpha}{2}} |u|}}{2a(\kappa^2 + r^2)^{\frac{\alpha}{2}}} r^{\frac{d}{2}} dr, \tag{42}$$

where J_b denotes the Bessel function of the first kind of order b. This model can be found in Jones & Zhang (1997). This is a symmetric non-separable model which follows a Matérn covariance structure in space and a mixture of exponentials in time. It is a well defined random function for $\alpha > d$. Following (39), the spatial trace of this model follows the SPDE over $\mathbb{R}^d$

$$\sqrt{2a}(\kappa^2 - \Delta)^{\frac{\alpha}{4}} U_S = W_S, \tag{43}$$

where W_S is a spatial White Noise. Since g_R and $\frac{1}{g_R}$ are in $\mathcal{O}_M(\mathbb{R}^d)$, by Theorem 2, if X is an arbitrary stationary GeRF, the unique stationary solution of the SPDE

$$\frac{\partial U}{\partial t} + a(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = X \tag{44}$$

has a covariance equal to the convolution of ρ_X and the covariance ρ_U in (42).

Example 6.2. Advection-Diffusion Equation. Sigrist et al. (2015) proposed estimation methods and simulation algorithms for the unique stationary solution of the SPDE over $\mathbb{R}^d \times \mathbb{R}$:

$$\frac{\partial U}{\partial t} + \kappa^2 U + v^T \nabla U - \boldsymbol{\nabla} \cdot (\Sigma \nabla U) = X_S \otimes W_T, \tag{45}$$

where $\kappa > 0$ is a damping parameter, $v \in \mathbb{R}^d$ is a velocity and Σ is a symmetric positive-definite matrix controlling non-isotropic diffusion. W_T is a temporal White Noise and X_S is a stationary spatial random field. This equation, known as a *Advection-Diffusion Equation*, is a particular First Order Evolution Model. Its spatial symbol function is $g(\xi) = \kappa^2 + \xi^T \Sigma \xi + i v^T \xi$. It satisfies the sufficient conditions for existence and uniqueness of a stationary solution. Without advection ($v = 0$), this equation was studied in Whittle (1963) in a non-generalized framework . Sigrist et al. (2015) considers a Matérn Model for X_S, with smoothness parameter equals to 1, corresponding for example to $\alpha = 2$ in (22) when $d = 2$. The spatial behavior of this particular model is described by the SPDE (39).

Example 6.3. A Langevin's Equation.

Using linear response theory, Hristopulos & Tsantili (2016) propose stationary random fields which are solutions to the Langevin equation

$$\frac{\partial U}{\partial t} + \frac{D}{2k^d \eta_0}\left(1 - \eta_1 k^2 \Delta + \nu k^4 \Delta^2\right) U = W, \tag{46}$$

with $D, k, \eta_0 > 0$, $\eta_1, \nu \geqslant 0$. For this first order evolution model, the spatial symbol function is $g(\xi) = \frac{D}{2k^d \eta_0}\left(1 + \eta_1 k^2 |\xi|^2 + \nu k^2 |\xi|^4\right)$, which satisfies the SCEU stated in 2. Hence, (46) has a unique stationary solution, whose spectral measure can be obtained using the general expression of first order evolution model in (35). Hristopulos & Tsantili (2016) provides expressions of the related covariance structures, which are functions for $d \leqslant 3$, and which can be obtained through formulas similar to (36) in combination with Fourier transforms of radial functions. The spatial behavior of this model can be described following equation (39), with spatial White Noise, $X_S = W_S$. In general, this Langevin equation model is not an evolution of a Matérn model. It is the particular case when the parameter ν, called *curvature coefficient*, equals to 0.

Example 6.4. The Heat Equation. We now consider the stochastic Heat (or Diffusion) Equation over $\mathbb{R}^d \times \mathbb{R}$

$$\frac{\partial U}{\partial t} - a\Delta U = X, \tag{47}$$

where $a > 0$ is the *diffusivity* parameter. It is a first order evolution model with spatial symbol $g(\xi) = a|\xi|^2$. In this case, the spatio-temporal symbol function $(\xi, \omega) \mapsto i\omega + a|\xi|^2$ is not strictly positive, the origin being the only zero of g. There is thus no uniqueness of stationary solutions, if they exist. Using similar arguments as those used in Section 5.2, one can see that the only stationary solutions of the homogeneous Heat Equation

$$\frac{\partial U_H}{\partial t} - a\Delta U_H = 0, \tag{48}$$

are random constants. Because of the singularity at the origin of the function $|g|^{-2}$, the existence condition (7) does not always hold. Existence needs to be checked for each source term X. Let us first consider that the source term is a spatio-temporal White Noise. Equation (47) becomes

$$\frac{\partial U}{\partial t} - a\Delta U = W. \tag{49}$$

Using Theorem 1, one concludes (see Appendix C.1) that *there exists stationary solutions to the Stochastic Heat equation (49) only for spatial dimensions* $d \geqslant 3$, and in those cases, they can only be conceived as GeRFs and never as Random Functions continuous in mean-square. When $d = 3$, computations reported in Appendix C.2 show that the covariance structure is

$$\rho_U^W(h,u) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{\pi}{2a|h|} \operatorname{erf}\left(\frac{|h|}{2\sqrt{a|u|}}\right). \tag{50}$$

This covariance must be interpreted in a distributional sense, since it is not defined at $|h| = |u| = 0$. The spatial trace of the stationary field associated to (50), U_S, can be described using equation (39) to obtain

$$\sqrt{2}(-\Delta)^{\frac{1}{2}} U_S \overset{2nd\ o.}{=} W_S, \tag{51}$$

where W_S is a spatial White Noise. In other words, U_S is a Matérn model without range parameter presented in Section 5.2, as can be seen when evaluating equation (50) at $u = 0$, with $h \neq 0$.

When X is an arbitrary source term, Theorem 2 cannot be applied for spatial dimensions smaller that 3. For $d = 3$, a convolvability condition between ρ_X and (50) must be verified. Nevertheless, the existence of a solution can be ensured independently on existence of solutions with White Noise source term by imposing necessary conditions on μ_X such that the existence criteria (7) holds. For example, one could require μ_X to be null in some neighborhood of the origin.

6.2.2 Second Order Evolution Models

Second order evolution models are solutions of (32) with $\beta = 2$. The spatio-temporal symbol function is

$$(\xi,\omega) \mapsto -\omega^2 + g(\xi). \tag{52}$$

The null-set of this function is $\mathcal{N}_2 = \{(\xi,\omega) \in \mathbb{R}^d \times \mathbb{R} \mid g_R(\xi) = \omega^2 \text{ and } g_I(\xi) = 0\}$. A sufficient condition to have a unique stationary solution is thus that g_R is a strictly negative function satisfying the SCEU, which is from now on supposed to hold. Let us first consider the case $X = W$,

$$\frac{\partial^2 U}{\partial t^2} + \mathcal{L}_g U = W. \tag{53}$$

The spectral measure of the unique stationary solution to 53 is

$$d\mu_U^W(\xi,\omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{(\omega^2 - g_R(\xi))^2 + g_I^2(\xi)}, \tag{54}$$

and its covariance distribution ρ_U^W is the Fourier transform of μ_U^W. To simplify the notation, consider the non-null complex spatial function

$$\gamma(\xi) = \sqrt{\frac{|g(\xi)| + g_R(\xi)}{2}} + i\sqrt{\frac{|g(\xi)| - g_R(\xi)}{2}}. \tag{55}$$

