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Abstract: We study in this report a model of optimal Carbon Capture and Storage in which the reservoir of sequestered carbon is leaky, and pollution eventually is released into the atmosphere. We formulate the social planner problem as an optimal control program and we describe the optimal consumption paths as a function of the initial conditions, the physical constants and the economical parameters. In particular, we show that the presence of leaks may lead to situations which do not occur otherwise, including that of non-monotonous price paths for the energy.

Key-words: Carbon Sequestration and Storage, Optimal Control

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Politiques optimales de séquestration du CO2 dans des réservoirs qui fuient

Résumé : A écrire...

Mots-clés : Capture et stockage du CO2, contrôle optimal

Chapter 1

Introduction

1.1 The Economic Relevance of Carbon Storage and Sequestration

The fact that the carbon emissions generated by the use of the fossil fuels could be captured and sequestered is now well documented both empirically and theoretically, and it is now included in the main empirical models of energy uses. Were this option open at a sufficiently low cost for the most potentially polluting primary resource, that is coal, its competitive full cost, including the shadow cost of its pollution power, could be drastically reduced given that coal is abundant at a low extraction cost and can be transformed into energy ready to use for final users at moderately transformation costs. The main problem concerning its future competitiveness is the cost at which its pollution damaging effects can be abated.

Abating the emissions involves two different types of costs. The first one is a monetary cost : capturing, compressing and transporting the captured CO_2 into reservoirs involves money outlays. The second one is a shadow cost because this type of garbage has to be stockpiled somewhere. This problem has been attacked in Lafforgue, Magné & Moreaux (2008*a*), Lafforgue, Magné & Moreaux (2008*b*). It is not quite clear that sufficient storage capacities would be available for low CO_2 capture and storage (CCS) costs, in which case the reservoir capacities themselves could have to be seen as scarce resources to which some rents should have to be imputed along an optimal or equilibrium path.

As far as equilibrium paths are concerned there is a very difficult problem about property rights. The reservoirs into which the captured CO_2 is assumed to be confined are in underground places, on which property rights are more or less defined, and differently defined all over the world.

Even if sufficiently large reservoirs are available there exists another problem concerning the security of such reservoirs. Most reservoirs are leaking in the long run, a well-known problem in engineering. The fact that captured CO_2 will eventually return into the atmosphere cannot be ignored when assessing the economic relevance of CCS.

A first investigation of this last problem has been given by Ha-Duong & Keith (2003). Their main conclusion is that "leakage rates on order of magnitude below the discount rate are negligible" (p. 188). Hence leakage is a second order problem as far as the rate of discount is sufficiently high, and probably that other characteristics of the empirical model they use are sufficiently well profiled.

A second batch of investigations has recently been conducted by Gerlagh, Smekens and Van der Zwaan.[1] These papers are mainly empirical papers using and comparing DEMETER and MARKAL models to assess the usefulness of CCS policies. Their results are twofold. First using CCS policies with leaky reservoirs does not permit to escape a big switch to renewable non polluting primary resources if a 450ppmv atmospheric pollution ceiling has to be enforced. But CCS with leaky reservoirs is smoothing the optimal path. A second point concerns the

[1] c.f. Van der Zwaan (2005), Van der Zwaan & Gerlagh (2009) and Van der Zwaan & Smekens (2009).

relative competitiveness of coal : "The large scale application of CCS needed for a significantly lower contribution of renewable would be consistent, in terms of climate change control, with the growing expectation that fossil fuels, and in particular coal, will continue to be a dominant form of energy supply during the twenty-first century" Van der Zwaan & Gerlagh (2009, p. 305). As they point out "The economic implications of potential CO_2 leakage associated with the large scale development of CCS have so far been researched in a few studies" (ibidem, p. 306). To our knowledge theoretical studies are even fewer.

The objective of this paper is to try to elucidate some theoretical features of optimal CCS policies with leaky reservoirs and specifically the dynamics of the shadow cost of both carbon stocks and their relation with the mining rent of the nonrenewable resource, determining the long run relative competitiveness of coal and solar energies. The paper has to be seen as mainly exploratory. To conduct the inquiry we adopt the most simple model permitting to isolate the dynamics of captured CO_2, leakage and atmospheric pollution.

Naturally, the presence of leaks, producing an additional flow of pollutant, makes the pressure on the atmospheric stock larger than when there is none, and should favor capture. On the other hand, for the same reason, it is not necessarily good to sequestrate too much pollution, since this will make economic conditions worse in the future.

The results presented in this paper show how the optimal consumption paths are modified with respect to the benchmark situation where there are no leaks. In particular, it turn out that over some optimal path, the price of energy is not necessarily monotonous. Non-monotonous price paths in the exploitation of nonrenewable resources have been described before: for a first paper in this direction, see for instance Livernois & Martin (2001). In the present situation, the lack of monotonicity results from a combination of a constraint on the present atmospheric stock of pollution, and a lag effect for the sequestered stock of pollution; such an effect has not been reported in the literature, to the best of our knowledge.

Our analysis reveals other interesting features. First of all, not every possible configuration of atmospheric and sequestered stock is acceptable, thus causing a possible *viability* problem. Other results quantitatively confirm that the presence of leakage does reduce the economic incentive to sequestrate pollution.

1.2 Technical Contribution

The model we develop conceals several technical features that are seldom encountered in the literature. First of all, il involves three state variables and three controls, with constraints on the three states and constraints on two of the controls. We are nevertheless able to provide a complete parametric description of solutions when one of the state variables is "saturated", and a quite complete one when all three state variables are present.

In the course of the solution, we identify the presence of a "hidden" viability or controllability constraint, and a "singular" point in the state space. In the vicinity of the viability constraint and of the singular point, optimal trajectories have an unusual behavior, and some costate variables (economically interpreted as hidden prices) may be discontinuous.

The report is organized as follows. We develop the model, its assumptions and notations in Chapter 2. In particular, in Section 2.1.6 we state the mathematical optimization program representing the social planner problem, and derive the necessary optimality conditions.

In Chapter 3, we prepare the construction of solutions by studying the behavior of optimal trajectories within *phases* characterized by a constant status (free or bound) of the different constraints on states and controls. This allows in particular to eliminate several configurations which cannot be optimal.

In Chapter 4, we construct the solutions of the optimization problem in the situation where the stock of polluting carbon energy is assumed to be infinite (that is, the resource is assumed to be renewable). While not economically relevant, this analysis provides essential insight in the behavior of solutions and the complexity of the problem.

Chapter 2

The Model

2.1 Model and Assumptions

We consider a global economy in which the energy consumption can be supplied by two primary resources: a nonrenewable polluting source like coal and a clean renewable one as solar plants.

2.1.1 Energy consumption and gross surplus

Let us denote by q the instantaneous energy consumption rate of the final users and by $u(q)$ the instantaneous gross surplus thus generated. The gross surplus function is assumed to satisfy the following standard assumptions:

Assumption 1. The function $u : [0, \infty) \to \mathbb{R}$ is a function of class C^2, strictly increasing and strictly concave, and which satisfies the first Inada condition $\lim_{q \to 0} u'(q) = +\infty$.

The function $u'(q)$, the inverse demand function, is also denoted by $p(q)$ and its inverse, the direct demand function, is denoted by $q^d(p)$. Under Assumption 1, the function q^d is strictly positive and strictly decreasing.

2.1.2 The non renewable polluting resource

Let $X(t)$ be the stock of coal available at time t, $X^0 = X(0)$ be its initial endowment, and $x(t)$ be the instantaneous extraction rate: $\dot{X}(t) = -x(t)$. The current average transformation cost of coal into useful energy is assumed to be constant and is denoted by c_x. We denote by $\widetilde{x}$ the non renewable energy consumption when its marked price is equal to c_x and coal is the only energy supplier: $u'(\widetilde{x}) = c_x$.

Burning coal for producing useful energy implies a flow of pollution emissions proportional to the coal thus burned. Let ζ be the unitary pollution contents of coal so that the gross emission flow amounts to $\zeta x(t)$. This gross emission flow can be either freely relaxed into the atmosphere or captured to be stockpiled into underground reservoirs however at some cost.

Let c_s be the average capturing and sequestrating cost of the potential pollution generated by the exploitation of coal. Let us denote by $s(t)$ this part of the potential flow $\zeta x(t)$ which is captured and sequestered. Then the sequestration cost amounts to $c_s s(t)$. The remaining flow of carbon $\zeta x(t) - s(t) \geq 0$ goes directly into the atmosphere.

2.1.3 Pollution stocks and leakage effects

We take two pollution stocks explicitly into account, the atmospheric stock denoted by $Z(t)$ and the sequestered stock denoted by $S(t)$.

The atmospheric stock Z is first fed by the non-captured pollution emissions, resulting from the use of coal, that is $\zeta x(t) - s(t)$. This atmospheric stock is self-regenerating at some constant proportional rate α.[1] However, Z is also fed by the leaks of the sequestered pollution stock S. We assume that leaks are proportional to the stock and denote by β the leakage rate. In total:

$$\dot{S}(t) \;=\; s(t) - \beta S(t)\;.$$

We assume that the sequestering capacities are sufficiently large to be never saturated and that no cost has to be incurred for maintaining the captured stock S into reservoirs. The only costs are the above capture costs $c_s s(t)$.

Taking into account both this leakage effect and the above self-regeneration effect, we get the dynamics of the atmospheric stock:

$$\dot{Z}(t) \;=\; \zeta x(t) - s(t) + \beta S(t) - \alpha Z(t)\;.$$

The flows and stocks of energy and pollution are illustrated in Figure 2.1.

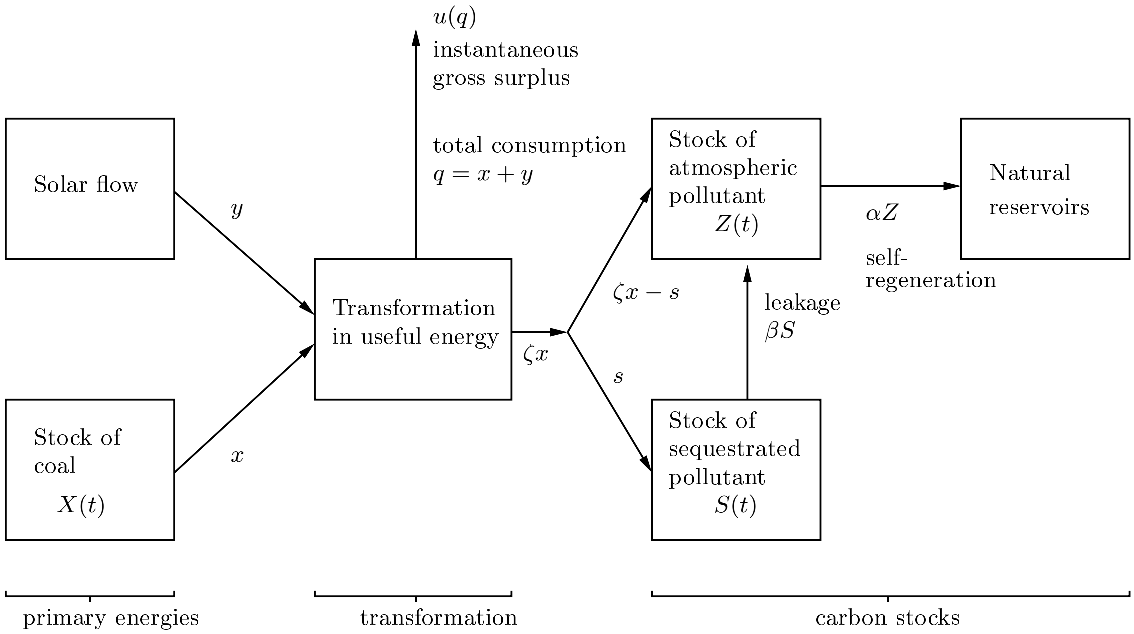

Figure 2.1: Flows and stocks of energy and pollution

2.1.4 Atmospheric pollution damages

There are two main ways for modeling the atmospheric pollution damages. A most favored way by some economists is to postulate some damage function, the higher is the atmospheric pollution stock $Z(t)$, the larger are the current damages at the same time t. Generally, this function is assumed to be convex. The other way is to assume that, as far as the atmospheric pollution stock is kept under some critical level $\overline{Z}$, the damages are not so large. However, around the critical level $\overline{Z}$, the damages are strikingly increasing, so that, whatever what could have been gained by following a path generating an overrun at $\overline{Z}$, the damages would counterbalance the gains.[2] We assume that the loss generated by Z are negligible provided that Z be maintained under some level $\overline{Z} \geq Z^0 \geq 0$, $Z^0 \equiv Z(0)$, but is infinitely costly once $Z(t)$ overruns $\overline{Z}$.

[1] This self-regeneration effect may be seen as some kind of leakage of the atmosphere reservoir towards some other natural reservoirs not explicitly modeled in the present setting. For models taking explicitly into account such questions, see for example Lontzek and Rickels (2008) or Rickels and Lontzek (2008).

[2] Some authors use simultaneously both approaches.

We denote by $\overline{x}$ the maximum coal consumption when the atmospheric pollution stock is at its ceiling $\overline{Z}$, no part of the gross pollution flow ζx is captured ($s = 0$) and the stock of sequestered pollution is nil:

$$\dot{Z} = 0 = \zeta\overline{x} - \alpha\overline{Z} \qquad \Longrightarrow \qquad \overline{x} = \frac{\alpha}{\zeta}\overline{Z} .$$

We denote by $\overline{p}$ the corresponding energy price assuming that coal is the only energy supplier: $\overline{p} \equiv q^d(\overline{x})$.

Clearly there exists an effective constraint on coal consumption if and only if $\overline{p} > c_x$ or equivalently $\overline{x} < \widetilde{x}$ and simultaneously the coal initial endowment X^0 is sufficiently large.

2.1.5 The renewable clean energy

The other primary resource is a renewable clean energy. Let $y(t)$ be its instantaneous consumption rate. We assume that its average cost, denoted by c_y, is constant. We denote by $\widetilde{y}$ the renewable energy consumption when the renewable one is the only energy supplier: $u'(\widetilde{y}) = c_y$.

Both c_x and c_y include all that has to be supported to supply ready to use energy to the final users. Hence, once these costs are supported the two types of energy are perfect substitutes for the final user, and we may define $q(t)$ as the sum of $x(t)$ and $y(t)$.

2.1.6 The Social Planner Problem

The social planner problem is to maximize the social welfare. The social welfare W is the sum of the discounted net current surplus, taking into account the gross surplus $u(q)$ and the production or capture costs. We assume that the social rate of discount ρ, $\rho > 0$, is constant throughout time.

Accordingly, the social planner faces the following optimization problem:

$$\max_{s(\cdot),x(\cdot),y(\cdot)} \int_0^\infty \left[u(x(t)+y(t)) - c_s s(t) - c_x x(t) - c_y y(t)\right] e^{-\rho t} \mathrm{d}t \tag{2.1.1}$$

given the controlled dynamics:

$$\left\{\begin{array}{rcl} \dot{X} &=& -x \\ \dot{Z} &=& -\alpha Z + \beta S + \zeta x - s \\ \dot{S} &=& -\beta S + s \ , \end{array}\right. \tag{2.1.2}$$

the initial conditions $(X(0), Z(0), S(0)) = (X^0, Z^0, S^0)$, and the constraints on state variables and controls:

$$Z(t) \leq \overline{Z} \tag{2.1.3}$$
$$S(t) \leq \overline{S} \tag{2.1.4}$$
$$X(t) \geq 0 \tag{2.1.5}$$
$$\overline{y} \geq y(t) \geq 0 \tag{2.1.6}$$
$$x(t) \geq 0 \tag{2.1.7}$$
$$s(t) \geq 0 \tag{2.1.8}$$
$$s(t) \leq \zeta x(t) \tag{2.1.9}$$

for all t. Other physically relevant constraints ($S \geq 0$, $Z \geq 0$) are automatically satisfied by the dynamics and are not explicitly taken into account. This follows from the fact that $Z = 0$ implies $\dot{Z} = \beta S + \zeta x - s \geq 0$ and likewise, $S = 0$ implies $\dot{S} \geq 0$.

2.1.7 Assumptions on costs

We assume not only that the cost of the renewable energy is higher than the cost of the nonrenewable one, but furthermore that c_y is higher than $\overline{p}$. In summary:

Assumption 2. It is assumed that $c_s > 0$, and

$$c_x \; < \; \overline{p} \; < \; c_y \; . \tag{2.1.10}$$

Equivalently, under Assumption 1, $\widetilde{y} < \overline{x} < \widetilde{x}$.

It turns out that Assumption 1 is unnecessarily strong, although it provides the convenience to separate assumptions made on $u(\cdot)$ and assumptions made on other parameters. The results we obtain are valid under the weaker composite assumption:

Assumption 3. The function $u : [0, \infty) \to \mathbb{R}$ is a function of class C^2, strictly increasing and strictly concave. It is assumed that $c_s > 0$, and

$$\lim_{x\to\infty} u'(x) \; < \; c_x \; < \; \overline{p} \; < \; c_y \; < \; u'(0) \; , \tag{2.1.11}$$

or equivalently, $0 < \widetilde{y} < \overline{x} < \widetilde{x}$.

These assumptions on the cost parameters of the model are summarized in Figure 2.2, which also recapitulates the notation

$$\overline{x} \; = \; \frac{\alpha \overline{Z}}{\zeta} \quad \widetilde{y} \; = \; q^d(c_y) \quad \widetilde{x} \; = \; q^d(c_x) \quad \overline{p} \; = \; u'(\overline{x}) \; .$$

The following unit system proves useful in calculations and interpretations (see Section 2.2.1 for the missing notation):

α, β, ρ	in	s^{-1}	Z, S, X	in	T
$u(\cdot)$	in	$\$/s$	$u'(\cdot)$	in	$\$/T$
$x(\cdot), y(\cdot), s(\cdot)$	in	T/s	$q^d(\cdot)$	in	T/s
c_x, c_y, c_s	in	$\$/T$	$\lambda_X, \lambda_Z, \lambda_S$	in	$\$/T$

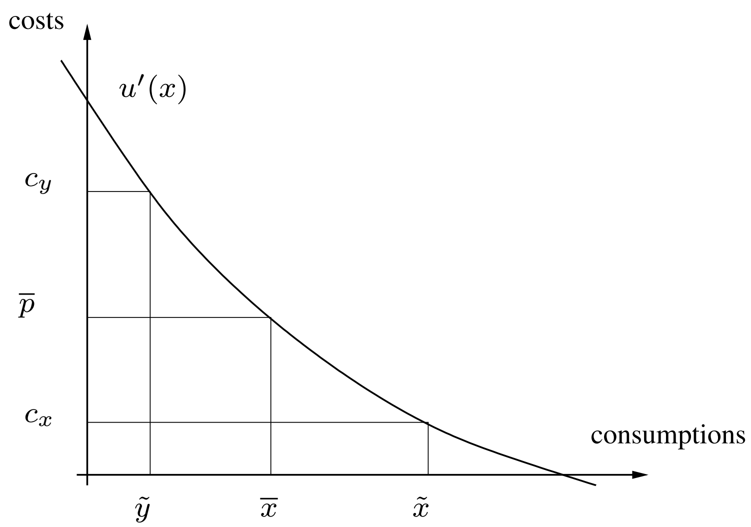

Figure 2.2: Graphical representation of the assumptions on marginal costs

2.1.8 Litterature and particular cases

The model generalizes several previous models of the literature, which can be recovered using particular values of the parameters.

No reservoirs, no capture The model where capture is not possible has been studied in Chakravorty, Magné & Moreaux (2006).

When $\beta \to \infty$ in the present model, then whatever is captured in the stock is immediately leaked into the atmosphere. The model therefore reduces to the case without reservoir and without capture (since capturing is more costly than not capturing).

The model without capture also shows up when the capture cost c_s is very large so as to make sequestration economically suboptimal (see Section 4.4.4). The difference with $\beta = \infty$ is however that the standing stock of sequestered carbon will empty only progressively. If the initial condition is an empty stock, then there is no difference.

No leakage The case $\beta = 0$ models the situation where reservoirs do not leak.

This model is studied in Lafforgue et al. (2008*a*), which actually considers the case of multiple reservoirs with different sequestration costs. Each reservoir has a finite capacity. The flow of clean energy $\overline{y}$ is never binding, which is equivalent to assuming that $\overline{y} \geq \widetilde{y}$.

In Lafforgue et al. (2008*b*), only one reservoir is considered, it has a finite capacity $\overline{S}$, and in addition the maximally available flow of clean energy $\overline{y}$ is possibly binding.

In both papers, an additional assumption is made: $c_s < (c_x - \overline{p})/\xi$. In the forthcoming analysis, this situation will be called "c_s small", see Section 4.4.1.

2.2 First elements of solution

We shall use the maximum principle in order to identify the solutions to this optimization problem. In this paragraph, we first state the first-order conditions for the problem, next review the theorems on which we base the method.

2.2.1 First order conditions

Let us denote by L the current-value Lagrangian of the problem:

$$\begin{aligned} L(y,x,s,X,Z,S) \;=\; & u(x+y) - c_s s - c_x x - c_y y \\ & + \lambda_X[-x] + \lambda_Z[-\alpha Z + \beta S + \zeta x - s] + \lambda_S[-\beta S + s] \\ & + \nu_Z[\overline{Z} - Z] + \nu_S[\overline{S} - S] + \nu_X X \\ & + \gamma_s s + \gamma_{sx}(\zeta x - s) + \gamma_y y + \gamma_Y(\overline{y} - y) \ . \end{aligned} \tag{2.2.1}$$

The first order conditions are then the following. First, optimality of the control yields:

$$\frac{\partial L}{\partial s} = 0 \iff 0 = -c_s - \lambda_Z + \lambda_S + \gamma_s - \gamma_{sx} \tag{2.2.2}$$

$$\frac{\partial L}{\partial x} = 0 \iff 0 = u'(x+y) - c_x - \lambda_X + \zeta\lambda_Z + \zeta\gamma_{sx} \tag{2.2.3}$$

$$\frac{\partial L}{\partial y} = 0 \iff 0 = u'(x+y) - c_y + \gamma_y - \gamma_Y \ , \tag{2.2.4}$$

together with the constraints:

$$\gamma_{sx} \geq 0, \quad \zeta x - s \geq 0 \quad \text{and} \quad \gamma_{sx}[\zeta x - s] = 0 \tag{2.2.5}$$

$$\gamma_s \geq 0, \quad s \geq 0 \quad \text{and} \quad \gamma_s s = 0 \tag{2.2.6}$$

$$\gamma_y \geq 0, \quad y \geq 0 \quad \text{and} \quad \gamma_y y = 0 \tag{2.2.7}$$

$$\gamma_Y \geq 0, \quad \overline{y} - y \geq 0 \quad \text{and} \quad \gamma_Y[\overline{y} - y] = 0 \tag{2.2.8}$$

$$\nu_X \geq 0, \quad X \geq 0 \quad \text{and} \quad \nu_X \ X = 0 \tag{2.2.9}$$

$$\nu_Z \geq 0, \quad \overline{Z} - Z \geq 0 \quad \text{and} \quad \nu_Z[\overline{Z} - Z] = 0 \tag{2.2.10}$$

$$\nu_S \geq 0, \qquad \overline{S} - S \geq 0 \quad \text{and} \quad \nu_S[\overline{S} - S] = 0 \ . \tag{2.2.11}$$

Next, the dynamics of the costate variables are

$$\dot{\lambda}_X = \rho\lambda_X - \frac{\partial L}{\partial X} \iff \dot{\lambda}_X = \rho\lambda_X - \nu_X \tag{2.2.12}$$

$$\dot{\lambda}_Z = \rho\lambda_Z - \frac{\partial L}{\partial Z} \iff \dot{\lambda}_Z = (\rho + \alpha)\lambda_Z + \nu_Z \tag{2.2.13}$$

$$\dot{\lambda}_S = \rho\lambda_S - \frac{\partial L}{\partial S} \iff \dot{\lambda}_S = (\rho + \beta)\lambda_S - \beta\lambda_Z + \nu_S \ . \tag{2.2.14}$$

Finally, we have the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_X X = 0 \tag{2.2.15}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_Z Z = 0 \tag{2.2.16}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_S S = 0 \ . \tag{2.2.17}$$

2.2.2 Sufficient optimality conditions

We will base our solution on the following result, which gives a sufficient condition for optimality. The statement is that of Seierstad & Sydsæter (1999, Theorem 11, p. 385).

Theorem 2.1 (Seierstad & Sydsæter (1999), Theorem 11)**.** *Consider the infinite-horizon optimal control problem:*

$$\max_{\mathbf{u}(\cdot)} \int_0^\infty f_0(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

where the state vector is $\mathbf{x}(\cdot)$ *belongs to* $\mathbb{R}^n$*, the control vector* $\mathbf{u}(\cdot)$ *belongs to some fixed set* $U \subset \mathbb{R}^r$*, and* $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$*. Assume that admissible trajectories must satisfy the vector of* s *constraints:*

$$g_j(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, j = 1, \ldots, s', \qquad g_j(\mathbf{x}(t), \mathbf{u}(t), t) = \bar{g}_j(\mathbf{x}(t)) \geq 0, j = s' + 1, \ldots, s,$$

as well as the terminal conditions

$$\liminf_{t\to\infty} x_i(t) = x_i^1, i = 1, \ldots, \ell, \quad \liminf_{t\to\infty} x_i(t) \geq x_i^1, i = \ell + 1, \ldots, m,$$

and no condition for $i = m + 1, \ldots, n$*.*

Assume that:

a) f_0*,* f *and* g_j *for* $j = 1, \ldots, s'$ *have derivatives w.r.t.* $\mathbf{x}$ *and* $\mathbf{u}$*, and that these derivatives are continuous.*

b) $\bar{g}_j$ *is* C^2 *for* $j = s' + 1, \ldots, s$*.*

If there exists an admissible pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$*, together with a piecewise continuous and piecewise continuously differentiable vector function* $\mathbf{p}(t)$ *with jump points* $0 < \tau_1 < \ldots < \tau_N$*, a piecewise-continuous function* $\mathbf{q}(t)$ *and* $2N$ *vectors* β_k^-*,* β_k^+*,* $k = 1, \ldots, N$ *in* $\mathbb{R}^s$ *such that, defining*

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) := f_0(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$L(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), \mathbf{q}(t), t) := H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) + \mathbf{q}(t) \cdot g(\mathbf{x}(t), \mathbf{u}(t), t) \ ,$$

c) for virtually all t*, and all* $\mathbf{u} \in U$*,* $\frac{\partial L}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t) \cdot (\mathbf{u} - \mathbf{u}^*) \leq 0$*,*

d) for virtually all t*,* $\dot{\mathbf{p}}(t) = -\frac{\partial L}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t)$*,*

e) the Hamiltonian is a concave function of $(\mathbf{x}(t), \mathbf{u}(t))$*, for all* t*,* *f) $q_j(i) \geq 0$ and $= 0$ if $g_j(\mathbf{x}^*(t), \mathbf{u}^*(t), t) > 0$, for all t and $j = 1, \ldots, s$,*

g) q_j is a quasi-concave function of $(\mathbf{x}(t), \mathbf{u}(t))$, for all t and $j = 1, \ldots, s$,

h) for each $i = 1, \ldots, n$ and $k = 1, \ldots, N$,

$$p_i(\tau_k^-) - p_i(\tau_k^+) \;=\; \sum_{j=1}^{s} \beta_{kj}^+ \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^+), \tau_k) + \beta_{kj}^- \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^-), \tau_k) \;, \tag{2.2.18}$$

i) for each $k = 1, \ldots, N$ and $\mathbf{u} \in U$, $\beta_k^\pm \cdot \dfrac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^\pm), \tau_k) \cdot (\mathbf{u} - \mathbf{u}^*(\tau_k^\pm)) \leq 0$,

j) for each $j = 1, \ldots, s$ and $k = 1, \ldots, N$, $\beta_{kj}^\pm \geq 0$, and $= 0$ if $g_j(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^\pm), \tau_k) > 0$,

k) and for all admissible $\mathbf{x}$, $\liminf_{t\to\infty} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \geq 0$,

then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ is cathing-up-optimal.

Applied to our problem, this theorem provides the following corollary. In order to state it, we first give the detail of the correspondence between notation.

We have a state $\mathbf{x} = (X, Z, S)$ $(n = 3)$ and a control $\mathbf{u} = (y, x, s)$ $(r = 3)$. The cost function is $f_0 = e^{-\rho t}(u(x+y) - c_s s - c_x x - c_y y)$ and the dynamics f are specified by (2.1.2). The constraints are enumerated as (omitting the argument (X, Z, S, y, x, s)):

$$g_1 = y, \quad g_2 = \overline{y} - y, \quad g_3 = s, \quad g_4 = \zeta x - s,$$

$$\bar{g}_5 = X, \quad \bar{g}_6 = \overline{Z} - Z, \quad \bar{g}_7 = \overline{S} - S.$$

These correspond, respectively, to constraints (2.1.6) (g_1 and g_2), (2.1.8) and (2.1.9) (g_3 and g_4), (2.1.3), (2.1.4) and (2.1.5). The constraint (2.1.7) is implied by the others, and is ommitted here. We have $s' = 4$ and $s = 7$. There are no constraints *a priori* on the behavior of the state trajectory as $t \to \infty$. In other words, we take $\ell = m = 0$.

The constraints have some specific features: they are all linear, and they depend either on control variables, or state variables, but not both. As a consequence, partial derivatives are constant, some being null. Also, the constraints expressed in (2.2.18) and requirement i) involve disjoint sets of parameters $\beta_{kj}^\pm$: those can therefore be chosen independently.

When applied to (2.2.18), we obtain the simpler requirement: for $i = 1, 2, 3$,

$$p_i(\tau_k^-) - p_i(\tau_k^+) \;=\; \sum_{j=5}^{7} (\beta_{kj}^+ + \beta_{kj}^-) \frac{\partial g_j}{\partial x_i} \;. \tag{2.2.19}$$

Each state variable appears in exactly one of the constraints g_5, g_6 and g_7, which leads to:

$$p_1(\tau_k^-) - p_1(\tau_k^+) \;=\; (\beta_{k5}^+ + \beta_{k5}^-), \quad p_2(\tau_k^-) - p_2(\tau_k^+) \;=\; -(\beta_{k6}^+ + \beta_{k6}^-), \quad p_3(\tau_k^-) - p_3(\tau_k^+) \;=\; -(\beta_{k7}^+ + \beta_{k7}^-).$$

Equivalently, since $\beta_{kj}^\pm \geq 0$ according to requirement j),

$$p_1(\tau_k^-) - p_1(\tau_k^+) \;\geq\; 0, \quad p_2(\tau_k^-) - p_2(\tau_k^+) \;\leq\; 0, \quad p_3(\tau_k^-) - p_3(\tau_k^+) \;\leq\; 0. \tag{2.2.20}$$

On the other hand, the requirement i) boils down to:

$$\beta_k^\pm \cdot \frac{\partial g}{\partial \mathbf{u}} \cdot (\mathbf{u} - \mathbf{u}^*(\tau_k^\pm)) \leq 0 \;, \tag{2.2.21}$$

and this is satisfied with equality, choosing $\beta_{kj}^\pm = 0$, $j = 1, \ldots, 4$.

Corollary 2.1. *Assume there exist:*

- *a vector of continuous function* $(X, Z, S)(t)$*, a vector function* $(y, x, s)(t)$*, satisfying equations (2.1.2)– (2.1.9),*
- *a vector function* $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ *such that* λ_X *and* λ_S *are continuous and continuously differentiable, and* λ_Z *piecewise continuously differentiable, a piecewise-continuous vector function* $\gamma(t) = (\gamma_y, \gamma_Y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_S, \nu_X)(t)$*, satisfying equations (2.2.2)– (2.2.11) for all* t*,* (2.2.12)– (2.2.14) *for virtually every* t*, and conditions (2.2.15)– (2.2.17),*
- *a sequence of time instants* $0 < \tau_1 < \ldots < \tau_N$*, where* $Z(\tau_k^-) < \overline{Z}$ *and* $Z(\tau_k^+) = \overline{Z}$*, such that,*

$$\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+) \ \leq \ 0. \tag{2.2.22}$$

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal for the criterion (2.1.1).*

Proof. We shall check the conditions of Theorem 2.1, using the correspondence of notation detailed above. The set U of Theorem 2.1 is chosen as $U = \{(y, x, s) \in \mathbb{R}_+^3 \mid x + y > 0\}$. The restriction on $x + y$ is not part of the optimization problem (2.1.1): we have therefore to show that, if (y^*, x^*, s^*) denotes an optimal trajectory, then $x^*(t) + y^*(t) > 0$ for virtually all t. By contradiction, assume that $x^*(t) + y^*(t) = 0$ for $t \in I$, some nonempty interval. Then, modifying this strategy into: $x^\dagger(t) = 0$, $y^\dagger(t) = \widetilde{y}$ for $t \in I$, while not changing $s(t)$ nor the strategy outside of interval I, yields a larger profit. Indeed, the difference in profits is

$$J^* - J^\dagger \ = \ \int_I [u(0) - u(\widetilde{y}) + c_y \widetilde{y}]\, e^{-\rho t} \mathrm{d}t \ .$$

The function $u(0) - u(y) - c_y y$ has derivative $c_y - u'(y)$. By Assumption 3, this is negative for $0 < y < \widetilde{y}$, and this is 0 for $y = \widetilde{y}$, by definition of $\widetilde{y}$. As a consequence, $J^* - J^\dagger < 0$ and the strategy (y^*, x^*, s^*) cannot be optimal.

