# Probability Weighting in Recursive Evaluation of Two-Stage Prospects

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# Outline



**1/ Introductory remarks**

**2/ Conceptual framework**

**3/ Experimental study**

**4/ Concluding remarks**

# Why investigating the evaluation of Two-stage prospects?

# Two-stage prospect and reduction of compound lotteries



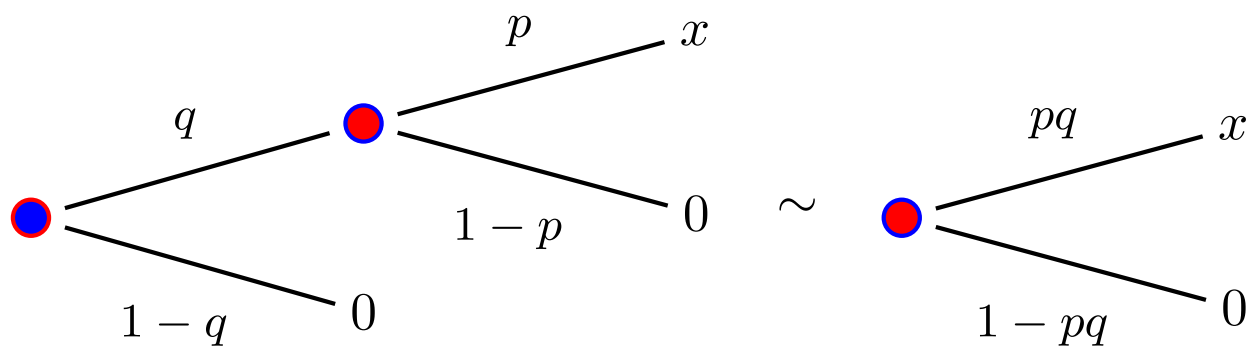



Table 1: compound risk and its reduced one-stage lottery

- ☐ Rational Decision makers (DMs) reduce compound risks, represented by compound lotteries, into single stage lotteries by using the Reduction of compound prospects axiom (RCP).

⇒ Rational DMs should exhibit a perfect neutrality toward compound risk.

However...

# Three observations on compound risk



1. Reduction of compound prospects have been descriptively challenged in many empirical investigations:
   - Bar Hillel (1973), Bernasconi & Loomes (1992), Budescu & Fisher (2001), Abdellaoui, Klibanoff & Placido (2014), Nebout & Dubois (2014)...

2. Following Becker & Brownson (1964) and Yates and Zukowski (1976), Segal (1987, 1990) represented ambiguous bets as two-stage prospects.

3. Prospect Theory (PT) is the most succesful descriptive model of decision making under risk and ambiguity.

$\Rightarrow$ Would it still be the case when dealing with attitudes toward two-stage prospects?

# Theoretical and Empirical Background

# Background: One-stage prospects



- $(x, p; y)$ denotes the one-stage *prospect* resulting in outcome $x$ with probability $p$ and in outcome $y$ with probability $1 - p$ with $x \geq y \geq 0$.

  – Probability $p$ is generated using a known urn containing 100 balls numbered from 1 to 100, i.e. drawing a ball which has a number between 1 and $p \times 100$.

- $(x, E_p; y)$ denotes the corresponding ambiguous prospect. The probability $P(E_p)$ is unknown to the DM.

  – We use an unknown urn containing 100 balls numbered from 1 to 100 in unknown proportions, i.e. drawing a ball which has a number between 1 and $p \times 100$. Symmetry arguments imply $P(E_p) = p$. (Chew & Sagi, 2006, 2008)

# Background: Evaluation of one-stage prospects



- Under Expected Utility (EU), prospects are evaluated as follows:

$$EU(x, p; y) = pu(x) + (1 - p)u(y)$$

  - Where $u$ is the utility function (and a risk attitude index).
  - Violations of EU popularized by Kahneman & Tversky.

- Under Prospect Theory (PT), prospects are evaluated as follows in the gain domain:

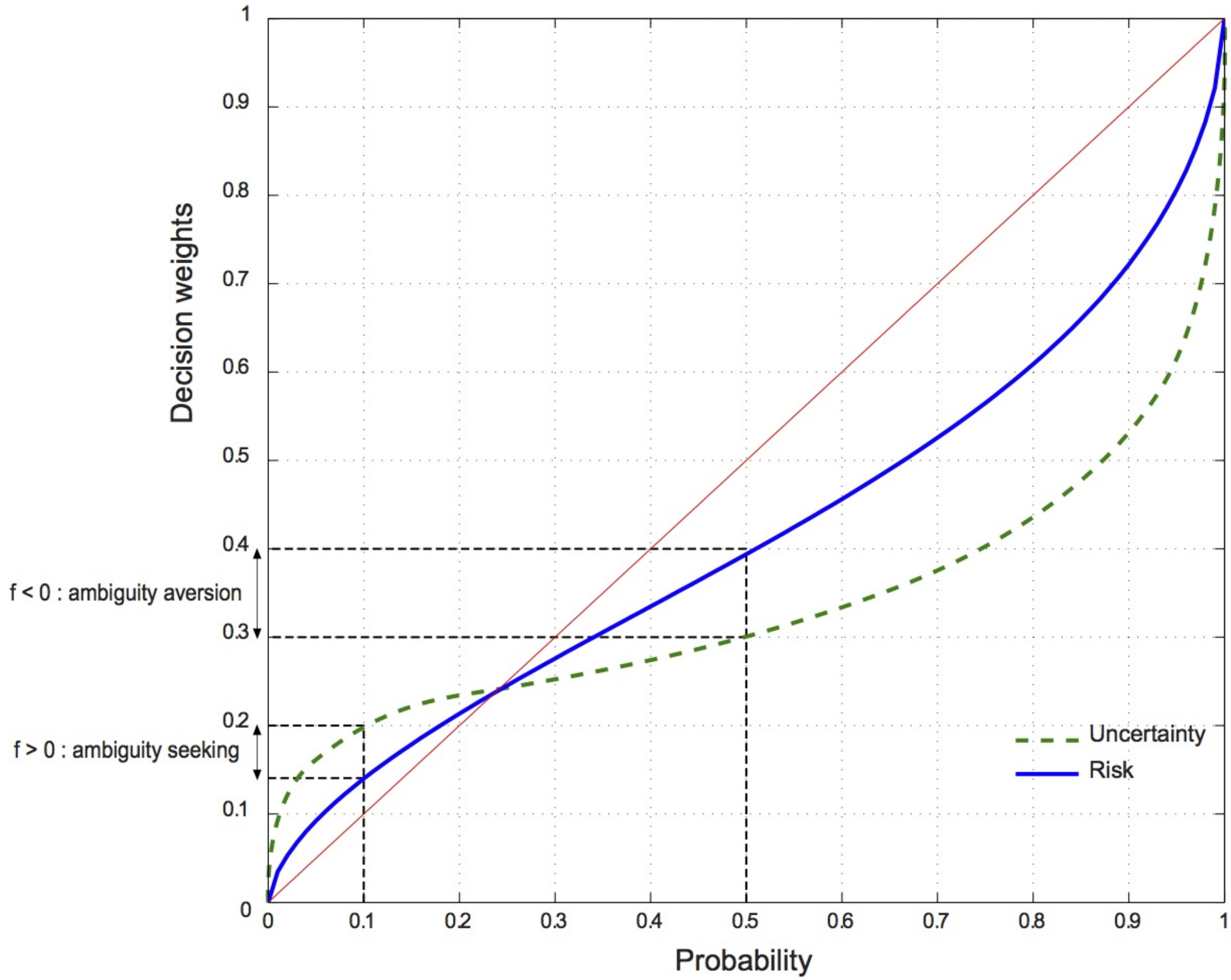
$$PT(x, p; y) = w(p)u(x) + (1 - w(p))u(y)$$