Let us denote γ_R and γ_I the real and imaginary parts of γ respectively. The covariance ρ_U^W can be expressed as

$$\rho_U^W(h,u) = \mathscr{F}_S\left(\xi \mapsto \frac{e^{-(|\gamma_I(\xi)|+i\gamma_R(\xi))|u|}}{(2\pi)^{\frac{d}{2}}8|\gamma_I(\xi)|^2}\left[\frac{1}{|\gamma_I(\xi)| + i\gamma_R(\xi)} + \frac{e^{i2\gamma_R(\xi)|u|}}{|\gamma_I(\xi)| - i\gamma_R(\xi)} + \frac{e^{i2\gamma_R(\xi)|u|} - 1}{i\gamma_R(\xi)}\right] d\xi\right)(h). \tag{56}$$

This positive-definite distribution has a functional meaning if the function $|\gamma_I|^{-1}|\gamma|^{-2}$ is integrable over $\mathbb{R}^d$, which is equivalent to require that the function $|g|^{-1}(|g|-g_R)^{-\frac{1}{2}}$ is integrable over $\mathbb{R}^d$. The term $\frac{e^{i2\gamma_R(\xi)|u|}-1}{i\gamma_R(\xi)}$ is interpreted as $2|u|$ when $\gamma_R(\xi) = 0$, which is the case when $g_I(\xi) = 0$. Contrarily to first order evolution models, this model is always symmetric. The spatial covariance structure is determined by both g_R and g_I, and we describe it by evaluating (56) at $u = 0$,

$$\rho_U^W(h,0) = \mathscr{F}_S\left(\xi \mapsto \frac{d\xi}{(2\pi)^{\frac{d}{2}}4|\gamma_I(\xi)||\gamma(\xi)|^2}\right)(h) = \mathscr{F}_S\left(\xi \mapsto \frac{d\xi}{(2\pi)^{\frac{d}{2}}2\sqrt{2}|g(\xi)|\sqrt{|g(\xi)| - g_R(\xi))}}\right)(h). \tag{57}$$

The spectral measure of a spatial trace of the unique stationary solution to (53) is thus

$$d\mu_{U_S}^W(\xi) = \frac{d\xi}{(2\pi)^{\frac{d}{2}}2\sqrt{2}|g(\xi)|\sqrt{|g(\xi)| - g_R(\xi))}}.$$

We consider next the separable source term $X = X_S \otimes W_T$. In this case, the covariance of the solution is

$$\rho_U(h,u) = \mathscr{F}_S\left(\xi \mapsto \frac{e^{-(|\gamma_I(\xi)|+i\gamma_R(\xi))|u|}}{(2\pi)^{\frac{d}{2}}8|\gamma_I(\xi)|^2}\left[\frac{1}{|\gamma_I(\xi)| + i|\gamma_R(\xi)|} + \frac{e^{i2\gamma_R(\xi)|u|}}{|\gamma_I(\xi)| - i|\gamma_R(\xi)|} + \frac{e^{i2\gamma_R(\xi)|u|} - 1}{i\gamma_R(\xi)}\right] d\mu_{X_S}(\xi)\right)(h). \tag{58}$$

Evaluating at $u = 0$ leads to the covariance of the spatial trace U_S, with spectral measure

$$d\mu_{U_S}(\xi) = \frac{d\mu_{X_S}(\xi)}{2\sqrt{2}|g(\xi)|\sqrt{|g(\xi)| - g_R(\xi))}}. \tag{59}$$

Observing that this is a measure of the form (8), we obtain that U_S satisfies the SPDE over $\mathbb{R}^d$

$$\sqrt{2\sqrt{2}}\mathscr{L}_{\sqrt{|g|\sqrt{|g|-g_R}}}U_S \overset{2nd\ o.}{=} X_S \tag{60}$$

When X is a general spatio-temporal stationary GeRF, a sufficient condition to apply Theorem 2 is that g_R, $\frac{1}{g_R}$ and g_I are in the space $\mathcal{O}_M(\mathbb{R}^d)$. In this case, the only stationary solution to the SPDE (32) with $\beta = 2$ has a covariance of the form $\rho_U = \rho_U^W * \rho_X$, where ρ_U^W is given by (56).

Example 6.5. Second Order Evolution of Matérn Model.

A *second order evolution of Matérn model* is the unique stationary solution to the SPDE over $\mathbb{R}^d \times \mathbb{R}$

$$\frac{\partial^2 U}{\partial t^2} - a(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = W, \tag{61}$$

with $a, \kappa > 0$ and $\alpha \in \mathbb{R}$. The spatial symbol function is $g(\xi) = -a(\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}}$, which satisfies SCUE. Its spectral measure is

$$d\mu_U(\xi,\omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{(\omega^2 + a(\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}})^2}. \tag{62}$$

Here, the function γ defined in (55) is purely imaginary, with $\gamma(\xi) = i\sqrt{a}(\kappa^2 + |\xi|^2)^{\frac{\alpha}{4}}$. Following equation (56) and using the expression of the Fourier transform of a radial function, the covariance is

$$\rho_U(h,u) = \frac{1}{(2\pi)^{\frac{d}{2}}|h|^{\frac{d-2}{2}}} \int_0^\infty J_{\frac{d-2}{2}}(|h|r) \frac{e^{-\sqrt{a}(\kappa^2+r^2)^{\frac{\alpha}{4}}|u|}(1 + \sqrt{a}(\kappa^2 + r^2)^{\frac{\alpha}{4}}|u|)}{4a\sqrt{a}(\kappa^2 + r^2)^{\frac{3\alpha}{4}}} r^{\frac{d}{2}} dr, \tag{63}$$

which has a meaning as a continuous function when $\alpha > \frac{2d}{3}$. Following (60), a spatial trace of this model is the unique solution of the SPDE over $\mathbb{R}^d$

$$2\sqrt{a\sqrt{a}}(\kappa^2 - \Delta)^{\frac{3\alpha}{8}} U_S \overset{2nd\ o.}{=} W_S, \tag{64}$$

where W_S is a spatial White Noise. As a consequence it is a Matérn Model. Since $\frac{1}{g_R} \in \mathcal{O}_M(\mathbb{R}^d)$, we conclude by Theorem 2 that the unique stationary solution of the SPDE

$$\frac{\partial^2 U}{\partial t^2} - a(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = X \tag{65}$$

has a covariance which is the convolution between ρ_X and $\rho_U(h,u)$ given in (63).