The pair $((X, Z, S), (y, x, s))$ is admissible, by assumption. In addition, we define the vector functions $\mathbf{p}(t) = e^{-\rho t}\lambda(t)$ and $\mathbf{q} = e^{-\rho t}\gamma(t)$. By assumptions on λ and γ, $\mathbf{p}$ is piecewise continuous and piecewise continuously differentiable, and $\mathbf{q}$ is piecewise-continuous. We now check *a*) to *k*).

a): given the definition of f_0, we have

$$\frac{\partial f_0}{\partial y} = e^{-\rho t}(u'(x+y) - c_y), \quad \frac{\partial f_0}{\partial x} = e^{-\rho t}(u'(x+y) - c_x), \quad \frac{\partial f_0}{\partial s} = e^{-\rho t}(-c_s).$$

By Assumption 3, and thanks to the fact that $x + y > 0$ on the set U, these derivatives exist and are continuous; f is linear hence C^∞; this is the case also for constraints g_j, $j = 1, \ldots, 4$;

b): constraints $\bar{g}_j$, $j = 5, 6, 7$ are also linear, hence C^∞;

c): is satisfied with equality, by the assumption $\partial L / \partial \mathbf{u} = 0$;

d): is also satisfied by assumption;

e): the hamiltonian of the problem is given by the two first lines in the Lagrangean (2.2.1). It is a linear function of the state (X, Z, S) (although not strictly concave), and a concave function of the control (y, x, s), thanks to the concavity of the utility function $u(\cdot)$ in Assumption 3. The hamiltonian is therefore a concave function of $(\mathbf{x}, \mathbf{u})$;

f): is satisfied, consequence of conditions (2.2.5)– (2.2.11);

g): the constraints are all linear, hence concave, hence quasi-concave;

h): by assumption, λ_X and λ_S are continuous, hence (2.2.18) holds for $i = 1, 3$ by choosing $\beta_{kj}^{\pm} = 0$. By assumption (3.1.1) on λ_Z, jumps of this function are negative. It is then sufficient to choose

$$\beta_{k6}^+ = -e^{-\rho \tau_k}(\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+)), \quad \beta_{k6}^- = 0$$

in order to comply with (2.2.18);

i): is satisfied with equality by setting $\beta^{\pm}_{kj} = 0$, $j = 1, \ldots, 4$ (see the preliminary discussion);

j): is satisfied trivially for $j = 1, \ldots, 4$ by the choice made in i). Likewise for $j = 5, 7$ by picking $\beta^{\pm}_{kj} = 0$. Given the choices of $\beta^{\pm}_{k6}$ in h), and the assumption on jump instants τ_k which specifies that the constraint is always bound after the jump, we indeed have $\beta^{\pm}_{k6} \geq 0$ and $\beta^{-}_{k6} = 0$ since $Z(\tau_k^-) < \overline{Z}$;

k): since the state variables X, Z and S are bounded by the system of constraints,[3] Conditions (2.2.15)– (2.2.17) imply respectively

$$\lim_{t\to\infty} e^{-\rho t}\lambda_X(t) = \lim_{t\to\infty} p_1(t) = 0, \quad \lim_{t\to\infty} p_2(t) = 0, \quad \lim_{t\to\infty} p_3(t) = 0.$$

This in turn implies that $\lim_{t\to\infty} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) = 0$ for every admissible trajectory $\mathbf{x}$, since the difference $\mathbf{x}(t) - \mathbf{x}^*(t)$ is also bounded.

□

Our task is therefore to exhibit solutions to the first-order conditions which are continuous, or if not continuous, which satisfy the jump condition (2.2.18).

2.2.3 The admissible domain of S and Z

When $\beta > 0$, the model exhibits a *viability* or *controllability* problem that we study in this section.

Assume that for some reason, $x(t) = s(t) = 0$ over some interval of time. Then the dynamics of S and Z are given by:

$$\dot{S}(t) \;=\; -\,\beta S(t) \quad \text{and} \quad \dot{Z}(t) \;=\; \beta S(t) - \alpha Z(t)\ .$$

Let t_0 be some time instant in this interval and let us denote by S_0 and Z_0 the stocks of S and Z at this time: $S_0 \equiv S(t_0)$ and $Z_0 \equiv Z(t_0)$. Integrating the above system, we obtain for all t (in the case $\alpha \neq \beta$; see Section 3.2 on page 16 for the case $\alpha = \beta$):

$$\begin{aligned} S(t) &= S^0 e^{-\beta(t-t^0)} \\ Z(t) &= Z^0 e^{-\alpha(t-t^0)} - S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\alpha(t-t^0)} - e^{-\beta(t-t^0)}\right)\ . \end{aligned}$$

Eliminating t, we get the family of trajectories in the (S, Z) space:

$$Z(S; S_0, Z_0) \;=\; \left(\frac{S}{S_0}\right)^{\alpha/\beta}\left(Z_0 - \frac{\beta}{\alpha-\beta} S_0\right) \;+\; \frac{\beta}{\alpha-\beta} S\ .$$

These curves depend upon α and β and, structurally, only upon α/β. As a function of S, Z is first increasing and next decreasing whatever $\alpha > 0$ and $\beta > 0$ may be. The maximum is attained when $Z = (\alpha/\beta)S$. The family of these curves is illustrated in Figure 2.3. The movement is going from the right to the left though time. Under the line $Z = (\alpha/\beta)S$, the leaks flow βS is higher than the self-regeneration flow αZ so that the atmospheric stock of pollutant increases, whereas above the line the reverse holds and the atmospheric stock decreases.

Among these trajectories, let $Z_M(S)$ be the one, the maximum of which is equal to $\overline{Z}$, S_m the value of S for which this maximum is attained, and S_M the (strictly) positive value of S for which $Z_M(S) = 0$. Clearly, $S_M > S_m$. Given that the maxima of $Z(\cdot)$ are located along the line $Z = (\alpha/\beta)S$, we get for $Z = \overline{Z}$: $S_m = (\alpha/\beta)\overline{Z}$. Then

$$Z_M(S) \;=\; Z(S; S_m, \overline{Z}) \;=\; \frac{\beta}{\alpha-\beta}\left(S - \overline{Z}\left(\frac{S}{S_m}\right)^{\alpha/\beta}\right)\ .$$

[3] This argument holds *stricto sensu* when $\overline{S}$ and $\overline{Z}$ are finite. However, it holds also when $\overline{S} = +\infty$ and $\beta > 0$, because there is a finite admissible domain, see Section 2.2.3.

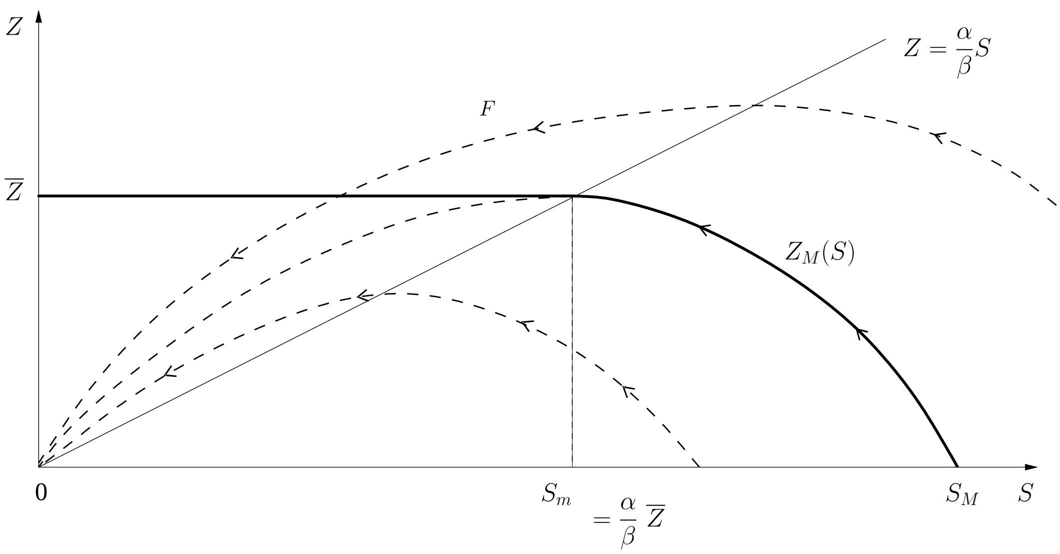

Figure 2.3: Admissible (S, Z) pairs

It follows that $S_M = \overline{Z}(\alpha/\beta)^{\alpha/(\alpha-\beta)}$, and it can be verified that $S_M > S_m$ for all values of α and β.

For any $S \in (S_m, S_M]$, the control vector $(s, \zeta x - s)$ points *outwards*, and it is easy to see that for any initial position located above the curve $Z = Z_M(S)$, and for any control, the trajectory will necessarily exit the domain $\{Z \leq \overline{Z}\}$. Such a trajectory is not viable. Likewise, if a non-zero control is applied at any point of the curve $(S, Z_M(S))$, then the trajectory will necessarily exit the domain $\{Z \leq \overline{Z}\}$, whatever control is applied later on.

Therefore, the set of *viable* initial states (S^0, Z^0) is delimited by the constraints $Z \leq \widetilde{Z}(S)$, where the function $\widetilde{Z}$ is defined on $[0, S_M]$ as:

$$\widetilde{Z}(S) \;=\; \begin{cases} \overline{Z}, & 0 \leq S \leq S_m \\ Z_M(S), & S_m \leq S \leq S_M. \end{cases} \tag{2.2.23}$$

Chapter 3

Preparation

3.1 Introduction to the solution

The central object of our analysis is the "phase", which we define as a piece of optimal path for which the set of active constraints on states or controls is constant. A complete optimal trajectory is necessarily decomposed into a succession of such phases. The method consists then in "glueing" together pieces of trajectory, each one being in some phase.

This chapter is devoted to the individual analysis of the different possible phases. The assembly of pieces of trajectories will be done in Chapter 4 for a simplification of the model. The complete solution for the model presented in Chapter 2 will be presented in a future version of this report.

The combinatorics of the exploration of phases is quite large *a priori*. Constraints (2.1.3)–(2.1.5) provide 2 situations each, constraint (2.1.6) provides 3, and the set of constraints (2.1.7)–(2.1.9) provide 4 distinct situations, for a potential total of 96 phases.

We choose to disregard the limit on the flow of renewable resource y, as well as capacity constraints $\overline{S}$ on the reservoir S. This simplification will allow us to concentrate on the importance of the self-regeneration rate α, the leakage rate β and the capture cost c_s on the shape of optimal extraction paths. However, most of what is reported in this report would remain true if the sequestered stock would be assumed to have a maximal capacity $\bar{S}$, as long as $\bar{S} > S_m = (\alpha/\beta)\overline{Z}$. The situation where $\beta = 0$ is the one studied in Lafforgue et al. (2008*a*) and Lafforgue et al. (2008*b*).

Ignoring the constraints $\overline{y}$ and $\overline{S}$ reduces the number of possible phases to 32. We will see however than only 8 phases are actually useful in the construction of optimal trajectories.

For this restricted problem, Corollary 2.1 takes the following form. The proof for this result is easily adapted from the proof of Corollary 2.1, with the aid of Footnote 3.

Corollary 3.1. *Assume there exist:*

- *a vector of continuous function* $(X, Z, S)(t)$*, a vector function* $(y, x, s)(t)$*, satisfying equations* (2.1.2), (2.1.3), (2.1.5), (2.1.6) *with* $\overline{y} = +\infty$, (2.1.8) *and* (2.1.9)*,*

- *a vector function* $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ *such that* λ_X *and* λ_S *are continuous and continuously differentiable, and* λ_Z *piecewise continuously differentiable, a piecewise-continuous vector function* $\gamma(t) = (\gamma_y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_X)(t)$*, satisfying equations (2.2.2)– (2.2.10) for all* t *(with* $\gamma_Y = 0$*),* (2.2.12)– (2.2.14) *for virtually every* t *(with* $\nu_S = 0$*), and conditions (2.2.15)–(2.2.17),*

- *a sequence of time instants* $0 < \tau_1 < \ldots < \tau_N$*, where* $Z(\tau_k^-) < \overline{Z}$ *and* $Z(\tau_k^+) = \overline{Z}$*, such that,*

$$\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+) \;\leq\; 0. \tag{3.1.1}$$

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal for the criterion (2.1.1).*

In the different sections of this chapter, we analyze separately the dynamics of each phase. We adopt the following common notation: t^0 denotes an arbitrary time instant at which the trajectory is the phase under study. The corresponding values of the state, costate variables and multipliers are denoted with the same superscript as in X^0, S^0, Z^0, λ_Z^0 etc. We express the value of the different relevant trajectories as a function of t and these "initial" values. They hold whether t is smaller or larger than t^0, as long as both time instants lie in an interval where the system stays in the phase without interruption.

We begin with the analysis of the "free" system, not submitted to any control (Section 3.2). Then we study phases which are "interior" with respect to state constraints (Section 3.3 to Section 3.5). Finally, we describe phases such that the atmospheric stock has reached its ceiling (Section 3.6).

In the course of analysis, the following threshold value for c_s will appear:

$$\hat{c}_s = \frac{\rho}{\rho+\beta}\frac{\overline{p}-c_x}{\zeta} \tag{3.1.2}$$

3.2 The uncontrolled system

In some cases, the state of the system is "free" in the sense that no control is applied to it: $x = s = 0$. We study this situation here.

Clearly, is $x = 0$, the $X(t) = X^0$ is constant.

Integrating the dynamical system:

$$\begin{cases} \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S \end{cases}$$

under initial conditions at t^0 yields:

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} - S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\alpha(t-t^0)} - e^{-\beta(t-t^0)}\right) \tag{3.2.1}$$

$$S(t) = S^0 e^{-\beta(t-t^0)} . \tag{3.2.2}$$

Eliminating the time variable, one finds that the trajectories can be written as curves in the domain (S, Z), parametrized by (S^0, Z^0):

$$Z = Z(S) = Z^0\left(\frac{S}{S^0}\right)^{\alpha/\beta} - \frac{\beta}{\alpha-\beta}\left(S^0\left(\frac{S}{S^0}\right)^{\alpha/\beta} - S\right) . \tag{3.2.3}$$

Observe that these curves depend only on the ratio α/β. As a function of S curves are increasing then decreasing. At the point where the maximum is reached, $Z = \beta S/\alpha$.

In particular, the curve which is such that the maximum is reached at point $(S_m, \overline{Z})$ is obtained by setting $S^0 = S_m$ and $Z^0 = \overline{Z}$. The equation obtained is:

$$\begin{aligned} Z = Z_M(S) &= \left(\frac{S}{S_m}\right)^{\alpha/\beta}\left(\overline{Z} - \frac{\beta}{\alpha-\beta}S_m\right) + \frac{\beta}{\alpha-\beta}S \\ &= \frac{\beta}{\alpha-\beta}\left(S - \overline{Z}\left(\frac{S}{S_m}\right)^{\alpha/\beta}\right) . \end{aligned} \tag{3.2.4}$$

The curve defined this way intercepts the S-axis at $S = 0$ and

$$S = S_M := S_m \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha-\beta}} = \overline{Z}\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\alpha-\beta}} .$$

These formulas must be modified in the limit case $\alpha = \beta$. In that case, we have $\overline{Z} = S_m$, then

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} + S^0\alpha(t-t^0)e^{-\alpha(t-t^0)} \qquad S(t) = S^0 e^{-\alpha(t-t^0)}$$

and

$$Z_M(S) = S - S \log \frac{S}{S_m} .$$

The value where this function vanishes is $S_M = eS_m$.

3.3 The system in the interior

When no state constraint is active, the dynamics of the adjoint variables take a particularly simple form, which yields closed-form expressions.

The interior of the domain, which we will denote by $\mathcal{D}$, is defined by the set of strict inequalities:

$$\mathcal{D} = \left\{ (X, S, Z) \in \mathbb{R}^3 \mid 0 < X(t) , \quad 0 < S(t) < \overline{S} , \quad 0 < Z(t) < \widetilde{Z}(S(t)) \right\} , \tag{3.3.1}$$

where the function $\widetilde{Z}$ has been defined in (2.2.23). For such time instants, the adjoint variables ν_X, ν_S and ν_Z vanish, and the dynamics of adjoint variables reduce to

$$\left\{ \begin{array}{lcl} \dot{\lambda}_X & = & \rho\lambda_X \\ \dot{\lambda}_Z & = & (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S & = & (\rho+\beta)\lambda_S - \beta\lambda_Z . \end{array} \right. \tag{3.3.2}$$

The dynamics on λ_X can clearly be separated from the rest. When $X > 0$, then $\lambda_X = 0$ because of Equation (2.2.9). Then, according to (2.2.12), $\dot{\lambda}_X = \rho\lambda_X$. It follows that for every t, t^0 in the period where $X > 0$,

$$\lambda_X(t) = \lambda_X^0 \, e^{\rho(t-t^0)} .$$

3.3.1 Dynamics of the adjoint variables

We concentrate now on λ_S and λ_Z. Integrating the dynamical system:

$$\left\{ \begin{array}{lcl} \dot{\lambda}_Z & = & (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S & = & (\rho+\beta)\lambda_S - \beta\lambda_Z \end{array} \right.$$

under initial conditions at t^0 yields:

$$\lambda_Z(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \tag{3.3.3}$$

$$\lambda_S(t) = \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\alpha-\beta}\lambda_Z^0 \left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)} \right) . \tag{3.3.4}$$

The pair $(\lambda_S(t), \lambda_Z(t))$ therefore lies on the curve:

$$\lambda_S = \lambda_S^0 \left(\frac{\lambda_Z}{\lambda_Z^0} \right)^{\frac{\rho+\beta}{\rho+\alpha}} - \frac{\beta}{\alpha-\beta} \left(\lambda_Z - \lambda_Z^0 \left(\frac{\lambda_Z}{\lambda_Z^0} \right)^{\frac{\rho+\beta}{\rho+\alpha}} \right) .$$

When $\alpha = \beta$, these formulas must be modified as follows:

$$\begin{array}{lcl} \lambda_S(t) & = & \left(\lambda_S^0 + \alpha(t-t^0)\lambda_Z^0 \right) e^{(\rho+\alpha)(t-t^0)} \\ \lambda_S & = & \dfrac{\lambda_Z}{\lambda_Z^0} \left(\lambda_S^0 - \lambda_Z^0 \dfrac{\alpha}{\rho+\alpha} \log\left(\dfrac{\lambda_Z}{\lambda_Z^0} \right) \right) . \end{array}$$

3.3.2 Dynamics of ratios

Define the ratio variables:

$$r(t) \ := \ \frac{Z(t)}{S(t)} \qquad r_\lambda(t) \ := \ \frac{\lambda_S(t)}{\lambda_Z(t)} \ .$$

It is straightforward to check that they satisfy the autonomous, first-order diferential equations:

$$\dot{r} \ = \ (\beta-\alpha)r+\beta \qquad \dot{r}_\lambda \ = \ (\beta-\alpha)r_\lambda-\beta \ ,$$

which do not depend on ρ. Integrating leads to the solutions:

$$\begin{aligned} r(t) \ &= \ \left(r(t_0)+\frac{\beta}{\beta-\alpha}\right)e^{(\beta-\alpha)(t-t_0)} \ - \ \frac{\beta}{\beta-\alpha} \\ r_\lambda(t) \ &= \ \left(r_\lambda(t_0)-\frac{\beta}{\beta-\alpha}\right)e^{(\beta-\alpha)(t-t_0)} \ + \ \frac{\beta}{\beta-\alpha} \ . \end{aligned}$$

When $\alpha=\beta$, these formulas take the form:

$$r(t) \ = \ r(t^0)+\alpha(t-t^0) \qquad r_\lambda(t) \ = \ r_\lambda(t^0)-\alpha(t-t^0) \ .$$

As an application of these formulas, observe that the time necessary for the system to go from a position (S^0,Z^0) to (S^1,Z^1) depends only on the ratios $r^0=Z^0/S^0$ and $r^1=Z^1/S^1$. The value of this duration is given by:

$$t_1-t_0 \ = \ \frac{1}{\beta-\alpha}\log\left(\frac{r^1+\frac{\beta}{\beta-\alpha}}{r^0+\frac{\beta}{\beta-\alpha}}\right) \ = \ \frac{1}{\beta-\alpha}\log\left(\frac{(\beta-\alpha)r^1+\beta}{(\beta-\alpha)r^0+\beta}\right) \ ,$$

when $\alpha\neq\beta$, and $t_1-t_0=(r^1-r^0)/\alpha$ when $\alpha=\beta$. In particular, when $Z(t_0)=0$, we have $r^0=0$ and:

$$t_1-t_0 \ = \ \frac{1}{\beta-\alpha}\log\left(\frac{\beta-\alpha}{\beta}r^1+1\right) \ .$$

Likewise for costate variables: the time necessary for the system to go from a position where the ratio is $r^0_\lambda=\lambda^0_S/\lambda^0_Z$ to one where the ratio is $r^1_\lambda=\lambda^1_S/\lambda^1_Z$ is given by:

$$t_1-t_0 \ = \ \frac{1}{\beta-\alpha}\log\left(\frac{(\beta-\alpha)r^1_\lambda-\beta}{(\beta-\alpha)r^0_\lambda-\beta}\right) \ ,$$

when $\alpha\neq\beta$, and $t_1-t_0=-(r^1_\lambda-r^0_\lambda)/\alpha$ when $\alpha=\beta$.

Observe also that the line $\{r_\lambda=\beta/(\beta-\alpha)\}=\{(\beta-\alpha)\lambda_S=\beta\lambda_Z\}$ is invariant. If $\beta>\alpha$, trajectories staring with $r_\lambda(t^0)>\beta/(\beta-\alpha)$ go to $+\infty$, and trajectories with $r_\lambda(t^0)<\beta/(\beta-\alpha)$ go to $-\infty$, as $t\to+\infty$. All trajectories tend do $\beta/(\beta-\alpha)>0$ when $t\to-\infty$. If $\beta<\alpha$, the converse situation occurs: all trajectories tend to $\beta/(\beta-\alpha)<0$ when $t\to+\infty$, and the limit when $t\to-\infty$ is $\pm\infty$ with the sign of $r_\lambda(t^0)-\beta/(\beta-\alpha)$.

3.3.3 Invariant

The following quantity is constant on pieces of trajectories in the interior of the domain $\mathcal{D}$:

$$(S(t)\lambda_S(t)+Z(t)\lambda_Z(t))\,e^{\rho t} \ .$$

3.4 Elimination of impossible phases

When the state of the system is not bound by a constraint, the structure of the cost function allows to eliminate controls that are necessarily suboptimal. This allows to eliminate certain phases from the construction of a solution.

Our first result is a sort of "bang-bang" principle for the capture control s in the interior of the domain.

Lemma 3.1. *Assume that $c_s > 0$. Consider a piece of optimal trajectory located in the interior of the domain $\mathcal{D}$, such that $x(t) > 0$. Then for every time instant t, either $s(t) = 0$, or $s(t) = \zeta x(t)$.*

Proof. Assume by contradiction that $0 < s(t) < \zeta x(t)$. Then by (2.2.5) and (2.2.6), we have $\gamma_s(t) = \gamma_{sx}(t) = 0$. Then, (2.2.2) reduces to:

$$-c_s - \lambda_Z(t) + \lambda_S(t) \;=\; 0\;. \tag{3.4.1}$$

Differentiating, we must have, over some time interval, $\dot{\lambda}_Z(t) = \dot{\lambda}_S(t)$. Using (2.2.13) and (2.2.14), this implies in turn that

$$(\rho+\alpha)\lambda_Z \;=\; (\rho+\beta)\lambda_S - \beta\lambda_Z \tag{3.4.2}$$

because $\nu_Z = 0$. Finally, solving (3.4.1)–(3.4.2), we find that the adjoint variables are necessarily constant and equal to:

$$\lambda_Z \;=\; \frac{\rho+\beta}{\alpha}\,c_s \qquad \lambda_S \;=\; \frac{\rho+\beta+\alpha}{\alpha}\,c_s\;.$$

However, these functions do not solve the differential system (3.3.2), unless $c_s = 0$. This is excluded by Assumption 2, hence the contradiction. □

Next, we rule out the possibility that both the renewable resource and the non-renewable resource be used at the same time.

Lemma 3.2. *Consider a piece of optimal trajectory located in the interior of the domain. Then either $x(t) > 0$ or $y(t) > 0$ but not both.*

Proof. Assume by contradiction that $x(t) > 0$ and $y(t) > 0$. Then $\gamma_x(t) = \gamma_y(t) = 0$ and the first-order conditions (2.2.2)– (2.2.4) reduce to: $x + y = \widetilde{y}$ and

$$0 \;=\; -c_s - \lambda_Z + \lambda_S + \gamma_s - \gamma_{sx} \tag{3.4.3}$$

$$0 \;=\; c_y - c_x - \lambda_X + \zeta\lambda_Z + \zeta\gamma_{sx}\;. \tag{3.4.4}$$

According to Lemma 3.1, either $s = 0$ and $\gamma_{sx} = 0$, or $s = \zeta x$ and $\gamma_s = 0$. In the first case, differentiating Equation (3.4.4) gives $\dot{\lambda}_X = \zeta\dot{\lambda}_Z$ or equivalently with (3.3.2): $\rho\lambda_X = \zeta(\rho+\alpha)\lambda_Z$. Then the adjoint variables are necessarily constant and equal to

$$\lambda_Z \;=\; \frac{c_y - c_x}{\alpha\zeta} \qquad \lambda_X \;=\; \frac{\rho+\alpha}{\rho}\,\frac{c_y - c_x}{\alpha}\;.$$

However, these functions do not solve the differential system (3.3.2): a contradiction.

In the second case, Equation (3.4.3) provides the identity $\lambda_Z + \gamma_{sx} = \lambda_S - c_s$, and replacing this into (3.4.4) yields:

$$0 \;=\; c_y - c_x - \zeta c_s - \lambda_X + \zeta\lambda_S\;.$$

Then the previous reasoning also leads to a contradiction. □

3.5 Dynamics in interior phases

Given Lemmas 3.1 and 3.2, the optimal control on an interior piece of trajectory therefore reduces to one of the three alternatives: either $y = 0$, $s = 0$, $x > 0$, or $y = 0$, $s = \zeta x$, $x > 0$, or $y = \widetilde{y}$, $x = s = 0$.

We name the first situation Phase “A”: it is characterized by the absence of constraints on the state, zero capture and exclusive consumption of nonrenewable energy.

We name the second situation Phase “B”: it is characterized by the absence of constraints on the state, total capture of the emissions due to nonrenewable energy.

The third situation is encountered when $X > 0$ but also when $X = 0$. We call it respectively: Phase “L” and Phase “T”.

We analyze the dynamics of the system in these three phases.

3.5.1 Dynamics in Phase A

Phase A corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is not reached ($Z(t) < \overline{Z}$), and no sequestration occurs ($s(t) = 0$). See Appendix A.1 on page 67.

Consumption is directly given by the first order equations:

$$x = q^d(c_x + \lambda_X - \zeta\lambda_Z) \tag{3.5.1}$$

and the value of the adjoint variable $\lambda_Z(t)$ is obtained using the "free" form (3.3.3), that is:

$$\lambda_Z(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)}$$

The integration of the dynamical system for the state variables gives:

$$X(t) = X^0 - \int_t^{t^0} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \tag{3.5.2}$$

$$\begin{aligned} Z(t) &= Z^0 e^{-\alpha(t-t^0)} + S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) \\ &\quad + \zeta \int_{t^0}^{t} e^{-\alpha(t-u)} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \end{aligned} \tag{3.5.3}$$

$$S(t) = S^0\, e^{-\beta(t-t^0)} . \tag{3.5.4}$$

3.5.2 Dynamics in Phase B

Phase B corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is not reached ($Z(t) < \overline{Z}$), and maximal sequestration occurs ($s(t) = \zeta x(t)$). See Appendix A.2 on page 68.

Consumption is directly given by the first order equations:

$$x = q^d(c_x + \lambda_X - \zeta\lambda_S + \zeta c_s) \tag{3.5.5}$$

and the value of the multiplier $\lambda_S(t)$ is obtained using the "free" form (3.3.4), that is:

$$\lambda_S(t) = \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\alpha-\beta}\lambda_Z^0\left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)}\right) .$$

The integration of the dynamical system for the state variables gives:

$$X(t) = X^0 - \int_t^{t^0} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_S(u) + \zeta c_s)\mathrm{d}u \tag{3.5.6}$$

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} + \beta \int_{t^0}^{t} e^{-\alpha(t-u)} S(u)\mathrm{d}u \tag{3.5.7}$$

$$S(t) = S^0\, e^{-\beta(t-t^0)} + \zeta \int_{t^0}^{t} e^{\beta(u-t)} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_S(u) + \zeta c_s)\mathrm{d}u . \tag{3.5.8}$$

3.5.3 Dynamics in Phase L and Phase T

Phase L corresponds to the situation where $x = s = 0$, and $y = \tilde{y}$ (see Appendix A.3 on page 69). This is also the definition of Phase T (see Appendix A.8 on page 74) with the difference that $X = 0$ in Phase T, and $X > 0$ in Phase L. This difference has no impact on the dynamics.

The trajectories of both the state and the adjoint variables follow the "free" forms (3.2.1)–(3.2.2) and (3.3.3)– (3.3.4), that is:

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} - S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\alpha(t-t^0)} - e^{-\beta(t-t^0)}\right)$$

$$
\begin{aligned}
S(t) &= S^0 e^{-\beta(t-t^0)} \\
\lambda_Z(t) &= \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \\
\lambda_S(t) &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\alpha-\beta}\lambda_Z^0 \left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)}\right) ,
\end{aligned}
$$

together with

$$X(t) = X^0 .$$

In Phase T, we have $X = 0$ in addition. Then for all t in the phase,

$$
\begin{aligned}
\lambda_Z(t) &= \lambda_S(t) = 0 \\
\lambda_X(t) &= c_y - c_x .
\end{aligned}
$$

Since $y > 0$, we have $\gamma_y = 0$. From the first-order equations (2.2.2) and (2.2.3), the other multipliers satisfy the following constraints:

$$0 = c_s - \gamma_s + \gamma_{sx}, \quad 0 = \zeta\gamma_{sx} + \gamma_x .$$

Since $\gamma_x \geq 0$ and $\gamma_{sx} \geq 0$, the second one implies $\gamma_x = 0$ and $\gamma_{sx} = 0$. Replacing in the first one, we have $\gamma_s = c_s$.

3.6 Boundary Phases (Phases with $Z = \overline{Z}$)

When $Z(t) = \overline{Z}$ over some interval of time, the dynamics (2.1.2) imply that the control is constrained by

$$\zeta x - s = \alpha\overline{Z} - \beta S = \beta(S_m - S) . \tag{3.6.1}$$

We analyze the consequences in this section, depending on whether s is constrained at 0, interior ($0 < s < \zeta x$) or constrained at its maximum ($s = \zeta x$).

3.6.1 Dynamics in Phase P (Constrained atmospheric stock and no capture)

If capture is further constrained to be 0, this actually determines the consumption

$$x(t) = \beta(S_m - S(t)) . \tag{3.6.2}$$

We call this situation Phase "P".

In such a phase, the values of the costate variables can be directly deduced from the first order conditions (2.2.2)–(2.2.4) and the dynamical system (2.2.13)–(2.2.14) (see Appendix A.4 on page 70).

$$\lambda_Z = \frac{1}{\zeta}\left(c_x + \lambda_X^0 e^{\rho t} - u'\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right)\right) \tag{3.6.3}$$

$$\lambda_S = \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta\int_{t^0}^{t} e^{(\rho+\beta)(t-u)}\lambda_Z(u)\mathrm{d}u \tag{3.6.4}$$

$$
\begin{aligned}
\nu_Z = {} & \frac{\rho}{\zeta}\lambda_X + \frac{\beta^2}{\zeta^2}Su''\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right) \\
& -(\rho+\alpha)\left(c_x + \lambda_X - u'\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right)\right) .
\end{aligned} \tag{3.6.5}
$$

The state variables are:

$$X(t) = X^0 + \frac{S^0}{\zeta}(1 - e^{-\beta(t-t^0)}) - \overline{x}(t - t^0) \tag{3.6.6}$$

$$S(t) \;=\; S^0\, e^{-\beta(t-t^0)}\ . \tag{3.6.7}$$

Along every optimal path in this phase, the fact that $s(t) = 0$ must imply by (2.2.6) that $\gamma_s(t) = c_s + \lambda_Z(t) - \lambda_S(t) \geq 0$.

3.6.2 Dynamics in Phase Q

Phase Q corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), and sequestration occurs, but not all emissions are sequestered ($0 < s(t) < \zeta x(t)$).

The use of the first order conditions and the dynamical system (see Appendix A.5 on page 71) leads to the following derivation. First, the first-order condition for s provides the identity:

$$\lambda_S(t) = \lambda_Z(t) + c_s\ . \tag{3.6.8}$$

Then, differentiating and using the dynamics on λ_Z, we obtain:

$$\dot\lambda_S \;=\; \dot\lambda_Z \;=\; \rho\lambda_Z + (\rho+\beta)c_s\ .$$

The adjoint variable for S is obtained by integrating Equation (2.2.14). The value of λ_Z is then deduced from (3.6.8). These are:

$$\lambda_Z(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z(t^0) + c_s\frac{\rho+\beta}{\rho}\right) \;-\; c_s\frac{\rho+\beta}{\rho} \tag{3.6.9}$$

$$\lambda_S(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z(t^0) + c_s\frac{\rho+\beta}{\rho}\right) \;-\; c_s\frac{\beta}{\rho}\ . \tag{3.6.10}$$

Finally, we also have the following expressions for ν_Z:

$$\begin{aligned}\nu_Z(t) \;&=\; (\rho+\beta)\lambda_S(t) - (\rho+\alpha+\beta)\lambda_Z(t)\\ &=\; (\rho+\beta)c_s - \alpha\lambda_Z(t) \;=\; (\rho+\alpha+\beta)c_s - \alpha\lambda_S(t)\end{aligned}$$

Let us focus on the trajectory of the co-state variable vector $(\lambda_Z(t), \lambda_S(t))$. If it happens that

$$0 \;=\; \lambda_Z(t^0) + c_s\frac{\rho+\beta}{\rho}\ , \tag{3.6.11}$$

then both quantities are constant and the system (4.2.4)– (4.2.5) is stationary at point

$$\Omega \;=\; \left(\;-\;c_s\frac{\rho+\beta}{\rho},\;-\;c_s\frac{\beta}{\rho}\right)\ . \tag{3.6.12}$$

If Condition (3.6.11) is not satisfied, then the vector $(\lambda_Z(t), \lambda_S(t))$ moves away from Ω on the line $\lambda_S = \lambda_Z + c_s$. In that case, whatever the value of λ_Z^0, we have:

$$\lim_{t\to-\infty}\lambda_Z(t) \;=\; -\,c_s\,\frac{\rho+\beta}{\rho}\ .$$

The dynamics for X and S are given by:

$$\dot X = -x \qquad \dot S = \zeta(x-\overline{x})\ .$$

Since the the values of consumption and capture are respectively given by:

$$x(t) \;=\; q^d(c_x + \lambda_X^0 e^{\rho t} - \zeta\lambda_Z(t)) \tag{3.6.13}$$

$$s(t) \;=\; \zeta x(t) - \beta(S_m - S(t)) \;=\; \zeta(x(t)-\overline{x}) + \beta S(t)\ , \tag{3.6.14}$$

they are integrated as:

$$X(t) \;=\; X^0 \;-\; \int_{t^0}^{t} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \tag{3.6.15}$$

$$S(t) \;=\; S^0 \;+\; \zeta\int_{t^0}^{t} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \;-\; \zeta\overline{x}(t-t^0)\;, \tag{3.6.16}$$

with $\lambda_Z(t)$ given by (3.6.9). From the analysis of the function λ_Z, we deduce that

$$\lim_{t\to-\infty} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u)) \;=\; q^d(c_x + \zeta\frac{\rho+\beta}{\rho}c_s)\;.$$

The comparison of this limit with $\overline{x}$ amounts to comparing c_s with some critical value. Indeed:

$$q^d(c_x + \zeta\frac{\rho+\beta}{\rho}c_s) \;\le\; \overline{x} \quad\Longleftrightarrow\quad c_x + \zeta\frac{\rho+\beta}{\rho}c_s \;\ge\; \overline{p} \quad\Longleftrightarrow\quad c_s \;\ge\; \hat{c}_s\;,$$

where $\hat{c}_s$ has been defined by Equation (3.1.2).