  - $u$ is the utility function.
  - $w$ is the probability weighting function. $w$ is strictly increasing and satisfies $w(0) = 0$ and $w(1) = 1$.

⇒ Many experimental evidence on RDU under risk and ambiguity.

# Probability weighting under risk and ambiguity

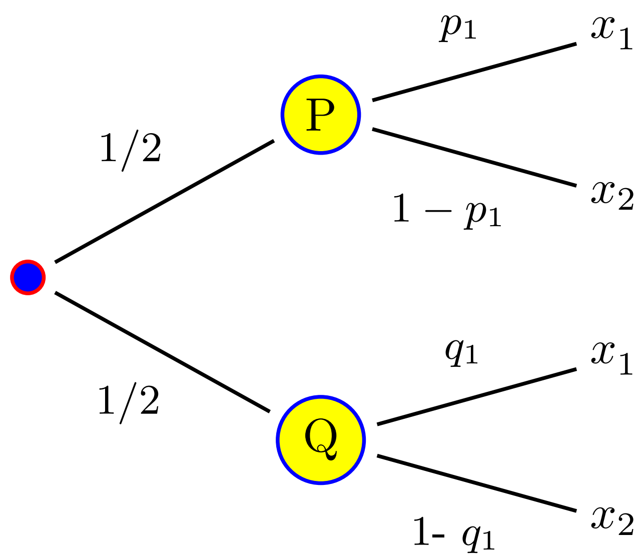





⇒ Ambiguity increases **likelihood insensitivity.**

# How to evaluate two-stage prospects?



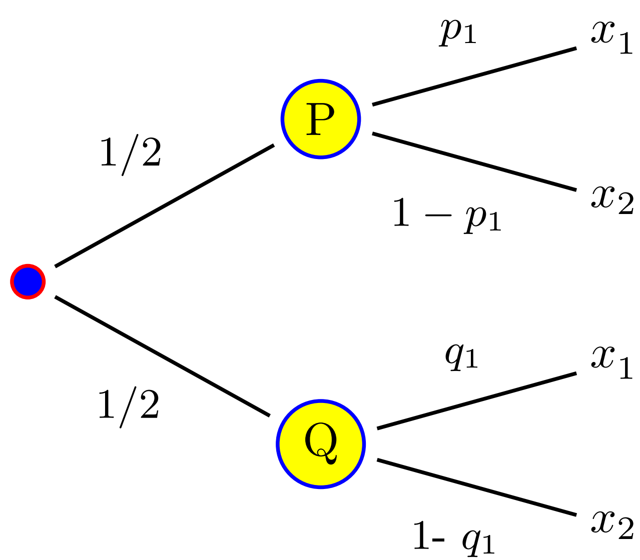



- Traditional Recursive Expected Utility (TREU):

$$\Rightarrow \frac{1}{2} \times EU(P) + \frac{1}{2} \times EU(Q)$$

# How to evaluate two-stage prospects?






- Recursive Expected Utility without RCP (REU):

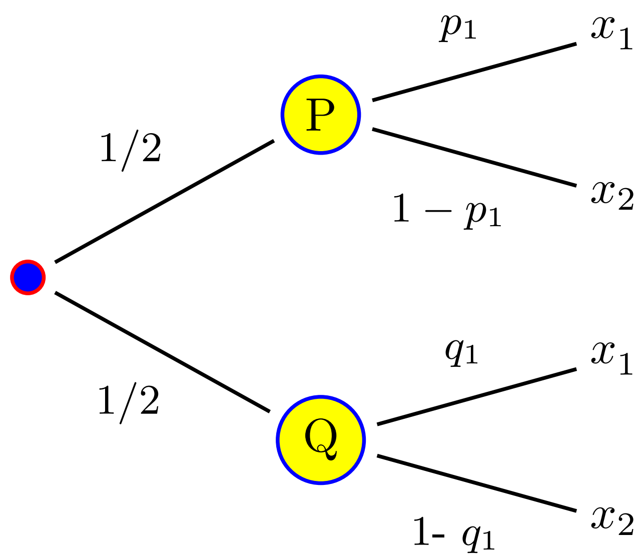
$$\Rightarrow \frac{1}{2} \times \phi\left[EU(P)\right] + \frac{1}{2} \times \phi\left[EU(Q)\right]$$

  – Kreps & Porteus (1978) introduced this transformed EU functionnal to account for delayed resolution of uncertainty.

  – Klibanof & al. (2005) used the same preference functional to model ambiguity. (Seo, 2009, Ergin & Gul (2008),...)

# How to evaluate two-stage prospects?






- Recursive Prospect Theory (RPT):

$$\Rightarrow \pi_1 \times \phi\left[PT(P)\right] + \pi_2 \times \phi\left[PT(Q)\right]$$

  - Segal (1987) suggested this recursive form of RDU to model ambiguity attitudes through second-order probabilities.
  - Abdellaoui & Zank (2014) provide the first axiomatization of this general form of Prospect Theory.