Example 6.6. The Wave Equation. As next example we consider the stochastic wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = X, \tag{66}$$

where X is a stationary random field and $c > 0$ is the *propagation velocity*. This is a second order evolution model with spatial symbol function $g(\xi) = c^2|\xi|^2$. The null-set of the associated spatio-temporal symbol function $(\xi, \omega) \mapsto -\omega^2 + c^2|\xi|^2$ is the spatio-temporal cone $\mathcal{C} = \{(\xi, \omega) \in \mathbb{R}^d \times \mathbb{R} \mid |\omega| = c|\xi|\}$. As a consequence uniqueness of a potential stationary solution does not hold. Following Remark 3, stationary solutions to the homogeneous wave equation

$$\frac{\partial^2 U_H}{\partial t^2} - c^2 \Delta U_H = 0 \tag{67}$$

are found by studying covariance structures associated to spectral measures supported on the cone $\mathcal{C}$. A spectral measure μ_{U_H} over $\mathbb{R}^d \times \mathbb{R}$, supported on $\mathcal{C}$, can be described trough its action to test functions $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R})$ by

$$\langle \mu_{U_H}, \psi \rangle = \int_{\mathbb{R}^d} \frac{\psi(\xi, c|\xi|) + \psi(\xi, -c|\xi|)}{2} d\nu(\xi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \psi(\xi, \omega) d\left(\frac{\delta_{-c|\xi|} + \delta_{c|\xi|}}{2}\right)(\omega) d\mu_{U_H^S}(\xi) \tag{68}$$

where $\mu_{U_H^S}$ is a spectral measure over $\mathbb{R}^d$. In the right hand side of (68), the disintegration language is used for the measure μ_{U_H}, which can then be expressed as $d\mu_{U_H}(\xi, \omega) = d(\frac{\delta_{-c|\xi|} + \delta_{c|\xi|}}{2})(\omega) d\mu_{U_H^S}(\xi)$. Hence, all stationary solutions of (67) have a spectral measure of this form. After applying a temporal Fourier transform, we obtain that the associated covariance structure is

$$\rho_{U_H}(h, u) = \mathscr{F}_S\left(\xi \mapsto \frac{\cos(c|\xi||u|)}{\sqrt{2\pi}} d\mu_{U_H^S}(\xi)\right)(h). \tag{69}$$

Evaluating (69) at $u = 0$, we get that the measure $(2\pi)^{-\frac{1}{2}} \mu_{U_H^S}$ describes the spatial structure of the solution U_H, which can be chosen arbitrarily. The covariance (69) is a continuous function if $\mu_{U_H^S}$ is a finite measure over $\mathbb{R}^d$. Thus, we can use any spatial stationary model to construct a spatio-temporal stationary solution to (67) maintaining its spatial behavior. As an example, the *waving Matérn model*, would correspond to $d\mu_{U_H^S}(\xi) = (2\pi)^{-\frac{d-1}{2}} a(\kappa^2 + |\xi|^2)^{-\alpha} d\xi$, with $a, \kappa > 0$ and $\alpha \in \mathbb{R}$. The associated covariance would then be

$$\rho(h, u) = \mathscr{F}_S\left(\frac{\cos(c|\xi||u|)}{(2\pi)^{\frac{d}{2}} a(\kappa^2 + |\xi|^2)^{\alpha}}\right)(h). \tag{70}$$

Let us now go back to the existence of stationary solutions of (66) with $X = W$, i.e.

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = W. \tag{71}$$

Since the function $(\xi, \omega) \mapsto (-\omega^2 + c^2|\xi|^2)^{-2}$ is not locally integrable, by applying Theorem 1 we conclude that *there are no stationary solutions to the stochastic wave equation (71)*. Hence, we cannot apply Theorem 2 to relate the covariance of a possible stationary solution of (66) to the covariance of the solution with White Noise source term. The existence of a stationary solution to (66) must be then studied for every particular case of X. Notice however that the existence is guaranteed when the spectral measure of the source term has a support which is strongly disjunct with the spatio-temporal cone $\mathcal{C}$, that is, there exists a neighborhood of the support that is disjoint with the cone.

6.2.3 Fractional Order Evolution Models

We close this Section with a class of operators defined through a symbol which can be interpreted as fractional differential operators in time. Let $\beta > 0$ a positive real number. We define

$$\frac{\partial^\beta}{\partial t^\beta} := \mathscr{F}_T^{-1}((i\omega)^\beta \mathscr{F}_T(\cdot)), \tag{72}$$

where we have used the symbol function over $\mathbb{R}$

$$\omega \mapsto (i\omega)^\beta := |\omega|^\beta e^{i\,\mathrm{sgn}(\omega)\frac{\beta\pi}{2}}. \tag{73}$$

The function (73) is continuous, Hermitian-symmetric and bounded by a polynomial for every $\beta > 0$, so it is indeed a symbol function. The operator (72) coincides with a classical differential operator for $\beta \in \mathbb{N}$. Similar definitions of a fractional differential operator can be found in Maniardi et al. (2001).

A *fractional order evolution model* is a spatio-temporal stationary solution of the SPDE (32) with $\beta \notin \mathbb{N}$. The associated spatio-temporal symbol function is

$$(\xi, \omega) \mapsto (i\omega)^\beta + g(\xi) = |\omega|^\beta \cos\left(\frac{\beta\pi}{2}\right) + g_R(\xi) + i\left(\mathrm{sgn}(\omega)|\omega|^\beta \sin\left(\frac{\beta\pi}{2}\right) + g_I(\xi)\right). \tag{74}$$

Since the null-set of (74) is

$$\mathcal{N}_\beta = \left\{(\xi, \omega) \in \mathbb{R}^d \times \mathbb{R} \mid g_R(\xi) = -|\omega|^\beta \cos\left(\frac{\beta\pi}{2}\right) \text{ and } g_I(\xi) = -\,\mathrm{sgn}(\omega)|\omega|^\beta \sin\left(\frac{\beta\pi}{2}\right)\right\},$$

a sufficient condition to have a unique stationary solution is that g_R satisfies SCEU with $(-1)^{\lfloor\frac{\beta+1}{2}\rfloor} g_R > 0$.

We suppose this holds and we consider the case where X is a spatio-temporal White Noise, yielding

$$\frac{\partial^\beta U}{\partial t^\beta} + \mathcal{L}_g U = W. \tag{75}$$

The spectral measure of the unique stationary solution of (75) is

$$d\mu_U^W(\xi,\omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{|\omega|^{2\beta} + 2|\omega|^\beta \left(g_R(\xi)\cos\left(\frac{\beta\pi}{2}\right) + \text{sgn}(\omega) g_I(\xi) \sin\left(\frac{\beta\pi}{2}\right)\right) + |g(\xi)|^2}. \tag{76}$$

This model is in general non-separable, and it can be non-symmetrical, depending on the function g_I. A symmetrical model is obtained when $g_I = 0$. If $\beta \leqslant \frac{1}{2}$, the measure (76) is not finite, in which case the solution is a GeRF. An important property of this model is that the temporal regularity of the Random Field can easily be controlled with the parameter β, thereby obtaining a large variety of models that are not necessarily symmetric, in a striking contrast with the Stein model presented in Section 6.1. For instance, when $\beta > \frac{1}{2}$, the solution of (75) is mean-square continuous in time. Notice that Theorem 2 cannot be applied since the function $(i\omega)^\beta$ is not smooth for a non-integer values of β.

When $\beta > \frac{1}{2}$, expressions of the spectral measure of a spatial trace of this model are possible but rather complicated. Example 6.7 details a particular case of g_R with $g_I = 0$. It can be generalized to other functions g_R following a similar approach.