Accordingly, we have the following property of the trajectories in Phase Q:

$$\lim_{t\to-\infty} \frac{\dot{X}(t)}{\dot{S}(t)} \qquad \begin{cases} >0 & \text{if} \quad c_s > \hat{c}_s \\ =0 & \text{if} \quad c_s = \hat{c}_s \\ <0 & \text{if} \quad c_s < \hat{c}_s. \end{cases} \tag{3.6.17}$$

3.6.3 Dynamics in Phase R

Phase R corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), no sequestration occurs, but there is mixed consumption of the renewable *and* nonrenewable resource ($x(t) > 0$ and $y(t) > 0$).

Given the first order conditions and the ceiling constraint (see Appendix A.6 on page 72), the consumptions are given by:

$$x \;=\; \frac{\beta}{\zeta}(S_m - S) \tag{3.6.18}$$

$$y \;=\; \frac{\beta}{\zeta}(S - S_{\tilde{y}})\;. \tag{3.6.19}$$

The dynamics of costate variables are integrated explicitly as:

$$\begin{aligned}
\lambda_Z &= \frac{c_y - c_x}{\zeta}(e^{\rho(t-T)} - 1) \\
\lambda_S &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta\int_0^t e^{(\rho+\beta)(t-u)}\lambda_Z(u)du \\
&= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} + \frac{c_y - c_x}{\zeta}e^{\rho(t-T)}(1 - e^{\beta(t-t^0)}) - \frac{c_y - c_x}{\zeta}\frac{\beta}{\rho+\beta}(1 - e^{(\rho+\beta)(t-t^0)}) \\
\nu_Z &= -\alpha\lambda_Z + \frac{\rho}{\zeta}(c_y - c_x)\;.
\end{aligned}$$

The value of λ_Z is clearly negative for $t \le T$, which implies that the value of ν_Z is positive.

Consider an initial condition $(X^0, \overline{Z}, S^0)$ at time t^0, such that $S^0 \in [S_{\tilde{y}}, S_m]$. The dynamics of Phase R imply that:

$$\begin{aligned}
S(t) &= S^0 e^{-\beta(t-t^0)} \\
X(t) &= X^0 \;-\; \int_{t^0}^{t}\left(\overline{x} - \frac{\beta}{\zeta}S(u)\right)\mathrm{d}u
\end{aligned}$$

$$= \quad X^0 \; - \; \overline{x}(t-t^0) \; + \; \frac{1}{\zeta}(S^0 - S(t)) \; .$$

Eliminating the variable t as:

$$\beta(t-t^0) \;=\; \log\frac{S^0}{S(t)} \;,$$

we see that the trajectory is the curve:

$$X \;=\; X^0 \;+\; \frac{\overline{x}}{\beta}\log\frac{S}{S^0} \;+\; \frac{1}{\zeta}(S^0 - S) \; . \tag{3.6.20}$$

Observe that these curves are increasing and concave in the interval $S \in [S_{\widetilde{y}}, S_m]$, and their derivative is 0 when $S = S_m$.

Let us now consider the multiplier:

$$\begin{aligned}\gamma_s(t) \;&=\; c_s - \frac{\rho}{\rho+\beta}\frac{c_y - c_x}{\zeta} \;+\; e^{(\rho+\beta)(t-t^0)}\left(-\lambda_S^0 + (e^{\rho(t^0-T)} - \frac{\beta}{\beta+\rho})\frac{c_y-c_x}{\zeta}\right) \\ &=\; c_s - \bar{c}_s \;+\; e^{(\rho+\beta)(t-t^0)}\left(-\lambda_S^0 + (e^{\rho(t^0-T)} - \frac{\beta}{\beta+\rho})\frac{c_y-c_x}{\zeta}\right) \; . \qquad (3.6.21)\end{aligned}$$

The constant $\bar{c}_s$ has been defined by Equation (4.0.3). Consequently, assuming that the term between the last parentheses is positive, there exists a finite value $\bar{t}_s$ at which $\gamma_s(\bar{t}_s) = 0$ if, and only if, $c_s < \bar{c}_s$.

3.6.4 Dynamics in Phase S

Phase S corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), maximal sequestration occurs ($s(t) = \zeta x(t)$). These conditions imply that $x = \overline{x}$ and the trajectory is stationary at the point $(S_m, \overline{Z})$. This phase is described in Appendix A.7 on page 73.

The integration of the dynamics of the costate variables yields the following expressions:

$$\lambda_Z(t) \;=\; \frac{\rho+\beta}{\beta}(c_s - \frac{\overline{p}-c_x}{\zeta}) \;+\; \frac{c_y-c_x}{\zeta}e^{\rho(t-T)} \tag{3.6.22}$$

$$\lambda_S(t) \;=\; c_s \;-\; \frac{\overline{p}-c_x}{\zeta} \;+\; \frac{c_y-c_x}{\zeta}e^{\rho(t-T)} \; . \tag{3.6.23}$$

This in turn provides the value of the multiplier: from the first-order condition

$$\lambda_S \;=\; c_s + \lambda_Z + \gamma_{sx} \;, \tag{3.6.24}$$

we obtain

$$\gamma_{sx} \;=\; \frac{\rho}{\beta\zeta}(\overline{p} - c_x) \;-\; \frac{\rho+\beta}{\beta}c_s \;=\; \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) \; . \tag{3.6.25}$$

This value is constant over time. It is positive if and only if $c_s \le \hat{c}_s$.

The state trajectory is simply given by:

$$X(t) = X^0 - \overline{x}(t-t^0) \qquad S(t) = S_m \; . \tag{3.6.26}$$

Chapter 4

Unexhaustible resources

We study in this chapter the case where the resource stock X is assumed to be infinite, and there are no constraints on the stock of sequestered carbon: $\overline{S} = +\infty$.

Formally, the problem is the same as exposed in Section 2.1.6, except that there is no dynamics of the stock X. The system is described by the two variables $Z(t)$ and $S(t)$.

The first-order conditions associated with this new problem are easily obtained from that of the general problem by setting formally $\lambda_X = 0$. In particular, we use the same phase names as in Chapter 3.

For future reference in this chapter, we introduce or recall some notation. We will need some *critical values* on the variable S:

$$S_m = \frac{\alpha \overline{Z}}{\beta} \quad (\text{or:} \quad \overline{x} = \frac{\beta S_m}{\zeta}) \qquad S_{\widetilde{y}} = \frac{\zeta}{\beta}(\overline{x} - \widetilde{y}) \ , \tag{4.0.1}$$

with the equivalent:

$$S_m = \frac{\zeta \overline{x}}{\beta} \qquad S_m - S_{\widetilde{y}} = \frac{\zeta}{\beta}\widetilde{y} \ .$$

On the parameter c_s, critical values will be:

$$\hat{c}_s = \frac{\rho}{\rho+\beta}\frac{\overline{p} - c_x}{\zeta} \tag{4.0.2}$$

$$\bar{c}_s = \frac{\rho}{\rho+\beta}\frac{c_y - c_x}{\zeta} \ , \tag{4.0.3}$$

with the identity:

$$\bar{c}_s - \hat{c}_s = \frac{\rho}{\rho+\beta}\frac{c_y - \overline{p}}{\zeta} \ .$$

A consequence of Assumption 3 is that:

$$0 < \hat{c}_s < \bar{c}_s \ . \tag{4.0.4}$$

4.1 Terminal phases

The first question we address is that of the behavior of the trajectory when $t \to \infty$. As a consequence of the first-order conditions and the transversality conditions (2.2.16)– (2.2.17), only a few phases are consistent with the infinite part of the trajectory.

In this report, we call "terminal phase" a phase for which there exists an optimal trajectory and some t^0 for which the trajectory is within the phase for all $t \geq t^0$.

We stick to the convention of Chapter 3 that t^0 denotes the arbitrary time instant inside the phase currently under study.

4.1.1 Terminal P phase

In Phase P (see Appendix A.4 on page A.4), $Z = \overline{Z}$, $s = 0$, $y = 0$ and $x = \overline{x} - \beta S/\zeta$. The evolution of S is "free", and $S(t) = S^0 e^{-\beta(t-t^0)}$.

The first-order equations provide the value of λ_Z (see (3.6.3)):

$$\lambda_Z(t) \;=\; \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S(t))\right) \;=\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(t-t^0)})\right) \;. \tag{4.1.1}$$

In this last expression, both terms are negative. The second one tends to 0 as $t \to +\infty$. Accordingly,

$$\lim_{t\to+\infty} \lambda_Z(t) \;=\; \frac{c_x - \overline{p}}{\zeta} < 0.$$

Next, the expression found for λ_S in (3.6.4) is:

$$\begin{aligned}
\lambda_S &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta \int_{t^0}^{t} e^{(\rho+\beta)(t-v)} \lambda_Z(v)\mathrm{d}v \\
&= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\zeta} \int_{t^0}^{t} e^{(\rho+\beta)(t-v)} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(v-t^0)})\right) \mathrm{d}v \\
&= e^{(\rho+\beta)(t-t^0)} \left[\lambda_S^0 \;-\; \frac{\beta}{\zeta}\int_0^{t-t^0} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta v})\right)\mathrm{d}v\right] \;.
\end{aligned} \tag{4.1.2}$$

Invoking the transversality condition (2.2.17), that is:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_S S \;=\; 0 \;,$$

with $S(t) = S^0 e^{-\beta(t-t^0)}$, we get *for* $S^0 \neq 0$,

$$\lambda_S^0 \;=\; \int_0^{\infty} \frac{\beta}{\zeta} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta v})\right) \mathrm{d}v \;. \tag{4.1.3}$$

Finally, replacing in (4.1.2), we obtain the value for the λ_S:

$$\begin{aligned}
\lambda_S &= \frac{\beta}{\zeta} e^{(\rho+\beta)(t-t^0)} \int_{t-t^0}^{\infty} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta v})\right) \mathrm{d}v \\
&= \frac{1}{\zeta}\; L(S^0 e^{-\beta(t-t^0)}) \\
&= \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\; M(S^0 e^{-\beta(t-t^0)}) \;,
\end{aligned}$$

where we have defined the functions

$$L(S) \;=\; \beta \int_0^{\infty} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S e^{-\beta v})\right) \mathrm{d}v \tag{4.1.4}$$

$$M(S) \;=\; \beta \int_0^{\infty} e^{-(\rho+\beta)v} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S e^{-\beta v})\right) \mathrm{d}v \;. \tag{4.1.5}$$

The properties of these functions are studied in Appendix B. In particular, $M(S) \leq 0$, so that the formula for λ_S above gives a negative value because both terms in its right-hand side are negative.

The value of γ_s can be written as:

$$\begin{aligned}
\gamma_s(t) &= \lambda_Z(t) - \lambda_S(t) + c_s \\
&= c_s \;+\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(t-t^0)})\right) \\
&\qquad -\; \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} \;-\; \frac{1}{\zeta}M(S^0 e^{-\beta(t-t^0)}) \\
&= c_s - \hat{c}_s \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(t-t^0)})\right) \;-\; \frac{1}{\zeta}M(S^0 e^{-\beta(t-t^0)}) \;.
\end{aligned} \tag{4.1.6}$$

The case with an empty reservoir. The previous reasoning applies only to $S^0 \neq 0$, when the value of λ_S is computed. Assume now that $S^0 = 0$, so that $S(t) = 0$ for all t in the phase. This is the case without capture, which has been studied in Chakravorty et al. (2006).

The transversality condition (2.2.17) is automatically satisfied. In that case, from the solutions obtained in Section 3.6.1, and given that $\beta S_m/\zeta = \overline{x}$, we obtain:

$$\lambda_Z(t) = \frac{c_x - \overline{p}}{\zeta} \tag{4.1.7}$$

$$\begin{aligned}\lambda_S(t) &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} + \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}(1 - e^{(\rho+\beta)(t-t^0)}) \\ &= e^{(\rho+\beta)(t-t^0)}\left(\lambda_S^0 - \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right) + \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} .\end{aligned} \tag{4.1.8}$$

Finally, the function γ_s is:

$$\begin{aligned}\gamma_s(t) &= c_s + \frac{c_x - \overline{p}}{\zeta}\frac{\rho}{\rho+\beta} - \\ &= c_s - \hat{c}_s + e^{(\rho+\beta)(t-t^0)}\left(\lambda_S^0 - \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right) \\ &= c_s - \hat{c}_s + e^{(\rho+\beta)(t-t^0)}\left(\lambda_S^0 - \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right) .\end{aligned}$$

Since the system is motionless, it is expected that the function $\gamma_s(\cdot)$ will be positive, whatever the value of t and t^0, since t^0 has been arbitrarily chosen within the phase. The only way this can happen is to chose

$$\lambda_S^0 = \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} ,$$

which implies, for all t:

$$\lambda_S(t) = \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} \qquad \gamma_s(t) = c_s - \hat{c}_s .$$

Finally, the formulas established for λ_S, λ_Z and γ_s hold for all $S^0 \geq 0$. We can now prove the following result.

Lemma 4.1. *Phase P can be terminal only if $c_s > \hat{c}_s$. In that case, the entry point in Phase P is such that $S(t^0) \leq S_{\tilde{y}}$. Under this condition, the following configuration is a solution to the first order equations and the system of constraints: $x(t) = \overline{x} - \frac{\beta}{\zeta}S(t)$, $s(t) = y(t) = 0$, and*

$$S = S(t^0)e^{-\beta(t-t^0)} \tag{4.1.9}$$

$$Z = \overline{Z} \tag{4.1.10}$$

$$\lambda_Z = \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) = \frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) \tag{4.1.11}$$

$$\lambda_S = \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S) \tag{4.1.12}$$

$$\gamma_x = 0$$

$$\gamma_s = c_s - \hat{c}_s + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) - \frac{1}{\zeta}M(S) \tag{4.1.13}$$

$$\gamma_{sx} = 0$$

$$\gamma_y = c_y - u'(\overline{x} - \frac{\beta}{\zeta}S) \tag{4.1.14}$$

on the interval $t \in [t^0, \infty)$.

Proof. If the phase is permanent, then the conditions $\nu_Z(t) \geq 0$, $\gamma_s(t) \geq 0$ and $\gamma_y(t) \geq 0$ must hold for all value of t.

From (2.2.13), we have $\nu_Z = \dot{\lambda}_Z - (\rho+\alpha)\lambda_Z$. From (4.1.1), we obtain

$$\dot{\lambda}_Z(t) \;=\; -\frac{\beta^2}{\zeta^2} S^0 u"(\overline{x} - \frac{\beta}{\zeta} S(t)) \;,$$

and since $u"(\cdot) \leq 0$, $\dot{\lambda}_Z \geq 0$. Therefore λ_Z is increasing and since its limit as $t \to +\infty$ is negative, it is always negative. As a consequence, $\nu_Z \geq 0$.

Given that $\gamma_y(t) = u'(x(t)) - c_y$ and since $u'(\cdot)$ is decreasing, we have: $\gamma_y \geq 0 \iff x \geq \widetilde{y} \iff \overline{x} - \beta S/\zeta \geq \widetilde{y} \iff S \leq S_{\widetilde{y}}$ (see the definition in (4.0.1)).

Turning now to $\gamma_s(t)$, we see that the two last terms in (4.1.6) both tend to 0 as $t \to \infty$, since $u'(\overline{x}) = \overline{p}$ and $M(0) = 0$ (see Appendix B). Therefore, $\lim_{t\to+\infty} \gamma_s(t) = c_s - \hat{c}_s$ and a *necessary condition* for $\gamma_s(s)$ to be positive for all $t \geq t^0$ is:

$$c_s \;\geq\; \hat{c}_s \;.$$

On the other hand, the condition $c_s < \hat{c}_s$ is sufficient for the existence of a t^0 such that $\gamma_s(t) > 0$ for all t, since in that case $\lim_{t\to\infty} \gamma_s(t) > 0$.

□

Actually, we can show that $\gamma_s(\cdot)$ is increasing under the additional condition that u' is convex. Indeed, according to (4.1.13), we have:

$$\dot{\gamma_s}(t) \;=\; \dot{S}(t)\frac{1}{\zeta}\;\left(\frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) - M'(S)\right) \;.$$

We know that $\dot{S} < 0$. On the other hand, it is shown in the proof of Lemma B.3 (page 77) that

$$M'(S) \;\geq\; \frac{\beta}{\rho+2\beta}\;\frac{\beta}{\zeta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;.$$

Therefore,

$$\frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) - M'(S) \;\leq\; -\frac{\beta}{\zeta}\;\frac{\rho+\beta}{\rho+2\beta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right)$$

is negative. As a consequence, $\dot{\gamma}_s$ is positive.

4.1.2 Terminal S phase

The assumptions made in Phase S are that: $Z = \overline{Z}$, $s = \zeta x$ and $y = 0$.

Since Z is constant, $\dot{Z} = 0$ and therefore from (2.1.2), it is necessary that $\beta S = \alpha\overline{Z}$, that is, $S = S_m$.

Next, since S is constant as well, $\dot{S} = 0$, and it is necessary that $\zeta x = \beta S_m$, that is, $x = \overline{x}$. This implies $\gamma_x = \gamma_s = 0$.

Turning to first-order conditions, we find with (2.2.7) that $\gamma_y = c_y - u'(\overline{x}) = c_y - \overline{p}$, which is positive. Next, from (2.2.6) we obtain $\lambda_Z + \gamma_{sx} = \lambda_S - c_s$, and reporting this in (**??**), we find that:

$$\overline{p} - c_x - \zeta c_s + \zeta\lambda_S \;=\; 0,$$

so that the value of λ_S is constant. Using (2.2.14), the value of λ_Z is constant as well. Finally, their values and that of γ_{sx} are:

$$\lambda_Z = \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \qquad \lambda_S = c_s + \frac{c_x - \overline{p}}{\zeta} \qquad \gamma_{sx} = \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) \;.$$

Clearly, $\gamma_{sx} \geq 0$ if and only if $c_s \leq \hat{c}_s$.

Finally, from (2.2.13), we get $\nu_Z = -(\rho+\alpha)\lambda_Z$. If $c_s \leq \hat{c}_s$, then $c_s + (c_x - \overline{p})/\zeta = c_s - (1 + \beta/\rho)\hat{c}_s < 0$. We then have $\lambda_Z < 0$ and $\nu_Z > 0$.

We have proved the following result:

Lemma 4.2. *Phase S can be terminal if and only if $c_s \leq \hat{c}_s$. In that case, the following configuration is a solution to the first order equations and the system of constraints: $S(t) = S_m$, $Z(t) = \overline{Z}$, $x(t) = \overline{x}$, $s(t) = \zeta\overline{x}$, $y(t) = 0$ and*

$$\lambda_Z = \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \quad (4.1.15)$$

$$\lambda_S = c_s + \frac{c_x - \overline{p}}{\zeta} \quad (4.1.16)$$

$$\gamma_x = 0$$

$$\gamma_s = 0$$

$$\gamma_{sx} = \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) \quad (4.1.17)$$

$$\gamma_y = c_y - \overline{p} \quad (4.1.18)$$

on any time interval.

4.1.3 Terminal Q phase

Phase Q may be terminal in the very specific case $c_s = \hat{c}_s$, see in Section 4.2.2.

4.2 Non-terminal Phases

We now show that phases A, B, L, Q and R cannot be terminal. Doing so, we obtain some insight on the way these phases may begin or end.

4.2.1 Phases A, B and L

The common feature of these three phases is that the adjoint variables evolve "freely" according to the equations (3.3.2) analyzed in Section 3.3.

It can be verified, for instance using the results of Section 3.3.2, that

$$\lim_{t\to\infty} \lambda_Z(t) = -\infty \qquad \lim_{t\to\infty} \lambda_S(t) = +\infty \qquad \lim_{t\to\infty} \lambda_S(t) - \lambda_Z(t) = +\infty$$

under the following conditions: $\lambda_Z^0 < 0$, $\lambda_S^0 < 0$ and either (a) $\beta \leq \alpha$ or (b) $\beta > \alpha$, and $\lambda_S^0 > \beta/(\beta-\alpha)\lambda_Z^0$.

According to first-order condition (2.2.2), we have

$$\gamma_{sx}(t) - \gamma_s(t) = \lambda_S(t) - \lambda_Z(t) - c_s \quad \to \quad +\infty$$

as $t \to \infty$. If $x(t) > 0$ (Phase A or B), then only one of γ_s and γ_{sx} can be different from 0. Since both are positive, it means that eventually $\gamma_{sx}(t) > 0$ and $\gamma_s = 0$. In other words, the trajectory cannot stay in Phase A forever, and must necessarily enter Phase B, unless the state variable hits the boundary first.

When the trajectory is in Phase B, the consumption is given (see (3.5.5)) by:

$$x = q^d(c_x + \zeta c_s - \zeta\lambda_S) .$$

Then, when $t \to \infty$, $x(t)$ becomes necessarily strictly larger than $\overline{x}$, according to Assumption 3. It is actually possible that $x(t)$ tends to infinity if $\lim_{x\to\infty} u'(x)$ is finite. In every situation, we have (see Appendix A.2):

$$\dot{Z} + \dot{S} = \zeta x - \alpha Z > \zeta x - \alpha\overline{Z} = \zeta(x - \overline{x}) > \zeta(x - \widetilde{x}) > 0 .$$

As a consequence, we have $\lim_{t\to\infty}(Z(t)+S(t)) = +\infty$, but this is not possible because the domain of Phase B is bounded. So Phase B must end in finite time, when the trajectory hits the boundary or, as we shall see, if $\gamma_y(t) = 0$.

Finally, consider a trajectory perpetually in Phase L. According to Conditions (2.2.2) and (2.2.3) (see also Appendix A.3), given that $y = \widetilde{y}$, we must have:

$$\begin{aligned}\gamma_s - \gamma_{sx} &= \lambda_Z - \lambda_S + c_s \\ \zeta\gamma_{sx} + \gamma_x &= c_x - c_y - \zeta\lambda_Z \ ,\end{aligned}$$

and all three γ_x, γ_s and γ_{sx} must be positive (Conditions (2.2.5)– (**??**)). But by a linear combination of these two equations, we obtain:

$$\zeta\gamma_s + \gamma_x \;=\; -\zeta\lambda_S + \zeta c_s + c_x - c_y \quad \to \quad -\infty$$

as $t \to \infty$. This is a contradiction. Phase L cannot be terminal. It is necessary that the consumption x becomes nonnegative at some point in time.

We have therefore proved that none of the three "interior" phases can be terminal.

4.2.2 Phase Q

In Phase Q, characterized by $Z = \overline{Z}$, $y = 0$ and $0 < s < \zeta x$, the dynamics of the state are $\dot{Z} = 0$ and $\dot{S} = \zeta(x - \overline{x})$. The first-order equations imply the relationship

$$\lambda_Z - \lambda_S + c_s = 0. \tag{4.2.1}$$

The values of consumption and capture, specialized from Section 3.6.2, are respectively given by:

$$x(t) \;=\; q^d(c_x - \zeta\lambda_Z(t)) \tag{4.2.2}$$

$$s(t) \;=\; \zeta x(t) - \beta(S_m - S(t)) \;=\; \zeta(x(t) - \overline{x}) + \beta S(t) \ , \tag{4.2.3}$$

and the constraints $s > 0$ and $s < \zeta x$ are satisfied as long as, respectively, $x > \overline{x} - \beta S/\zeta$ and $S < S_m$.

The adjoint variables are given by Equations (3.6.9) and (3.6.10) which we recall here:

$$\lambda_Z(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) \;-\; c_s\frac{\rho+\beta}{\rho} \tag{4.2.4}$$

$$\lambda_S(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) \;-\; c_s\frac{\beta}{\rho} \ . \tag{4.2.5}$$

As observed in Section 3.6.2 (on page 22), if

$$0 \;=\; \lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}$$

then $(\lambda_Z(t), \lambda_S(t))$ is stationary at point Ω defined by (3.6.12). In that case, consumption is $x = q^d(c_x + \zeta c_s(\rho+\beta)/\rho)$ (is constant as well) and $\dot{S} = \zeta(x - \overline{x})$. The value of γ_y is:

$$\gamma_y = c_y - c_x - \zeta c_s\frac{\rho+\beta}{\rho} \;=\; \zeta\frac{\rho+\beta}{\rho}(\bar{c}_s - c_s) \ .$$

This is positive as long as $c_s \le \bar{c}_s$. In the special case $c_s = \hat{c}_s$, then $x = \overline{x}$ and $S(t)$ is constant. The phase can therefore *a priori* be terminal.

In other cases, $(\lambda_Z(t), \lambda_S(t))$ moves away from Ω and tends to infinity. If $\lambda_Z^0 < -c_s(\rho+\beta)/\rho$, then $\lambda_Z(t)$ tends to $-\infty$ when $t \to \infty$, so that the first-order condition on y (2.2.4):

$$0 \le \gamma_y = c_y - u'(x) = c_y - c_x + \zeta\lambda_Z$$

is eventually violated. If $\lambda_Z^0 > -c_s(\rho+\beta)/\rho$, then $\lambda_Z(t)$ tends to ∞ when $t \to \infty$, so $c_x - \zeta\lambda_Z(t)$ tends to $-\infty$. According to Assumption 3, the value of $x(t) = q^d(c_x - \zeta\lambda_Z(t))$ tends to infinity, possibly in finite time. Since $\dot{S}(t) = \zeta(x(t) - \overline{x})$, this implies that $S(t)$ tends to infinity, which is clearly not possible.

The results can be summarized as:

Lemma 4.3. *Phase Q is terminal if, and only if $c_s = \bar{c}_s$. In that case, the following constant trajectory is a solution of the first-order equations and the system of constraints: $S(t) = S^0$, $Z(t) = \overline{Z}$, $x(t) = \overline{x}$, $s(t) = \beta S^0$*

$$\begin{aligned} \lambda_Z &= -\frac{\rho+\beta}{\rho}\hat{c}_s = \frac{c_x - \overline{p}}{\zeta} \\ \lambda_S &= -\frac{\beta}{\rho}\hat{c}_s = \frac{\beta}{\rho+\beta}\,\frac{c_x-\overline{p}}{\zeta} \\ \gamma_y &= c_y - \overline{p} \end{aligned}$$

and $\gamma_x = \gamma_s = \gamma_{sx} = 0$ on any time interval and for any $0 \le S^0 \le S_m$.

4.2.3 Phase R

In Phase R, $Z = \overline{Z}$, $y > 0$ and $s = 0$. The dynamics of this phase can be specialized from the equations of Section 3.6.3.

In particular, we have $S(t) = S^0 e^{-\beta(t-t^0)}$ but also, according to (3.6.19), $y(t) = (\beta/\zeta)(S(t) - S_{\tilde{y}})$. Therefore, as $t \to \infty$, the value of y cannot remain positive. Another possibility is that γ_s may become negative. In any case, Phase R cannot be terminal.

We can state the following result:

Lemma 4.4. *The following configuration is a solution to the first order equations and the system of constraints:*

$$\begin{aligned} S &= S^0 e^{-\beta(t-t^0)} \\ Z &= \overline{Z} \\ \lambda_Z &= -\frac{c_y - c_x}{\zeta} \qquad (4.2.6) \\ \lambda_S &= \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y-c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\rho+\beta}\,\frac{c_y-c_x}{\zeta} \qquad (4.2.7) \\ x &= \frac{\beta}{\zeta}(S_m - S) \\ y &= \tilde{y} - x = \frac{\beta}{\zeta}(S - S_{\tilde{y}}) \\ \gamma_s &= c_s - \bar{c}_s - \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y-c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)} \end{aligned}$$

together with $\gamma_x = \gamma_{sx} = \gamma_y = 0$, as long as $S(t) \ge S_{\tilde{y}}$ and $\gamma_s(t) \ge 0$.

4.3 Junction between phases

We examine in this section how phases can be connected together. As a result of the analysis of Section 4.1, we know that all final phases (that is, the last phase in which an optimal trajectory enters) are located on the boundary: either the limit point is $(0, \overline{Z})$ when $c_s < \hat{c}_s$, or every point $(S, \overline{Z})$ with $0 \le S \le S_m$ is a limit point if $c_s = \hat{c}_s$, or the limit point is $(S_m, \overline{Z})$ if $c_s > \hat{c}_s$.

For this reason, we first look at the trajectories which follow the boundary of the domain (in Section 4.3.1). Next, we have a closer look at families of trajectories which pass through, or end up in, point $(S_m, \overline{Z})$ (in Section 4.3.3.5). We recapitulate the situation on the boundary in Section 4.3.2. Next, we discuss how trajectories coming from the inside of the domain can connect to the boundary (Section 4.3.3). Section 4.5.2 contains a local analysis of optimal curves when they connect to the boundary; this part is useful to assess the global consistency of the family of optimal curves.

4.3.1 Junction between phases on the boundary

The analysis of terminal phases reveals that *whatever the value of c_s*, all optimal trajecories eventually end up on one boundary of the domain, namely, the curve defined as:

$$B(S) \;=\; \begin{cases} \overline{Z} & \text{if } S \le S_m \\ Z_M(S) & \text{if } S_m \le S \le S_M . \end{cases} \tag{4.3.1}$$

Observe that the function $B(\cdot)$ is continuous, and it is differentiable because $Z'_M(S_m) = 0$. It is decreasing and concave.

The computation of optimal trajectories can be decomposed in two sub-problems: A) computing the optimal trajectory on the curve $Z = B(S)$, and B) computing the optimal way to join the curve. This section addresses the first problem. Section 4.3.3 is devoted to the second problem.

The following convention is adopted throughout: when a function $f(\cdot)$ of time (state, adjoint variable, Lagrange multiplier) refers to a generic trajectory in Phase ϕ, it will be denoted as $f^{(\phi)}$.

4.3.1.1 Phases Q/P

Assume that a trajectory begins at time t^0 in state $(S^0, \overline{Z})$ and in phase Q, then enters phase P at time t^{QP}, then stays in that phase forever. Denote $S^{QP} = S(t^{QP})$.

In Phase Q, the equations of the state and the multipliers are given in Section 4.2.2. In Phase P, they are given in Section 4.1.1.

We try to construct a trajectory such that the multipliers $\lambda_Z(\cdot)$ and $\lambda_S(\cdot)$ are continuous at $t = t^{QP}$. For $t < t^{QP}$, these functions are given by formulas for Phase Q, and for $t > t^{QP}$, they are given by formulas for terminal Phase P. Therefore: equating (4.2.4) and (4.1.12) on the other hand (after the appropriate change of variable in the formulas for the Phase P), we obtain the continuity equations (using the functions $L(\cdot)$ and $M(\cdot)$ which have been defined in equations (4.1.4) and (4.1.5) on p. 26):

$$S^{QP} := S^{(Q)}(t^{QP}) \;=\; S^0 \;+\; \zeta \int_{t^0}^{t^{QP}} q^d(c_x - \zeta\lambda_Z^{(Q)}(u))\mathrm{d}u \;-\; \zeta\overline{x}(t^{QP} - t^0) \tag{4.3.2}$$

$$\begin{aligned} \lambda_Z^{(Q)}(t^{QP}) \;&=\; e^{\rho(t^{QP}-t^0)}\left(\lambda_Z^0 + \frac{\rho+\beta}{\rho}c_s\right) - \frac{\rho+\beta}{\rho}c_s \\ &=\; \lambda_Z^{(P)}(t^{QP}) \;=\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right) \end{aligned} \tag{4.3.3}$$

$$\begin{aligned} \lambda_S^{(Q)}(t^{QP}) \;&=\; c_s \;+\; e^{\rho(t^{QP}-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) \\ &=\; \lambda_S^{(P)}(t^{QP}) \;=\; \frac{1}{\zeta}\;L(S^{QP}) \;. \end{aligned} \tag{4.3.4}$$

The unknown quantities in these equations are: $t^{QP} - t^0$, S^{QP} and λ_Z^0. We have to discuss under which conditions there exists a solution to this system.

We first determine S^{QP}. Eliminating the factor of $e^{\rho(t^{QP}-t^0)}$ between Equations (4.3.3), (4.3.4), we obtain the equality:

$$\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right) \;+\; c_s \;=\; \frac{1}{\zeta}L(S^{QP}) \;=\; \frac{\beta}{\rho+\beta}\,\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^{QP}) \;.$$

This is actually equivalent to require that the function $\gamma_s^{(P)}$ given by Equation (4.1.13) is equal to 0 at $t = t^{QP}$, which gives directly this formula. Rewriting this equation gives the form:

$$\zeta(c_s - \hat{c}_s) \;+\; \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP}) \;=\; M(S^{QP}) \;. \tag{4.3.5}$$

The unique unknown quantity in this equation is S^{QP}. The existence of solutions to this equation is the topic of the following lemma. Define the critical cost:

$$c_{sm} \;:=\; \frac{c_y - c_x}{\zeta} + \frac{1}{\zeta} L(S_{\widetilde{y}}) \;=\; \hat{c}_s + \frac{c_y - \overline{p}}{\zeta} + \frac{1}{\zeta} M(S_{\widetilde{y}}) \;. \tag{4.3.6}$$

The constant c_{sm} defined in (4.3.6) is such that $c_{sm} > \bar{c}_s$, as proved in (B.0.4).

