Inference for diffusions with small diffusions coefficient. Application to epidemics

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Inference for diffusions with small diffusion coefficient and application to epidemics

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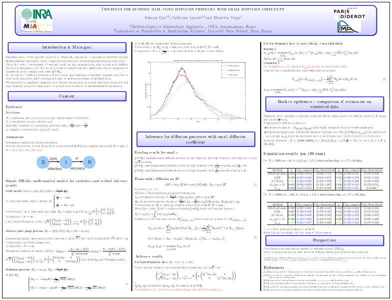
Outline

1. Parametric inference for discretely observed diffusion processes
2. Application to Epidemics



Settle the context

Classical SIR Epidemics S.D.E.

$$\begin{aligned} ds_t &= -\lambda s_t i_t dt + \frac{1}{\sqrt{N_{pop}}}\sqrt{\lambda s_t i_t}\, dB_1(t) \\ di_t &= (\lambda s_t i_t - \gamma i_t)dt - \frac{1}{\sqrt{N_{pop}}}\sqrt{\lambda s_t i_t}\, dB_1(t) + \frac{1}{\sqrt{N_{pop}}}\sqrt{\gamma i_t}\, dB_2(t) \end{aligned}$$

Specificity : Multidimensionnal process, small diffusion coefficient, parameters (λ, γ) both in drift and diffusion function, (few observations available)

Theoretical Context

- $dX_t^\epsilon = b(\alpha, X_t^\epsilon)dt + \epsilon\sigma(\beta, X_t^\epsilon)dB_t, \;\; X_0 = x_0 \in \mathbb{R}^p$
- Discrete observation : X_t^ϵ at times $t_k = k\Delta$ on a fixed interval $[0, T]$ $(T = n\Delta)$
- $\sigma(\beta, x) \in M_p(\mathbb{R}), b(\alpha, x) \in \mathbb{R}^p, \Sigma(\beta, x) = \sigma(\beta, x)\, {}^t\sigma(\beta, x)$ invertible.

Existing results : Maximum Contrast Estimators for high frequency data (linking ϵ and n)

(Sørensen-Uchida Bernoulli 2003) $\frac{1}{\epsilon n} \to 0$, $\frac{1}{\epsilon\sqrt{n}}$ bounded

(Gloter Sørensen S.P.A. 2009) $\exists \rho > 0$, $\frac{1}{\epsilon n^\rho}$ bounded, for a class of contrast processes depending on ρ

Main idea of our inference approach (extension of Genon-Catalot(90))

Taylor's stochastic expansion formula (Azencott (82),Wentzell-Freidlin(79))

$$X_t^\epsilon = x_\alpha(t) + \epsilon g_{\alpha,\beta}(t) + \epsilon^2 R^\epsilon_{\alpha,\beta}(t)$$

where $x_\alpha(t)$ is the deterministic solution $\frac{dx_\alpha(t)}{dt} = b(\alpha, x_\alpha(t)),\ x(0) = x_0 \in \mathbb{R}^p$,

$$dg_{\alpha,\beta}(t) = \frac{\partial b}{\partial x}(\alpha, x_\alpha(t))g_{\alpha,\beta}(t)dt + \sigma(\beta, x_\alpha(t))dB_t,\ \ g_{\alpha,\beta}(0) = 0_{\mathbb{R}^p}$$

and $R^\epsilon_{\alpha,\beta}$ satisfies

$$\sup_{t\in[0,T]} \{\|\epsilon R^\epsilon_{\alpha,\beta}(t)\|\} \underset{\mathbb{P},\epsilon\to 0}{\longrightarrow} 0,\ \ \mathbb{E}\left[\|R^\epsilon_{\alpha,\beta}(t+h) - R^\epsilon_{\alpha,\beta}(t)\|^2\right] \leq Ch.$$

Main Idea

- Compute the likelihood of the Gaussian process $Y_t^\epsilon = x_\alpha(t) + \epsilon g_{\alpha,\beta}(t)$
- Derive a Contrast process for X_t^ϵ from it.

Properties of $g_{\alpha,\beta}$

Φ_α the Resolvent matrix of the linearized ODE

Let Φ_α be the invertible matrix solution of
$\frac{d\Phi_\alpha}{dt}(t, t_0) = \frac{\partial b}{\partial x}(\alpha, x_\alpha(t))\Phi_\alpha(t, t_0)$,
with $\Phi_\alpha(t_0, t_0) = I_p$.