# Experimental Design

62 subjects:

- Parts 1 & 2: certainty equivalents for known and unknown Ellsberg urns ($2 \times 13$ equivalents).
- Parts 3 & 4: matching probabilities for two-stage prospects ($2 \times 10$ equivalents).
- Payment: show up 10 euros + RIS (max 50 euros).
- Individual interviews (about 45 minutes).

Under RPT, we aim to elicit the following functionals:

| | Elicitation 1: Risk | Elicitation 2: Ambiguity |
|---|---|---|
| Attitudes towards 1-stage prospects | $u$<br>$w$ | $\tilde{u}$<br>$\tilde{w}$ |
| Attitudes towards 2-stage prospects | $w^*$ | $\tilde{w}^*$ |

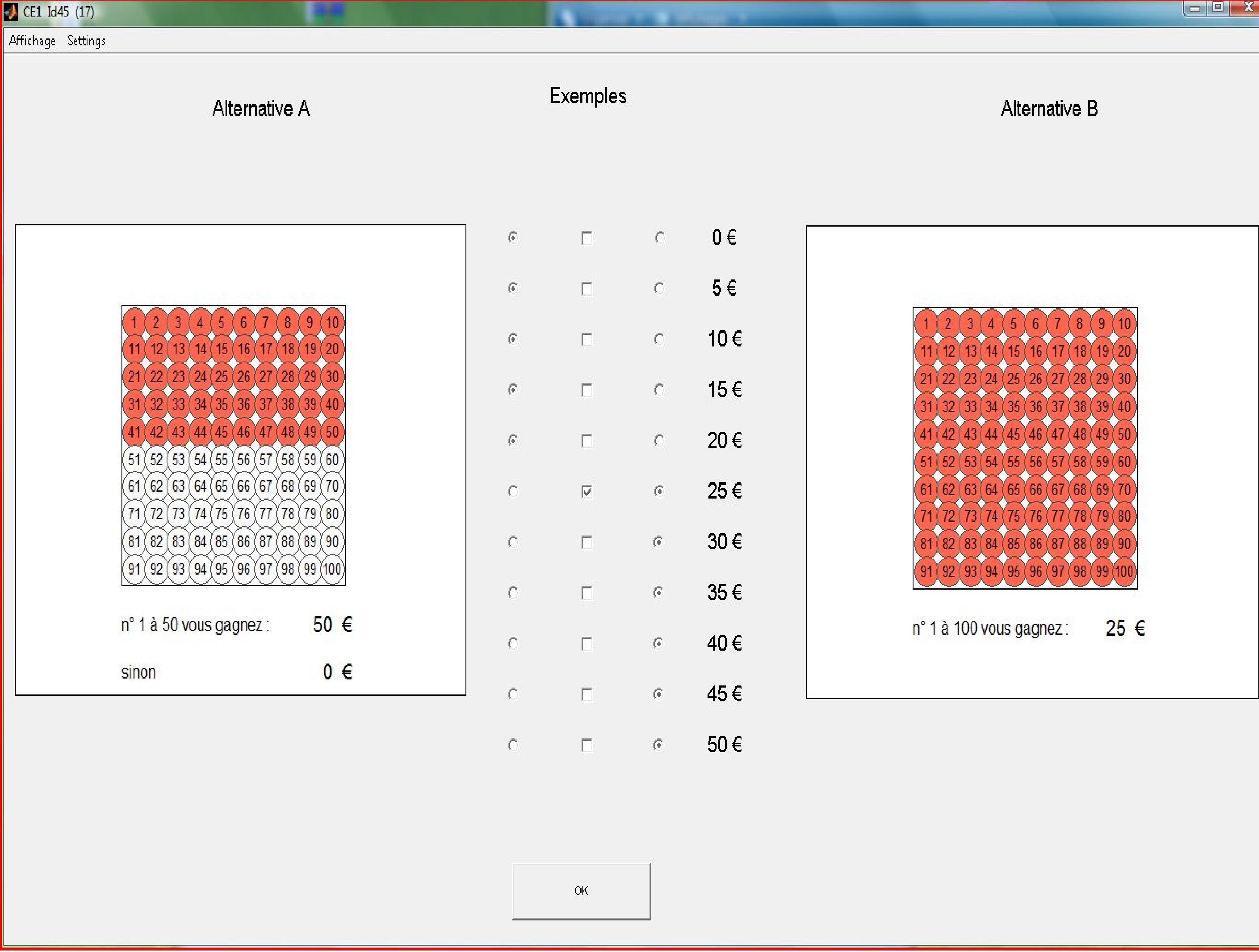
# Certainty equivalents for one-stage prospects