Lemma 4.5. *We have the following properties, under Assumption 3:*

(i) if $c_s < \hat{c}_s$, then Equation (4.3.5) *has no positive solution;*

(ii) if in addition $u'(\cdot)$ is convex, and if

$$\hat{c}_s \le c_s \le c_{sm} \tag{4.3.7}$$

then Equation (4.3.5) *has a unique solution $S^{QP} \in [0, S_{\widetilde{y}}]$;*

(iii) if in addition $u'(\cdot)$ is convex, and if $c_s > c_{sm}$, then Equation (4.3.5) *has no solution in $[0, S_{\widetilde{y}}]$.*

Proof. Statement (i) is proved as Lemma B.2 in Appendix B.

For (ii), Lemma B.3 states that there is at most one intersection. When $S^{QP} = 0$, the left-hand side of (4.3.5) is $\zeta(c_s - \hat{c}_s) \ge 0$ whereas the right-hand side is 0. There will necessarily be a solution in the interval $[0, S_{\widetilde{y}}]$ if the left-hand side evaluated at $S^{QP} = S_{\widetilde{y}}$, that is, $\zeta(c_s - \hat{c}_s) + \overline{p} - c_y$, is smaller than the right-hand side evaluated at the same point, that is, $M(S_{\widetilde{y}})$. This condition is exactly $c_s \le c_{sm}$. We have therefore existence and uniqueness in this case.

Finally, (iii) is also a consequence of the proof of Lemma: the function called $h(\cdot)$ in this lemma is decreasing, and its value at $S = S_{\widetilde{y}}$ is strictly positive if $c_s > c_{sm}$. Therefore, this function has no zero, that is, Equation (4.3.5) has no solution. □

Now that S^{QP} has been determined, the remaining unknowns can be computed as well. First, from (4.3.3):

$$e^{\rho(t^{QP} - t^0)} \left(\lambda_Z^0 + \frac{\rho+\beta}{\rho} c_s \right) \;=\; \frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right) \;.$$

Then, in Phase Q (that is, for $t < t^{QP}$), the function $\lambda_Z^{(Q)}$ can be written as:

$$\lambda_Z^{(Q)}(t) \;=\; e^{\rho(t - t^{QP})} \left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right) \right] \;-\; \frac{\rho+\beta}{\rho} c_s \;. \tag{4.3.8}$$

Under the condition (4.3.7), the term inside brackets is positive (Lemma B.4). Then the function $\lambda_Z^{(Q)}(t)$ is negative and increasing, and it is bounded on the interval $(-\infty, t^{QP}]$: its limit when $t \to -\infty$ is $-(\rho+\beta)c_s/\rho$. This limit is the point Ω introduced in Section 3.6.2.

The condition $\gamma_y(t) \ge 0$, or equivalently, $\lambda_Z(t) \ge (c_x - c_y)/\zeta$ is required for Phase Q. Given the value of $\lambda_Z(t)$ in (4.3.8), this condition is equivalent to:

$$\frac{c_x - c_y}{\zeta} \;\le\; e^{\rho(t - t^{QP})} \left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right) \right] \;-\; \frac{\rho+\beta}{\rho} c_s$$

$$\frac{c_x - c_y}{\zeta} \;+\; \frac{\rho+\beta}{\rho} c_s \;\le\; e^{\rho(t - t^{QP})} \left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right) \right] \;.$$

The left-hand side of this inequality is $(\rho+\beta)(c_s - \bar{c}_s)/\rho$. The inequality is therefore automatically satisfied if $c_s \le \bar{c}_s$. If $c_s > \bar{c}_s$, it is equivalent to:

$$t - t_{QP} \;\ge\; \frac{1}{\rho} \log \left(\frac{\frac{c_x - c_y}{\zeta} + c_s \frac{\rho+\beta}{\rho}}{\frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta} S^{QP} \right) \right) + c_s \frac{\rho+\beta}{\rho}} \right) \;. \tag{4.3.9}$$

Since the function $\lambda_Z^{(Q)}$ is increasing, then $\lambda_Z^{(Q)}(t) \leq \lambda_Z^{(Q)}(t^{QP})$, then it follows that

$$x^{(Q)}(t) \;=\; q^d(c_x - \zeta\lambda_Z^{(Q)}(t)) \;<\; \overline{x} - \frac{\beta}{\zeta}S^{QP} \;<\; \overline{x}\,.$$

As a consequence, $\dot{S}^{(Q)}(t) = \zeta(x(t) - \overline{x}) < -\beta S^{QP}/\zeta < 0$. This property implies that equation (4.3.2) can be solved for every value of $S^0 \in [S^{QP}, S_m]$: the solution gives the value of $t^{QP} - t^0$.

We summarize the solution just constructed in the following result.

Lemma 4.6. *Under Assumption 3 and supposing $u'(\cdot)$ convex, assume also that $\hat{c}_s \leq c_s \leq c_{sm}$. Then, denoting with S^{QP} the unique solution to Equation* (4.3.5)*, and t^{QP} the unique solution to equation* (4.3.2)*, the following configurations satisfy the first-order conditions and the system of constraints.*

for $S^0 \leq S^{QP}$: *the trajectory in Phase P, starting from $S(t^0) = S^0$, as described in Lemma 4.1;*

for $S^0 > S^{QP}$: *the trajectory in Phase Q, starting from $S(t^0) = S^0$, described by Equations* (4.2.2)– (4.2.5)*, for $t \in [t^0, t^{QP}]$ (equivalently, Equations* (4.3.8) *and* (4.2.1) *for costate variables), and for $t \in [t^{QP}, \infty)$, the trajectory in Phase P, starting from $S(t^{QP}) = S^{QP}$, as described in Lemma 4.1, for every value of t^0 satisfying Condition* (4.3.9) *in case $\bar{c}_s < c_s \leq c_{sm}$.*

Proof. The only constraint not checked yet is $\nu_Z \geq 0$. From (2.2.13), $\nu_Z = \dot{\lambda}_Z - (\rho + \alpha)\lambda_Z$. We have observed that λ_Z is negative and increasing. This difference is therefore always positive. □

The result is not explicit on the exact range of values for which the trajectory starts in Phase Q. We come back to this point in Section 4.3.1.3.

4.3.1.2 Phases R/P

Assume the system is in phase R at time t^0, with initial position $(S^0, \overline{Z})$, and that it passes from phase R to phase P at time t^{RP}. When the transition occurs, the position is necessarily $S(t^{RP}) = S_{\widetilde{y}}$, as proved in (4.3.10) below.

Since in Phase R, $S(t) = S^0 e^{-\beta(t-t^0)}$, the condition $S(t^{RP}) = S^1$ provides the value of $t^{RP} - t^0 = -\beta^{-1}\log(S^1/S^0)$.

When in Phase R, the evolution of state and multipliers is given by (see (4.2.6) and (4.2.7)):

$$\begin{aligned}
S^{(R)}(t) &= S^{(R)}(t^0)e^{-\beta(t-t^0)} \\
\lambda_Z^{(R)}(t) &= -\frac{c_y - c_x}{\zeta} \\
\lambda_S^{(R)}(t) &= \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right)e^{(\rho+\beta)(t-t^0)} \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\,.
\end{aligned}$$

On the other hand, the equations for a terminal Phase P, starting in $S(t^{RP}) = S^1$ are (see (4.1.11) and (4.1.12) on page 27):

$$\begin{aligned}
\lambda_Z^{(P)} &= \frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^1 e^{(t-t^{RP})})\right) \\
\lambda_S^{(P)} &= \frac{\beta}{\rho+\beta}\,\frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S^1 e^{-\beta(t-t^{RP})})\,.
\end{aligned}$$

The continuity of the multipliers imposes that $\lambda_Z^{(P)}(t^{RP}) = \lambda_Z^{(R)}(t^{RP})$ and $\lambda_S^{(P)}(t^{RP}) = \lambda_S^{(R)}(t^{RP})$. The first condition implies:

$$\begin{aligned}
\frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^1)\right) &= -\frac{c_y - c_x}{\zeta} \\
\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^1)) &= \overline{p} - c_y
\end{aligned}$$

$$S^1 \;=\; \frac{\zeta}{\beta}(\overline{x} - \tilde{y}) \;=\; S_{\tilde{y}} \,. \tag{4.3.10}$$

The second condition implies:

$$\left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) \;=\; e^{-(\rho+\beta)(t^{RP}-t^0)}\left[\frac{\beta}{\rho+\beta}\frac{c_y - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^1)\right] \,,$$

from which the value of $\lambda_S^{(R)}(t)$ for $t \leq t^{RP}$ is derived as

$$\lambda_S^{(R)}(t) \;=\; e^{(\rho+\beta)(t-t^{RP})}\left[\frac{\beta}{\rho+\beta}\frac{c_y - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^1)\right] \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta} \,.$$

Finally, the value of $\gamma_s^{(R)}$ is computed as:

$$\begin{aligned}\gamma_s^{(R)} \;&=\; \lambda_Z^{(R)} - \lambda_S^{(R)} + c_s \\ &=\; c_s \;-\; \bar{c}_s \;-\; \left[\frac{\beta}{\rho+\beta}\frac{c_y - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S_{\tilde{y}})\right] e^{(\rho+\beta)(t-t^{RP})} \,.\end{aligned} \tag{4.3.11}$$

The term inside brackets in (4.3.11) is positive, as a consequence of Inequality (B.0.3). The function $\gamma_s^{(R)}$ is therefore decreasing on the interval $(-\infty, t^{RP}]$. Its limit when $t \to -\infty$ is $c_s - \bar{c}_s$ and its value at $t = t^{RP}$ is, using (4.3.11):

$$\gamma_s^{(R)}(t^{RP}) \;=\; c_s \;-\; \bar{c}_s \;-\; \frac{\beta}{\rho}(\bar{c}_s - \hat{c}_s) - \frac{1}{\zeta}M(S_{\tilde{y}}) \,.$$

The function $\gamma_s^{(R)}$ is positive on the interval $(-\infty, t^{RP}]$ if and only if this value is positive, and this is equivalent to:

$$c_s \;-\; \bar{c}_s \;-\; \frac{\beta}{\rho}(\bar{c}_s - \hat{c}_s) - \frac{1}{\zeta}M(S_{\tilde{y}}) \;\geq\; 0 \qquad \Longleftrightarrow \qquad c_s \;\geq\; c_{sm}$$

(after rearrangements), where c_{sm} is defined in (4.3.7).

The results are summarized as follows.

Lemma 4.7. *Under Assumption 3, assume also that $c_s \geq c_{sm}$. Then, for every $S^0 \in [S_{\tilde{y}}, S_m]$, the following configuration satisfies the first-order conditions and the system of constraints. Let $t^{RP} = t^0 - \beta^{-1}\log(S_{\tilde{y}}/S^0)$. For $t \in [t^0, t^{RP}]$: the trajectory is in Phase R, as described in Lemma 4.4; for $t \in [t^{RP}, \infty)$: the trajectory in Phase P, as decribed in Lemma 4.1.*

4.3.1.3 Phases R/Q

In Section 4.3.1.1, we have left open the issue of whether Phase Q can start from any initial $S^0 \in [S^{QP}, S_m]$, where S^{QP} solves Equation (4.3.5). We resolve this issue by considering the possibility that a Phase R precedes Phase Q.

Assume the system is in phase R at time t^0, with initial position $(S^0, \overline{Z})$, and that it passes from phase R to phase Q at time t^{RQ} and location $S^{RQ} = S(t^{RQ})$.

Since in Phase R, $S(t) = S^0 e^{-\beta(t-t^0)}$, the condition $S(t^{RQ}) = S^{RQ}$ provides the value of $t^{RQ} - t^0 = -\beta^{-1}\log(S^{RQ}/S^0)$.

When in Phase R, the evolution of state and the multipliers is given by (see (4.2.6) and (4.2.7)):

$$\begin{aligned}S^{(R)}(t) \;&=\; S^{(R)}(t^0)e^{-\beta(t-t^0)} \\ \lambda_Z^{(R)}(t) \;&=\; -\frac{c_y - c_x}{\zeta} \\ \lambda_S^{(R)}(t) \;&=\; \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)} \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta} \,.\end{aligned}$$

On the other hand, assuming that Phase Q is followed by Phase P, we have the form (4.3.8) for $\lambda_Z^{(Q)}$, and we have the relation $\lambda_S^{(Q)} = \lambda_Z^{(Q)} + c_s$ which characterizes Phase Q. The continuity of the multipliers imposes that $\lambda_Z^{(Q)}(t^{RQ}) = \lambda_Z^{(R)}(t^{RQ})$ and $\lambda_S^{(Q)}(t^{RQ}) = \lambda_S^{(R)}(t^{RQ})$. The first condition writes just as:

$$\lambda_Z^{(Q)}(t^{RQ}) \;=\; -\frac{c_y - c_x}{\zeta}$$

From this equation (see (4.3.9) on page 33) and the reasoning preceding it), this equation can be solved only for $c_s > \bar{c}_s$ and we get:

$$t_{RQ} - t_{QP} \;=\; \frac{1}{\rho}\,\log\left(\frac{\frac{c_x - c_y}{\zeta} + c_s\frac{\rho+\beta}{\rho}}{\frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta}S^{QP}\right)\right) + c_s\frac{\rho+\beta}{\rho}}\right)\;. \tag{4.3.12}$$

Remember that S^{QP} itself depends on c_s since it is defined as the solution of (4.3.5). The continuity of λ_S at $t = t^{RQ}$ provides the value of λ_S^0, and then:

$$\lambda_S^{(R)}(t) \;=\; (c_s - \bar{c}_s)\, e^{(\rho+\beta)(t-t^{RQ})} \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\;. \tag{4.3.13}$$

Finally, dynamics of the state in Phase Q are given by (3.6.16), which yields

$$\begin{aligned}
S(t) \;&=\; S^{QP} + \zeta\int_{t^{QP}}^{t}(x^{(Q)}(t) - \overline{x})\mathrm{d}t \;=\; S^{QP} + \zeta\int_{t^{QP}}^{t}(q^d(c_x - \lambda_Z^{(Q)}(t)) - \overline{x})\mathrm{d}t \\
&=\; S^{QP} - \zeta\overline{x}(t - t^{QP}) \\
&\quad +\zeta\int_{t^{QP}}^{t} q^d\left(c_x - \zeta\left(e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}c_s + \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)\right] \;-\; \frac{\rho+\beta}{\rho}c_s\right)\right)\mathrm{d}t \\
&=\; S^{QP} - \zeta\overline{x}(t - t^{QP}) \\
&\quad +\zeta\int_0^{t-t^{QP}} q^d\left(c_x + \frac{\rho+\beta}{\rho}\zeta c_s - e^{\rho v}\left[\frac{\rho+\beta}{\rho}\zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right]\right)\mathrm{d}v\;.
\end{aligned}$$

In particular, the value of the stock S at the time the system passes from Phase R to Phase Q is given by:

$$\begin{aligned}
S^{RQ} \;&=\; S^{QP} - \zeta\overline{x}(t^{RQ} - t^{QP}) \\
&\quad +\zeta\int_0^{t^{RQ}-t^{QP}} q^d\left(c_x + \frac{\rho+\beta}{\rho}\zeta c_s - e^{\rho v}\left[\frac{\rho+\beta}{\rho}\zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right]\right)\mathrm{d}v\;.
\end{aligned} \tag{4.3.14}$$

Depending on the value of c_s, this value S^{RQ} is smaller than S_m or not.

This leads us to introduce a new threshold for c_s: this value c_{sQ} is such that phase R "just disappears" at the stock value $S = S_m$. More precisely, we have simultaneously:

$$S^{(Q)}(t^{RQ}) \;=\; S_m \qquad \lambda_Z^{(Q)}(t^{RQ}) \;=\; \frac{c_x - c_y}{\zeta}\;.$$

Given the formula above for S^{RQ}, we have the equivalent form:

$$\begin{aligned}
S_m \;&=\; S^{QP} - \zeta\overline{x}(t^{RQ} - t^{QP}) \\
&\quad +\zeta\int_0^{t^{RQ}-t^{QP}} q^d\left(c_x + \frac{\rho+\beta}{\rho}\zeta c_s - e^{\rho v}\left[\frac{\rho+\beta}{\rho}\zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right]\right)\mathrm{d}v\;.
\end{aligned} \tag{4.3.15}$$

The values of S^{QP} and $t^{RQ} - t^{QP}$ are given respectively by (4.3.5) and (4.3.12). The number c_{sQ} is the unique solution of this equation; it belongs to the interval $[\hat{c}_s, c_{sm}]$.

We are now in position to complete Lemma 4.6.

Lemma 4.8. *Under Assumption 3 and supposing $u'(\cdot)$ convex, assume also that $\hat{c}_s \leq c_s \leq c_{sQ}$. Denote with S^{QP} the unique solution to Equation* (4.3.5)*. Then for every $S^0 \in [S^{QP}, S_m]$, the trajectory described in Lemma 4.6 satisfies the first-order conditions and the system of constraints.*

Lemma 4.9. *Under Assumption 3 and supposing $u'(\cdot)$ convex, assume also that $c_{sQ} < c_s \leq c_{sm}$. Denote with S^{QP} the unique solution to Equation* (4.3.5)*. Let S^{RQ} be defined by* (4.3.14)*: then $S^{RQ} \leq S_{\widetilde{y}}$.*

Then the following configurations satisfy the first-order conditions and the system of constraints.

for $S^0 \in [S^{QP}, S^{RQ}]$**:** *the trajectory in Phase Q, as described in Lemma 4.6;*

for $S^0 \in [S^{RQ}, S_m]$**:** *the trajectory in Phase R, starting from $S(t^0) = S^0$ for $t \in [t^0, t^{RQ}]$ (where $t^{RQ} = t^0 - \beta^{-1} \log(S^{RQ}/S^0)$), and for $t \in [t^{RQ}, \infty)$, the trajectory in Phase Q, starting from $S(t^{RQ}) = S^{RQ}$, as described in Lemma 4.6.*

Proof. Again, it is sufficient to check that $\nu_Z \geq 0$. It follows from the fact that λ_Z is constant and negative in Phase R. □

4.3.2 Synthesis on the boundary

The situation of phases is summarized in Figure 4.1 (page 38). This figure depicts the optimal consumption $x(t)$, $y(t)$ and capture $s(t)$ as a *state feedback*. As a function of time, $S(t)$ is decreasing (or constant if $c_s = \hat{c}_s$) so that the evolution occurs from right to left. Capture is represented as $s(t)/\zeta$ in order to make an easier comparison with its maximum value $x(t)$.

The different cases are detailed as follows. The trajectory of interest is starting at $S = S_M$ and $Z = 0$.

Case $c_s \geq c_{sm}$**.** In this situation, the sequence of phases is $L/R/P$ (Lemma 4.7 on Page 35). Capture $s(t)$ is zero at all times. Both paths $x(t)$ and $y(t)$ are continuous. The value function $V(S)$ is continuous with a continuous derivative.

Case $c_{sm} \geq c_s \geq c_{sQ}$**.** In this situation, the sequence of phases is $L/R/Q/P$ (Lemma 4.9 on Page 37). The consumption/capture paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{RQ}$ (i.e. when $S = S^{RQ}$). The function $x(t) + y(t)$ is continuous everywhere. The value function $V(S)$ is continuous with a continuous derivative.

Case $c_{sQ} \geq c_s > \hat{c}_s$**.** In this situation, the sequence of phases is reduced to $L/Q/P$ (Lemma 4.8 on Page 37). The paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{LQ}$ (i.e. when $S = S^{LQ}$). The function $x(t) + y(t)$ is also discontinuous at that point. The value function $V(S)$ is continuous with a continuous derivative, except at $S = S_m$.

Case $c_s = \hat{c}_s$**.** In this particular situation, the sequence of phases is L/Q, but all points in phase Q are stationary (Lemma 4.3 on Page 31). The paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{LQ}$ (i.e. when $S = S_m$). The function $x(t) + y(t)$ is also discontinuous at that point. The value function $V(S)$ is continuous with a continuous derivative, except at $S = S_m$.

4.3.3 Junction with the boundary of the domain

We now study how trajectories inside the domain join the boundary. It turns out that, depending on the value of the parameters, two types of junctions take place. One is a "standard" junction, with continuity of state and costate variables: we will show that it takes place with the boundary phases called P, Q, R and L. The second one is a junction at the particular location $(S_m, \overline{Z})$, with a discontinuity in the costate variable λ_Z. We analyze this specific situation in Section 4.3.3.5.

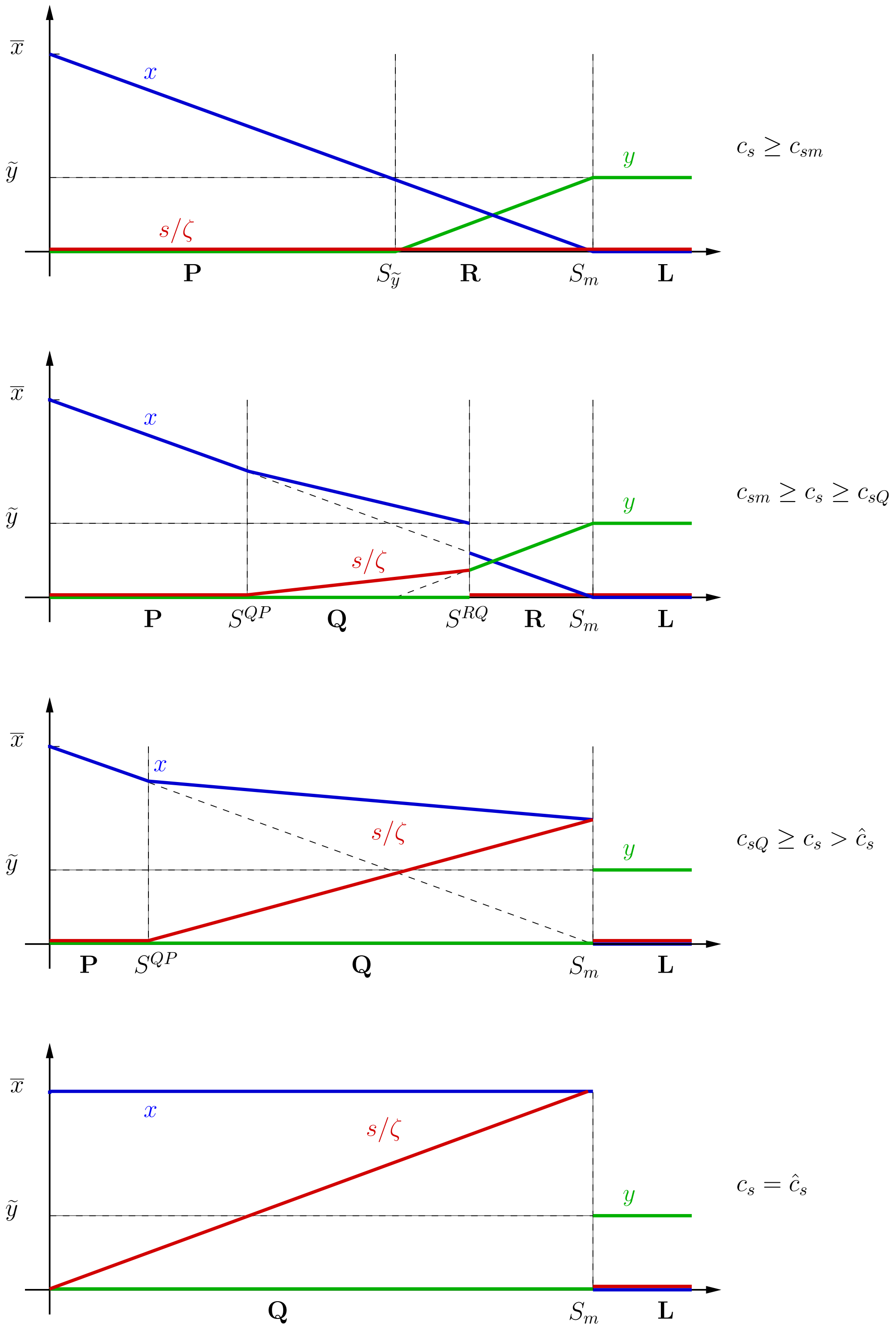

Figure 4.1: Phases on the boundary: optimal controls as state feedback

4.3.3.1 Junction on P

Imposing the continuity of λ_S at the junction point is sufficient for obtaining a solution.

Lemma 4.10. *Under Assumption 3, $u'(\cdot)$ convex and $c_s > \hat{c}_s$, the following configuration satisfies the first-order conditions and the system of constraints: the trajectory is in Phase A, characterized by*

$$x(t) = q^d(c_x - \zeta\lambda_Z(t)), \quad s(t) = y(t) = 0, \quad S(t) = S^0 e^{-\beta(t-t^0)},$$

$$\begin{aligned} Z(t) &= Z^0 e^{-\alpha(t-t^0)} + S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) \\ &+ \zeta \int_{t^0}^{t} e^{-\alpha(t-u)} q^d(c_x - \zeta\lambda_Z(u))du \end{aligned} \tag{4.3.16}$$

and λ_S, λ_Z given by Equations (3.3.3) *and* (3.3.4)*, for $t \in [t^0, t^{AP}]$, where t^{AP} solves the equation $Z(t^{AP}) = \overline{Z}$. Then the trajectory continues in Phase P as described in Lemma 4.1.*

Proof. It is necessary to check that the constraints $\gamma_s(t) \geq 0$ and $\gamma_y(t) \geq 0$ are satisfied on the trajectory constructed with continuous λ_S and λ_Z.

Observe that in both Phases A and P, $\gamma_y = c_y - c_x + \zeta\lambda_Z$. It is therefore a continuous function on the trajectory. In Phase P, $\gamma_y \geq 0$ so that $\gamma_y(t^{AP}) \geq 0$. Since λ_Z is decreasing in Phase A, so is γ_y and we have for all $t \in [t^0, t^{AP}]$: $\gamma_y(t) \geq 0$.

Likewise, $\gamma_s = \lambda_Z - \lambda_S + c_s$ in both phases A and P, and it is positive in Phase P, hence at time $t = t^{AP}$. A straightforward variation analysis based on observations in Section 3.3.1 reveals that the general behavior of $\gamma_s(t)$ is as follows. Starting from $t \to -\infty$, γ_s starts from c_s then increases, then decreases, goes through 0 and tends to $-\infty$ when $t \to +\infty$. Therefore, it is necessarily positive on the interval $[t^0, t^{PA}]$ since it is positive at the end of the interval. □

Observe also that when in Phase A, we have $\dot{Z} > 0$, which justifies the idea that there are initial values (S^0, Z^0), $Z^0 < \overline{Z}$, such that $Z(t) = \overline{Z}$ ($Z(t)$ given by (4.3.16)) has actually a solution. Since in Phase A we have $x(t) = q^d(c_x - \zeta\lambda_Z(t))$ and λ_Z is decreasing, $x(t)$ is decreasing as well. Its value at t^{AP} is $x^{(P)}(t^{AP}) = \overline{x} - \beta S^{AP}/\zeta$. Then we can write (remember that $\alpha\overline{Z} = \zeta\overline{x}$):

$$\dot{Z} = -\alpha Z + \beta S + \zeta x = \alpha(\overline{Z} - Z) + \beta(S - S^{AP}) + \zeta(x - \overline{x} + \beta S^{AP}/\zeta) \ .$$

Since S is also decreasing in Phase A, all three terms in this expression are positive.

The set of initial positions (S^0, Z^0) of trajectories which satisfy Lemma 4.10 is limited by the particular trajectory which joins point $(S^{QP}, \overline{Z})$ (when $c_s \in (\hat{c}_s, c_{sm}]$) or point $(S_{\widetilde{y}}, \overline{Z})$ (when $c_s \geq c_{sm}$).

4.3.3.2 Junction on Q

The geometric position of $(\lambda_Z^{(Q)}(t), \lambda_S^{(Q)}(t))$ also allows to construct consistent continuous trajectories where Phase A joins Phase Q.

Lemma 4.11. *Under Assumption 3, $u'(\cdot)$ convex and $\hat{c}_s \leq c_s \leq c_{sm}$, the following configuration satisfies the first-order conditions and the system of constraints: the trajectory is in Phase A (see its equations in Lemma 4.10) for $t \in [t^0, t^{AQ}]$, where t^{AQ} solves the equation $Z(t^{AQ}) = \overline{Z}$. Then the trajectory continues in Phase Q as described in Lemma 4.3 (if $c_s = \hat{c}_s$), or Lemmas 4.6 and 4.8 (if $\hat{c}_s < c_s \leq c_{sm}$).*

Proof. The proof is similar to that of Lemma 4.10, with the difference that $\gamma_s(t^{AQ}) = 0$ instead of being positive. One concludes nevertheless that γ_y and γ_s are both positive. □

4.3.3.3 Junction on R

It is also possible to construct consistent continuous trajectories where Phase A joins Phase R.

Lemma 4.12. *Under Assumption 3, $u'(\cdot)$ convex and $c_s \geq c_{sQ}$, the following configuration satisfies the first-order conditions and the system of constraints: the trajectory is in Phase A (see its equations in Lemma 4.10) for $t \in [t^0, t^{AR}]$, where t^{AR} solves the equation $Z(t^{AR}) = \overline{Z}$. Then the trajectory continues in Phase R as described in Lemma 4.7 or Lemma 4.9.*

Proof. In Phase R, λ_Z is constant and λ_S, given by Equation (4.3.13), is increasing. According to Lemmas 4.7 and 4.9, the trajectory in Phase R finishes either in Phase Q of in Phase P, with the continuity of λ_S and λ_Z, and therefore of $\gamma_s = \lambda_Z - \lambda_S + c_s$. This function either vanishes at $t = t^{RQ}$ or is positive at $t = t^{RP}$. It is decreasing, therefore it is positive in Phase R (see an illustration in Figure 4.13 or Figure 4.16). The same reasoning as for the proof of Lemma 4.10 can be applied, to conclude that γ_y and γ_s are both positive. □

4.3.3.4 Junction on L

AJM: Change this, because it relies on the scrap value approach, which has been removed.

Assume that the optimal trajectory joins the curve $Z = B(S)$ at some $S(T^*) \geq S_m$, and therefore $Z(T^*) = Z_M(S(T^*))$. In that case $V(S) = V_L(S)$ is given by (4.5.2). Then, it is straightforward to compute the identity

$$V'(S) \;=\; -\frac{e^{-\rho\tau_L(S)}}{\beta S}\left(\rho V(S_m) - u(\widetilde{y}) + c_y\widetilde{y}\right)\ . \tag{4.3.17}$$

On the other hand, we have the other identity:

$$Z'_M(S) \;=\; -\frac{\dot{Z}}{\dot{S}} \;=\; \frac{-\alpha Z + \beta S}{-\beta S} \;=\; \frac{\alpha Z}{\beta S} \;-\; 1\ . \tag{4.3.18}$$

Replacing these expressions in (4.3.27) yields:

$$\begin{aligned} \lambda_S \;&=\; -\frac{e^{-\rho\tau_L(S)}}{\beta S}\left(\rho V(S_m) - u(\widetilde{y}) + c_y\widetilde{y}\right) \;-\; \frac{-\alpha Z + \beta S}{-\beta S}\lambda_Z \\ -\beta S\lambda_S + (-\alpha Z + \beta S)\lambda_Z \;&=\; e^{-\rho\tau_L(S)}\left(\rho V(S_m) - u(\widetilde{y}) + c_y\widetilde{y}\right)\ . \end{aligned} \tag{4.3.19}$$

Next, replacing in (4.3.28), one gets the equivalences:

$$\begin{aligned} &(\rho V(S_m) - u(\widetilde{y}) + c_y\widetilde{y})e^{-\rho\tau_L(S)} + u(\widetilde{y}) - c_y\widetilde{y} \\ &\quad= \; u(x+y) - c_s s - c_x x - c_y y + \lambda_Z(\zeta x - s) + s\lambda_S + e^{-\rho\tau_L(S)}\ (\rho V(S_m) - u(\widetilde{y}) + c_y\widetilde{y}) \\ u(\widetilde{y}) - c_y\widetilde{y} \;&=\; u(x+y) - c_s s - c_x x - c_y y + \lambda_Z[\zeta x - s] - s\lambda_S\ . \end{aligned} \tag{4.3.20}$$

Two configurations satisfying this condition have been identified at this point (not meaning that no other are possible):

junction in phase A: take $y = s = 0$, $x = \widetilde{y}$. Then (4.3.20) simplifies into:

$$\begin{aligned} u(\widetilde{y}) - c_y\widetilde{y} \;&=\; u(\widetilde{y}) - c_x\widetilde{y} + \lambda_Z\zeta\widetilde{y} \\ \lambda_Z \;&=\; \frac{c_x - c_y}{\zeta}\ . \end{aligned} \tag{4.3.21}$$

junction in phase B: take $y = 0$, $x = \widetilde{y}$, $s = \zeta x$. Then (4.3.20) simplifies into:

$$\begin{aligned} u(\widetilde{y}) - c_y\widetilde{y} \;&=\; u(\widetilde{y}) - c_s\zeta\widetilde{y} - c_x\widetilde{y} - \zeta\widetilde{y}\lambda_S \\ \lambda_S \;&=\; c_s \;+\; \frac{c_x - c_y}{\zeta}\ . \end{aligned} \tag{4.3.22}$$

The second one does not seem to occur.

4.3.3.5 Phases connecting at point $(S_m, \overline{Z})$

The location $(S_m, \overline{Z})$ of the boundary has a particular status. For one thing, we have seen in Lemma 4.2 (page 28) that there exist stationary optimal trajectories staying at that point: what we have called Phase S. This situation happens if and only if $c_s \leq \hat{c}_s$. When $c_s > \hat{c}_s$, this location is not stationary anymore, but may retain its "non-standard" character.

4.3.3.5.1 Necessary conditions. In order to better focus on trajectories going through (or ending at) point $(S_m, \overline{Z})$, we use a scrap value approach. The idea is the same as followed until here: our aim is to find necessary conditions for the different possible phase junctions to take place.