Example 6.7. Fractional Evolution of Matérn Model. A *fractional evolution of Matérn model* is the (unique) stationary solution of the equation over $\mathbb{R}^d \times \mathbb{R}$

$$\frac{\partial^\beta U}{\partial t^\beta} + (-1)^{\lceil \frac{\beta}{2} \rceil + 1} a(\kappa^2 - \Delta)^{\frac{\alpha}{2}} U = W, \tag{77}$$

with $\beta \notin \mathbb{N}$, $a, \kappa > 0$ and $\alpha \in \mathbb{R}$. It is a fractional order evolution model with spatial symbol function $g(\xi) = (-1)^{\lceil \frac{\beta}{2} \rceil + 1} a(\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}}$. Its spectral measure is

$$d\mu_U(\xi,\omega) = \frac{1}{(2\pi)^{\frac{d+1}{2}}} \frac{d\xi d\omega}{|\omega|^{2\beta} + (-1)^{\lceil \frac{\beta}{2} \rceil + 1} 2a|\omega|^\beta (\kappa^2 + |\xi|^2)^{\frac{\alpha}{2}} \cos\left(\frac{\beta\pi}{2}\right) + a^2(\kappa^2 + |\xi|^2)^\alpha}. \tag{78}$$

The spatial structure of this model can be described when $\beta > \frac{1}{2}$ by computing the integral over the temporal frequency domain of the measure (78). The spectral measure of the spatial trace is then

$$d\mu_{U_S}(\xi) = \frac{\sigma_{a,\beta}^2 d\xi}{(2\pi)^{\frac{d}{2}} (\kappa^2 + |\xi|^2)^{\alpha - \frac{\alpha}{2\beta}}}, \tag{79}$$

with $\sigma_{a,\beta}^2 = \frac{-a^{\frac{1}{\beta}-2}\sin\left(\frac{\pi}{2}\beta\right)\sin\left(\frac{\pi}{\beta}\right)}{\beta\cos\left(\frac{\pi}{2}\beta+\frac{\pi}{2}\left(\lceil\frac{\beta}{2}\rceil+1\right)\right)}$. The spatial trace is therefore the solution of the SPDE over $\mathbb{R}^d$

$$\frac{1}{\sigma_{a,\beta}}(\kappa^2-\Delta)^{\frac{\alpha}{2}\left(1-\frac{1}{2\beta}\right)}U_S \overset{2nd\ o.}{=} W_S, \tag{80}$$

where W_S is a spatial White Noise. Direct identification in (80) indicates that the spatial covariance is thus a Matérn covariance with functional meaning when $\alpha\left(1-\frac{1}{2\beta}\right) > \frac{d}{2}$.

7 Conclusion

We have proposed a very general setting that allows to relate a SPDE to spatial and spatio-temporal covariance structures through the specification of symbol functions. It is grounded on Schwartz's theory of distribution Schwartz (1966), as already proposed in Itô (1954) and Matheron (1965). This setting offers a convenient framework to build and characterize models of random fields that are stationary solutions, when they exist, of a very large class of SPDEs. Their covariance structure is in direct relationship with the symbol function thanks to Theorem 1. In particular, this setting allows to handle relatively easily SPDEs with fractional behavior, in time, in space, and in both spatial and temporal dimensions. Thanks to this framework, we were able to construct very general models, that include and encompass existing models, as shown in details in Section 5 and Section 6.

Theorem 2 establishes that the covariance of the stationary solution of a given SPDE for general random source term with covariance ρ_X is the convolution between the covariance of the same SPDE with White Noise source term and ρ_X. This results is a powerful tool for easily characterizing solutions of very general SPDEs. It also emphasizes the central role played by White Noise source term.

We envision this work as a contribution strengthening the SPDE paradigm shift for analyzing spatial and spatio-temporal data as initiated in Lindgren et al. (2011). Our contribution offers the possibility to build and characterize models far beyond the Matérn family which is currently the covariance model considered within most SPDE implementations.

Efficient simulation of our models can be easily conceived using Fourier analysis based PDE-solvers as proposed in Lang & Potthoff (2011). Inference and simulation methods presented in Sigrist et al. (2015) can be easily adapted to any first order evolution models presented in Section 6.2.1. Since the linear operators considered in this work are not strictly speaking differential operators, methods inspired by the Finite Elements Method or by the Finite Difference Method are not applicable without specific adaptation. For instance, Bolin & Kirchner (2017) proposed adaptations of finite elements methods for Matérn Model with fractional regularity.

Appendix

A Reminders on tempered distributions

Here we give a brief overview of the main definitions and results regarding Schwartz's distribution theory in a tempered framework. For a more detailed presentation, the reader is referred to Donoghue (1969) and, of course, to Schwartz (1966). For a brief introduction with geostatistical purposes, we suggest Matheron (1965), appendix A.

Let $\mathcal{S}(\mathbb{R}^d)$ be the set of all complex, smooth and fast decreasing functions over $\mathbb{R}^d$,

$$\mathcal{S}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \text{ such that } \|x^\alpha D^\beta \varphi\|_\infty = \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta \varphi(x)| < \infty, \ \forall \alpha, \beta \in \mathbb{N}^d\},$$

where the multi-index notation for the power x^α and the differential operator D^β for $\alpha, \beta \in \mathbb{N}^d$ is used, meaning respectively $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}$, with $|\beta| = \beta_1 + ... + \beta_d$. Equipped with a particular topology, $\mathcal{S}(\mathbb{R}^d)$ is a complete metric space, known as the Schwartz space of *test functions*. Its dual space, i.e. the space of all continuous linear functionals from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{C}$, is called the space of *tempered distributions* and it is denoted $\mathcal{S}'(\mathbb{R}^d)$. In order to emphasize the dual aspect of tempered distributions and test functions, we will denote $\langle T, \varphi\rangle$ the action of $T \in \mathcal{S}'(\mathbb{R}^d)$ on $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Tempered distributions can be seen as a generalization of functions, on which Fourier transform and differentiation of any order can be properly defined. Polynomials, continuous and bounded functions or functions $f \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty]$ can be interpreted as tempered distributions through the integral $\langle f, \varphi\rangle := \int_{\mathbb{R}^d} f(x)\varphi(x)dx$ which is well defined for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Similarly, a finite measure μ over $\mathbb{R}^d$ can also define a tempered distribution through the integral $\langle \mu, \varphi\rangle := \int_{\mathbb{R}^d} \varphi(x)d\mu(x)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Tempered distributions can be differentiated any number of times. Let D^α be a differential operator with $\alpha \in \mathbb{N}^d$. Inspired by the integration by parts formula, the derivative of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is defined as a new tempered distribution $D^\alpha T \in \mathcal{S}'(\mathbb{R}^d)$ through $\langle D^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The Fourier transform and its inverse are defined for any test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\mathscr{F}(\varphi)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi^T x}\varphi(x)dx, \qquad \mathscr{F}^{-1}(\varphi)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi^T x}\varphi(x)dx. \tag{81}$$

For tempered distributions, the Fourier transform is defined as a new tempered distribution through the transfer formula

$$\langle \mathscr{F}(T), \varphi\rangle := \langle T, \mathscr{F}(\varphi)\rangle; \quad \langle \mathscr{F}^{-1}(T), \varphi\rangle := \langle T, \mathscr{F}^{-1}(\varphi)\rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d), \ T \in \mathcal{S}'(\mathbb{R}^d). \tag{82}$$

The Fourier transform is a bijective endomorphism over $\mathcal{S}(\mathbb{R}^d)$ and over $\mathcal{S}'(\mathbb{R}^d)$. The classical property of the Fourier transform with respect to the differentiation, $\mathscr{F}(D^\alpha T) = i^{|\alpha|}\xi^\alpha \mathscr{F}(T)$, where ξ denotes the variable in the frequency space, holds also for every tempered distribution T.