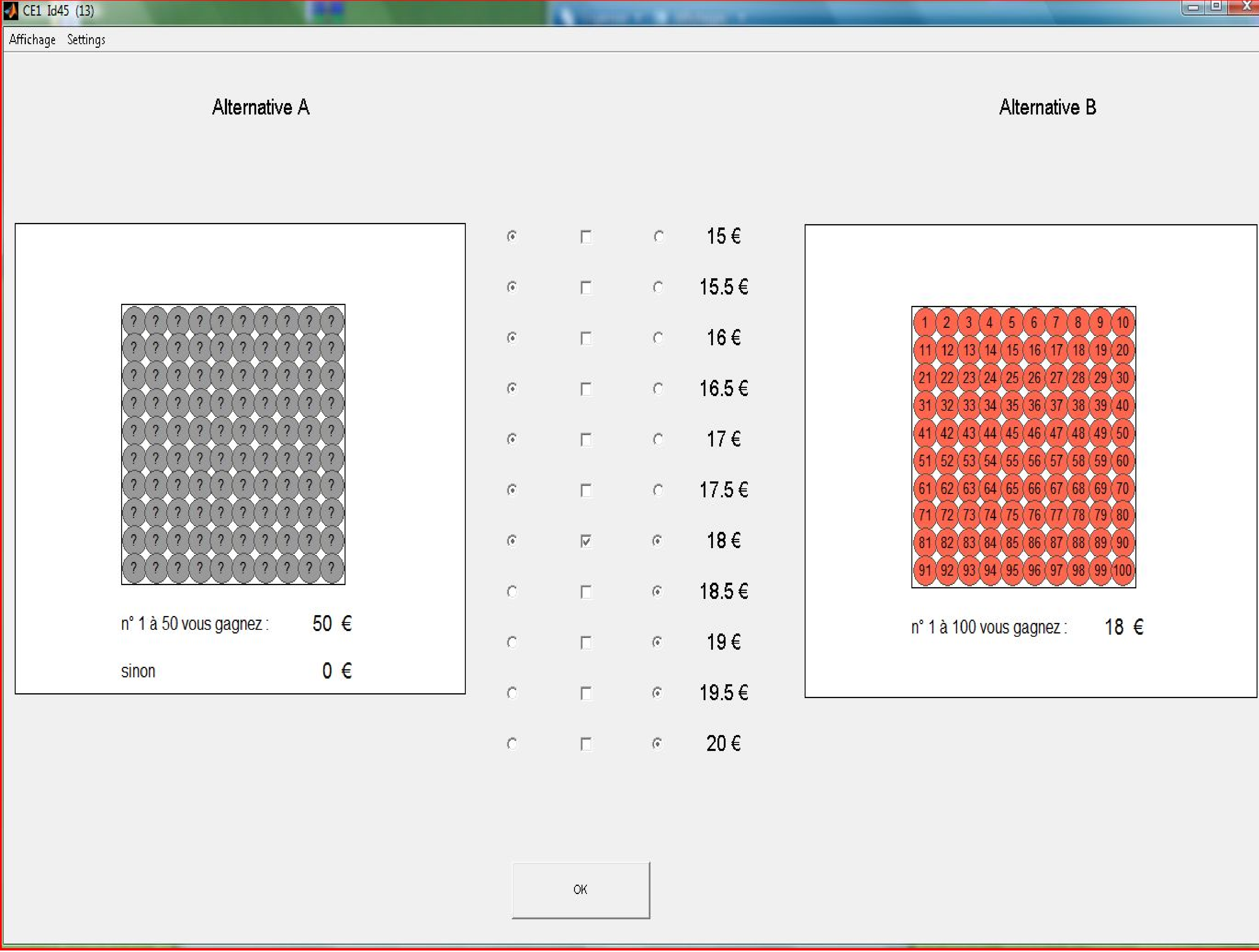


| | $c \sim (x, p; y)$ and $\tilde{c} \sim (x, E_p; y)$ | | | | | | $c_p \sim (x, p; y)$ and $\tilde{c}_p \sim (x, E_p; y)$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 50 | 40 | 50 | 50 | 25 | 10 | 50 | 50 | 50 | 50 | 50 | 50 | 50 |
| $y$ | 25 | 20 | 10 | 35 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $p$ | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.02 | 0.06 | 0.17 | 0.33 | 0.50 | 0.67 | 0.94 |
| $E_p$ | $E_{30}$ | $E_{30}$ | $E_{30}$ | $E_{30}$ | $E_{30}$ | $E_{30}$ | $E_2$ | $E_6$ | $E_{17}$ | $E_{33}$ | $E_{50}$ | $E_{67}$ | $E_{94}$ |

- Under risk, we elicit $c$ and $c_p$ following Abdellaoui & al. (2008).
- Under ambiguity, we elicit $\tilde{c}$ and $\tilde{c}_p$ following Abdellaoui & al. (2011).

⇒ We can compare our results to these benchmark studies.


CE1 Id45 (17)
Affichage Settings
Alternative A
Exemples
Alternative B
0 €
5 €
10 €
15 €
20 €
25 €
30 €
35 €
40 €
45 €
50 €
n° 1 à 50 vous gagnez : 50 €
sinon 0 €
n° 1 à 100 vous gagnez : 25 €
OK


CE1 Id45 (13)
Affichage Settings
Alternative A
Alternative B
15 €
15.5 €
16 €
16.5 €
17 €
17.5 €
18 €
18.5 €
19 €
19.5 €
20 €
n° 1 à 50 vous gagnez : 50 €
sinon 0 €
n° 1 à 100 vous gagnez : 18 €
OK

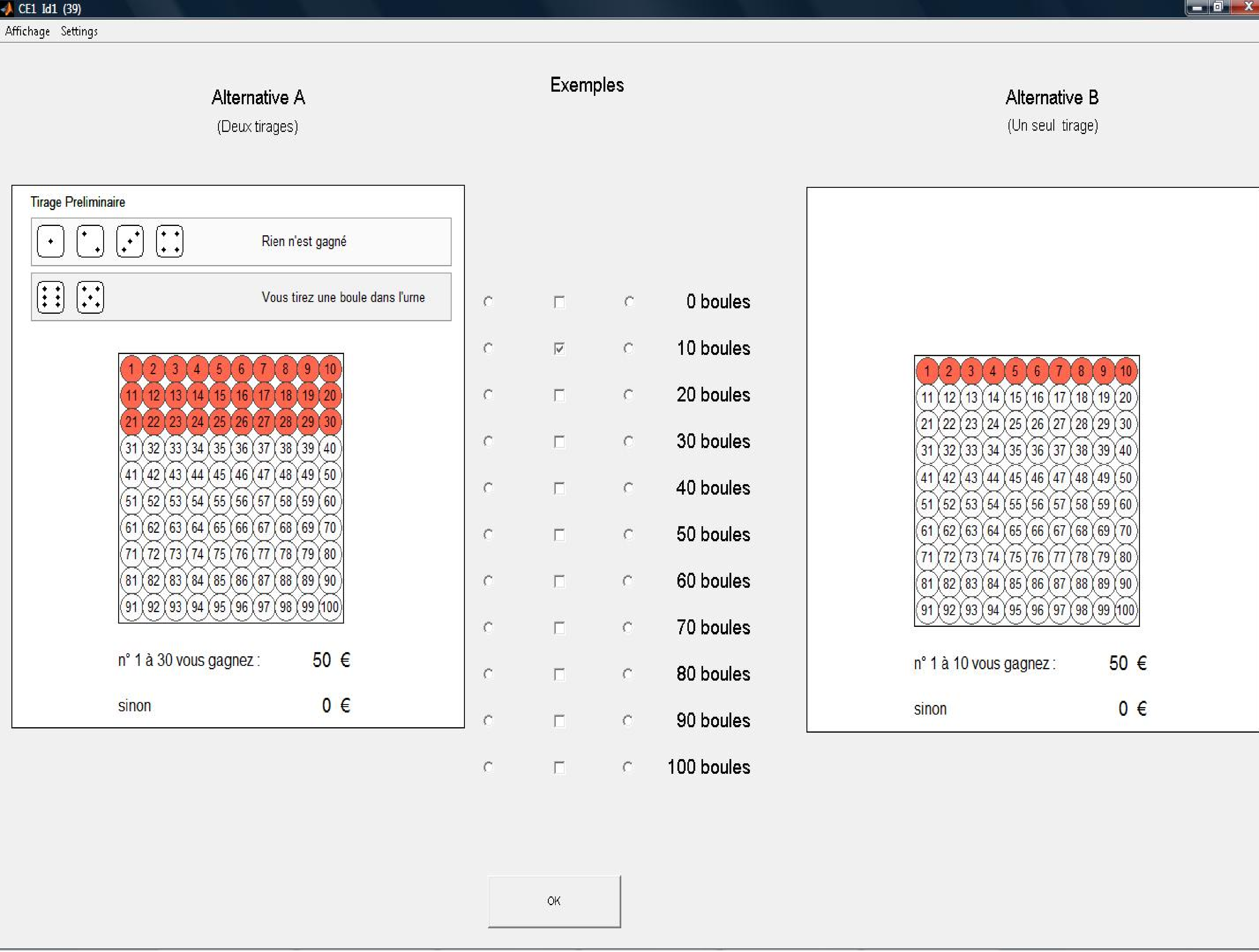
# Matching probabilities for two-stage prospects