Properties of $g_{\alpha,\beta}$

- $g_{\alpha,\beta}$ is a Gaussian process for which we can obtain the analytic expression.
- $g_{\alpha,\beta}(t_k) = \Phi_\alpha(t_k, t_{k-1})g_{\alpha,\beta}(t_{k-1}) + \sqrt{\Delta}Z_k^{\alpha,\beta}$
- $Z_k^{\alpha,\beta}$ are independent $\mathcal{N}\left(0, S_k^{\alpha,\beta}\right)$ variables.
- $S_k^{\alpha,\beta} = \frac{1}{\Delta}\int_{t_{k-1}}^{t_k} \Phi_\alpha(t_k, s)\Sigma(\beta, x_\alpha(s))\, {}^t\Phi_\alpha(t_k, s)ds$

Likelihood of the Gaussian process and consequences of the approach

log-likelihood of the Gaussian process Y_t^ϵ

$$
\begin{aligned}
L_{\Delta,\epsilon}(\alpha,\beta) &= \epsilon^2 \sum_{k=1}^{n} log\left[det\left(S_k^{\alpha,\beta}\right)\right] \\
&+ \frac{1}{\Delta}\sum_{k=1}^{n} {}^tN_k(\alpha)(S_k^{\alpha,\beta})^{-1}N_k(\alpha) \\
\text{with } N_k(\alpha) &= Y_{t_k} - x_\alpha(t_k) - \Phi_\alpha(t_k, t_{k-1})\left[Y_{t_{k-1}} - x_\alpha(t_{k-1})\right].
\end{aligned}
$$

Define MLE estimators

$$
(\hat{\alpha}_{\epsilon,\Delta}, \hat{\beta}_{\epsilon,\Delta}) = \underset{(\alpha,\beta)\in\Theta}{argmin}\; U_{\Delta,\epsilon}(\alpha,\beta)
$$

Remark on low frequency data

Δ (and n) is fixed : no asymptotic result on $\hat{\beta}_{\epsilon,\Delta}$
$\Rightarrow$ Adaptation needed

Construct the contrast for low frequency data (Δ and n fixed)

Contrast process for the diffusion X_t^ϵ (β unknown)

$$
\begin{aligned}
\bar{U}_\epsilon(\alpha) &= \cancel{\epsilon^2 \sum_{k=1}^{n} log\left[det\left(S_k^{\alpha,\beta}\right)\right]} \\
&+ \frac{1}{\Delta}\sum_{k=1}^{n} {}^tN_k(\alpha)\cancel{(S_k^{\alpha,\beta})^{-1}}N_k(\alpha) \\
\text{with } N_k(\alpha) &= X_{t_k} - x_\alpha(t_k) - \Phi_\alpha(t_k, t_{k-1})\left[X_{t_{k-1}} - x_\alpha(t_{k-1})\right].
\end{aligned}
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Define MCE estimator

$$\bar{\alpha}_\epsilon = \underset{(\alpha)\in\Theta_\alpha}{argmin}\ \bar{U}_\epsilon(\alpha)$$

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Define MCE estimator

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Results for low frequency data (Δ and n fixed)

Contrast process for the diffusion X_t^ϵ (β unknown)

$$\bar{U}_\epsilon\left(\alpha, (X_{t_k})_{k\in\{1,..,n\}}\right) = \frac{1}{\Delta}\sum_{k=1}^{n} {}^tN_k(\alpha)N_k(\alpha).$$

Results for low frequency data and β unknown

Under classical regularity assumptions on b and σ,
and identifiability assumption : $\alpha \neq \alpha' \Rightarrow \{\exists k, \quad 1 \leq k \leq n, \quad x_\alpha(t_k) \neq x_{\alpha'}(t_k)\}$.

$\bar{\alpha}_\epsilon = \underset{\alpha\in\Theta_a}{argmin}\ \bar{U}_\epsilon(\alpha)$ satisfies

$$\epsilon^{-1}\left(\bar{\alpha}_\epsilon - \alpha_0\right) \underset{\epsilon\to 0}{\longrightarrow} \mathcal{N}(0, J_\Delta^{-1}(\alpha_0, \beta_0))$$

where $J_\Delta(\alpha_0, \beta_0)$ do not converges toward $I_b(\alpha_0, \beta_0)$ as $\Delta \to 0$(the Fisher Information Matrix) .

Additionnal information on β for low frequency data

In SIR-Epidemics $\alpha = (\lambda, \gamma) = \beta$

Case of useful additionnal information

$\beta = f(\alpha)$
$\Sigma(\beta, x) = f(\beta)\Sigma_0(x)$

Return on the log-likelihood of the Gaussian process Y_t^ϵ

$$\begin{aligned} L_{\Delta,\epsilon}(\alpha,\beta) &= \epsilon^2 \sum_{k=1}^{n} log\left[det\left(S_k^{\alpha,\beta}\right)\right] \\ &+ \frac{1}{\Delta}\sum_{k=1}^{n} {}^tN_k(\alpha)(S_k^{\alpha,\beta})^{-1}N_k(\alpha) \\ \text{with } N_k(\alpha) &= Y_{t_k} - x_\alpha(t_k) - \Phi_\alpha(t_k, t_{k-1})\left[Y_{t_{k-1}} - x_\alpha(t_{k-1})\right]. \end{aligned}$$

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Additionnal information on β for low frequency data

In SIR-Epidemics $\alpha = (\lambda, \gamma) = \beta$

Case of useful additionnal information

$$\left.\begin{array}{l} \beta = f(\alpha) \\ \Sigma(\beta, x) = f(\beta)\Sigma_0(x) \end{array}\right\} S_k^{\alpha,\beta} = \tilde{S}_k^{\alpha}$$