Let us also define the space $\mathcal{O}_M(\mathbb{R}^d)$ of complex smooth functions defined over $\mathbb{R}^d$ such that all of its derivatives are polynomially bounded. Explicitly,

$$\mathcal{O}_M(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \; : \; \forall \alpha \in \mathbb{N}^d \; \exists C > 0 \; \exists N \in \mathbb{N} \text{ such that } |D^\alpha f(x)| \leqslant C(1+|x|^2)^N \; \forall x \in \mathbb{R}^d\}.$$

This space is known as *the space of multiplicators* of the Schwartz space. If $f \in \mathcal{O}_M(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $f\varphi \in \mathcal{S}(\mathbb{R}^d)$. If $T \in \mathcal{S}'(\mathbb{R}^d)$, the multiplication $fT \in \mathcal{S}'(\mathbb{R}^d)$ is defined through $\langle fT, \varphi \rangle = \langle T, f\varphi \rangle$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. If $f \in \mathcal{O}_M(\mathbb{R}^d)$, then its Fourier transform $\mathscr{F}(f)$ is convolvable with any tempered distribution, and the exchange formula for the Fourier transform holds: $\mathscr{F}(fT) = (2\pi)^{-\frac{d}{2}}\mathscr{F}(f) * \mathscr{F}(T)$ for every $T \in \mathcal{S}'(\mathbb{R}^d)$. See Schwartz (1966), chapter VII, section 5 and Theorem XV in section 8.

B Proof of Proposition 2

The main difficulty of this Proposition lies in a proper definition of the multiplication gZ as a GeRF. Indeed, we could simply write $\langle gZ, \varphi \rangle := \langle Z, g\varphi \rangle$, but Z is only defined over functions in $\mathcal{S}(\mathbb{R}^d)$, and $g\varphi$ is not in general in $\mathcal{S}(\mathbb{R}^d)$. Nevertheless, we will show that we can define $\langle Z, f \rangle$ if Z is a slow-growing random measure and f is a continuous function with fast decreasing behavior .

We define $C_{FD}(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) \; | \; \|(1+|x|^2)^N f\|_\infty < \infty \; \forall N \in \mathbb{N}\}$, the space of all continuous functions with fast decreasing behavior, equipped with the following topology: a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset C_{FD}(\mathbb{R}^d)$ converges to $f \in C_{FD}(\mathbb{R}^d)$, denoted $f_n \overset{C_{FD}}{\to} f$, if for all $N \in \mathbb{N}$ we have that $\|(1+|x|^2)^N(f_n - f)\|_\infty \to 0$. For this topological vector space, the following two lemmas hold. They will be proven later.

Lemma B.1. *$\mathcal{S}(\mathbb{R}^d) \subset C_{FD}(\mathbb{R}^d)$, and it is a dense sub-space (with the topology of C_{FD}).*

Lemma B.2. *$\mathcal{M}_{SG}(\mathbb{R}^d) = C'_{FD}(\mathbb{R}^d)$, that is, every measure $\mu \in \mathcal{M}_{SG}(\mathbb{R}^d)$ defines a continuous linear functional T over $C_{FD}(\mathbb{R}^d)$ through the integral*

$$\langle T, f \rangle = \int_{\mathbb{R}^d} f(x) d\mu(x), \quad \forall f \in C_{FD}(\mathbb{R}^d). \tag{83}$$

Conversely, for every continuous linear functional $T : C_{FD}(\mathbb{R}^d) \to \mathbb{C}$ there exists a unique $\mu \in \mathcal{M}_{SG}(\mathbb{R}^d)$ such that (83) holds.

We now prove Proposition 2. If g is a continuous function bounded by a polynomial, then $g\varphi \in C_{FD}(\mathbb{R}^d)$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since, as stated in Lemma B.1, $\mathcal{S}(\mathbb{R}^d)$ is dense in $C_{FD}(\mathbb{R}^d)$, we can construct the random variable $\langle Z, g\varphi \rangle$ as a limit in a mean-square sense. Let $(g_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ be a sequence such that $g_n \overset{C_{FD}}{\to} g\varphi$. Consider the sequence of square-integrable (centred) random variables $(\langle Z, g_n \rangle)_{n\in\mathbb{N}}$. We obtain by linearity that

$$\mathbb{E}(|\langle Z, g_n \rangle - \langle Z, g_m \rangle|^2) = \mathbb{E}(|\langle Z, g_n - g_m \rangle|^2) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (g_n - g_m)(x)\overline{(g_n - g_m)}(y) dC_Z(x, y). \tag{84}$$

Since the sequence $(g_n)_n$ converges in $C_{FD}(\mathbb{R}^d)$, it is straightforward that the sequence $(g_n \otimes \overline{g_n})_n$ converges in $C_{FD}(\mathbb{R}^d \times \mathbb{R}^d)$. Since $C_Z \in \mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R}^d)$, by Lemma B.2 the integral in (84) goes to zero as n and m grow, due to continuity. The sequence $(\langle Z, g_n \rangle)_{n\in\mathbb{N}}$ is thus a Cauchy sequence in L^2. Hence, it is convergent, and we write

$$\langle gZ, \varphi \rangle := \langle Z, g\varphi \rangle := \lim_{n\to\infty} \langle Z, g_n \rangle, \tag{85}$$

where the limit is taken in the sense of L^2. The covariance structure of gZ can be easily obtained as a limit of covariances, obtaining for all $\varphi, \phi \in \mathcal{S}(\mathbb{R}^d)$

$$\mathbb{C}ov(\langle gZ, \varphi \rangle, \langle gZ, \phi \rangle) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x)\overline{\phi}(y)g(x)\overline{g}(y) dC_Z(x, y) = \langle (g \otimes \overline{g})C_Z, \varphi \otimes \overline{\phi} \rangle. \tag{86}$$

The result of Corollary 1, which describes the case of a slow-growing orthogonal random measure, follows from 86. Details are left to the reader. ■

Proof of Lemma B.1. It is clear that $\mathcal{S}(\mathbb{R}^d) \subset C_{FD}(\mathbb{R}^d)$. To prove the density, we first prove that if $f \in C_{FD}(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $f * \varphi \in \mathcal{S}(\mathbb{R}^d)$, where $*$ is the convolution product. It is clear that f is integrable and bounded, as well as φ which, in addition, is smooth. Thus $f * \varphi$ is a smooth integrable, continuous and bounded function, and its Fourier transform satisfies $\mathscr{F}(f * \varphi) = (2\pi)^{\frac{d}{2}} \mathscr{F}(f)\mathscr{F}(\varphi)$. We have that $\mathscr{F}(\varphi) \in \mathcal{S}(\mathbb{R}^d)$ since $\mathscr{F}$ is a bijective endomorphism of $\mathcal{S}(\mathbb{R}^d)$. Since $f \in C_{FD}(\mathbb{R}^d)$ we conclude by Riemann-Lebesgue lemma that $\mathscr{F}(f)$ is a smooth function with all derivatives vanishing at infinity. Thus $\mathscr{F}(f) \in \mathcal{O}_M(\mathbb{R}^d)$, which implies that $(2\pi)^{\frac{d}{2}} \mathscr{F}(f)\mathscr{F}(\varphi) \in \mathcal{S}(\mathbb{R}^d)$. This proves that $f * \varphi = \mathscr{F}^{-1}\left((2\pi)^{\frac{d}{2}} \mathscr{F}(f)\mathscr{F}(\varphi) \right) \in \mathcal{S}(\mathbb{R}^d)$.