The problem is formulated as:

$$\max_{s(\cdot),x(\cdot),y(\cdot),T} \int_0^T [u(x(t)+y(t)) - c_s s(t) - c_x x(t) - c_y y(t)]\, e^{-\rho t} \mathrm{d}t \;+\; e^{-\rho T} V_S \tag{4.3.23}$$

given the controlled dynamics (2.1.2):

$$\begin{cases} \dot{Z} &= -\alpha Z + \beta S + \zeta x - s \\ \dot{S} &= -\beta S + s \end{cases}$$

with the constraints on state variables and controls:

$$R(S(t), Z(t)) := B(S(t)) - Z(t) \;\geq\; 0\,, \quad t \in [0, T] \tag{4.3.24}$$

$$R(S(T), Z(T)) \;=\; 0\,, \tag{4.3.25}$$

the initial and terminal conditions

$$S(0) = S^0, \quad Z(0) = Z^0, \quad S(T) = S_m, \quad Z(T) = \overline{Z}\,, \tag{4.3.26}$$

and the usual constraints on $x(t)$, $s(t)$ and $y(t)$. The scrap value V_S is the value function of the problem restricted to the curve $Z = B(S)$. Its exact form varies depending on the value of c_s, see Section 4.5.1 on page 58.

According to Seierstad & Sydsæter (1999, Theorem 13, p. 350) (generalized to the situation of a free terminal time),[1] a sufficient condition for an optimal trajectory is the existence of continuous multipliers λ_S and λ_Z (written in current value) and real numbers $\beta \geq 0$ and γ, such that, T^* being the final optimal time:

$$\begin{aligned} \lambda_S(T^*) &= \beta \frac{\partial R}{\partial S}(S(T^*), Z(T^*)) \;+\; V'(S(T^*)) \;+\; \gamma \frac{\partial R}{\partial S}(S(T^*), Z(T^*)) \\ &= V'(S(T^*)) \;+\; (\beta+\gamma) B'(S(T^*)) \\ \lambda_Z(T^*) &= \beta \frac{\partial R}{\partial Z}(S(T^*), Z(T^*)) \;+\; \gamma \frac{\partial R}{\partial Z}(S(T^*), Z(T^*)) \\ &= -(\beta+\gamma)\,. \end{aligned}$$

These two conditions are satisfied if

$$\lambda_S(T^*) \;=\; V'(S(T^*)) \;-\; B'(S(T^*))\, \lambda_Z(T^*)\,. \tag{4.3.27}$$

Given the definition of $B(\cdot)$ in (4.3.1), this last condition is in turn refined into:

$$\lambda_S(T^*) \;=\; \begin{cases} V'(S(T^*)) & \text{if } S(T^*) \leq S_m \\ V'(S(T^*)) \;-\; Z'_M(S(T^*))\lambda_Z(T^*) & \text{if } S(T^*) \geq S_m. \end{cases}$$

[1] The theorem also requires that $R(S, Z)$ be quasiconcave, which is true, and that $V(\cdot)$ be concave, which is *not* satisfied here.

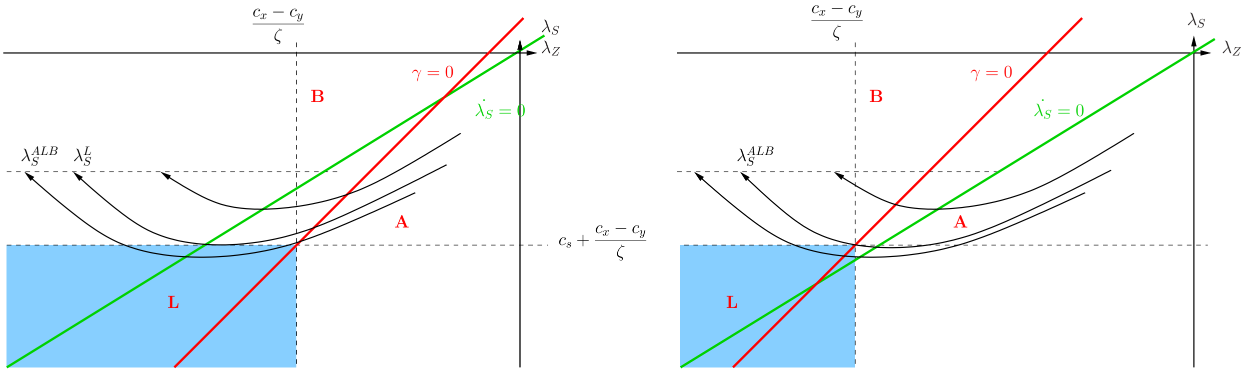

Figure 4.2: Trajectories of adjoint variables through phases A, B and L

Next, since the terminal time is free, Seierstad & Sydsæter (1999, Theorem 16, p. 398), claims that a sufficient condition is that, evaluated at $t = T^*$,

$$H(S, Z, y, x, s, \lambda_S, \lambda_Z) \;=\; \rho V_S \;.$$

In extensive form, this amounts to requiring (see Equation (2.2.1)):

$$\begin{aligned} & u(x(T^*) + y(T^*)) - c_s s(T^*) - c_x x(T^*) - c_y y(T^*) \\ & + \lambda_S(T^*)(-\beta S_m + s(T^*)) + \lambda_Z(T^*)(\zeta x(T*) - s(T^*)) \;=\; \rho V_S \;. \end{aligned} \tag{4.3.28}$$

The set of conditions

$$y(T^*) = 0, \quad x(T^*) = \overline{x}, \quad s(T^*) = \zeta \overline{x},$$

turns out to solve this equation, independently of the values of $\lambda_S(T^*)$ and $\lambda_Z(T^*)$. Assuming the continuity of controls, we see that this set of controls correspond to Phase B since $s = \zeta x$. Inside Phase B, the value of the consumption $x(t)$ is given by: $x(t) = q^d(c_x + \zeta c_s - \zeta \lambda_S(t))$. The continuity of controls is then equivalent to the continuity of $\lambda_S(\cdot)$. The value of $\lambda_Z(T^*)$ remains undetermined, except that it must satisfy some inequality. We examine this situation next.

AJM: To be completed

4.3.3.5.2 Adjoint variables and the phases A, B, L. When setting $\lambda_Z(T^*)$ to all possible values in $(-\infty, \lambda_Z^{max})$ (the value of λ_Z^{max} depends on the situation), we obtain a family of trajectories. For all of them, the state variables end up at point $(S_m, \overline{Z})$ in Phase B, and the costate variables end up at point $(\lambda_Z^{(}T^*), \lambda_S(T^*))$. These trajectories may actually be in one of three possible phases, according to the sign of $\lambda_S - \lambda_Z - c_s$ (Phases A or B), and to whether the consumption x is larger or smaller than $\widetilde{y}$ (Phase L).

The situation is represented in Figure 4.2. It is assumed that a family of trajectories of $(\lambda_Z(t), \lambda_S(t))$ terminate at some time T with the same value of $\lambda_S(T)$, represented as a horizontal dashed line. The zones corresponding to Phases A and B are delimited by the red line $\gamma = \lambda_S - \lambda_Z - c_s = 0$. Phase A is below the line, Phase B is above it. The zone corresponding to Phase L is represented in blue. It is separated from Phase A by the line $\lambda_Z = (c_x - c_y)/\zeta$ (corresponding to $q^d(c_x - \zeta\lambda_Z) = \widetilde{y}$) and from Phase B by the line $\lambda_S = c_s + (c_x - c_y)/\zeta$ (corresponding to $q^d(c_x + \zeta c_s - \zeta \lambda_S) = \widetilde{y}$). The green line represents the locus of points where $\dot{\lambda}_S = 0$, that is, $(\rho+\beta)\lambda_S = \beta\lambda_Z$. When above this curve, $\lambda_S(t)$ increases, and it decreases below. In both sides of Figure 4.2, there is a particular value λ_Z^{ABL} which is such that when $\lambda_Z(T) = \lambda_Z^{ALB}$, the trajectory goes precisely through the corner of Phase L.

On the left-hand side of Figure 4.2, we have represented the situation where the curve $\dot{\lambda}_S = 0$ enters the "Phase L" zone by intersecting its horizontal boundary. In other terms, the corner of the Phase L zone is below the line, which translates as:

$$(\rho + \beta)\left(c_s + \frac{c_x - c_y}{\zeta}\right) \le \beta\, \frac{c_x - c_y}{\zeta} \quad\iff\quad c_s \le \bar{c}_s,$$

where $\bar{c}_s$ has been defined in (4.0.3). There exists a critical value $\lambda_Z^L > \lambda_Z^{ABL}$ such that, for all $\lambda_Z(T) > \lambda_Z^L$, the trajectory of $(\lambda_Z(t), \lambda_S(t))$ never enters the L zone, whereas it does for $\lambda_Z(T) < \lambda_Z^L$ (and it just touches it for $\lambda_Z(T) = \lambda_Z^L$). Trajectories that do not enter Phase L just pass from Phase A to Phase B. Then when $\lambda_Z(T) \in (\lambda_Z^{ALB}, \lambda_Z^L)$, trajectories go through phases A, B, L then B again. When $\lambda_Z(T) < \lambda_Z^{ALB}$, trajectories go through phases A, L and B.

On the righ-hand side of Figure 4.2, we have represented the situation where $c_s > \bar{c}_s$, and the green line $\dot{\lambda}_S = 0$ enters the "Phase L" zone by intersecting its vertical boundary. In that case, whenever the point $(\lambda_Z(t), \lambda_S(t))$ is in Phase B, $\lambda_S(t)$ is increasing. It follows that there are only two cases left: either $\lambda_Z(T) > \lambda_Z^{ALB}$ and Phase A is followed by Phase B, or $\lambda_Z(T) < \lambda_Z^{ALB}$ and the phases are A, then L, then B.

4.3.3.5.3 Junction with Phase S. Assume that an optimal trajectory enters Phase S at time T. According to Lemma 4.2, this happens for $c_s \le \hat{c}_s$. Then the total gain on this trajectory, evaluated from instant T on, is, since the control is $x = \overline{x}$ and $s = \zeta\overline{x}$,

$$V_S \;=\; \int_0^\infty e^{-\rho t}\,(u(\overline{x}) - (c_x + \zeta c_s)\overline{x})\,\mathrm{d}t \;=\; \frac{1}{\rho}(u(\overline{x}) - (c_x + \zeta c_s)\overline{x})\;. \tag{4.3.29}$$

We use this value when solving the finite-horizon problem with scrap value and free terminal time (4.3.23) subject to the usual dynamics and constraints (2.1.2)– (2.1.9), and the initial and terminal conditions (4.3.26). The first-order conditions for this problem include (2.2.2)– (2.2.17), and in addition we have the optimality condition for the terminal time (4.3.28).

As observed in Section 4.3.3.5.1, the continuity of controls and of λ_S are equivalent. We can then assume that the terminal values of λ_S and λ_Z satisfy

$$\lambda_S(T) \;=\; \lambda_S^{(S)} \;:=\; c_s + \frac{c_x - \overline{p}}{\zeta}\;, \qquad \lambda_Z(T) \;\le\; \lambda_Z^{(S)} := \frac{\rho+\beta}{\beta}\,\lambda_S^{(S)}\;.$$

The typical situation is depicted in Figure 4.3, where the three curves represented are $s(t) = \zeta x(t)$, $\beta S(t)$ and $\alpha Z(t)$. These three functions take the same value at time $t^0 = T$ where the point $(S_m, \overline{Z})$ is reached. The diagram assumes that the trajectory is in Phase B throughout for the purpose of illustration. However, it is not possible that the trajectory be in this phase for all $t \le T$. Depending on the value of $\lambda_Z(T)$, the phase is limited by one of the events: (a) $Z = \overline{Z}$; (b) $Z = 0$; (c) $\gamma_{sx} = 0$; (d) $x(t) = \widetilde{y}$. This last situation has been discussed in Section 4.3.3.5.2.

The remainder of this section is devoted to a proof that the general scheme of Figure 4.3 is correct, at least for a set of trajectories "close" to the point $(S_m, \overline{Z})$. The result is stated as Lemma 4.13 next. As a corollary, we state in Lemma 4.15 that some optimal trajectories consist in a Phase B followed by the Phase S.

Lemma 4.13. *Consider the dynamical system characteristic of Phase B, under Assumption 3 and $c_s < \hat{c}_s$. There exists a constant $\overline{\ell}$ such that, for all $\ell \in (0, \overline{\ell}]$, the trajectories which terminate at $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$, have the following property: there exist $\tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \tau_6 < T$ such that the table of variation in Table 4.1 holds.*

The time instants τ_i are illustrated in Figure 4.3. The proof uses in part the following intermediate result:

Lemma 4.14. *Consider the dynamical system characteristic of Phase B, for $c_s < \hat{c}_s$. There exist constants, C_1, C_2, C_3 and $\overline{\ell}$ such that, for all $\ell \in (0, \overline{\ell}]$, the trajectories which terminate at $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$, are such that:*

$$\zeta x(T - C_1\ell) \;>\; \beta S(T - C_1\ell) \tag{4.3.30}$$
$$\alpha Z(T - C_2\ell) \;>\; \beta S(T - C_2\ell) \tag{4.3.31}$$
$$\alpha Z(T - C_3\ell) \;>\; \alpha\overline{Z}\;. \tag{4.3.32}$$

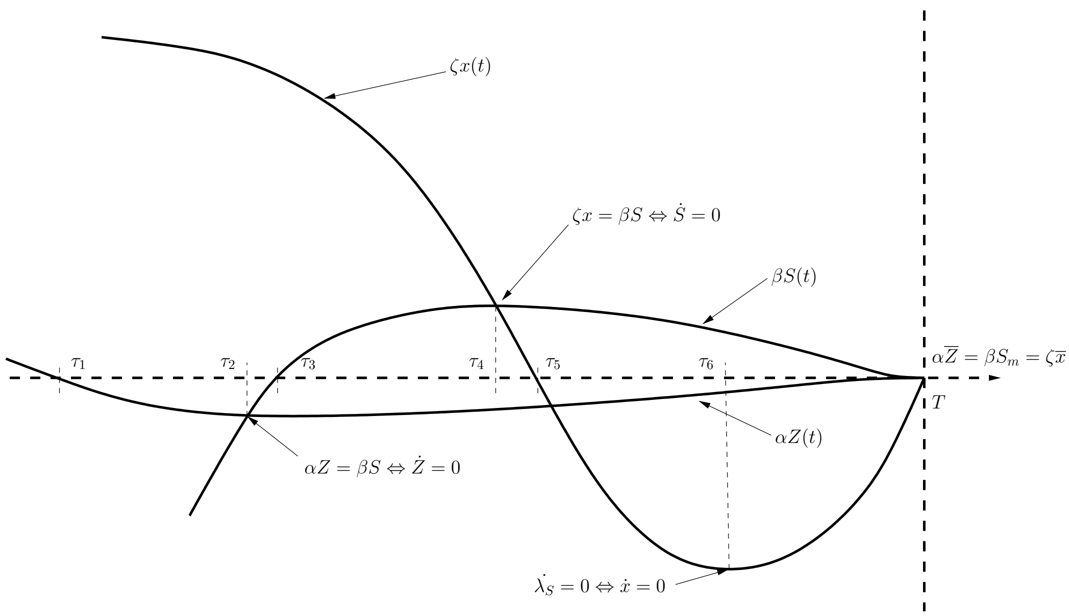

Figure 4.3: Trajectories of state and control just before joining Phase S

	τ_1		τ_2		τ_3		τ_4		τ_5		τ_6		T	
					$-$						0	$+$		$\zeta\dot{x}$
				$\searrow$					$\zeta\overline{x}$	$\searrow$		$\nearrow$	$\zeta\overline{x}$	ζx
			$+$				0			$-$			0	$\beta\dot{S}$
		$\nearrow$			βS_m	$\nearrow$				$\searrow$			βS_m	βS
	$-$		0					$-$					0	$\alpha\dot{Z}$
$\searrow$	$\alpha\overline{Z}$	$\searrow$						$\nearrow$					$\alpha\overline{Z}$	αZ

Table 4.1: Table of variation of trajectories in Phase B

Proof. The proof consists in computing Taylor expansions of the three different functions $\zeta x(t)$, $\beta S(t)$ and $\alpha Z(t)$ around $t = T$, while at the same time considering $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$. In a second phase, the value of ℓ is linked appropriately to the time parameter in the expansion.

We start with $\lambda_S(t)$, the formula of which is given in (3.3.4). Using the boundary conditions, and the fact that $\beta\lambda_Z^{(S)} = (\rho+\beta)\lambda_S^{(S)}$, we have:

$$\begin{aligned}
\lambda_S(T+u) &= \lambda_S^{(S)} e^{(\rho+\beta)u} - \frac{1}{\alpha-\beta}\Big((\rho+\beta)\lambda_S^{(S)} - \beta\ell\Big)\Big(e^{(\rho+\alpha)u} - e^{(\rho+\beta)u}\Big) \\
&= \lambda_S^{(S)} + u\beta\ell + \frac{1}{2}Au^2 + O(u^3)\,,
\end{aligned}$$

where we have used the shorthand notation $A := \beta\ell(2\rho+\alpha+\beta) - (\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}$. The function $O(u^3)$ in this expansion is bounded by Mu^3, for some constant M, uniformly for ℓ in any compact containing 0. Next, consider the expansion of $x(t)$:

$$\begin{aligned}
x(T+u) &= q^d(c_x + \zeta c_s - \zeta\lambda_S(T+u)) = q^d(\overline{p} + \zeta(\lambda_S^{(S)} - \lambda_S(T+u))) \\
&= \overline{x} + (q^d)'(\overline{p})\zeta(\lambda_S^{(S)} - \lambda_S(T+u)) \\
&\quad + \frac{1}{2}(q^d)''(\overline{p})\zeta^2(\lambda_S^{(S)} - \lambda_S(T+u))^2 + O(|\lambda_S^{(S)} - \lambda_S(T+u)|^3) \\
&= \overline{x} - (q^d)'(\overline{p})\zeta\left(\beta\ell + \frac{1}{2}uA\right)u + \frac{1}{2}(q^d)''(\overline{p})\zeta^2\beta^2\ell^2u^2 + O(u^3).
\end{aligned}$$

Again, the "$O(u^3)$" term is uniform for ℓ in a compact, assuming that q^d admits a bounded third derivative. The expansion for $S(\dot{})$ is derived from that of x, through the integral formula (3.5.8). After a change of variables:

$$\begin{aligned}
S(T+u) &= S_m e^{-\beta u} + \zeta e^{-\beta u}\int_0^u e^{\beta w}x(T+w)\mathrm{d}w \\
&= e^{-\beta u}\Bigg(S_m + \zeta\overline{x}\,\frac{e^{\beta u}-1}{\beta} - \zeta^2\beta\ell(q^d)'(\overline{p})\int_0^u we^{\beta w}\mathrm{d}w \\
&\quad - \frac{1}{2}\zeta^2\left((q^d)'(\overline{p})A - (q^d)''(\overline{p})\zeta\beta^2\ell^2\right)\int_0^u w^2e^{\beta w}\mathrm{d}w + \int_0^u O(w^3)e^{\beta w}\mathrm{d}w\Bigg) \\
&= e^{-\beta u}\Bigg(S_m e^{\beta u} - \frac{1}{2}(q^d)'(\overline{p})\zeta^2\beta\ell u^2 - \frac{1}{3}(q^d)'(\overline{p})\zeta^2\beta^2\ell u^3 \\
&\quad - \frac{1}{6}(q^d)'(\overline{p})\zeta^2Au^3 + \frac{1}{6}(q^d)''(\overline{p})\zeta^3\beta^2\ell^2u^3\Bigg) + O(u^4)\,.
\end{aligned}$$

Finally, the expansion for $Z(\cdot)$ follows from (3.5.7):

$$\begin{aligned}
Z(T+u) &= \overline{Z}e^{-\alpha u} + \beta e^{-\alpha u}\int_0^u e^{\alpha w}S(T+w)\mathrm{d}w \\
&= \overline{Z}e^{-\alpha u} + \beta e^{-\alpha u}\int_0^u e^{\alpha w}S_m\mathrm{d}w \\
&\quad - e^{-\alpha u}\int_0^u e^{(\alpha-\beta)w}(q^d)'(\overline{p})\zeta^2w^2\left(\frac{1}{2}\beta\ell + \frac{1}{6}A + lO(w) + O(w^2)\right)\mathrm{d}w \\
&= \overline{Z} - (q^d)'(\overline{p})\zeta^2\beta\left(\frac{1}{6}\beta\ell - \frac{1}{24}(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}u\right)u^3 + \ell O(u^4) + O(u^5)\,.
\end{aligned}$$

If we choose now to set $u = -C\ell$ for some positive constant C, we get the expansions:

$$x(T - C\ell) = \overline{x} + (q^d)'(\overline{p})\zeta C\left(\beta + \frac{1}{2}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^2 + O(\ell^3)$$

$$
\begin{aligned}
S(T-C\ell) &= S_m - (q^d)'(\overline{p})\zeta^2 C^2 \left(\frac{1}{2}\beta + \frac{1}{6}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^3 + O(\ell^4) \\
Z(T-C\ell) &= \overline{Z} - (q^d)'(\overline{p})\zeta^2 C^3 \beta \left(\frac{1}{6}\beta + \frac{1}{24}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^4 + O(\ell^5)\ .
\end{aligned}
$$

By assumption, $(q^d)' < 0$. If the constants C_1, C_2 and C_3 are chosen such that

$$
C_1 > 2C_0, \quad C_2 > 3C_0, \quad C_3 > 4C_0, \qquad C_0 := -\frac{\beta}{(\rho+\alpha)(\rho+\beta)}\,\frac{1}{\lambda_S^{(S)}},
$$

then the different orders of the expansions allow to conclude that for ℓ sufficiently close to 0, $\zeta x(t-C_1\ell) > \beta S(t-C_1\ell)$, $\beta S(t-C_2\ell) > \alpha Z(t-C_2\ell)$ and $\alpha Z(t-C_3\ell) > \alpha\overline{Z}$. □

Proof of Lemma 4.13. We begin with $x(t)$ and its related function $\lambda_S(t)$, since $x(t) = q^d(c_x+\zeta c_s - \zeta\lambda_S(t))$. Using the results of Section 3.3, it is straightforward to show that there exists $\tau_6 < T$ such that $\dot\lambda_S(\tau_6) = 0$. Indeed, $\dot\lambda_S(t) = 0$ iff $(\rho+\beta)\lambda_S(t) = \beta\lambda_Z(t)$, and from the observations in Section 3.3.2, the ratio $r_\lambda = \lambda_S/\lambda_Z$ is decreasing on the interval $t \in (-\infty, T]$. There exists therefore a unique τ_6 where $\lambda_S(\tau_6)$ is minimal: $\lambda_S(t)$ is decreasing up to τ_6, then increasing.

Next, we have

$$
\dot x(t) = -\zeta\dot\lambda_S\ (q^d)'(c_x+\zeta c_s-\zeta\lambda_S)
$$

and since $(q^d)' < 0$ under Assumption 3, the variation of $\zeta\dot x$ is as in Table 4.1. When $t \to -\infty$, $\lambda_S(t) \to 0$ so that $x(t) \to q^d(c_x+\zeta c_s)$. Under the assumption that $c_s < \hat c_s$, we find that $q^d(c_x+\zeta c_s) > \overline{x}$. This implies the existence of $\tau_5 < \tau_6$ such that $x(\tau_5) = 0$. The variation of $\zeta x(t)$ is therefore as claimed in Table 4.1.

Consider now the function $\beta S(t)$. According to the development close to $t = T$ computed in the proof of Lemma 4.14 (see also Section 4.5.2), $\beta S(t) > \beta S_m = \zeta\overline{x} > \zeta x(t)$ for t sufficiently close to T. On the other hand, from Lemma 4.14, there exists $\overline{\ell}$ such that for all $\ell \in (0, \overline{\ell}]$, there is a time τ such that $\beta S(\tau) < \zeta x(\tau)$. By continuity, this implies the existence of at least one t such that $\beta S(t) = \zeta x(t)$. Let τ_4 be the largest of them. Necessarily, $x(\tau_4) > \overline{x}$ because $\dot S(\tau_4) = -\beta S(\tau_4)+\zeta x(\tau_4) = \beta(S_m - S(\tau_4))+\zeta(x(\tau_4)-\overline{x}) = 0$, and because $\dot S(t) < 0$ for $t \in (\tau_4, T)$ implies $S(\tau_4) > S_m$. From the variation of $x(t)$, this implies in turn that $\tau_4 < \tau_5$.

We argue now that $\dot S(t) > 0$ for all $t < \tau_4$, so that the variation of $\dot S$ is as claimed in Table 4.1. Assume by contradiction that $\dot S(\tau) = 0$ for some $\tau < \tau_4$, and consider the largest of such values. Then $\dot S(t) > 0$ for all t in the interval (τ, τ_4). Then, since $\ddot S = -\beta\dot S+\zeta\dot x$, and since $\dot x(t) < 0$ on the interval, according to the variation of $\dot x$, we conclude that $\ddot S(t) < 0$ over the interval. We reach a contradiction with the fact that $\dot S = 0$ at both extremities.

Finally, according again to Lemma 4.14, there exists a τ such that $\beta S(\tau) < \alpha Z(\tau)$. Similarly as above, this implies the existence of a unique τ_2 such that $\beta S(\tau_2) = \alpha Z(\tau_2)$. Clearly, Z is increasing on the interval $[\tau_2, T]$ so that $\beta S(\tau_2) = \alpha Z(\tau_2) < \alpha Z(T) = \alpha\overline{Z} = \beta S_m$. This implies in turn: on the one hand that $\tau_2 < \tau_4$, and on the other hand that there exists τ_3 such that $S(\tau_3) = S_m$ and $\tau_2 < \tau_3 < \tau_4$. This concludes the proof that the variation of S is as in Table 4.1.

There remains to complete the analysis of $Z(t)$. By the same convexity argument, $\alpha Z(t)$ cannot cross twice $\beta S(t)$ because $\ddot Z = -\alpha\dot Z + \beta\dot S$ is positive on any interval ending at τ_2. Therefore, $\dot Z$ cannot vanish on interval $(-\infty, \tau_2)$ and the variation of $\dot Z$ is as shown in Table 4.1.

Using Lemma 4.14 a last time, we conclude that there exists a value τ_1 such that $Z(\tau_1) = \overline{Z}$. The function $Z(\cdot)$ therefore evolves as described in Table 4.1. This concludes the proof. □

We now conclude with the construction of optimal trajectories for the case $c_s < \hat c_s$.

Lemma 4.15. *Under Assumption 3, assume also that $c_s < \hat c_s$. Then there exists a constant $\overline{\ell}$ such that for all $\ell \in (0, \overline{\ell}]$, the following configuration satisfy the first-order conditions and the system of constraints:*

in a time interval $[\tau_1, T]$, the system is in Phase B, with $S(\tau_1) \in (0, S_m)$, $Z(\tau_1) = \overline{Z}$, $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$.

in the time interval $[T, \infty)$, the trajectory is stationary with $S(t) = S_m$, $Z(t) = \overline{Z}$, $\lambda_S(t) = \lambda_S^{(S)}$ and $\lambda_Z(t) = \lambda_Z^{(S)}$ (Phase S).

Proof. By construction, both pieces of this trajectory satisfy the differential equations of the first-order conditions, and the control constraints on x, s and y. Also by construction, the trajectories are continuous everywhere, except for $\lambda_Z(\cdot)$ which has a discontinuity at $t = T$. The jump at $t = T$ satisfies the condition (2.2.18). There remains to check the constraints on states and multipliers.

Using Lemma 4.13, for each $\ell \in (0, \overline{\ell}]$ there exists τ_1 such that $Z(\tau_1) = \overline{Z}$ and $Z(t) < \overline{Z}$ for $t \in (\tau_1, T)$. Applying Grönwall's lemma to the differential equation $\dot{S} = -\beta S + \zeta x$, with the bound $x(t) \leq \hat{x} := q^d(c_x + \zeta c_s)$, we conclude that $S(t) > 0$ for all $t > \tau_3 + \beta^{-1}\log(1 - \overline{x}/\hat{x})$. Likewise, since $\dot{Z} = -\alpha Z + \beta S \leq \beta S_M$, $Z(t) > 0$ for all $t > T - S_m/(\beta S_M)$. Since τ_1 can be bounded by $C_3\ell$ (Lemma 4.14), we conclude that $\overline{\ell}$ can be chosen so that the trajectory in Phase B satisfies all state constraints $Z \leq \overline{Z}$, $Z \geq 0$ and $S \geq 0$ in the interval $[\tau_1, T]$.

We now turn to the constraints on multipliers. Clearly, in Phase B, $x > 0$ and $s > 0$ so that $\gamma_s = \gamma_x = 0$ and it remains to check that $\gamma_{sx} \geq 0$ and $\gamma_y \geq 0$. These are equivalent to

$$\begin{aligned} \lambda_S - \lambda_Z - c_s &\geq 0 \\ \lambda_S &\geq c_s + \frac{c_x - c_y}{\zeta} . \end{aligned}$$

This second inequality is satisfied when $\lambda_Z(T) \in [\lambda_S^L, \lambda_Z^{(S)}]$ (see Section 4.3.3.5.2). The constant $\overline{\ell}$ can be chosen such that this is the case for all ℓ. For the first inequality, it is easily shown that the function $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$ is increasing. Therefore, the instant τ at which $\gamma(\tau) = 0$ is an increasing function of ℓ. Since the value of $\gamma(0)$ is strictly positive in Phase (see Lemma 4.2 on page 28), the value of τ can never approach 0, so that the value of $\overline{\ell}$ can be chosen so that, for all $\ell \leq \overline{\ell}$, $\gamma_{sx}(t) = \gamma(t) > 0$ for $t \in [\tau_1, T]$. □

Informally, we now describe what happens as $\lambda_Z(T)$ describes the rest of the interval $(-\infty, \lambda_Z^{(S)})$. The situation is depicted in Figure 4.4 on Page 50.

When $\lambda_Z(T) \in (\lambda_Z^{ABL}, \lambda_Z^L)$, the minimum of the curve $x(t)$ (Figure 4.3) is below $\widetilde{y}$. There is a period in Phase L inserted inside Phase B, between time instants τ_2 and T. This essentially does not modify the behavior in Figure 4.3 because $S(t)$ is decreasing and $Z(t)$ is increasing in Phase L, given that $\beta S(t) > \alpha Z(t)$.

In all situations where $\lambda_Z(T) > \lambda_Z^{ABL}$, the first (or unique) Phase B is always preceded by a Phase A. When $\lambda_Z(T) < \lambda_Z^{ABL}$, this is different since the Phase L is directly preceded by a Phase A. Observe that in Phase A, $Z(t)$ is increasing and $S(t)$ is decreasing (see Section 4.3.3.1). Therefore, the behavior represented in Figure 4.3 is not possible in Phase A. There exists a critical value of λ_Z which the trajectory $(\lambda_Z(t), \lambda_S(t))$ switches from Phase A to Phase B at the exact moment where $Z(t)$ reaches $\overline{Z}$. This critical value may or may not be larger than λ_Z^L.

4.3.3.5.4 Impossibility of joining Phase Q and Phase S. In this paragraph, we develop the argument that no optimal trajectory consists in Phase Q joining Phase S. We have seen that Phase S can be terminal only if $c_s \leq \hat{c}_s$. Moreover, we know that if $c_s = \hat{c}_s$, every point $(S, \overline{Z})$ with $S \leq S_m$ is stationary, so that Phase Q cannot be followed by Phase S. We therefore assume that $c_s < \hat{c}_s$.

Assume an initial condition $(S^0, \overline{Z})$ at some arbitrary time t^0, with $S^0 < S_m$. Let t^{QS} be such that $S^{(Q)}(t^{QS}) = S_m$. We have:

$$\lambda_Z(t^{QS-}) \; - \; \lambda_S(t^{QS-}) \; - \; c_s \; = \; \lambda_Z^{(Q)}(t^{QS-}) \; - \; \lambda_S^{(Q)}(t^{QS-}) \; - \; c_s \; = \; 0 \, . \tag{4.3.33}$$

On the other hand, given the values (4.1.15) and (4.1.16), we have:

$$\begin{aligned} \lambda_Z(t^{QS+}) \; - \; \lambda_S(t^{QS+}) \; - \; c_s \; &= \; \lambda_Z^{(S)}(t^{QS-}) \; - \; \lambda_S^{(S)}(t^{QS-}) \; - \; c_s \\ &= \; \frac{\rho + \beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \; - \; \left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \; - \; c_s \end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho+\beta}{\beta} c_s \; + \; \frac{\rho}{\beta}\frac{c_x - \overline{p}}{\zeta} \\
&= \frac{\rho+\beta}{\beta}(c_s - \hat{c}_s) \; . \qquad (4.3.34)
\end{aligned}
$$

According to Seierstad & Sydsæter (1999, Theorem 1, p. 317), it is possible for multipliers to have jumps at the final time $t_1 = t^{QS}$, provided that the jumps are in the "right direction". More precisely, we have:[2] if $g(S,Z) = B(S) - Z \geq 0$ is the state constraint (where the function $B(\cdot)$ is defined in (4.3.1)), then there exits $\gamma \geq 0$ such that:

$$
\begin{aligned}
\lambda_S(t^{QS-}) - \lambda_S(t^{QS}) &= \gamma \, \frac{\partial g}{\partial S}(S(t^{QS}), Z(t^{QS})) \; = \; \gamma B'(S(t^{QS})) \\
\lambda_Z(t^{QS-}) - \lambda_Z(t^{QS}) &= \gamma \, \frac{\partial g}{\partial Z}(S(t^{QS}), Z(t^{QS})) \; = \; -\gamma \; ,
\end{aligned}
$$

Since we assume that $S(t^{QS}) = S_m$, then $B'(S(t^{QS})) = 0$. This implies that λ_S is actually continuous at $t = t^{QS}$. On the other hand, we have the possibility that λ_Z has a jump, with:

$$
\lambda_Z(t^{QS-}) \; - \; \lambda_Z(t^{QS}) \; \leq \; 0 \; . \qquad (4.3.35)
$$

However, using the continuity of $\lambda_S(t)$ in (4.3.33) and (4.3.34) we obtain, by difference,

$$
\lambda_Z(t^{QS-}) \; - \lambda_Z(t^{QS+}) \; = \; - \; \frac{\rho+\beta}{\beta}(c_s - \hat{c}_s) \; > \; 0 \; ,
$$

a contradiction.

4.4 Synthesis

We are now in position to describe the optimal trajectories in the different cases.

First of all, recall that the analysis has revealed several threshold values for c_s, which are ordered as:

$$
\hat{c}_s \; < \; \bar{c}_s \; < \; c_{sQ} \; < \; c_{sm} \; .
$$

Some qualitative features of the optimal trajectories are summarized in the following table, according to the intervals where c_s lies.