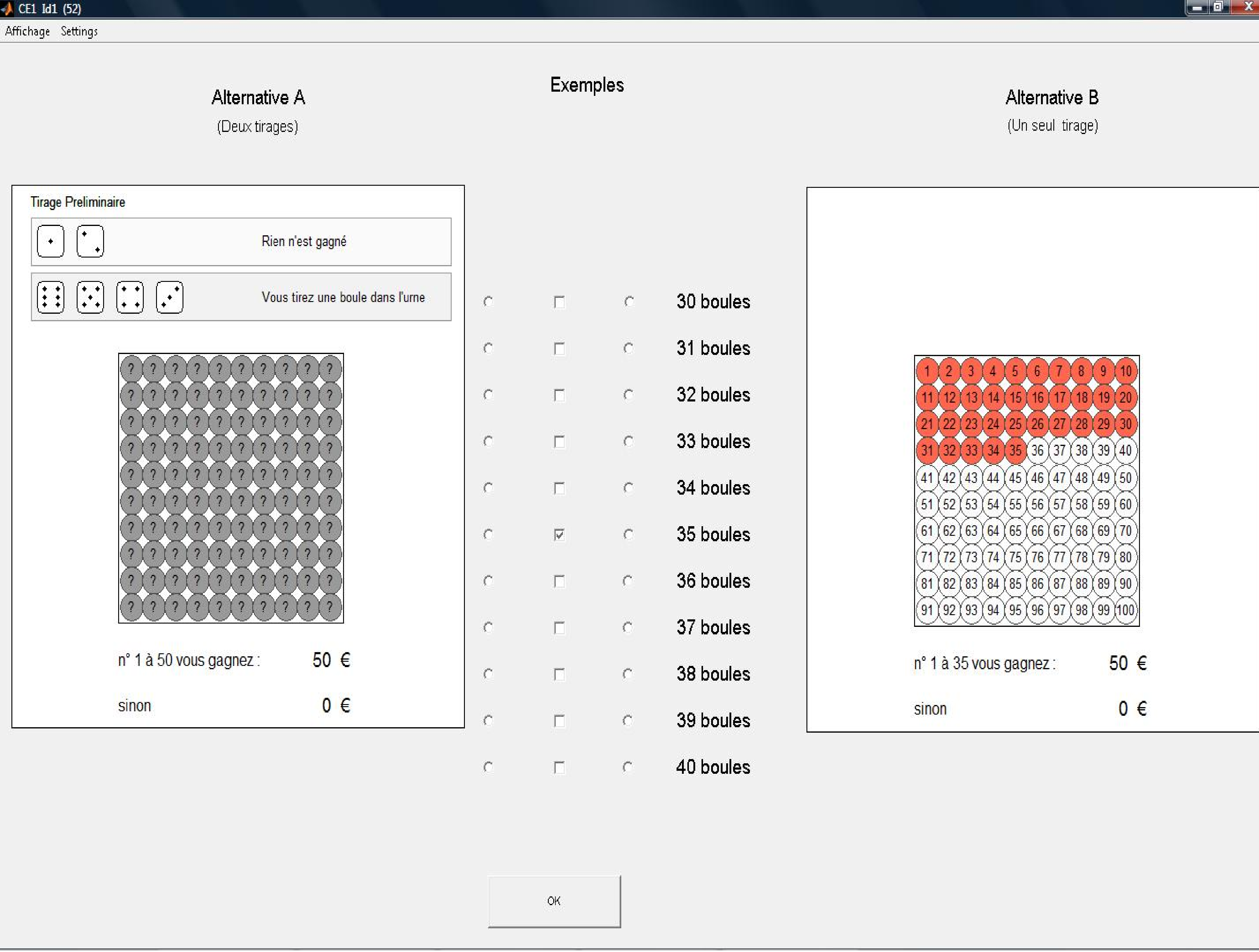


| $((\bar{x}, p), q) \sim (\bar{x}, m)$ and $((\bar{x}, E_p), q) \sim (\bar{x}, \tilde{m})$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | 1/3 | 1/3 | 1/3 | 1/3 | 1/3 | 2/3 | 2/3 | 2/3 | 2/3 | 2/3 |
| $p$ | 0.06 | 0.17 | 0.33 | 0.50 | 0.94 | 0.06 | 0.17 | 0.50 | 0.75 | 0.94 |
| $E_p$ | $E_6$ | $E_{17}$ | $E_{33}$ | $E_{50}$ | $E_{94}$ | $E_6$ | $E_{17}$ | $E_{50}$ | $E_{75}$ | $E_{94}$ |

- $\bar{x} = 50$.
- 10 matching probabilities for risky second stage stage, $r$ and 10 for ambiguous second stage $\tilde{r}$.
- 2 first stage probability levels: 1/3 and 2/3.

$\Rightarrow$ Elicitation, comparison and test of 4 second stage probability weighting functions.