Let $(\phi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ be a regularizing sequence of positive smooth functions with compact support, such that $supp(\phi_n) = \overline{B_{1/n}(0)}$ and $\int_{\overline{B_{1/n}(0)}} \phi_n(x) dx = 1$ for all $n \in \mathbb{N}$. Here $B_r(0) \subset \mathbb{R}^d$ denotes the open ball with center 0 and radius $r \geqslant 0$. We consider the sequence of functions $f_n = f * \phi_n$, which are all in $\mathcal{S}(\mathbb{R}^d)$. We will prove that $f_n \overset{C_{FD}}{\to} f$. Let $m \in \mathbb{N}$ be fixed. We must show that $\|(1 + |x|^2)^m (f_n - f)\|_\infty \to 0$ as $n \to \infty$. Let $\epsilon > 0$. As $f \in C_{FD}(\mathbb{R}^d)$, we can take $R > 0$ large enough such that for every x such that $|x| > R - 1$, $(1 + 2|x|^2)^m |f(x)| < \frac{\epsilon}{3(2^{m-1} + 2^{2m-1})}$ holds. Notice that in this case, $(1 + |x|^2)^m |f(x)| < \frac{\epsilon}{3}$. Since f is continuous, it is uniformly continuous over the compact set $\overline{B_{R+1}(0)}$. Thus, there exists $\delta > 0$

such that if $|x-y|<\delta$, then $|f(x)-f(y)|<\frac{\epsilon}{3(1+R^2)^m}$ for all $x,y\in\overline{B_{R+1}(0)}$. Consider $n_0\in\mathbb{N}$ such that $\frac{1}{n_0}<\delta$. Then, for all $n\geqslant n_0$,

$$\begin{aligned}\|(1+|x|^2)^m(f-f_n)\|_\infty = & \sup_{x\in\mathbb{R}^d}\left|\int_{B_{1/n}(0)}(1+|x|^2)^m(f(x)-f(x-y))\phi_n(y)dy\right| \\ \leqslant & \sup_{x\in\overline{B_R(0)}}\left|\int_{B_{1/n}(0)}(1+|x|^2)^m(f(x)-f(x-y))\phi_n(y)dy\right| \quad \text{(a)} \\ & +\sup_{x\in\overline{B_R(0)}^c}\left|\int_{B_{1/n}(0)}(1+|x|^2)^m(f(x)-f(x-y))\phi_n(y)dy\right|. \quad \text{(b)}\end{aligned} \tag{87}$$

For the first term (a) uniform continuity of f implies

$$\sup_{x\in\overline{B_R(0)}}\left|\int_{B_{1/n}(0)}(1+|x|^2)^m(f(x)-f(x-y))\phi_n(y)dy\right|\leqslant\int_{B_{1/n}(0)}(1+R^2)^m\frac{\epsilon}{3(1+R^2)^m}\phi_n(y)dy=\frac{\epsilon}{3}. \tag{88}$$

Regarding the second term (b), the integral is split to obtain

$$(b)\leqslant\sup_{x\in\overline{B_R(0)}^c}\{\underbrace{\int_{B_{1/n}(0)}(1+|x|^2)^m|f(x)|\phi_n(y)dy}_{\leqslant\frac{\epsilon}{3}}+\int_{B_{1/n}(0)}(1+|x|^2)^m|f(x-y)|\phi_n(y)dy\} \tag{89}$$

When applying Jensen's inequality twice, one shows that $(1+|x|^2)^m\leqslant 2^{m-1}[(1+2|x-y|^2)^m+2^m|y|^{2m}]$ for all x and y, and thus

$$\begin{aligned}\int_{B_{1/n}(0)}(1+|x|^2)^m|f(x-y)|\phi_n(y)dy\leqslant & \quad 2^{m-1}\Big[\int_{B_{1/n}(0)}\underbrace{(1+2|x-y|^2)^m|f(x-y)|}_{<\frac{\epsilon}{3(2^{m-1}+2^{2m-1})}\text{ from }|x-y|>R-1}\phi_n(y)dy \\ & +2^m\int_{B_{1/n}(0)}\underbrace{|f(x-y)|}_{<\frac{\epsilon}{3(2^{m-1}+2^{2m-1})}}\underbrace{|y|^{2m}}_{\leqslant 1}\phi_n(y)dy\Big] \\ < & \quad 2^{m-1}\left(\frac{\epsilon}{3(2^{m-1}+2^{2m-1})}+2^m\frac{\epsilon}{3(2^{m-1}+2^{2m-1})}\right)=\frac{\epsilon}{3}.\end{aligned} \tag{90}$$

Hence considering (89) and (90) we finally obtain $(b)<\frac{2\epsilon}{3}$. Putting together this result and (88) on equation

(87), we finally obtain that for all $n \geqslant n_0$,

$$\|(1+|x|^2)^m(f-f_n)\|_\infty < \epsilon, \tag{91}$$

hence $\|(1+|x|^2)^m(f-f_n)\|_\infty \to 0$. Since m was arbitrary, this result hods for all m. We therefore can conclude that $f_n \overset{C_{FD}}{\to} f$. Since for any arbitrary $f \in C_{FD}(\mathbb{R}^d)$ we can find a sequence contained in $\mathcal{S}(\mathbb{R}^d)$ which converges to f, we conclude that $\mathcal{S}(\mathbb{R}^d)$ is dense in $C_{FD}(\mathbb{R}^d)$. ■

Proof of Lemma B.2. Let $\mu \in \mathcal{M}_{SG}(\mathbb{R}^d)$. By definition, there exists $N \in \mathbb{N}$ such that $(1+|x|^2)^{-N}|\mu|$ is finite. Let $f \in C_{FD}(\mathbb{R}^d)$. We have that

$$\left|\int_{\mathbb{R}^d} f(x)d\mu(x)\right| \leqslant \int_{\mathbb{R}^d} |(1+|x|^2)^N f(x)|(1+|x|^2)^{-N} d|\mu|(x) \leqslant \|(1+|x|^2)^N f\|_\infty \left((1+|x|^2)^{-N}|\mu|\right)(\mathbb{R}^d) < \infty. \tag{92}$$

Thus, every $f \in C_{FD}(\mathbb{R}^d)$ is Lebesgue integrable with respect to μ, and the integral defines a linear functional on $C_{FD}(\mathbb{R}^d)$. If $f_n \overset{C_{FD}}{\to} 0$ then

$$|\langle \mu, f_n\rangle| \leqslant \left((1+|x|^2)^{-N}|\mu|\right)(\mathbb{R}^d)\|(1+|x|^2)^N f_n\|_\infty \to 0. \tag{93}$$

Hence, μ defines a continuous linear functional from $C_{FD}(\mathbb{R}^d)$ to $\mathbb{C}$.