Range of c_s	0	$\hat{c}_s$	$\bar{c}_s$	c_{sQ}	c_{sm} ∞
Value of $S(\infty)$	0	S_m	S_m	S_m	S_m
Continuity of λ_Z	n	n	n	y	y
Presence of a Phase R	n	n	n	y	y
Use of y inside the domain	n	n	n	y	y
Use of capture s	possible	possible	possible	possible	n

According to this classification, only four qualitative situations are relevant: those limited by the three thresholds

$$
\hat{c}_s \; < \; c_{sQ} \; < \; c_{sm} \; .
$$

We call these situations respectively: "c_s small", "c_s medium-inf", "c_s medium-sup" and "c_s large". We describe these four cases next, with the help of diagrams in the state space (S, Z) and in the co-state space (λ_Z, λ_S).

[2]The theorem addresses the case of a finite-horizon optimization problem with fixed terminal time: we extrapolate it here to the case with arbitrary final time, reducing Phase S as a scrap value. The proper setting is normally the infinite horizon case with jumps at arbitrary time instants. There is a theorem in S+S which addresses the case of finite horizon... to be checked.

These figures feature several common geometric elements. Points S^{xy} generally mark where the state moves from Phase x to Phase y. Point Ω, when it is present, represents the situation where $\dot{S} = 0$ while in Phase Q (see its definition in (3.6.12) on page 22). This is a repulsive point for the dynamics in Phase Q, which occurs on the red line $\gamma = 0$. Point P_∞, when it is present, represents the limit of the multipliers when $t \to \infty$; these are also the values of the multipliers when $S = 0$. Point P_{xy} generally represents the location of the multipliers when the state is $(S^{xy}, \overline{Z})$. Point P_S represents the location of the multipliers when the state trajectory passes through $(S_m, \overline{Z})$ at the junction between phases P and Q.

4.4.1 Small c_s ($c_s < \hat{c}_s$)

When $0 < c_s < \hat{c}_s$, the situation is represented in Figures 4.4 on page 50 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time), Figure 4.5 (for the evolution of $(S(t), Z(t))$ over time) and Figure 4.6 on page 51 for the correspondence between the evolution of λ_Z, λ_S and that of consumption. See also Figure 4.7 for the boundary case $c_s = \hat{c}_s$.

The typical situation can be summarized as follows:

Phase A A trajectory starting with $S(0)$ small enough will follow a state and a costate path as the ones labelled with **I** in Figures 4.4 and 4.5. The costate path and the consumption/capture path is represented in Figure 4.6. First, capture will be 0 so that Z will increase and S decrease, until Z hits the ceiling. Both λ_Z and λ_S are decreasing in this phase. At some point in time, simultaneously, $Z(t) = \overline{Z}$ and $\lambda_S(t) = \lambda_Z(t) + c_s$. The trajectory enters Phase Q.

Phase Q Next, the trajectory stays at the ceiling in Phase Q: capture occurs according to Equation (4.2.3):

$$s = \zeta x - \beta(S_m - S) .$$

Since $S(t)$ increases towards S_m, the gap between $\zeta x(t)$ and $s(t)$ decreases over time. It is not possible for the optimal trajectory to stay on the boundary $Z = \overline{Z}$ until $S = S_m$, as explained in Section 4.3.3.5.4, page 47. There exists therefore a point (labelled Υ in Figure 4.5) where the trajectory leaves the boundary and enters Phase B.

This particular trajectory is labelled as **(II)** and represented as a continuous blue line in Figures 4.4 and 4.5.

Phase B In this phase, capture is maximum, and the dynamics of Z is given by $\dot{Z} = \beta S - \alpha Z$. Initially, $S(t)$ is increasing and $Z(t)$ is decreasing, until $\beta S = \alpha Z$. Then $Z(t)$ is increasing again. The costate variable $\lambda_S(t)$ is decreasing then increasing, and so is the consumption $x(t)$. There happens a time at which $\dot{S} = \zeta x - \beta S$ becomes null then negative, and $S(t)$ decreases. The trajectory ends up at point $(S_m, \overline{Z})$ in Phase S.

Some trajectories, as the one labelled **(III)** in the figures, follow a sequence of phases A/B/S. They do not reach the ceiling $Z = \overline{Z}$ before the final phase S.

Phase L It may happen that consumption in Phase B falls below $\widetilde{y}$, or equivalently that λ_S falls below $c_s + (c_x - \overline{p})/\zeta$. In that case, a Phase L is inserted in the middle of this Phase B. This situation not represented in Figure 4.6), but in Figure 4.4, it corresponds to trajectories of the costate variable entering the zone colored in light blue. During this Phase L, $x = 0$ and $y = \widetilde{y}$.

Some trajectories, as the one labelled **(IV)** in the figures, follow a sequence of phases L/B/S.

Phase S All trajectories terminate at the point $(S_m, \overline{Z})$, where they stay forever. The values of (λ_Z, λ_S), as well as x, y and s are constant in that phase: they are given in Section 4.1.2. These terminal values correspond to the point marked as P_S in Figure 4.4.

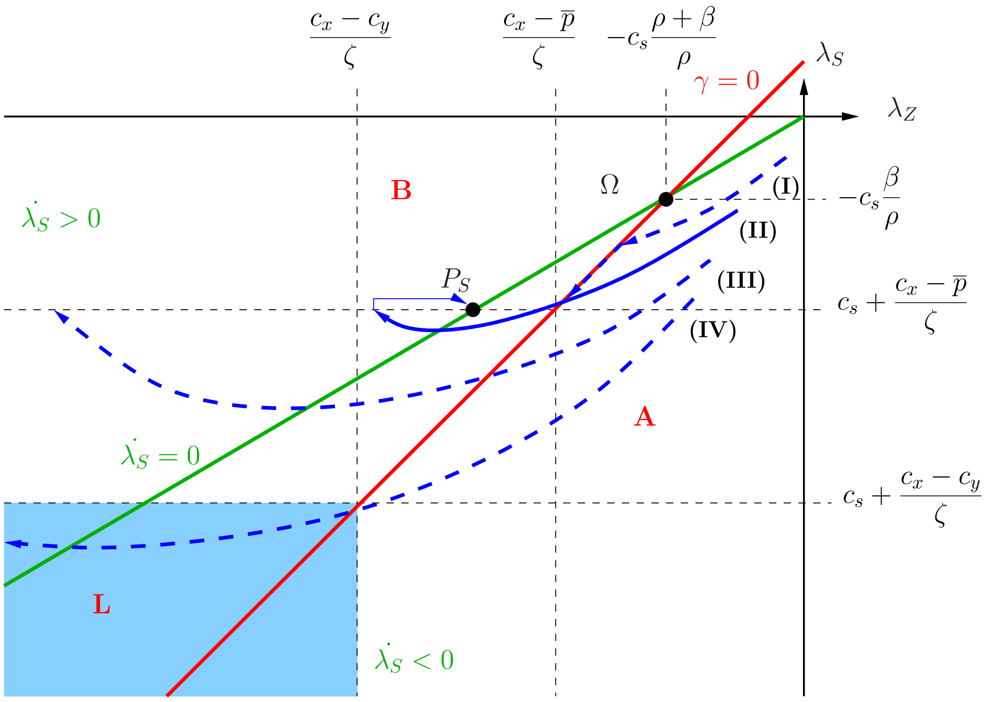

Figure 4.4: Evolution of (λ_Z, λ_S), case $c_s < \hat{c}_s$

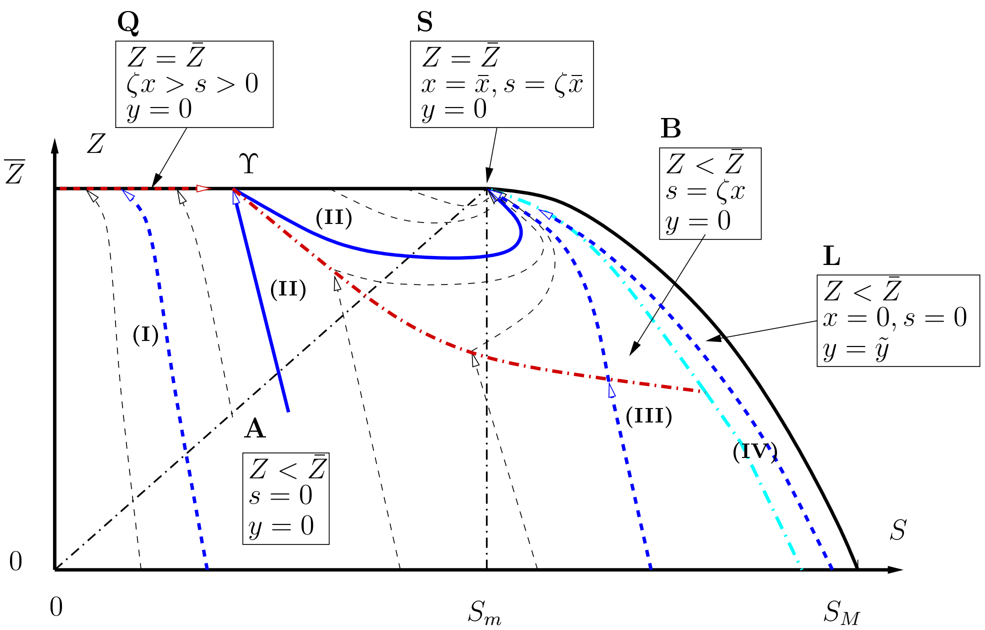

Figure 4.5: Evolution of (S, Z), case $c_s < \hat{c}_s$

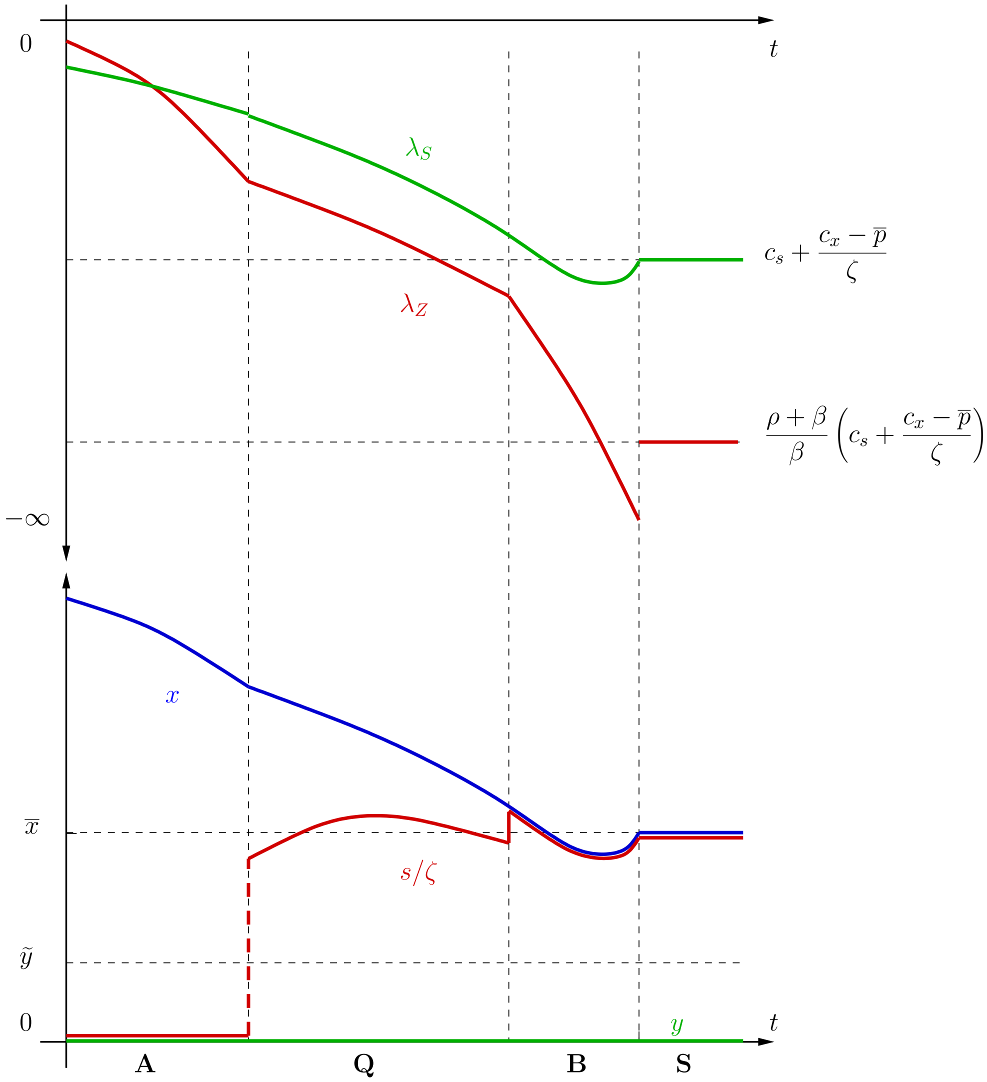

Figure 4.6: Evolution of λ_Z, λ_S, x, y and s, case $c_s < \hat{c}_s$

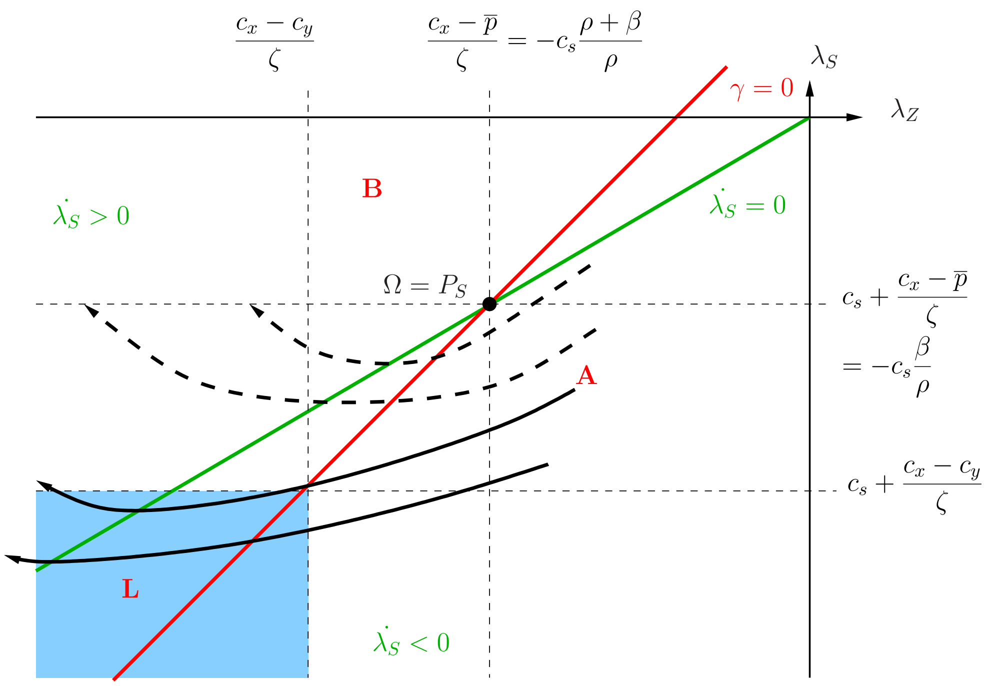

Figure 4.7: Evolution of (λ_Z, λ_S), case $c_s = \hat{c}_s$

4.4.2 Medium-inf c_s $(\hat{c}_s < c_s < c_{sQ})$

When $\hat{c}_s \le c_s < \bar{c}_s$, the situation is represented in Figures 4.8 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time) and 4.9 (for the evolution of $(S(t), Z(t))$ over time). See also Figure 4.7 for the boundary case $c_s = \hat{c}_s$ (in that case the points Ω and P_S coincide) and Figure 4.10 for the boundary case $c_s = \bar{c}_s$ (in that case, the point Ω enters the zone labelled as "L").

Figures 4.8 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time) and 4.9 ehibit four trajectories, labelled as **(I)** to **(IV)**. These trajectories go, respectively through phases A/P, A/Q/P, A/S/Q/P where Phase S is limited to a passage through point $(S_m, \overline{Z})$, and phases A/B/S/Q/P. The possibilities A/B/L/B/S/Q/P and A/L/B/S/Q/P also exist but are not represented. We now describe these curves.

The zone labelled as "**L**" corresponds to values of the multipliers result in a consumption $x(t)$ less than $\tilde{y}$, when $Z(t) < \overline{Z}$: both $\lambda_Z < (c_y - c_x)/\zeta$, which implies $x^{(A)} < \tilde{y}$ and $\lambda_S < c_s + (c_y - c_x)/\zeta$, which implies $x^{(B)} < \tilde{y}$. Note that when $c_s < \bar{c}_s$, the point Ω is located outside this zone.

A typical trajectory starting with a moderate value of $S(0)$ (labelled as **(II)**) has the following features.

Phase A It starts in the interior of the domain in Phase A. The evolution of $(\lambda_Z(t), \lambda_S(t))$ is that of the "free" trajectories (3.3.3)– (3.3.4). While λ_Z always decreases, λ_S decreases, then increases again.

Phase Q If the initial value of S is large enough, the value of $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$, which is negative in Phase A, eventually vanishes. At that moment, the value of $Z(t)$ hits the ceiling $\overline{Z}$. The trajectory then continues in Phase Q: atmospheric stock at the ceiling, with some capture $s(t)$.

In Figure 4.8, the point moves on the red line which represents $\gamma = 0$. It moves *upwards* because $\dot{\lambda_S} > 0$ since the point is located above the green line which represents $\dot{\lambda_S} = (\rho + \beta)\lambda_S - \beta\lambda_Z = 0$.

Eventually, the value of $s(t)$ vanishes and the trajectory enters Phase P.

Phase P Phase P is terminal: the states moves asymptotically to point $(0, \overline{Z})$; the multipliers move the point materialized as P_∞. At that location, we have simultaneously $\dot{\lambda}_S = 0$ and $\lambda_Z = (c_x - \overline{p})/\zeta$, corresponding to a consumption of $\overline{x}$ (see also Figure 4.1).

The line which passes through P_{QP} and P_∞ in Figure 4.8 is the trajectory of the multipliers in Phase P, which is actually independent of c_s.

A trajectory which starts with smaller values of $S(0)$ (labelled as **(I)** on the figures) will follow Phase A in the interior of the domain, but will enter directly Phase P. At the contact point with the boundary $Z = \overline{Z}$, the trajectory is tangent, as explained in Section 4.5.2.

On the other hand, a trajectory starting with a large value of $S(0)$ (labelled as **(IV)** on the figures) will get close to the boundary $Z = Z_M(S)$ and has the following features.

Phase A It starts in the interior of the domain in Phase A as before. However, either λ_Z reaches the critical value $(c_x - c_y)/\zeta$ or λ_S reaches the critical value $c_s + (c_x - c_y)/\zeta$. In the first event, the trajectory enters Phase L; in the second event, it enters directly Phase B.

Phase L Consumption $x(t)$ falls below the level $\widetilde{y}$. Consistent with Lemma 3.2 on page 19, it becomes optimal to set $x = 0$ and consume $y(t) = \widetilde{y}$. The state variables evolve along "free" trajectories, as well as costate variables. Eventually, $\gamma(t)$ becomes positive and λ_S increases to become equal to $c_s + (c_x - c_y)/\zeta$. At that moment, the trajectory enters Phase B.

Phase B Capture $s(t) = \zeta x(t)$ is maximal. This piece of trajectory ends up at point $(S_m, \overline{Z})$ with a value of $\lambda_S = (c_x - \overline{p})/\zeta$ corresponding to a consumption $x = \overline{x}$. The value of λ_Z however depends on the trajectory. The smaller it is, the closer the trajectory gets to the limit $Z = Z_M(S)$.

Phase Q From the point $(S_m, \overline{Z})$, the trajectory enters Phase Q. There is a *discontinuity* in the value of λ_Z (represented as a thin line in Figure 4.8) so that $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$, which is negative in Phase B, becomes null in Phase Q. The evolution is similar to the situation described previously. Eventually, the value of $s(t)$ vanishes and the trajectory enters Phase P.

Phase P As above.

One particular trajectory (labelled as **(III)** on the figures) joins with the boundary precisely at point $(S_m, \overline{Z})$. On this trajectory, the costate variables are continuous.

The distinction between cases $c_s < \bar{c}_s$ and $c_s > \bar{c}_s$ lies in the geometric position of the point Ω. In the second case, it is located inside the zone "**L**" (and located just at the border when $c_s = \bar{c}_s$). It becomes geometrically possible for the point P_S to move on the line $\gamma = 0$ to a position where $\lambda_Z = (c_y - c_x)/\zeta$. However, it does not do so as long as $c_s < c_{sQ}$. Indeed, the value of c_{sQ} is defined in Section 4.3.2 on p. 37 as the value of c_s such that point P_S is located both on the line $\gamma = 0$ and the boundary $\lambda_Z = (c_x - c_y)/\zeta$.

See also Figure 4.10 for the boundary case $c_s = \bar{c}_s$.

4.4.3 Medium-sup c_s ($c_{sQ} < c_s < c_{sm}$)

The situation is represented in Figure 4.14. In that case, the point P_S is located on the boundary $\lambda_Z = (c_x - c_y)/\zeta$ of the zone **L**, which corresponds to the fact that a Phase R appears on the boundary $Z = \overline{Z}$. In Figure 4.13, a point P_{RQ} appears.

In that case, the scenario above is modified as follows, for initial values of S large enough:

Phase A ends when λ_Z reaches $(c_y - c_x)/\zeta$ first. At that moment, $Z(t)$ reaches $Z_M(S(t))$ and consumption $x(t)$ reaches $\widetilde{y}$. Depending on whether $S(t)$ is larger or smaller than S_m, the trajectory continues in Phase L, or one of Phases R, Q or P, respectively.

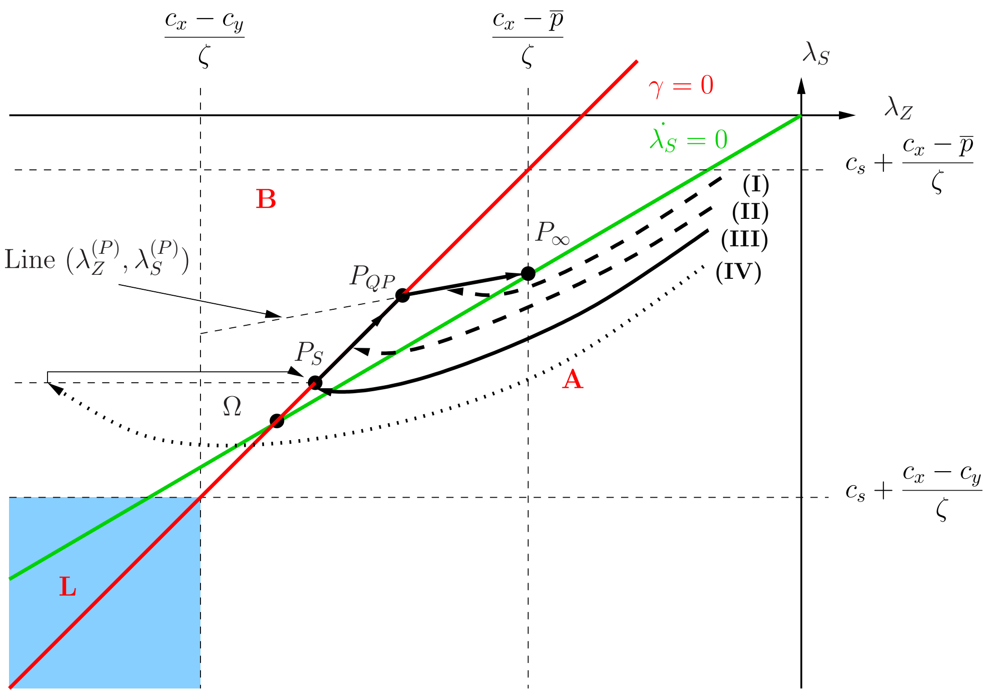

Figure 4.8: Evolution of (λ_Z, λ_S), case $\hat{c}_s \leq c_s \leq \bar{c}_s$

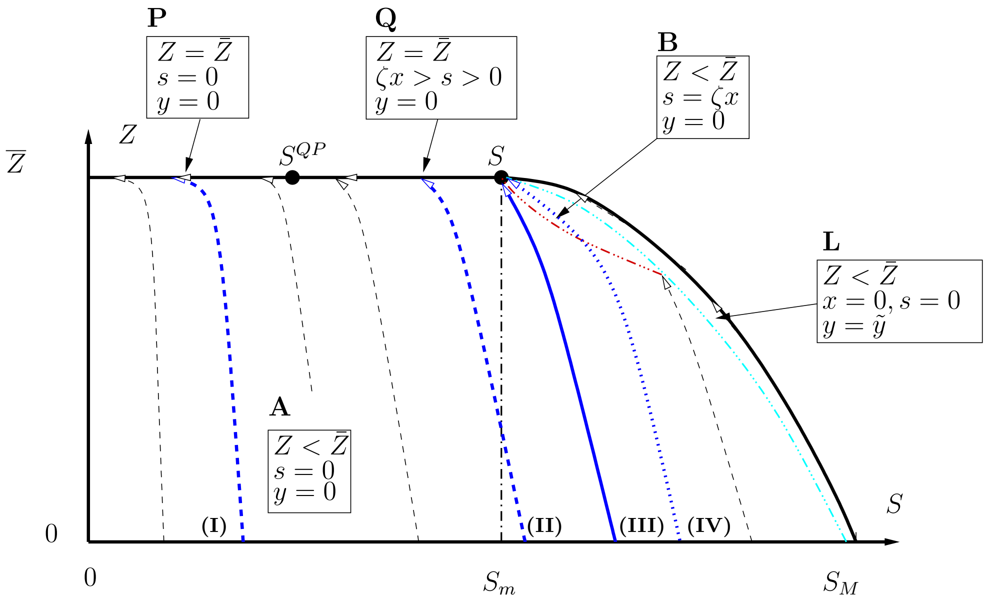

Figure 4.9: Evolution of (S, Z), case $\hat{c}_s \leq c_s \leq \bar{c}_{sQ}$

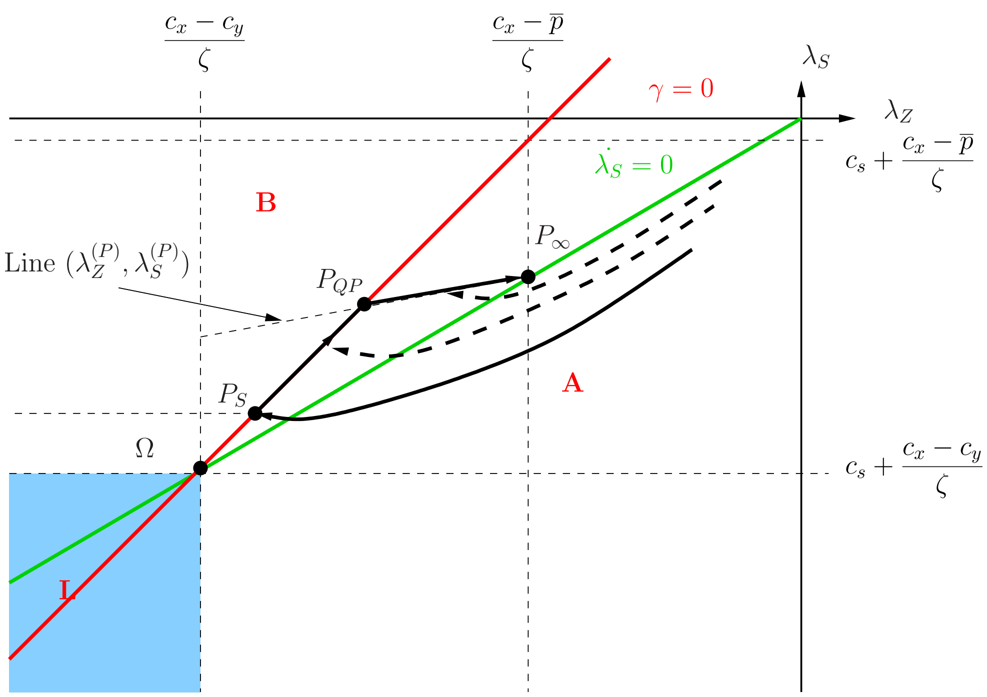

Figure 4.10: Evolution of (λ_Z, λ_S), case $c_s = \bar{c}_s$

Phase L The trajectory continues along $Z = Z_M(S)$ with $x = 0$ and $y = \widetilde{y}$. The value of λ_S is not determined by a differential equation, but rather by the junction condition (4.3.27).

$$\lambda_S(T^*) \;=\; V'(S(T^*)) - Z'_M(S(T^*))\frac{c_x - c_y}{\zeta} \;.$$

The trajectory eventually reaches $(S_m, \overline{Z})$. The location of (λ_Z, λ_S) corresponding to this time instant is labelled as P_S in Figure 4.13.

Phase R The trajectory in Phase R has been described in Figure 4.1: as S decreases from S_m to $S_{\widetilde{y}}$, consumption x increases from 0 to $\widetilde{y}$ while y decreases from $\widetilde{y}$ to 0, their sum being always $x + y = \widetilde{y}$. At some point, $\gamma(t) = 0$ and the trajectory enters Phase Q at point $(S^{RQ}, \overline{Z})$, see Figure 4.1.

Phase Q It becomes optimal to use capture. As $S(t)$ decreases, capture $s(t)$ decreases also and eventually vanishes: the trajectory enters Phase P at point $(S^{QP}, \overline{Z})$.

Phase P As before.

See Figure 4.12 for the boundary case $c_s = c_{sQ}$. In this last case, the points P_S and P_{RQ} coincide. Phase R just vanishes.

4.4.4 Large c_s $(c_s \geq c_{sm})$

When $c_s > c_{sm}$, Phase Q disappears completely, as well as Phase B. Actually, capture is so expensive in this case that $s(t) = 0$ at all times. The model is equivalent to one where capture is not possible at all.

The situation is represented in Figures 4.16 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time) and 4.17 (for the evolution of $(S(t), Z(t))$ over time). See also Figure 4.15 for the boundary case $c_s = \bar{c}_{sm}$.