CE1 Id1 (39)
Affichage Settings
Exemples
Alternative A
(Deux tirages)
Alternative B
(Un seul tirage)
Tirage Preliminaire
Rien n'est gagné
Vous tirez une boule dans l'urne
0 boules
10 boules
20 boules
30 boules
40 boules
50 boules
60 boules
70 boules
80 boules
90 boules
100 boules
n° 1 à 30 vous gagnez : 50 €
sinon 0 €
n° 1 à 10 vous gagnez : 50 €
sinon 0 €
OK


CE1 Id1 (52)
Affichage Settings
Exemples
Alternative A
(Deux tirages)
Tirage Preliminaire
Rien n'est gagné
Vous tirez une boule dans l'urne
n° 1 à 50 vous gagnez : 50 €
sinon 0 €
30 boules
31 boules
32 boules
33 boules
34 boules
35 boules
36 boules
37 boules
38 boules
39 boules
40 boules
Alternative B
(Un seul tirage)
n° 1 à 35 vous gagnez : 50 €
sinon 0 €
OK

# Model Specification

# Model specification: TREU and REU



Using the equivalence revealed by the elicitation of the matching probability $r$:

$$((\bar{x}, p), q) \sim (\bar{x}, m)$$

we infer the following equalities:

1. Under TREU, we have:

$$q \times p = m$$

2. Under REU, we have:

$$q \times \phi(p) = \phi(m)$$

Where $\phi$ is a transformation function.

Parametric specification: $\phi(x) = x^{1/\theta}$.

# Model specification RPT



- Under RPT, we have:

| Setup | RPT-$r$<br>RPT for "*risk-risk*" | RPT-$a$<br>RPT for "*risk-ambiguity*" |
|---|---|---|
| One-stage | $c \sim (x, p; y)$<br>$u(c) = w(p)u(x) + (1 - w(p))u(y)$ | $c \sim (x, p; y)$<br>$u(c) = w(p)u(x) + (1 - w(p))u(y)$<br>$\tilde{c} \sim (x, E_p; y)$<br>$u(\tilde{c}) = \tilde{w}(p)u(x) + (1 - \tilde{w}(p))u(y)$ |
| Two-stage | $((\bar{x}, p), q) \sim ((\bar{x}, m), 1)$<br>$w(q)w^\star(p) = w^\star(m)$ | $((\bar{x}, E_p), q) \sim ((\bar{x}, \tilde{m}), 1)$<br>$w(q)\tilde{w}^\star(p) = w^\star(\tilde{m})$ |

- Parametric specifications:

$$u(x) = x^\alpha \text{ and } w(p) = exp(-(-ln(p)^\gamma)^\delta.$$

# Results

# Results: RCP and TREU



| $p$ | $q = 1/3$ | | $q = 2/3$ | |
|---|---|---|---|---|
| | $\#(\Delta \gtrless 0)$ | $t$-test | $\#(\Delta \gtrless 0)$ | $t$-test |
| 0.06 | 60/2 | $8.42^{**}$ | 48/14 | $2.12^{**}$ |
| 0.17 | 38/24 | $3.65^{**}$ | 27/35 | $-0.22^{ns}$ |
| 0.33 | 26/28 | $2.26^{*}$ | - | - |
| 0.50 | 39/23 | $3.39^{**}$ | 24/38 | $-1.96^{*}$ |
| 0.75 | - | - | 24/25 | $0.33^{ns}$ |
| 0.94 | 27/35 | $0.69^{ns}$ | 40/22 | $1.99^{**}$ |

Table 2: RCP $(\Delta = m - pq)/pq)$

- RCP is globally violated, thus TREU is not descriptively valid for evaluating two-stage prospects.
- Overall, we observe preference for the compound prospect, especially for $q = 1/3$.

# Results: REU



| | Probability $q$ | | |
|---|---|---|---|
| | $1/3$ | $2/3$ | $\{1/3, 2/3\}$ |
| Mean | 0.89 | 1.03 | 0.91 |
| Median | 0.90 | 0.98 | 0.91 |
| Std | 0.18 | 0.35 | 0.18 |
| IQR | 0.75-0.98 | 0.83-1.23 | 0.79-1.01 |

Table 3: Parameter $\theta$ empirical distribution characteristics under REU

- $\phi$ is convex for $q = 1/3$ and linear for $q = 2/3$.
- The transformation function $\varphi$ in REU can not absorb the observed discrepancies from RCP.