In order to prove the converse, we first claim that if $T : C_{FD}(\mathbb{R}^d) \to \mathbb{C}$ is a linear functional, then it is continuous if and only if there exist $C > 0$ and $N_0 \in \mathbb{N}$ such that

$$|\langle T, f\rangle| \leqslant C\|(1+|x|^2)^{N_0} f\|_\infty \quad \forall f \in C_{FD}(\mathbb{R}^d). \tag{94}$$

The sufficiency of this claim is straightforward. Consider any sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \overset{C_{FD}}{\to} 0$. In particular this sequence satisfies $\|(1+|x|^2)^{N_0} f_n\|_\infty \to 0$. Thus $\langle T, f_n\rangle \to 0$, and T is continuous. Let us prove the necessity. Let us suppose that T is continuous but that (94) does not hold. Then, for all $C > 0$ and for all $N \in \mathbb{N}$ we can find a function $f_{C,N} \in C_{FD}(\mathbb{R}^d)$ such that $|\langle T, f_{C,N}\rangle| > C\|(1+|x|^2)^N f_{C,N}\|_\infty$. We consider $C = n^2$ and $N = n$ for all $n \in \mathbb{N}$. We obtain thus a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $C_{FD}(\mathbb{R}^d)$ such that

$$|\langle T, f_n\rangle| > n^2\|(1+|x|^2)^n f_n\|_\infty \quad \forall n \in \mathbb{N}. \tag{95}$$

Let us define the family of functions

$$\phi_n = \frac{f_n}{n\sum_{m\leqslant n}\|(1+|x|^2)^m f_n\|_\infty}, \quad n \in \mathbb{N}. \tag{96}$$

Clearly the sequence $(\phi_n)_{n\in\mathbb{N}}$ is in $C_{FD}(\mathbb{R}^d)$. Let $M \in \mathbb{N}$ arbitrary. By (96), we get that if $n \geqslant M$,

$$\|(1+|x|^2)^M \phi_n\|_\infty = \frac{\|(1+|x|^2)^M f_n\|_\infty}{n \sum_{m\leqslant n} \|(1+|x|^2)^m f_n\|_\infty} < \frac{1}{n}$$

and thus $\phi_n \overset{C_{FD}}{\to} 0$ as $n \to \infty$. On the other hand, by (95) we get

$$|\langle T, \phi_n\rangle| = \frac{1}{n} \frac{|\langle T, f_n\rangle|}{\sum_{m\leqslant n} \|(1+|x|^2)^m f_n\|_\infty} > \frac{n^2}{n} \frac{\|(1+|x|^2)^n f_n\|_\infty}{\sum_{m\leqslant n} \|(1+|x|^2)^m f_n\|_\infty} \geqslant 1$$

where we have used that $\sum_{m\leqslant n} \|(1+|x|^2)^m f_n\|_\infty \leqslant n\|(1+|x|^2)^n f_n\|_\infty$ as $\|(1+|x|^2)^m f_n\|_\infty \leqslant \|(1+|x|^2)^n f_n\|_\infty$ for all $m \leqslant n$. We conclude that $|\langle T, \phi_n\rangle|$ does not converge to 0 as n grows, and thus T is not sequentially continuous, which is a contradiction. Hence, our claim holds.

Let us now prove the second part of Lemma (B.2).

Let $T \in C'_{FD}(\mathbb{R}^d)$. There exists $C > 0$ and $N \in \mathbb{N}$ such that (94) holds. Let us define the linear functional $(1+|x|^2)^{-N}T : C_{FD}(\mathbb{R}^d) \to \mathbb{C}$ with

$$\langle (1+|x|^2)^{-N}T, f\rangle := \langle T, (1+|x|^2)^{-N} f\rangle. \tag{97}$$

Since for all $f \in C_{FD}(\mathbb{R}^d)$, $(1+|x|^2)^{-N} f$ is also in $C_{FD}(\mathbb{R}^d)$ this functional is well defined. Considering that $(1+|x|^2)^{-N}|f| \leqslant |f|$, it is easy to see that it is continuous. Using (94) we get

$$|\langle (1+|x|^2)^{-N}T, f\rangle| \leqslant C\|(1+|x|^2)^N(1+|x|^2)^{-N} f\|_\infty = C\|f\|_\infty, \tag{98}$$

for all $f \in C_{FD}(\mathbb{R}^d)$. In particular, (98) holds for all $f \in C_c(\mathbb{R}^d)$, the space of compactly supported continuous complex functions over $\mathbb{R}^d$. Consider $C_0(\mathbb{R}^d)$, the space of continuous complex functions defined over $\mathbb{R}^d$ vanishing at infinity, which is a Banach space with the supreme norm. As $C_c(\mathbb{R}^d)$ is a dense subspace of $C_0(\mathbb{R}^d)$, then by extension of bounded linear functionals, we obtain that $(1+|x|^2)^{-N}T$ is a bounded linear functional over $C_0(\mathbb{R}^d)$, for which (98) holds for every $f \in C_0(\mathbb{R}^d)$. By Riesz's Representation Theorem (see Rudin (1987), chapter 6) we conclude that $(1+|x|^2)^{-N}T$ is identified with a unique finite measure ν over $\mathbb{R}^d$. By defining $\mu = (1+|x|^2)^N\nu$, we obtain that μ is a well defined Radon measure which is in $\mathcal{M}_{SG}(\mathbb{R}^d)$, and considering that $(1+|x|^2)^N(1+|x|^2)^{-N}T = T$, it is straightforward that $\langle T, f\rangle = \langle \mu, f\rangle$ for every $f \in C_{FD}(\mathbb{R}^d)$. ■

C Proofs regarding the Stochastic Heat Equation (Example 6.4)

C.1 Existence of stationary solutions

According to Theorem 1, there exists a stationary solution to the Stochastic Heat Equation with White Noise source term (49) if and only if the spatio-temporal measure $(\omega^2 + a^2|\xi|^4)^{-1}d\xi d\omega$ is in $\mathcal{M}_{SG}(\mathbb{R}^d \times \mathbb{R})$, i.e. is it is a slow-growing measure. This is the case if the function $(\xi, \omega) \mapsto (\omega^2 + a^2|\xi|^4)^{-1}$ is locally integrable, in which case the slow-growing behavior is provided by the fact that this function is bounded outside a neighborhood around the origin. It suffices thus to study the integrability over subsets of $\mathbb{R}^d \times \mathbb{R}$ of the form $B_R^{(d)}(0) \times [-M, M]$ for $R, M > 0$, where $B_R^{(d)}(0) \subset \mathbb{R}^d$ is the ball of radius R centered in 0. Using integration with polar coordinates in the spatial domain and the symmetry in the time dimension, we obtain

$$\int_{B_R^{(d)}(0)\times[-M,M]} \frac{1}{\omega^2 + a^2|\xi|^4} d(\xi, \omega) = C \int_0^R \arctan\left(\frac{M}{ar^2}\right) r^{d-3} dr \tag{99}$$

for some positive constant C. As the function $r \mapsto \arctan\left(\frac{M}{ar^2}\right)$ is both inferiorly and superiorly bounded by a constant when $r \in [0, R]$, we conclude that the integral (99) is finite only for $d > 2$, from which we get that there exists stationary solutions to the equation (49) only for spatial dimensions $d \geqslant 3$. In these cases, solutions would have a functional meaning if the measure $(\omega^2 + a^2|\xi|^4)^{-1}d\xi d\omega$ was finite, which would hold if the limit when M and R go to ∞ would exist and was finite. However, by seeing that $\int_0^R \arctan\left(\frac{M}{ar^2}\right) r^{d-3} dr \geqslant \arctan\left(\frac{M}{aR^2}\right) \frac{R^{d-2}}{d-2}$, and by letting $M \to \infty$ first and $R \to \infty$ second, one gets that the limit is not finite. Hence, the stationary solutions to (49) only have a meaning as GeRFs and not as Random Functions.