Contrast process for the diffusion X_t^{ϵ} (with information on β)

$$\begin{aligned} \tilde{U}_\epsilon(\alpha) &= \cancel{\epsilon^2 \sum_{k=1}^{n} log\left[det\left(\tilde{S}_k^{\alpha}\right)\right]} \\ &+ \frac{1}{\Delta}\sum_{k=1}^{n} {}^{t}N_k(\alpha)(\tilde{S}_k^{\alpha})^{-1}N_k(\alpha) \\ \text{with } N_k(\alpha) &= X_{t_k} - x_\alpha(t_k) - \Phi_\alpha(t_k, t_{k-1})\left[X_{t_{k-1}} - x_\alpha(t_{k-1})\right]. \end{aligned}$$

Define MCE estimator

$$\tilde{\alpha}_\epsilon = \underset{(\alpha)\in\Theta_a}{argmin}\ \tilde{U}_\epsilon(\alpha)$$

Results for low frequency data (Δ and n fixed)

Contrast with information on β

$$\tilde{U}_\epsilon\left(\alpha\right) = \frac{1}{\Delta}\sum_{k=1}^{n} {}^tN_k(\alpha)(\tilde{S}_k^\alpha)^{-1} N_k(\alpha)$$

Results

Under the same assumptions (regularity and identifiability)
$\tilde{\alpha}_\epsilon = \underset{(\alpha)\in\Theta_a}{argmin}\ \tilde{U}_\epsilon\left(\alpha\right)$ satisfies

$$\epsilon^{-1}\left(\tilde{\alpha}_\epsilon - \alpha_0\right) \underset{\epsilon\to 0}{\longrightarrow} \mathcal{N}(0, I_\Delta^{-1}(\alpha_0, \beta_0))$$

with $I_\Delta(\alpha_0, \beta_0) \underset{\Delta\to 0}{\longrightarrow} I_b(\alpha_0, \beta_0)$

Results for high frequency data ($\Delta \to 0$) (without linking ϵ and n)

Contrast process

Using $\|S_k^{\alpha_0,\beta_0} - \Sigma(\beta_0, X_{t_{k-1}})\| \underset{\epsilon,\Delta\to 0}{\longrightarrow} 0$, we consider :

$$\begin{aligned} \check{U}_{\Delta,\epsilon}(\alpha,\beta)) &= \epsilon^2 \sum_{k=1}^{n} log \left[det \left(\Sigma(\beta, X_{t_{k-1}}) \right) \right] \\ &+ \frac{1}{\Delta} \sum_{k=1}^{n} {}^t N_k(\alpha) \Sigma^{-1}(\beta, X_{t_{k-1}}) N_k(\alpha) \end{aligned}$$

Asymptotic Normality

Under classical regularity and identifiability assumptions on b and σ $(\check{\alpha}_{\epsilon,\Delta}, \check{\beta}_{\epsilon,\Delta}) = \underset{(\alpha,\beta)\in\Theta}{argmin}\ \check{U}_{\Delta,\epsilon}(\alpha,\beta)$ satisfies

$$\begin{pmatrix} \epsilon^{-1}(\alpha_{\check{\epsilon},\Delta} - \alpha_0) \\ \sqrt{n}(\beta_{\check{\epsilon},\Delta} - \beta_0) \end{pmatrix} \underset{n\to\infty,\epsilon\to 0}{\longrightarrow} N\left(0, \begin{pmatrix} I_b^{-1}(\alpha_0,\beta_0) & 0 \\ 0 & I_\sigma^{-1}(\alpha_0,\beta_0) \end{pmatrix}\right)$$

.

Some generalities

Transmissible disease : a world of incomplete and aggregated data

- Date of infection and recovery of an infected individual unknown
- Total Number of new infected cases collected at regular time interval (days or week)

Modelisation : the simpler, the better

$$S \xrightarrow{\lambda I/N_{pop}} I \xrightarrow{\gamma} R$$

Ethier & Kurtz diffusion approximation

$$\begin{aligned} ds_t &= -\lambda s_t i_t dt + \frac{1}{\sqrt{N_{pop}}}\sqrt{\lambda s_t i_t}\, dB_1(t) \\ di_t &= (\lambda s_t i_t - \gamma i_t)dt - \frac{1}{\sqrt{N_{pop}}}\sqrt{\lambda s_t i_t}\, dB_1(t) + \frac{1}{\sqrt{N_{pop}}}\sqrt{\gamma i_t}\, dB_2(t) \end{aligned}$$

Estimation over simulations

Simulations using Matlab

Simulations over 1000 runs of a scenario close to influenza ($\lambda = 0.4$ $days^{-1}$, $\gamma = 1/3$ $days^{-1}$, $T = 50$)
Study of different scenarios : $N_{pop} \in [100; 10000]$, $\Delta \in [T/10; 1; T/100]$