# Results: RPT under risk



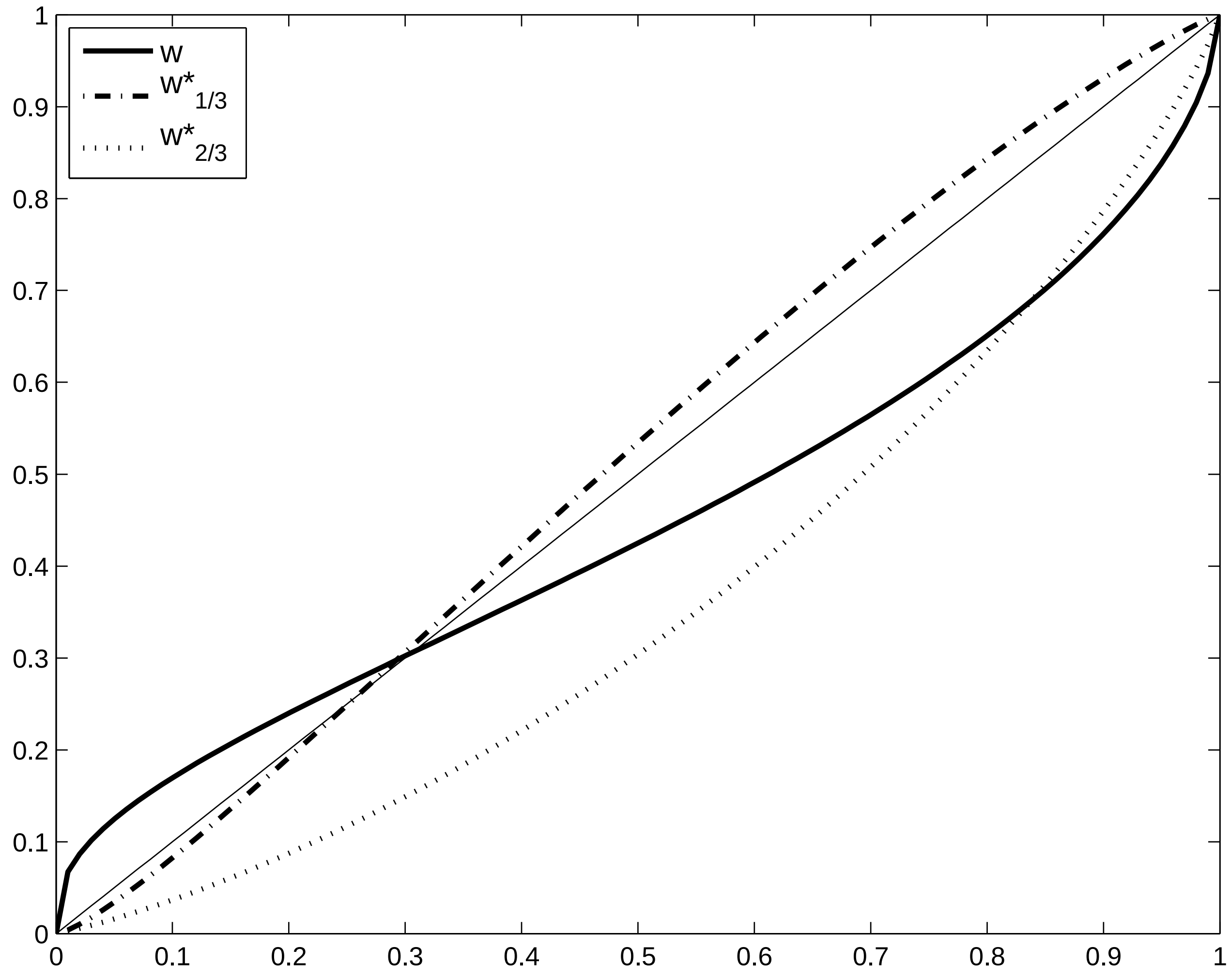
| Stage | func | param | Estimates | | | |
|---|---|---|---|---|---|---|
| | | | Mean | Median | Std | IQR |
| First | $u$ | $\alpha$ | 0.94 | 0.93 | 0.26 | 0.74-1.10 |
| | $w$ | $\gamma$ | 1.06 | 1.07 | 0.32 | 0.85-1.22 |
| | | $\delta$ | 0.62 | 0.61 | 0.20 | 0.48-0.77 |
| Second | $w^*_{1/3}$ | $\gamma^*_{1/3}$ | 1.01 | 0.96 | 0.42 | 0.74-1.18 |
| | | $\delta^*_{1/3}$ | 1.20 | 1.15 | 0.29 | 1.03-1.35 |
| | $w^*_{2/3}$ | $\gamma^*_{2/3}$ | 1.66 | 1.63 | 0.52 | 1.25-2.04 |
| | | $\delta^*_{2/3}$ | 0.90 | 0.85 | 0.28 | 0.69-1.07 |

- Function, $w^\star$, depends on probability $q$. While it is close to linearity for $q = 1/3$, it is convex for $q = 2/3$.

$\Rightarrow$ Inverse than for REU but same problem (differences both for elevation and curvature between $w^*_{1/3}$ and $w^*_{2/3}$).

# Results: RPT under risk





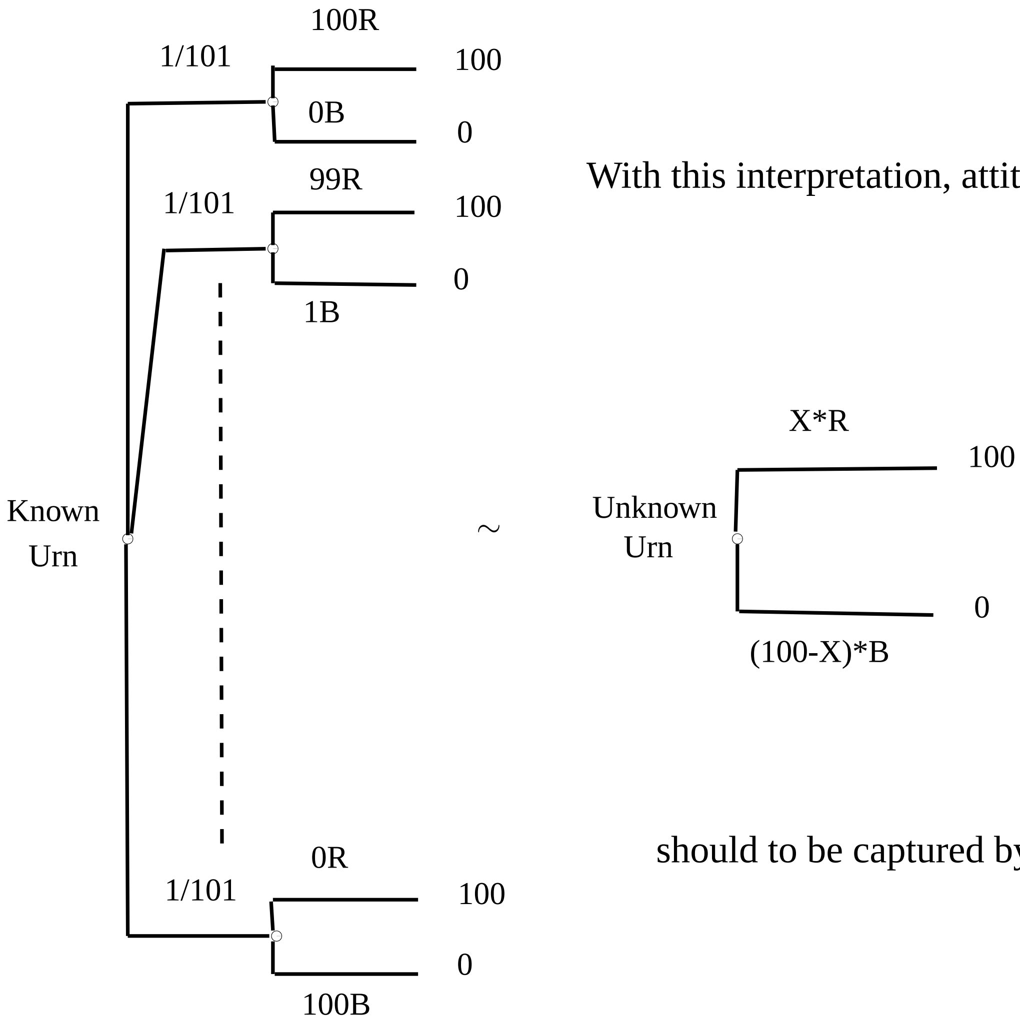

$\Rightarrow$

# Additional results



- Adding ambiguity (first and second stage) does not change our main results i.e.
  - Impact of probability $q$ on the shape of function $\phi$.
  - Stage dependent pwf.
  - Dependence of the second stage pwf on the first stage probability.
- Benchmark results are found for the single stage pwf under risk and ambiguity.
- No association between RCP and ambiguity attitudes ($\neq$ from Halevy, 2008 and Segal).