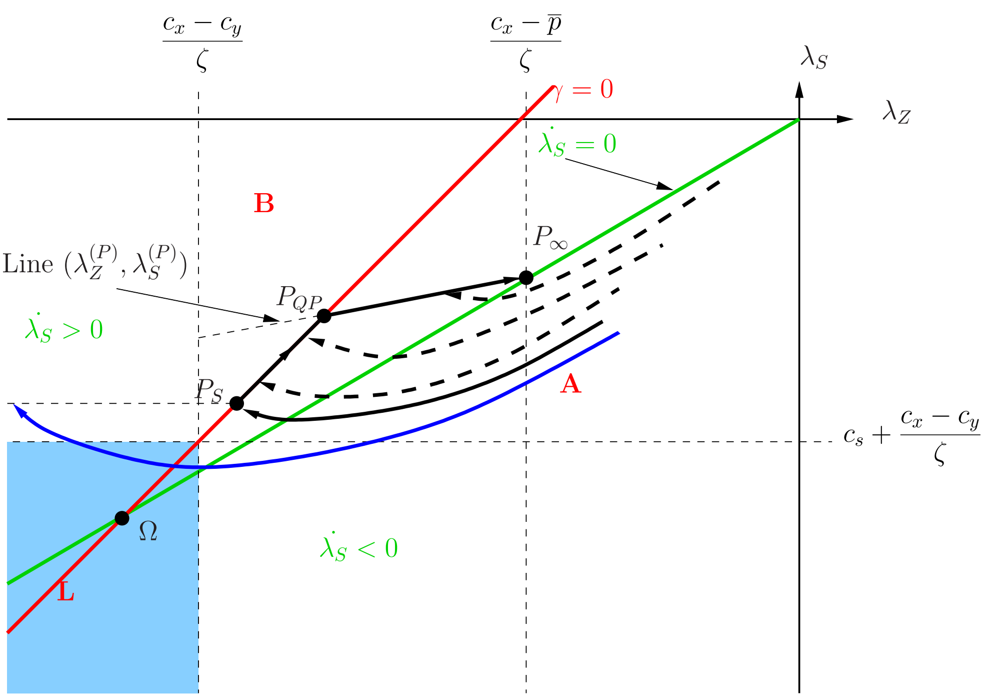

Figure 4.11: Evolution of (λ_Z, λ_S), case $\bar{c}_s \leq c_s \leq c_{sQ}$

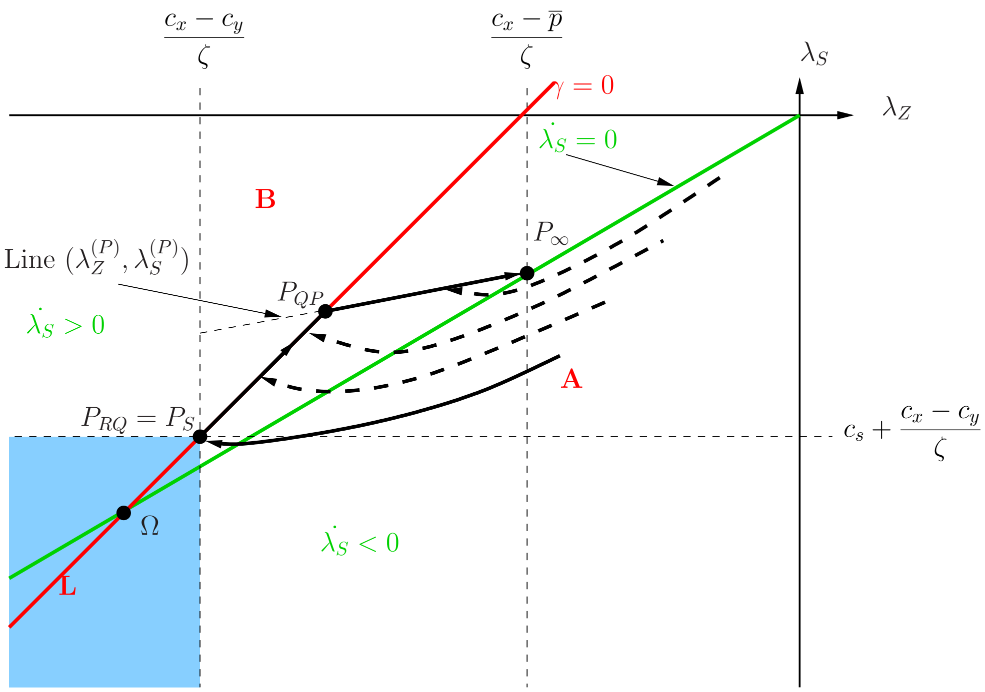

Figure 4.12: Evolution of (λ_Z, λ_S), case $c_s = c_{sQ}$

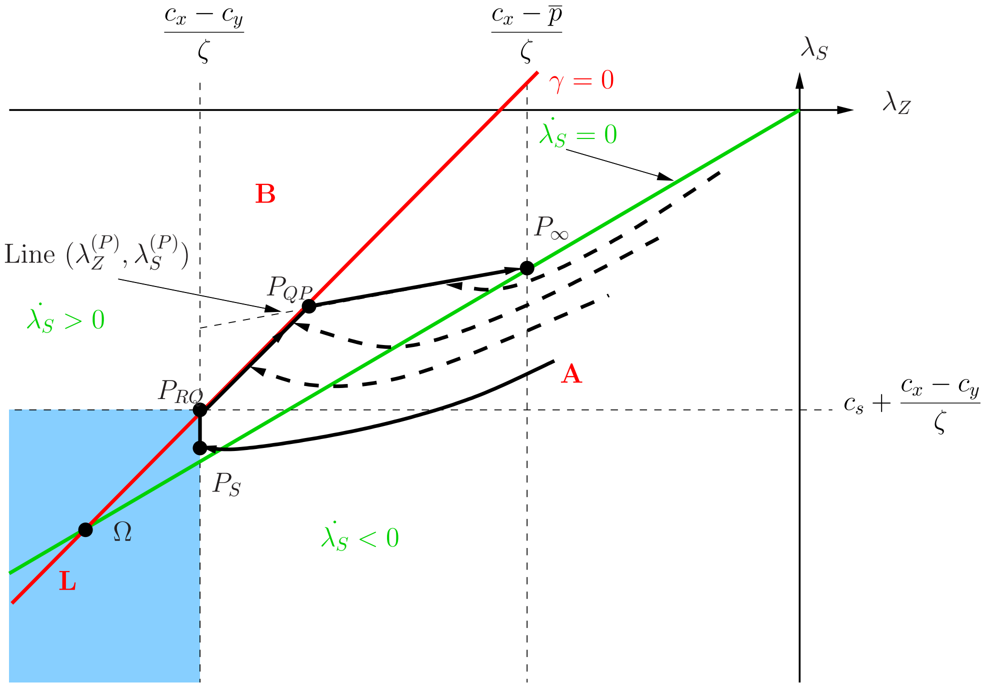

Figure 4.13: Evolution of (λ_Z, λ_S), case $c_{sQ} \leq c_s \leq c_{sm}$

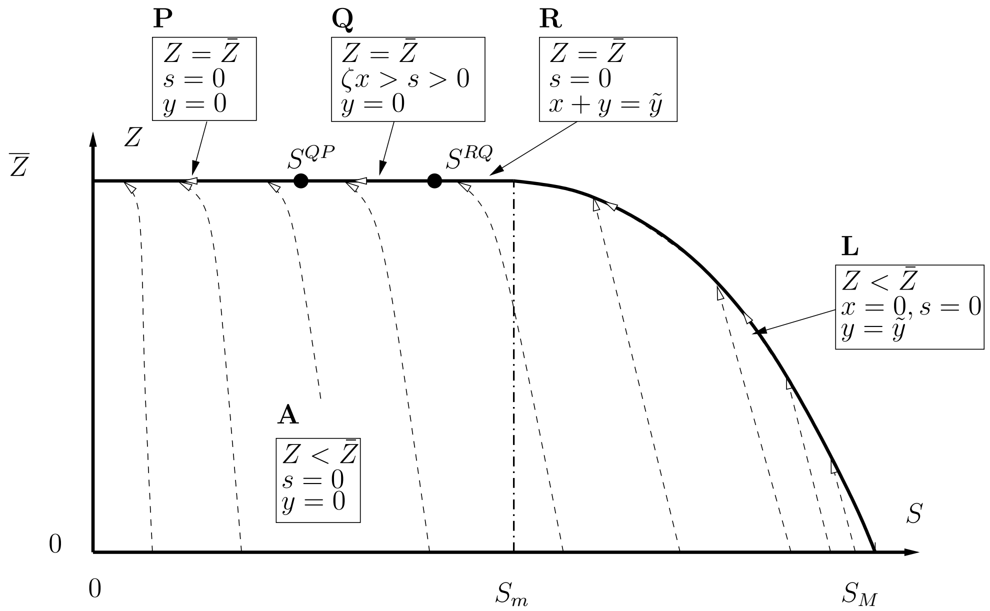

Figure 4.14: Evolution of (S, Z), case $c_{sQ} \leq c_s \leq c_{sm}$

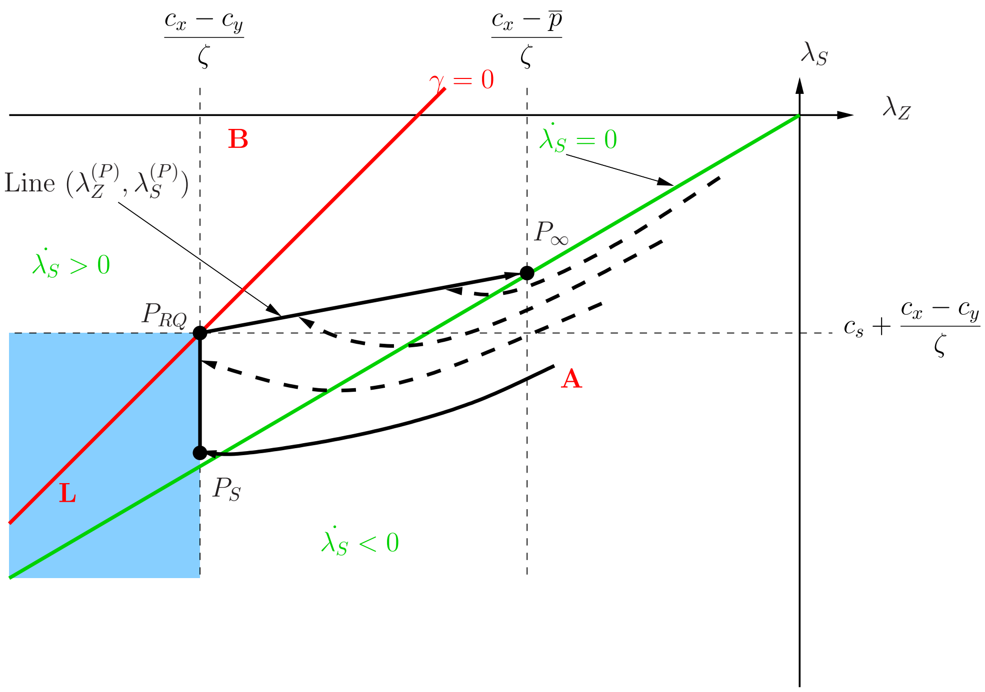

Figure 4.15: Evolution of (λ_Z, λ_S), case $c_s = c_{sm}$

The description of a typical trajectory is quite similar to the case $c_{sQ} < c_s < c_{sm}$ ("medium-sup c_s"), except that there is no Phase Q. When in Phase R, as S decreases from S_m to $S_{\widetilde{y}}$, consumption x increases from 0 to $\widetilde{y}$ while y decreases from $\widetilde{y}$ to 0, their sum being always $x + y = \widetilde{y}$. The trajectory then continues in Phase P as before.

Trajectories starting from smaller values of $S(0)$ will have a succession of phases A/R/P or just A/P.

4.5 Complements

4.5.1 Value on the boundary

We study in this section the value function of the problem, considered as depending on the variable S alone, on the boundary $Z = B(S)$.

Depending on the value of c_s, various phases are chained together. In all situations where $c_s > \hat{c}_s$, Phase P is terminal (and runs forever) and in all situations, Phase L finishes when $S = S_m$. In general, the sequence of phases is L/R/Q/P, with phase R possibly absent. We derive separate expressions for the value function in each phase, denoted respectively as V_L, V_R, V_Q and V_P.

We therefore have the following expressions in all configurations:

$$V_P(S) = \int_0^\infty e^{-\rho v} \left(u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) - c_x \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) \right) \mathrm{d}v \tag{4.5.1}$$

$$V_L(S) = V(S_m) e^{-\rho \tau_L(S)} + \left(1 - e^{-\rho \tau_L(S)} \right) \frac{u(\widetilde{y}) - c_y \widetilde{y}}{\rho} , \tag{4.5.2}$$

with:

$$\tau_L(S) := \frac{1}{\beta} \log \frac{S}{S_m} , \tag{4.5.3}$$

and $V(S_m)$ is the value at point S_m, the expression of which depends on the situation.

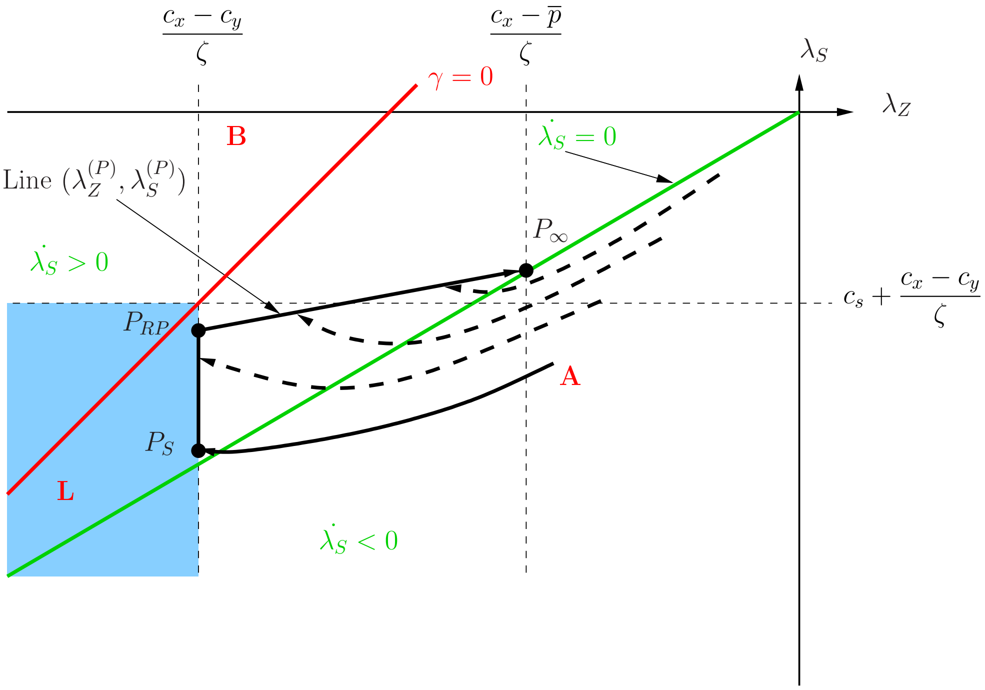

Figure 4.16: Evolution of (λ_Z, λ_S), case $c_s \geq c_{sm}$

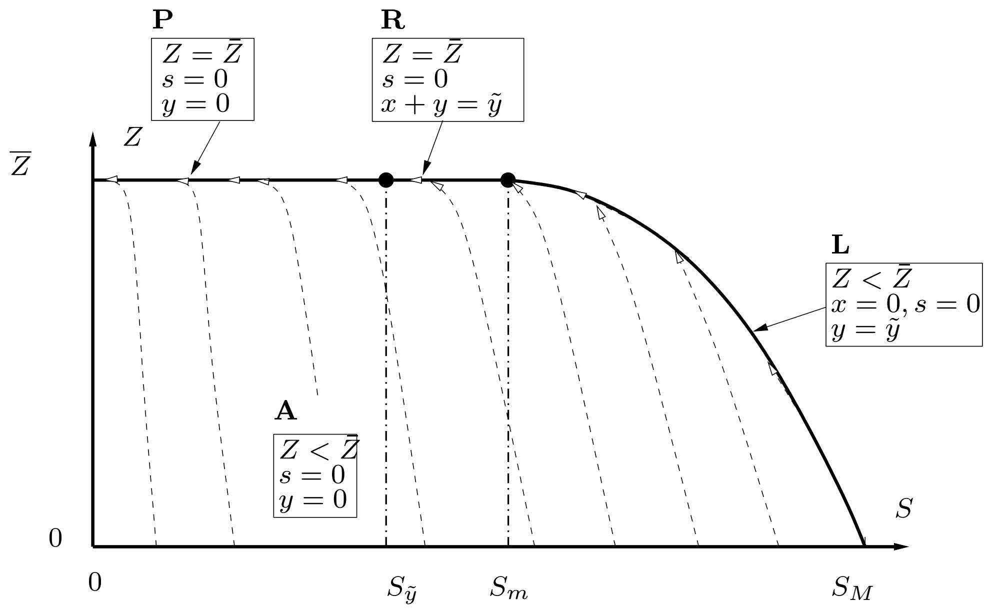

Figure 4.17: Evolution of (S, Z), case $c_s \geq c_{sm}$

By convention, assume that the change from phase R to Q occurs at time t^{RQ} when the state is S^{RQ}. Similarly, assume that the change from phase Q to P occurs at time t^{QP} when the state is S^{QP}. Accordingly, we have the expressions:

$$\begin{aligned} V_R(S) &= \int_0^{\tau_R(S)} e^{-\rho v}\left(u(\widetilde{y}) - c_x\frac{\beta}{\zeta}\left(S_m - Se^{-\beta v}\right) - c_y\frac{\beta}{\zeta}\left(Se^{-\beta v} - S_{\widetilde{y}}\right)\right)\mathrm{d}v \\ &\quad + V(S^{RQ})e^{-\rho\tau_R(S)} \qquad (4.5.4) \\ V_Q(S) &= \int_0^{\tau_Q(S)} e^{-\rho v}\left(u(x^{(Q)}(v)) - c_x x^{(Q)}(v) - c_s\left(\zeta(x^{(Q)}(v) - \overline{x}) + \beta S^{(Q)}(v)\right)\right)\mathrm{d}v \\ &\quad + V(S^{QP})e^{-\rho\tau_Q(S)}\ , \qquad (4.5.5) \end{aligned}$$

with the functions:

$$\begin{aligned} \tau_R(S) &:= \frac{1}{\beta}\log\frac{S}{S^{RQ}} \qquad (4.5.6) \\ x^{(Q)}(t) &:= q^d\left(\left(c_x + \zeta c_s\frac{\rho+\beta}{\rho}\right)\left(1 - e^{\rho(t-t^{QP})}\right) + e^{\rho(t-t^{QP})}u'\left(\overline{x} - \frac{\beta}{\zeta}S^{QP}\right)\right) \qquad (4.5.7) \end{aligned}$$

$$\tau_Q(S) \quad \text{solves} \qquad S + \int_{t^{QP}-\tau_Q(S)}^{t^{QP}} \zeta(x^{(Q)}(t) - \overline{x})\mathrm{d}t \;=\; S^{QP}\ . \qquad (4.5.8)$$

Observe that $V_R(S)$ can be expressed in a more explicit function of S, with, rewriting (4.5.4):

$$\begin{aligned} V_R(S) &= \left(u(\widetilde{y}) - c_x\frac{\beta S_m}{\zeta} + c_y\frac{\beta S_{\widetilde{y}}}{\zeta}\right)\frac{1-e^{-\rho\tau_R(S)}}{\rho} + \int_0^{\tau_R(S)} e^{-\rho v}\frac{\beta}{\zeta}Se^{-\beta v}(c_x - c_y)\mathrm{d}v \\ &\quad + V(S^{RQ})e^{-\rho\tau_R(S)} \\ &= \left(u(\widetilde{y}) - c_x\overline{x} + c_y(\overline{x} - \widetilde{y})\right)\frac{1-e^{-\rho\tau_R(S)}}{\rho} + \beta S\frac{c_x - c_y}{\zeta}\,\frac{1-e^{-(\rho+\beta)\tau_R(S)}}{\rho+\beta} \\ &\quad + V(S^{RQ})e^{-\rho\tau_R(S)}\ . \end{aligned}$$

Identities. The following identities can be checked directly (still to be done for (4.5.10)) or by applying Hamilton-Jacobi-Bellman. We use these identities in the following, in particular to prove the continuity of V'.

$$\begin{aligned} \beta S V_P'(S) + \rho V_P(S) &= u\left(\overline{x} - \frac{\beta}{\zeta}S\right) + \frac{c_x}{\zeta}\left(\beta S - \zeta\overline{x}\right) \qquad (4.5.9) \\ -\zeta(x-\overline{x})V_Q'(S) + \rho V_Q(S) &= u(x) - c_x x - c_s\left(\zeta(x-\overline{x}) + \beta S\right) \qquad (4.5.10) \\ \beta S V_R'(S) + \rho V_R(S) &= u(\widetilde{y}) - c_x\frac{\beta}{\zeta}\left(S - S_m\right) - c_y\frac{\beta}{\zeta}\left(S_{\widetilde{y}} - S\right) \qquad (4.5.11) \\ \beta S V_L'(S) + \rho V_L(S) &= u(\widetilde{y}) - c_y\widetilde{y}\ . \qquad (4.5.12) \end{aligned}$$

Continuity of the value function. The value function is defined as:

$$V(S) = \begin{cases} V_P(S) & \text{if } 0 \le S \le S^{QP} \\ V_Q(S) & \text{if } S^{QP} \le S \le S^{RQ} \\ V_R(S) & \text{if } S^{RQ} \le S \le S_m \\ V_L(S) & \text{if } S_m \le S \le S_M. \end{cases} \qquad (4.5.13)$$

When $c_s \ge c_{sm}$, we have $S^{QP} = S^{RQ} = S_{\widetilde{y}}$ so that phase Q disappears. In some situations, $S^{RQ} = S_m$ so that phase R disappears. See the discussion in Section 4.3.2.

It is clear from the definition of $V(S)$ that the value function is continuous (more precisely, its restriction to the boundary is continuous). Using identities (4.5.11) and (4.5.12), one concludes that $V'(S)$ is continuous at $S = S_m$ when phase R is present since $\beta/\zeta(S_{\widetilde{y}} - S_m) = \widetilde{y}$. Using identities (4.5.9) and (4.5.11), one concludes that $V'(S)$ is continuous at $S = S_{\widetilde{y}}$ when phase Q is absent. Also, using (4.5.9) and (4.5.10), one concludes that $V'(S)$ is continuous at $S = S^{QP}$ because at that point $\beta S^{QP} = \zeta(x - \overline{x})$. Finally, using the fact that $x(t^{RQ}) = \widetilde{y}$, the fact that $V'(S(t)) = \lambda_S(t)$ (see below) and identities (4.5.10) and (4.5.11), it is seen that the continuity of $\lambda_S(t)$ at $t = t^{RQ}$ and that of $V'(S)$ at $S = S^{RQ}$ are equivalent.

4.5.2 Local analysis of trajectories at junction points

The following analysis gives indications on the orientation of the state trajectory when it is in phase A or B, in particular at junction points.

4.5.2.1 Phase A.

The state and costate trajectories are solution to:

$$\left\{\begin{array}{rcl} \dot{Z} &=& -\alpha Z + \beta S + \zeta x \\ \dot{S} &=& -\beta S \end{array}\right. \qquad \left\{\begin{array}{rcl} \dot{\lambda}_Z &=& (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &=& (\rho+\beta)\lambda_S - \beta\lambda_Z \end{array}\right.$$

and $x(t) = q^d(c_x - \zeta\lambda_Z)$. It follows that

$$\ddot{S} = -\beta\dot{S} = \beta^2 S \qquad \dddot{S} = -\beta\ddot{S} = -\beta^3 S\ ,$$

and

$$\begin{array}{rcl} \ddot{Z} &=& -\alpha\dot{Z} + \beta\dot{S} + \zeta\dot{x} \\ &=& \alpha^2 Z - \beta(\alpha+\beta)S - \alpha\zeta x + \zeta\dot{x} \\ \dddot{Z} &=& -\alpha\ddot{Z} + \beta\ddot{S} + \zeta\ddot{x} \\ &=& -\alpha^3 Z + \beta(\alpha^2+\alpha\beta+\beta^2)S + \alpha^2\zeta x - \alpha\zeta\dot{x} + \zeta\ddot{x}\ . \end{array}$$

Finally, from the specific form of $x(t)$, we have:

$$\begin{array}{rcl} \dot{x} &=& -\zeta\dot{\lambda}_Z\ (q^d)'(c_x - \zeta\lambda_Z) \ = \ -\zeta(\rho+\alpha)\lambda_Z\ (q^d)'(c_x - \zeta\lambda_Z) \\ \ddot{x} &=& -\zeta\ddot{\lambda}_Z\ (q^d)'(c_x - \zeta\lambda_Z) + \zeta(\dot{\lambda}_Z)^2\ (q^d)''(c_x - \zeta\lambda_Z)\ . \end{array}$$

By assumption, $u'(\cdot)$ and $q^d(\cdot)$ are decreasing: $(q^d)' < 0$. There is no assumption on the sign of $(q^d)''$. By analysis, $\lambda_Z < 0$ so that $\ddot{\lambda}_Z < 0$ and $\dot{\lambda}_Z < 0$. Finally, $\dot{x} < 0$ but the sign of $\ddot{x}$ is not determined *a priori*. In the LQ case (see Section C), $(q^d)'' = 0$ and $\ddot{x} < 0$.

Junction with $Z = \overline{Z}$. Assume that the trajectory hits the state $(S, Z) = (\overline{Z}, S^0)$ at time $t = 0$. Then we have the Taylor expansion for Z:

$$Z(t) \ = \ \overline{Z} + t(\beta(S^0 - S_m) + \zeta x(0)) + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 - \alpha\zeta x(0) + \zeta\dot{x}(0)) + O(t^3)\ .$$

When the junction occurs in Phase P with continuity of λ_Z, we have from (4.1.1):

$$\lambda_Z(0) \ = \ \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0)\right)\ ,$$

or equivalently, $\zeta x(0) = \zeta q^d(u'(\overline{x} - \frac{\beta}{\zeta}S^0)) = \zeta\overline{x} - \beta S^0$ (see for instance Figure 4.1, top, on page 38). Replacing in the development, we get:

$$Z(t) \ = \ \overline{Z} + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 - \alpha\zeta\overline{x} + \alpha\beta S^0 + \zeta\dot{x}(0)) + O(t^3)$$

$$= \overline{Z} + \frac{t^2}{2}(-\beta^2 S^0 + \zeta\dot{x}(0)) + O(t^3) \ .$$

We have seen above that $\dot{x} < 0$ in general, so that in fact, $\ddot{Z} < 0$. On the other hand, the development for S is just:

$$S(t) \ = \ S^0 - \beta t S^0 + \frac{t^2}{2}\beta^2 S^0 + O(t^3) \ .$$

The conclusion is: at the junction of phases A/P, the trajectory is tangent to the line $Z = \overline{Z}$, coming from below and from the right.

When the junction occurs in Phase R, we have from (4.2.6),

$$\lambda_Z(0) \ = \ -\frac{c_y - c_x}{\zeta}, \quad \text{or equivalently} \quad x(0) \ = \ \widetilde{y} \ .$$

The development can be expressed as:

$$Z(t) \ = \ \overline{Z} + t(\beta S^0 + \zeta(\widetilde{y} - \overline{x})) + O(t^2) \ = \ \overline{Z} + t\beta(S^0 - S_{\widetilde{y}}) + O(t^2) \ .$$

Then, the trajectory hits the ceiling at an angle of direction $(-S^0, S^0 - S_{\widetilde{y}})$. At the triple point of phases A, R and P, we have $S^0 = S_{\widetilde{y}}$ and this direction is tangent to the line $Z = \overline{Z}$, in accordance with the junction in phase P, see above. At any other point $S_{\widetilde{y}} < S^0 \leq S_m$, this angle is sharp.

When junction occurs in Phase Q, then according to (4.2.3) we have: $s(0) = \zeta x(0) - \beta(S_m - S^0) = \zeta(x(0) - \overline{x}) + \beta S^0$. Replacing in the development of Z, we get:

$$Z(t) \ = \ \overline{Z} + ts(0) + O(t^2) \ ,$$

and again, the trajectory hits the line $Z = \overline{Z}$ with an angle of direction $(-\beta S^0, s(0))$. As the junction point S^0 moves from S^{QP} to S^{QR}, this angle moves continuously between the tangent to $Z = \overline{Z}$ to the same angle as in Phase R.

Junction on the curve $Z = Z_M(S)$. When an optimal trajectory joins the boundary curve at some point $(S, Z_M(S))$, its tangent vector is $(-\beta S, -\alpha Z + \beta S + \zeta x)$. The tangent vector to the boundary itself is, since the curve is a "free" trajectory: $(-\beta S, -\alpha Z + \beta S)$. The tangent vector of the optimal trajectory is therefore pointing "outwards" as required.

When the junction point is close to $S = S_m$, the tangent vector tends to $(-\beta S_m, \zeta\widetilde{y})$, This is the same limit as in Phase R: according to what was said above, the tangent vector in Phase R close to $S = S_m$ has the direction: $(-\beta S_m, \beta(S_m - S_{\widetilde{y}})) = (-\beta S_m, \zeta\widetilde{y})$ (see below Equation (4.0.1)). There is therefore continuity of directions at that point.

4.5.2.2 Phase B.

The state and costate trajectories are solution to:

$$\begin{cases} \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S + \zeta x \end{cases} \qquad \begin{cases} \dot{\lambda}_Z &= (\rho + \alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho + \beta)\lambda_S - \beta\lambda_Z \end{cases}$$

and $x(t) = q^d(c_s + \zeta c_s - \zeta\lambda_S)$. It follows that:

$$\begin{aligned} \ddot{Z} &= -\alpha\dot{Z} + \beta\dot{S} \\ &= \alpha^2 Z - \beta(\alpha + \beta)S + \beta\zeta x \\ \ddot{S} &= -\beta\dot{S} + \zeta\dot{x} \\ &= \beta^2 S - \beta\zeta x + \zeta\dot{x} \ , \end{aligned}$$

and

$$
\begin{aligned}
\dddot{Z} &= -\alpha \ddot{Z} + \beta \ddot{S} \\
&= -\alpha^3 Z + \beta(\alpha^2 + \alpha\beta + \beta^2) S - \beta(\alpha+\beta)\zeta x + \beta\zeta \dot{x} \\
\dddot{S} &= -\beta \ddot{S} + \zeta \ddot{x} \\
&= -\beta^3 S + \beta^2 \zeta x - \beta\zeta \dot{x} + \zeta \ddot{x} \ .
\end{aligned}
$$

Finally, from the specific form of $x(t)$, we have:

$$
\begin{aligned}
\dot{x} &= -\zeta \dot{\lambda}_S \ (q^d)'(c_x + \zeta c_s - \zeta \lambda_S) \\
\ddot{x} &= -\zeta \ddot{\lambda}_S \ (q^d)'(c_x + \zeta c_s - \zeta \lambda_S) + \zeta (\dot{\lambda}_S)^2 \ (q^d)''(c_x + \zeta c_s - \zeta \lambda_S) \ .
\end{aligned}
$$

We conclude that the sign of $\dot{x}$ is the same as the sign of λ_S, but the latter can be $+$ or - in Phase B. A more precise analysis in function of c_s is necessary.

Junction with $Z = \overline{Z}$. The analysis which follows suggests that only two possibilities occur for a junction in phase B: 1) either $c_s < \hat{c}_s$ and the trajectory may actually *leave* the line $Z = \overline{Z}$ to enter phase B; 2) the trajectory hits $(\overline{Z}, S_m)$ in phase B.

When the trajectory hits the point $(S^0, \overline{Z})$, the Taylor developments of the state variables are generally:

$$
\begin{aligned}
Z(t) &= \overline{Z} + t\beta(S^0 - S_m) + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 + \beta\zeta x(0)) + O(t^3) && (4.5.14) \\
S(t) &= S^0 + t(\zeta x(0) - \beta S^0) + \frac{t^2}{2}(\beta^2 S^0 - \beta\zeta x(0) + \zeta \dot{x}(0)) + O(t^3) \ . && (4.5.15)
\end{aligned}
$$

Assume first that $S^0 < S_m$. Then clearly $\dot{Z}(0) < 0$ and the trajectory *cannot arrive at* the line $Z = \overline{Z}$: it must be leaving. Its direction is $(\zeta x(0) - \beta S^0, S^0 - S_m)$.

Assume next that $S^0 = S_m$. Then the development is simplified into:

$$
\begin{aligned}
Z(t) &= \overline{Z} + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S_m + \beta\zeta x(0)) + O(t^3) \\
&= \overline{Z} + \frac{t^2}{2}\beta\zeta(x(0) - \overline{x}) + O(t^3) && (4.5.16) \\
S(t) &= S_m + t\zeta(x(0) - \overline{x}) + \frac{t^2}{2}(\beta\zeta\overline{x} - \beta\zeta x(0) + \zeta\dot{x}(0)) + O(t^3) \ . && (4.5.17)
\end{aligned}
$$

If $x(0) \neq \overline{x}$, by elimination of the time variable, one gets that

$$
t \sim \frac{S(t) - S_m}{\zeta(x(0) - \overline{x})}
$$

so that the trajectory is, asymptotically,

$$
\begin{aligned}
Z &= \overline{Z} + \frac{1}{2}\beta\zeta(x(0) - \overline{x}) \left(\frac{S - S_m}{\zeta(x(0) - \overline{x})} \right)^2 + o((S - S_m)^2) \\
&= \overline{Z} + \frac{1}{2}\frac{\beta}{\zeta}\frac{(S - S_m)^2}{x(0) - \overline{x}} + o((S - S_m)^2) \ .
\end{aligned}
$$

On the condition that $x(0) < \overline{x}$, this trajectory is tangent to the line $Z = \overline{Z}$ and arrives from below and from the right. If $x(0) > \overline{x}$, the trajectory arrives from above, which is not consistent.

However, if $x(0) = \overline{x}$, then we have $\dot{S} = \dot{Z} = \ddot{Z} = 0$, and the development of $Z(t)$ has to be refined to get, using the formula for $\dddot{Z}$:

$$
Z(t) \ = \ \overline{Z} \ + \ \frac{t^3}{6}\zeta\dot{x}(0) \ + \ O(t^4) \qquad (4.5.18)
$$

$$S(t) \;=\; S_m \;+\; \frac{t^2}{2}\zeta\dot{x}(0) \;+\; O(t^3) \;. \tag{4.5.19}$$

If $\dot{x}(0) > 0$, which happens when $\dot{\lambda}_S > 0$, then the trajectory is tangent to the line $Z = \overline{Z}$ and approaches it from below and from the right. In the case $\dot{x}(0) < 0$, it approaches it from above, and this is not consistent. In the first case, eliminating the time variable gives (remembering that $t \leq 0$):

$$t \;\sim\; -\;\left(\frac{2(S(t)-S_m)}{\zeta\dot{x}(0)}\right)^{1/2}$$

so that the trajectory is, asymptotically,

$$\begin{aligned} Z \;&=\; \overline{Z} - \frac{1}{6}\zeta\dot{x}(0)\left(\frac{2(S(t)-S_m)}{\zeta\dot{x}(0)}\right)^{3/2} + o((S-S_m)^{3/2}) \\ &=\; \overline{Z} - \frac{2^{3/2}}{6}\frac{(S-S_m)^{3/2}}{(\zeta\dot{x}(0))^{1/2}} + o((S-S_m)^{3/2}) \;. \end{aligned}$$

4.5.3 Discussion on the thesholds

Here we discuss the interpretation of some thresholds on c_s.

$\hat{\mathbf{c}}_{\mathbf{s}}$**:** an interpretation of this value derives from a local perturbation of trajectories close to the point $(S_m, \overline{Z})$, as follows.

Consider the reference situation where $Z(t) = \overline{Z}$, $S(t) = S_m$, $x(t) = \overline{x}$ and $s(t) = \zeta\overline{x}$ (see Section 4.1.2). Assume that on the time interval $[0, \Delta t]$, the consumption is modified into $x(t) = \overline{x} - \Delta x$ (constant over time) and the capture computed so that the constraint $Z(t) = \overline{Z}$ still holds. Then since $\dot{Z} = 0$, we must have:

$$0 \;=\; -\alpha\overline{Z} + \beta S(t) + \zeta(\overline{x} - \Delta x) - s(t) \;\;\Longrightarrow\;\; s(t) \;=\; \beta S(t) - \zeta\Delta x \;.$$

As a consequence, we have $\dot{S} = -\beta S + s = -\zeta\Delta x$ is constant on the interval, and $S(t) = S_m - \zeta\Delta x\, t$.

On interval $[\Delta t, \infty)$, capture is restored to the nominal level $\zeta\overline{x}$, and consumption is such that $Z = \overline{Z}$: it is therefore

$$x(t) \;=\; \overline{x} \;+\; \beta(S_m - S(t))/\zeta \;.$$

As a consequence, $\dot{S} = \beta(S_m - S)$ on the interval, and $S(t) = S_m + (S(\Delta t) - S_m)e^{-\beta(t-\Delta t)} = S_m - \zeta\Delta x\Delta t e^{-\beta(t-\Delta t)}$.