C.2 Covariance structure

The covariance structure (50) is the Fourier transform of the spatio-temporal spectral measure $d\mu_U(\xi, \omega) = (2\pi)^{-\frac{d+1}{2}}(\omega^2 + a^2|\xi|^4)^{-1}d\xi d\omega$ for $d = 3$. This measure is not finite. The Fourier transform $\rho_U = \mathscr{F}(\mu_U)$ is obtained as the limit, in a distributional sense, of continuous functions. Let $R > 0$ and let denote μ_U^R the restriction of the measure μ_U to $B_R(0) \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$, i.e. $d\mu_U^R(\xi, \omega) = (2\pi)^{-\frac{d+1}{2}}(\omega^2 + a^2|\xi|^4)^{-1}\mathbf{1}_{B_R(0)}(\xi) d\xi d\omega$. This measure is finite, so $\rho_U^R = \mathscr{F}(\mu_U^R)$ is a continuous positive-definite function. By the dominated convergence theorem, one gets that for every $\varphi \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$, $\langle \mu_U^R, \varphi \rangle \to \langle \mu_U, \varphi \rangle$ as $R \to \infty$. Thus, by continuity of the Fourier transform, we have $\rho_R^U \to \rho_U$, in distributional sense. Let us calculate $\rho_U^R(h,u)$ for $(h,u) \in \mathbb{R}^3 \times \mathbb{R}$.

$$\begin{aligned}
\rho_U^R(h,u) &= \frac{1}{(2\pi)^4} \int_{\mathbb{B}_R(0)} \int_{\mathbb{R}} \frac{e^{-iu\omega - ih^T\xi}}{\omega^2 + a^2|\xi|^4} d\omega d\xi \\
&= \frac{1}{(2\pi)^3} \frac{1}{2a} \int_{B_R(0)} e^{-ih^T\xi} \frac{e^{-a|\xi|^2|u|}}{a|\xi|^2} d\xi \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{2a} \sqrt{\frac{2}{\pi}} \int_0^R \frac{J_{\frac{1}{2}}(|h|r)}{\sqrt{r|h|}} e^{-a|u|r^2} dr \\
&= \frac{1}{(2\pi)^2} \frac{1}{a|h|} \int_0^R \frac{\sin(|h|r)}{r} e^{-a|u|r^2} dr. \qquad (100)
\end{aligned}$$

Let us evaluate $\lim_{R\to\infty} \rho_R(h,u)$ for $|h| \neq 0 \neq |u|$. Consider the function $f_R : \mathbb{R}^+ \to \mathbb{R}$ defined by $f_R(\lambda) = \int_0^R \frac{\sin(\lambda r)}{r} e^{-a|u|r^2} dr$ for $\lambda \geqslant 0$, with $f_R(0) = 0$. By the dominated convergence theorem, we have $f_R'(\lambda) = \int_0^R \cos(\lambda r) e^{-a|u|r^2} dr$. Using the expressions of the Fourier transform of a Gaussian function, one proves that $\lim_{R\to\infty} f_R'(\lambda) = \sqrt{\frac{\pi}{4a|u|}} e^{-\frac{\lambda^2}{4a|u|}}$. Using $f_R(\lambda) = \int_0^\lambda f_R'(s) ds$ and the dominated convergence theorem, we get

$$\lim_{R\to\infty} f_R(\lambda) = \int_0^\lambda \sqrt{\frac{\pi}{4a|u|}} e^{-\frac{s^2}{4a|u|}} ds = \frac{\pi}{2} \operatorname{erf}\left(\frac{\lambda}{2\sqrt{a|u|}}\right). \qquad (101)$$

Using this result in (100) with $\lambda = |h|$ and $R \to \infty$, we finally obtain the distribution *associated to the function*

$$\rho_U(h,u) = \frac{1}{(2\pi)^2} \frac{\pi}{2a|h|} \operatorname{erf}\left(\frac{|h|}{2\sqrt{a|u|}}\right). \qquad (102)$$

which is the expression in (50).

It is worth emphasizing that this expression is only valid in a distributional sense. The distribution ρ_U is only meaningful when applied to test functions, satisfying $\langle \rho_U, \psi \rangle = \lim_{R\to\infty} \langle \rho_U^R, \psi \rangle$ for all $\psi \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$. The expression *associated to the function (102)* refers to the fact that for every test function ψ such that its support does not contains the origin, we have $\langle \rho_U, \psi \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}} \frac{1}{(2\pi)^2} \frac{\pi}{2a|h|} \operatorname{erf}\left(\frac{|h|}{2\sqrt{a|u|}}\right) \psi(h,u) dh du$.

D Proofs of results in Example 6.7

We show the steps to calculate the spectral measure of the spatial trace of the unique stationary solution to (77) for $\beta > \frac{1}{2}$. As $\rho_U(h,u) = \mathscr{F}(\mu_U)$, we have explicitly (when we have a functional meaning)

$$\rho_{U_S}(h) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ih^T\xi}\frac{1}{(2\pi)^{\frac{d}{2}+1}}\int_{\mathbb{R}}\frac{d\omega}{|\omega|^{2\beta}+(-1)^{\lceil\frac{\beta}{2}\rceil+1}2a|\omega|^{\beta}(\kappa^2+|\xi|^2)^{\frac{\alpha}{2}}\cos\left(\frac{\beta\pi}{2}\right)+a^2(\kappa^2+|\xi|^2)^{\alpha}}d\xi \tag{103}$$

The difficulty is then to compute the temporal integral with respect to ω. Using the parity of the involved function and the change of variable $\omega = \left(a(\kappa^2+|\xi|^2)^{\frac{\alpha}{2}}\theta\right)^{\frac{1}{\beta}}$ one obtains the expression

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ih^T\xi}\frac{2\left(a(\kappa^2+|\xi|^2)^{\frac{\alpha}{2}}\right)^{\frac{1}{\beta}-2}}{(2\pi)^{\frac{d}{2}+1}\beta}\int_0^{\infty}\frac{\theta^{\frac{1}{\beta}-1}}{\theta^2+2\theta(-1)^{\lceil\frac{\beta}{2}\rceil+1}\cos\left(\frac{\pi}{2}\beta\right)+1}d\theta\, d\xi. \tag{104}$$

The integral with respect to θ is computable following Gradshteyn & Ryzhik (1994), 3.252.12. After a few adaptations for the expression "$(-1)^{\lceil\frac{\beta}{2}\rceil+1}\cos\left(\frac{\pi}{2}\beta\right)$" depending in the value of β, and by using basic properties of the trigonometric functions one proves that

$$\int_0^{\infty}\frac{\theta^{\frac{1}{\beta}-1}}{\theta^2+2\theta(-1)^{\lceil\frac{\beta}{2}\rceil+1}\cos\left(\frac{\pi}{2}\beta\right)+1}d\theta = \frac{-\pi\sin\left(\frac{\pi}{2}\beta\right)\sin\left(\frac{\pi}{\beta}\right)}{\cos\left(\frac{\pi}{2}\beta+\frac{\pi}{2}\left(\lceil\frac{\beta}{2}\rceil+1\right)\right)}.$$

Replacing this in 104, one obtains the expression (79) as a spectral measure of the spatial trace. This procedure also works without a functional meaning, as all we need to do is to calculate the density of the (not necessarily finite) spatial measure in (103).

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