## Proposition of Segal (1987) for the 2-colour example.




With this interpretation, attitude toward ambiguity in the unknown urn

should to be captured by attitude toward two-stage prospects.

# Concluding remarks



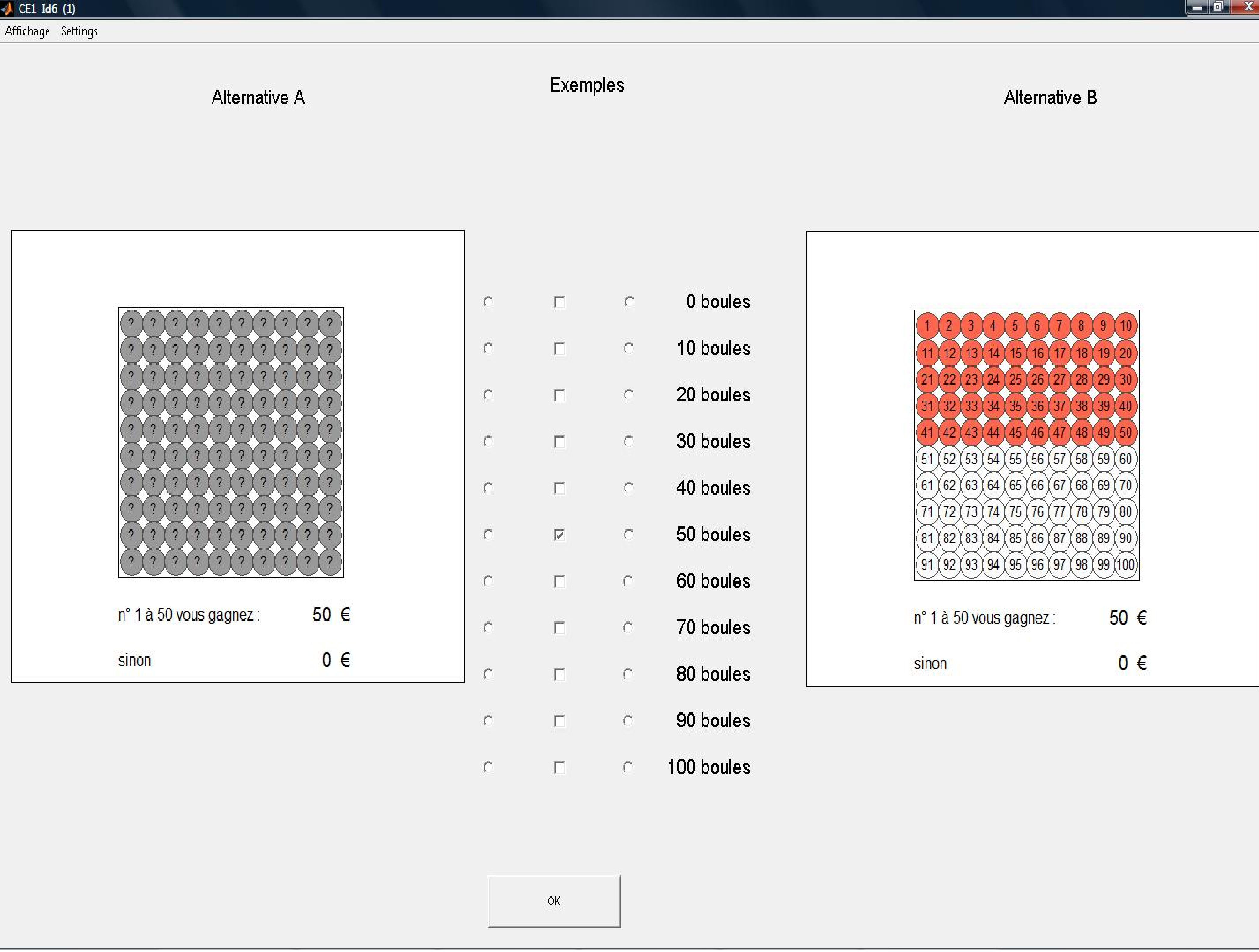
1. Recursive evaluation of two-stage prospects is more complex than allowed by any existing recursive model.

2. In Kreps and Porteus (or KMM) integral, function $\phi$ is sensitive to the first-stage probability of winning.

3. Second-stage probability weighting is very sensitive to the first-stage winning probability.

$\Rightarrow$ Abdellaoui & Zank (2014) first axiomatized a RPT model that could account for our experimental findings.

Thank you for your attention!


CE1 Id6 (1)
Affichage
Settings
Alternative A
Exemples
Alternative B
0 boules
10 boules
20 boules
30 boules
40 boules
50 boules
60 boules
70 boules
80 boules
90 boules
100 boules
n° 1 à 50 vous gagnez :
50 €
sinon
0 €
n° 1 à 50 vous gagnez :
50 €
sinon
0 €
OK

# Comparative ignorance, (Fox & Tversky, 1995)



Fox & Tversky (1995) introduced the *comparative ignorance* hypothesis as a condition of observability of ambiguity aversion. For this hypothesis they proposed the following conjecture :

> "When evaluating an uncertain event in isolation, people attempt to assess its likelihood – as a good bayesian would – paying relatively little attention to second-order characteristics such as vagueness or weight of evidence. However, when people compare two events about which they have different levels of knowledge, the contrast makes the less familiar bet less attractive or the more familiar bet more attractive". p 588.

⇒ Many experimental tests of this hypothesis. Non neutrality toward ambiguity is always observed! (Fox & Tversky (1995), Chow & Sarin (2001), Rubaltelli & al. (2010))

⇒Complete analysis in Nebout (2011).