On the interval $[0, \Delta t]$, the difference in profit between both trajectories is

$$\begin{aligned} D_1 \;&=\; \int_0^{\Delta t} e^{-\rho t}[u(\overline{x}) - u(\overline{x}-\Delta x) - c_x\Delta x - c_s(\zeta\overline{x} - \beta S + \zeta\Delta x)]\mathrm{d}t \\ &=\; \frac{1-e^{-\rho\Delta t}}{\rho}[u(\overline{x}) - u(\overline{x}-\Delta x) - (c_x + \zeta c_s)\Delta x] - c_s\int_0^{\Delta t} e^{-\rho t}\beta(S_m - S(t))\mathrm{d}t \\ &=\; \frac{1-e^{-\rho\Delta t}}{\rho}[u(\overline{x}) - u(\overline{x}-\Delta x) - (c_x + \zeta c_s)\Delta x] - \beta c_s\zeta\Delta x\int_0^{\Delta t} te^{-\rho t}\mathrm{d}t \;. \end{aligned}$$

On the interval $[\Delta t, \infty)$, this difference is:

$$\begin{aligned} D_2 \;&=\; \int_{\Delta t}^{\infty} e^{-\rho t}[u(\overline{x}) - u(\overline{x} + \beta(S_m - S(t))/\zeta) + c_x\beta(S_m - S)/\zeta]\mathrm{d}t \\ &=\; \int_{\Delta t}^{\infty} e^{-\rho t}[u(\overline{x}) - u(\overline{x} + \beta\Delta t\Delta x e^{\beta(t-\Delta t)}) + \beta c_x\Delta t\Delta x e^{\beta(t-\Delta t)}]\mathrm{d}t \;. \end{aligned}$$

When Δt tends to 0, we have

$$\begin{aligned} D_2 &= \int_{\Delta t}^{\infty} e^{-\rho t}[-\overline{p}\beta\Delta t\Delta x e^{\beta(t-\Delta t)} + \beta c_x \Delta t \Delta x e^{\beta(t-\Delta t)}]\mathrm{d}t \ + \ o(\Delta t) \\ &= \frac{\beta}{\rho+\beta}\Delta t \Delta x (c_x - \overline{p}) \ + \ o(\Delta t) \ . \end{aligned}$$

On the other hand, assuming that Δx is also small,

$$\begin{aligned} D_1 &= [u(\overline{x}) - u(\overline{x}-\Delta x) - (c_x + \zeta c_s)\Delta x]\Delta t \ + \ o(\Delta t) \\ &= [\overline{p}\Delta x - (c_x+\zeta c_s)\Delta x + o(\Delta x)]\Delta t \ + \ o(\Delta t) \\ &= (\overline{p} - c_x - \zeta c_s)\Delta x \Delta t + o(\Delta x)\Delta t \ + \ o(\Delta t) \ . \end{aligned}$$

If the reference trajectory is optimal, then $D_1 + D_2$ must be positive. Asymptotically when Δt and Δx tend to 0, this means:

$$\begin{aligned} 0 &\leq (\overline{p} - c_x - \zeta c_s)\Delta x \Delta t + \frac{\beta}{\rho+\beta}\Delta t \Delta x (c_x - \overline{p}) \\ c_s &\leq \frac{\rho}{\rho+\beta}\,\frac{\overline{p}-c_x}{\zeta} \ = \ \hat{c}_s \ . \end{aligned}$$

$\mathbf{c_{sQ}}$**:** this quantity is defined by Equation (4.3.15) on page 36. Interpretation with marginal "values" of the stock S at S_m^+ and S_m^-?

$\mathbf{c_{sm}}$**:** this quantity is defined by Equation (4.3.6) on page 33. Interpretation with the marginal "value" of the stock S at S_m?

AJM: To be developped.

Appendix A

Synthetic description of the different phases

A.1 Phase A (free extraction of the NRR; no sequestration)

Specification

s	x	y	X	Z	S
$=0$	>0	$=0$	>0	$<\overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S + \zeta x \\ \dot{S} &= -\beta S \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s &=& c_s + \lambda_Z - \lambda_S \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & x &=& q^d(c_x + \lambda_X - \zeta\lambda_Z) \\ u'(x) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{ll} X & X > 0 \\ Z & Z \leq \overline{Z} \\ s & s = 0 \\ x & c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \end{array}$$

A.2 Phase B (free extraction of the NRR; maximal sequestration)

Specification

s	x	y	X	Z	S
$=\zeta x$	>0	$=0$	>0	$<\overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S + \zeta x \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z + \gamma_{sx} & \Longrightarrow & \gamma_{sx} &=& \lambda_S - c_s - \lambda_Z \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & \Longrightarrow & x &=& q^d(c_x + \lambda_X - \zeta\lambda_S + \zeta c_s) \\ u'(x) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_S - \zeta c_s \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z \leq \overline{Z} \\ s & s = \zeta x \\ x & c_x + \lambda_X - \zeta\lambda_S + \zeta c_s \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s = 0 \\ \gamma_{sx} & \lambda_S - c_s - \lambda_Z \geq 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_S - \zeta c_s \geq 0 \end{array}$$

A.3 Phase L (zero extraction of the NRR)

Specification

s	x	y	X	Z	S
$=0$	$=0$	>0	>0	$<\overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho + \alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho + \beta)\lambda_S - \beta \lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} 0 &=& \lambda_S - c_s - \lambda_Z + \gamma_s - \gamma_{sx} & \Longrightarrow & \gamma_s - \gamma_{sx} &=& \lambda_Z - \lambda_S + c_s \\ u'(y) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} - \gamma_x & & & & \\ u'(y) &=& c_y & \Longrightarrow & y &=& \tilde{y} \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z \leq \overline{Z} \\ s & s = 0 \\ x & x = 0 \\ y & y > 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s \geq 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_x & \gamma_x \geq 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

A.4 Phase P (ceiling; no sequestration)

Specification

s	x	y	X	Z	S
$=0$	>0	$=0$	>0	$=\overline{Z}$	≥ 0

Ceiling constraint:

$$x \;=\; \overline{x} - \frac{\beta}{\zeta} S \;=\; \frac{\beta}{\zeta}(S_m - S)\;.$$

Dynamical system

$$\left\{\begin{array}{lcl} \dot{X} & = & -x \\ \dot{Z} & = & 0 \\ \dot{S} & = & -\beta S \\ \dot{\lambda}_X & = & \rho\lambda_X \\ \dot{\lambda}_Z & = & (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S & = & (\rho+\beta)\lambda_S - \beta\lambda_Z \end{array}\right.$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S & = & c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s & = & c_s + \lambda_Z - \lambda_S \\ u'(x) & = & c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & \lambda_Z & = & \frac{1}{\zeta}(c_x + \lambda_X - u'(\overline{x} - \frac{\beta S}{\zeta})) \\ u'(x) & = & c_y - \gamma_y & \Longrightarrow & \gamma_y & = & c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{ll} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = 0 \\ x & S \leq S_m \text{ et } c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \iff u'(x) \leq c_y \\ & \iff x \geq \widetilde{y} \iff S \leq S_{\widetilde{y}}. \end{array}$$

A.5 Phase Q (ceiling; sequestration)

Specification

s	x	y	X	Z	S
> 0 and $< \zeta x$	> 0	$= 0$	> 0	$= \overline{Z}$	≥ 0

Ceiling constraint:

$$s = \zeta(x - \overline{x}) + \beta S = \zeta x - \beta(S_m - S) .$$

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= 0 \\ \dot{S} &= \zeta(x - \overline{x}) \\ \dot{\lambda}_X &= \rho\lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z & & & & \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & x &=& q^d(c_x + \lambda_X - \zeta\lambda_Z) \\ u'(x) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z = \overline{Z} \\ s & S \leq S_m \text{ and } x \geq \frac{\beta}{\zeta}(S_m - S) \\ x & c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \lambda_Z \leq \frac{\rho+\beta}{\alpha}c_s \text{ ou } \lambda_S \leq \frac{\rho+\alpha+\beta}{\alpha}c_s \\ \gamma_s & c_s + \lambda_Z - \lambda_S = 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \end{array}$$

Observations. Conditions $\lambda_Z \leq 0$ and $c_s + \lambda_Z - \lambda_S \geq 0$ imply Conditions $\lambda_Z \leq \frac{\rho+\beta}{\alpha}c_s$ ou $\lambda_S \leq \frac{\rho+\alpha+\beta}{\alpha}c_s$.

If $c_s = 0$, then λ_Z cannot change sign.

A.6 Phase R (ceiling; no sequestration, double extraction)

Specification

s	x	y	X	Z	S
$=0$	>0	>0	>0	$=\overline{Z}$	≥ 0

Ceiling constraint:

$$x \;=\; \overline{x} - \frac{\beta}{\zeta} S \;=\; \frac{\beta}{\zeta}(S_m - S)\;.$$

Dynamical system

$$\begin{cases} \dot{X} &=& -x \\ \dot{Z} &=& 0 \\ \dot{S} &=& -\beta S \\ \dot{\lambda}_X &=& \rho\lambda_X \\ \dot{\lambda}_Z &=& (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &=& (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s &=& c_s + \lambda_Z - \lambda_S \\ u'(x+y) &=& c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & \lambda_Z &=& \frac{1}{\zeta}(c_x + \lambda_X - c_y) \\ u'(x+y) &=& c_y & \Longrightarrow & y &=& \widetilde{y} - \frac{\beta}{\zeta}(S_m - S) \;=\; \frac{\beta}{\zeta}(S - S_{\widetilde{y}})\;. \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = 0 \\ x & S \geq S_m \\ y & y > 0 \iff S(t) \geq S_{\widetilde{y}} \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

A.7 Phase S (ceiling for Z et S; maximal sequestration)

Specification

s	x	y	X	Z	S
$=\zeta\overline{x}$	$=\overline{x}$	$=0$	>0	$=\overline{Z}$	$=S_m$

Ceiling constraint: satisfied by construction.

Dynamical system

$$\begin{cases} \dot{X} &= -\overline{x} \\ \dot{Z} &= 0 \\ \dot{S} &= 0 \\ \dot{\lambda}_X &= \rho\lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z + \gamma_{sx} & \Longrightarrow & \gamma_{sx} + \lambda_Z &=& \lambda_S - c_s \\ u'(\overline{x}) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & \Longrightarrow & \gamma_{sx} + \lambda_Z &=& \frac{1}{\zeta}(c_x + \lambda_X - \overline{p}) \\ u'(\overline{x}) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - \overline{p}\ . \end{array}$$

Constraints

$$\begin{array}{ll} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = \zeta x \\ x & S \le S_m \\ y & y > 0 \iff S(t) \ge S_{\widetilde{y}} \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \ge 0 \\ \gamma_s & \gamma_s = 0 \\ \gamma_{sx} & \gamma_{sx} \ge 0 \\ \gamma_x & \gamma_x = 0 \\ \gamma_y & \gamma_y \ge 0 \end{array}$$

A.8 Phase T (terminal; no extraction of the NRR; extraction of the RR)

Specification

s	x	y	X	Z	S
$=0$	$=0$	>0	≥ 0	$\leq \overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} & = & 0 \\ \dot{Z} & = & -\alpha Z + \beta S \\ \dot{S} & = & -\beta S \\ \dot{\lambda}_X & = & \rho\lambda_X - \nu_X \\ \dot{\lambda}_Z & = & (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S & = & (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rcllrcl} \lambda_S & = & c_s + \lambda_Z - \gamma_s + \gamma_{sx} & \Longrightarrow & \gamma_s, \gamma_{sx} & = & ? \\ u'(y) & = & c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} - \gamma_x & \Longrightarrow & \gamma_x & = & ? \\ u'(y) & = & c_y & \Longrightarrow & y & = & q^d(c_y) \end{array}$$

Constraints

$$\begin{array}{cl} X & X = 0 \text{ ou } \nu_X = 0 \\ Z & Z < \overline{Z} \\ s & s = 0 \\ x & x = 0 \\ y & y > 0 \\ \nu_X & \nu_X = 0 \text{ ou } X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s \geq 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_x & \gamma_x \geq 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

Conditions de transversalité à l'infini :

$$\lambda_Z(T) = 0, \qquad \lambda_S(T) = 0.$$

Appendix B

The functions L and M

The functions $L(\cdot)$ and $M(\cdot)$ are defined in (4.1.4) and (4.1.5) as:

$$\begin{aligned} L(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v \\ M(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v \ . \end{aligned}$$

They differ by a constant and negative additive factor:

$$L(S) = \frac{\beta}{\rho+\beta}(c_x - \overline{p}) + M(S) \ .$$

The function M is clearly negative with $M(0) = 0$. It is decreasing: differentiating in its definition, one gets:

$$L'(S) = M'(S) = \frac{\beta^2}{\zeta} \int_0^\infty e^{-(\rho+2\beta)v} u'' \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) \mathrm{d}v \ , \tag{B.0.1}$$

which is negative because $u'' \leq 0$. The function L is therefore decreasing as well.

Lemma B.1. *We have the bounds, for all S:*

$$L(S) \geq \frac{\beta}{\rho+\beta} \left(c_x - u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \right) \tag{B.0.2}$$

$$M(S) \geq \frac{\beta}{\rho+\beta} \left(\overline{p} - u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \right) \ , \tag{B.0.3}$$

with equality if and only if $S = 0$.

Proof. This bound is proven with the following sequence of inequalities. Given that $u'(\cdot)$ is decreasing, then for all $v \geq 0$,

$$\begin{aligned} \frac{\beta}{\zeta} S e^{-\beta v} &\leq \frac{\beta}{\zeta} S \\ \overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} &\geq \overline{x} - \frac{\beta}{\zeta} S \\ u' \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) &\leq u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \\ \overline{p} - u' \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) &\geq \overline{p} - u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \end{aligned}$$

$$\int_0^\infty e^{-(\rho+\beta)v} \left[\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}\right)\right] \mathrm{d}v \quad \geq \quad \frac{1}{\rho+\beta} \left[\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S\right)\right] .$$

□

As a corollary, from the definition of c_{sm} in Equation (4.3.6), we have the inequality:

$$\begin{aligned} c_{sm} &= \frac{c_y - c_x}{\zeta} + L(S_{\widetilde{y}}) \\ &\geq \frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}}\right)\right) \\ &= \frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \frac{c_x - c_y}{\zeta} = \overline{c}_s . \end{aligned} \tag{B.0.4}$$

The following refines this reasoning. According to the definition of c_{sm} in Equation (4.3.6), and that of $L(S)$ in Equation (4.1.4), we have actually:

$$\begin{aligned} c_{sm} - \overline{c}_s &= \frac{c_y - c_x}{\zeta} + L(S_{\widetilde{y}}) - \left(\frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \frac{c_x - c_y}{\zeta}\right) \\ &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}} e^{-\beta v}\right) - u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}}\right)\right) \mathrm{d}v . \end{aligned} \tag{B.0.5}$$

This is positive, because u' is decreasing.

Alternate expressions exist for $L(\cdot)$ and $M(\cdot)$. For instance:

$$L(S) = \frac{\beta c_x}{\rho+\beta} + \frac{\zeta}{\beta S} u(\overline{x} - \frac{\beta}{\zeta} S) - \frac{\zeta\rho}{\beta S} \int_0^\infty e^{-\rho t} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta t}) \mathrm{d}t . \tag{B.0.6}$$

This expression is obtained from the definition in (4.1.4) and integration by parts as:

$$\begin{aligned} L(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v})\right) \mathrm{d}v \\ &= \frac{\beta c_x}{\rho+\beta} - \frac{\zeta}{\beta S} \int_0^\infty e^{-\rho v} \frac{\beta^2 S}{\zeta} e^{-\beta v} u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v \\ &= \frac{\beta c_x}{\rho+\beta} - \frac{\zeta}{\beta S} \left\{ \left[e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S)\right]_0^\infty + \int_0^\infty \rho e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v \right\} \\ &= \frac{\beta c_x}{\rho+\beta} + \frac{\zeta}{\beta S} u(\overline{x} - \frac{\beta}{\zeta} S) - \frac{\zeta\rho}{\beta S} \int_0^\infty e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v . \end{aligned}$$

We now prove results concerning the resolution of Equation (4.3.5), that is:

$$\zeta(c_s - \hat{c}_s) + \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S) = M(S) . \tag{B.0.7}$$

Lemma B.2. *Assume that $c_s < \hat{c}_s$. Then there is no solution of Equation* (B.0.7) *for $S \geq 0$.*

Proof. Denote with $\phi(S)$ the left-hand side of the equation. According to the bound (B.0.3), we have

$$M(S) \geq \frac{\beta}{\rho+\beta} \left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S\right)\right) .$$

If the right-hand side of this inequality is strictly larger than $\phi(S)$, then the theorem is proved. This sufficient condition writes as:

$$\frac{\beta}{\rho+\beta} \left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S\right)\right) > \zeta(c_s - \hat{c}_s) + \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S)$$

$$\iff \qquad \zeta(c_s - \hat{c}_s) \quad < \quad - \frac{\rho}{\rho+\beta}\left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right)$$

This last inequality indeed holds since $c_s - \hat{c}_s < 0$ by assumption, and the right-hand side is positive for $S \geq 0$. □

Lemma B.3. *Assume that $c_s \geq \hat{c}_s$, and that $u'(\cdot)$ is a convex function. Then the function*

$$h(S) \;=\; \zeta(c_s - \hat{c}_s) \;+\; \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S) \;-\; M(S)$$

is decreasing. As a consequence, there is at most one solution to Equation (B.0.7) *for $S \in [0, S_{\widetilde{y}}]$.*

Proof. If $u'(\cdot)$ is convex, then $u''(\cdot)$ is increasing. Then we have:

$$\begin{aligned}
\frac{\beta}{\zeta}Se^{-\beta v} &\leq \frac{\beta}{\zeta}S \\
\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v} &\geq \overline{x} - \frac{\beta}{\zeta}S \\
u''\left(\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v}\right) &\geq u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) .
\end{aligned}$$

Given Equation (B.0.1) for $M'(S)$, we have for all $S \geq 0$,

$$M'(S) \;\geq\; \frac{\beta}{\rho+2\beta}\,\frac{\beta}{\zeta}\,u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) .$$

On the other hand, we have

$$\begin{aligned}
h'(S) &= \frac{\beta}{\zeta}\,u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;-\; M'(S) \\
&\leq \frac{\beta}{\zeta}\,u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;-\; \frac{\beta}{\rho+2\beta}\,\frac{\beta}{\zeta}\,u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \\
&= \frac{\beta}{\zeta}\,\frac{\rho+\beta}{\rho+2\beta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;\leq\; 0 .
\end{aligned}$$

Therefore, h is decreasing. The solutions to (B.0.7) are the zeroes of $h(\cdot)$. As a consequence, there can be at most one solution. □

Lemma B.4. *Assume that $\hat{c}_s \leq c_s \leq c_{sm}$ and that $u'(\cdot)$ is convex. Then the unique solution S^{QP} of Equation* (B.0.7) *in the interval $[0, S_{\widetilde{y}}]$ is an increasing function of c_s, and the term*

$$\phi(c_s) \;:=\; \frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)$$

is positive.

Proof. Denote with $\sigma(c_s)$ the solution S^{QP} of Equation (B.0.7). By implicit differentiation with respect to c_s, we get

$$\zeta + \frac{\beta}{\zeta}\sigma'(c_s)u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right) \;=\; \sigma'(c_s)M'(\sigma(c_s))$$

hence

$$\sigma'(c_s) \;=\; \zeta\left(M'(\sigma(c_s)) - \frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right)\right)^{-1} .$$

The denominator is $-h'(\sigma(c_s))$ in the notation of the proof of Lemma B.3. It is therefore positive, and it has been proved that

$$M'(\sigma(c_s)) - \frac{\beta}{\zeta}u"\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right) \;\geq\; -\;\frac{\beta}{\zeta}\;\frac{\rho+\beta}{\rho+2\beta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right)\;.$$

Therefore, σ' is positive and σ is increasing. Moreover,

$$\sigma'(c_s) \;\geq\; \zeta\left(-\frac{\beta}{\zeta}\;\frac{\rho+\beta}{\rho+2\beta}u"\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right)\right)^{-1}\;.$$

The function $\phi(c_s)$ is such that $\phi(0) = 0$ and

$$\begin{aligned}\phi'(c_s) &= \frac{\rho+\beta}{\beta} + \frac{\beta}{\zeta^2}\sigma'(c_s)u"\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right)\\ &\geq \frac{\rho+\beta}{\beta} - \frac{\rho+\beta}{\rho+2\beta}\\ &= \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)} \;>\; 0\;.\end{aligned}$$

The function $h(\cdot)$ is therefore increasing, and it is positive. □

Appendix C

The LQ case

In this section, we develop explicit formulas for the case where $u(\cdot)$ is quadratic, in the situation where X is infinite.

In that case, $u'(\cdot)$ is linear. Let $-W$ denote its slope, with $W > 0$. Let us choose the form:

$$u'(x) = \overline{p} - W(x - \overline{x}) \tag{C.0.1}$$

$$u(x) = u(\overline{x}) + \overline{p}(x - \overline{x}) - \frac{1}{2}W(x - \overline{x})^2 \tag{C.0.2}$$

$$q^d(p) = \overline{x} - \frac{1}{W}(p - \overline{p}) \ . \tag{C.0.3}$$

Since $c_y = u'(\widetilde{y})$, and $c_x = u'(\widetilde{x})$, we have the alternate forms for W:

$$W = \frac{\overline{p} - c_y}{\widetilde{y} - \overline{x}} = \frac{\overline{p} - c_x}{\widetilde{x} - \overline{x}} = \frac{c_y - c_x}{\widetilde{x} - \widetilde{y}} = \frac{c_y - \overline{p}}{S_{\widetilde{y}}} \frac{\zeta}{\beta} \ . \tag{C.0.4}$$

Other formulas linking W and previously introduced quantities are:

$$\hat{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{x} - \overline{x}}{\zeta W} \tag{C.0.5}$$

$$\bar{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{x} - \widetilde{y}}{\zeta W} \ . \tag{C.0.6}$$

$$\hat{c}_s - \bar{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{y} - \overline{x}}{\zeta W} \ .$$

C.1 Phase P

The functions $M(\cdot)$ and $L(\cdot)$ are respectively given by:

$$M(S) = -\frac{W}{\zeta} \frac{\beta^2 S}{\rho + 2\beta} \tag{C.1.1}$$

$$L(S) = \frac{\beta}{\rho + \beta}(c_x - \overline{p}) - \frac{W}{\zeta} \frac{\beta^2 S}{\rho + 2\beta} \ . \tag{C.1.2}$$

The value S^{QP} solves equation (4.3.4) or (4.3.5), which gives:

$$\zeta(c_s - \hat{c}_s) - \frac{W\beta}{\zeta} S^{QP} = -\frac{W}{\zeta} \frac{\beta^2 S^{QP}}{\rho + 2\beta}$$

$$S^{QP} = (c_s - \hat{c}_s) \frac{\zeta^2}{W} \frac{\rho + 2\beta}{\beta(\rho + \beta)} \ . \tag{C.1.3}$$

One checks directly that $S^{QP} < S_{\widetilde{y}}$ when $c_s < \bar{c}_s$. Indeed, we have:

$$\begin{aligned} S^{QP} < S_{\widetilde{y}} &\iff (c_s - \hat{c}_s)\frac{\zeta^2}{W}\frac{\rho+2\beta}{\beta(\rho+\beta)} \leq (c_y - \overline{p})\frac{\zeta}{\beta W} \\ &\iff c_s - \hat{c}_s \leq \frac{\rho+\beta}{\rho+2\beta}\frac{c_y - \overline{p}}{\zeta} = \frac{\rho+\beta}{\rho+2\beta}(\bar{c}_s - \hat{c}_s) \\ &\iff c_s \leq \frac{\beta}{\rho+2\beta}\hat{c}_s + \frac{\rho+\beta}{\rho+2\beta}\bar{c}_s\,. \end{aligned}$$

The right-hand side is a convex combination of $\hat{c}_s$ and $\bar{c}_s$, and since $\hat{c}_s < \bar{c}_s$, it lies between these two values.

The multipliers in phase P are given by (4.1.1) and $\lambda_S^{(P)}(t) = L(S(t))/\zeta$. Therefore we have the formulas expressed as a state feedback:

$$\begin{aligned} \lambda_Z &= \frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right) \\ &= \frac{c_x - \overline{p}}{\zeta} + \frac{W}{\zeta}\left(\overline{x} - \frac{\beta}{\zeta}S - \overline{x}\right) \\ &= \frac{c_x - \overline{p}}{\zeta} - \frac{W\beta}{\zeta^2}S \qquad \text{(C.1.4)} \\ \lambda_S &= \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} - \frac{W}{\zeta^2}\frac{\beta^2 S}{\rho+2\beta} \qquad \text{(C.1.5)} \\ &= \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} - \frac{\beta}{\rho+2\beta}\left(\frac{c_x - \overline{p}}{\zeta} - \lambda_Z\right) \\ &= \frac{c_x - \overline{p}}{\zeta}\frac{\beta^2}{(\rho+\beta)(\rho+2\beta)} + \frac{\beta}{\rho+2\beta}\lambda_Z\,. \qquad \text{(C.1.6)} \end{aligned}$$

According to this last formula, the trajectory of $(\lambda_Z(t), \lambda_S(t))$ in the $\lambda_Z - \lambda_S$ plane is a straight line with a slope that does not depend on W.

When $S \to 0$, the point tends to:

$$(\lambda_Z(0), \lambda_S(0)) = \left(\frac{c_x - \overline{p}}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right)\,.$$

When $S \to S_{\widetilde{y}}$, it tends to:

$$\begin{aligned} (\lambda_Z(S_{\widetilde{y}}), \lambda_S(S_{\widetilde{y}})) &= \left(\frac{c_x - c_y}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} - \frac{\beta}{\rho+2\beta}\frac{c_y - \overline{p}}{\zeta}\right) \\ &= \left(\frac{c_x - c_y}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - c_y}{\zeta} + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y - \overline{p}}{\zeta}\right)\,. \qquad \text{(C.1.7)} \end{aligned}$$

Value of c_{cm}. By definition of c_{sm}, the point given by (C.1.7) is on the line $\lambda_S = \lambda_Z + c_{sm}$, because Phase Q occurs just at $S = S_{\widetilde{y}}$. Therefore, it follows that:

$$\begin{aligned} c_{sm} &= \frac{\rho}{\rho+\beta}\frac{c_y - c_x}{\zeta} + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y - \overline{p}}{\zeta} \qquad \text{(C.1.8)} \\ &= \bar{c}_s + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y - \overline{p}}{\zeta}\,. \end{aligned}$$

As expected, it follows from the last line that $c_{sm} > \bar{c}_s$.

Alternately, when $c_s = c_{sm}$, we must have $S^{QP} = S_{\widetilde{y}}$. Accordingly, using (C.1.3) and (C.0.4) and simplifying, we get the second identity:

$$c_{sm} = \hat{c}_s + \frac{\rho+\beta}{\rho+2\beta}\frac{c_y - \overline{p}}{\zeta}\,. \qquad \text{(C.1.9)}$$

Value function. Finally, the value function $V_P(S)$ is computed from its definition (4.5.1) as:

$$\begin{aligned} V_P(S) &= \int_0^\infty e^{-\rho v}\left(u(\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v}) - c_x\left(\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v}\right)\right)\mathrm{d}v \\ &= \int_0^\infty e^{-\rho v}\left(u(\overline{x}) + \overline{p}\left(-\frac{\beta}{\zeta}Se^{-\beta v}\right) - \frac{W}{2}\left(-\frac{\beta}{\zeta}Se^{-\beta v}\right)^2\right)\mathrm{d}v - \frac{c_x\overline{x}}{\rho} + \frac{c_x\beta S}{\zeta}\frac{1}{\rho+\beta} \\ &= \frac{u(\overline{x}) - c_x\overline{x}}{\rho} + \frac{\beta S}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} - \frac{W}{2}\frac{\beta^2 S^2}{\zeta^2}\frac{1}{\rho+2\beta}\ . \end{aligned} \tag{C.1.10}$$

It is possible to check the Hamilton-Jacobi-Bellman identity (4.5.9) from (C.1.10), as well as the identity $V_P' = \lambda_S$ from (C.1.10) and (C.1.5).

C.2 Phase Q

The value of λ_Z is expressed from (4.3.8) and the value of S^{QP} in (C.1.3) as:

$$\begin{aligned} \lambda_Z(t) &= e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) + \frac{W}{\zeta}\left(-\frac{\beta}{\zeta}S^{QP}\right)\right] - \frac{\rho+\beta}{\rho}c_s \\ &= e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) - \frac{W\beta}{\zeta^2}(c_s - \hat{c}_s)\frac{\zeta^2}{W}\frac{\rho+2\beta}{\beta(\rho+\beta)}\right] - \frac{\rho+\beta}{\rho}c_s \\ &= e^{\rho(t-t^{QP})}(c_s - \hat{c}_s)\left[\frac{\rho+\beta}{\rho} - \frac{\rho+2\beta}{\rho+\beta}\right] - \frac{\rho+\beta}{\rho}c_s \\ &= e^{\rho(t-t^{QP})}(c_s - \hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)} - \frac{\rho+\beta}{\rho}c_s\ . \end{aligned} \tag{C.2.1}$$

Next, the value of $x^{(Q)} = q^d(c_x - \zeta\lambda_Z)$ is, using (C.0.3),

$$\begin{aligned} x^{(Q)}(t) &= \overline{x} - \frac{1}{W}\left(c_x - \zeta e^{\rho(t-t^{QP})}(c_s - \hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)} + \zeta\frac{\rho+\beta}{\rho}c_s - \overline{p}\right) \\ &= \overline{x} - \frac{1}{W}\left(\zeta\frac{\rho+\beta}{\rho}\left(c_s + \frac{\rho}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right) - \zeta e^{\rho(t-t^{QP})}(c_s - \hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}\right) \\ &= \overline{x} - \frac{\zeta}{\rho W}(c_s - \hat{c}_s)\left(\rho+\beta - e^{\rho(t-t^{QP})}\frac{\beta^2}{\rho+\beta}\right)\ . \end{aligned} \tag{C.2.2}$$

As a particular value, we can evaluate $x^{(Q)}(t^{QP})$, see Figure 4.1. We have:

$$\begin{aligned} x^{(Q)}(t^{QP}) &= \overline{x} - \frac{\zeta}{\rho W}(c_s - \hat{c}_s)\left(\rho+\beta - \frac{\beta^2}{\rho+\beta}\right) = \overline{x} - \frac{\zeta}{\rho W}(c_s - \hat{c}_s)\frac{\rho(\rho+2\beta)}{\rho+\beta} \\ &= \overline{x} - \frac{\beta}{\zeta}S^{QP}\ , \end{aligned}$$

where we have used the value of S^{QP} obtained in (C.1.3). This is of course consistent with the general relationship which prevails in Phase P: $x = \overline{x} - \beta S/\zeta$. Next, the dynamics of $S(t)$ are integrated with (4.3.2) as:

$$\begin{aligned} S^{(Q)}(t) &= S^{QP} - \zeta\int_t^{t^{QP}}(x^{(Q)}(t) - \overline{x})\ \mathrm{d}t \\ &= S^{QP} + \frac{\zeta^2}{\rho W}(c_s - \hat{c}_s)\int_t^{t^{QP}}\left(\rho+\beta - e^{\rho(t-t^{QP})}\frac{\beta^2}{\rho+\beta}\right)\mathrm{d}t \\ &= S^{QP} + \frac{\zeta^2}{W}(c_s - \hat{c}_s)\frac{\rho+\beta}{\rho}(t^{QP} - t) - \frac{\beta^2\zeta^2}{\rho(\rho+\beta)W}(c_s - \hat{c}_s)\int_t^{t^{QP}} e^{\rho(u-t^{QP})}\ \mathrm{d}u \end{aligned}$$

$$= \quad S^{QP} \;+\; \frac{\zeta^2}{W}(c_s-\hat{c}_s)\frac{\rho+\beta}{\rho}(t^{QP}-t) \;-\; \frac{\beta^2\zeta^2}{W}(c_s-\hat{c}_s)\frac{1-e^{\rho(t-t^{QP})}}{\rho^2(\rho+\beta)}\;. \tag{C.2.3}$$

The value t^{RQ} satisfies $\lambda_Z^{(Q)}(t^{RQ}) = (c_x - c_y)/\zeta$. Accordingly, from (C.2.1) (see also (4.3.12)), we have:

$$\begin{aligned} t^{RQ} - t^{QP} &= \frac{1}{\rho}\log\left[\frac{\frac{\rho+\beta}{\rho}c_s + \frac{c_x-c_y}{\zeta}}{(c_s-\hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}}\right] \;=\; \frac{1}{\rho}\log\left[\frac{\frac{\rho+\beta}{\rho}(c_s-\bar{c}_s)}{(c_s-\hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}}\right] \\ &= \frac{1}{\rho}\log\left[\left(\frac{\rho+\beta}{\beta}\right)^2\frac{c_s-\bar{c}_s}{c_s-\hat{c}_s}\right]\;. \end{aligned} \tag{C.2.4}$$

It is easy to check with identities (C.1.8) and (C.1.9) that when $c_s = c_{sm}$, this quantity reduces to 0. This is of course consistent with the fact that Phase Q vanishes in that situation.

C.3 Phase A

Assuming that the system is in state $S^0 = S(t^0)$ at some arbitrary time instant t^0, we have: $\lambda_Z^{(A)}(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)}$ and consequently, since $x^{(A)}(t) = q^d(c_x - \zeta\lambda_Z)$,

$$x^{(A)}(t) \;=\; \overline{x} - \frac{c_x-\overline{p}}{W} + \frac{\zeta}{W}\lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \;=\; \widetilde{x} \;+\; \frac{\zeta}{W}\lambda_Z^0 e^{(\rho+\alpha)(t-t^0)}\;, \tag{C.3.1}$$

where we have used, from (C.0.4): $(c_x - \overline{p})/W = \overline{x} - \widetilde{x}$. Next, according to (3.5.3),

$$\begin{aligned} Z(t) &= Z^0 e^{-\alpha(t-t^0)} \;+\; S^0\frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) \;+\; \zeta\int_{t^0}^t e^{-\alpha(t-u)}x^{(A)}(u)\,\mathrm{d}u \\ &= Z^0 e^{-\alpha(t-t^0)} \;+\; S^0\frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) \\ &\quad +\; \zeta\widetilde{x}\frac{1-e^{-\alpha(t-t^0)}}{\alpha} \;+\; \frac{\zeta^2}{W}\lambda_Z^0\frac{e^{(\rho+\alpha)(t-t^0)} - e^{-\alpha(t-t^0)}}{\rho+2\alpha}\;. \end{aligned} \tag{C.3.2}$$

Using the dynamics of S: $S^{(A)}(t) = S^0 e^{-\beta(t-t^0)}$, it is possible to eliminate the time variable so as to obtain the equation of the trajectory in the (S,Z) space:

$$Z \;=\; Z_M(S) \;+\; \frac{\zeta\widetilde{x}}{\alpha}\left(1-\left(\frac{S}{S^0}\right)^{\alpha/\beta}\right) \;+\; \frac{\zeta^2}{W}\frac{\lambda_Z^0}{\rho+2\alpha}\left(\left(\frac{S}{S^0}\right)^{-(\rho+\alpha)/\beta} - \left(\frac{S}{S^0}\right)^{\alpha/\beta}\right)\;. \tag{C.3.3}$$

C.4 Phase B

Integrating Equations (3.5.8) then (3.5.7), we get:

$$\begin{aligned} S^{(B)}(t) &= S_m e^{-\beta t} + \frac{\zeta}{\beta}(\widetilde{x} - \zeta c_s/W)(1-e^{-\beta t}) \\ &\quad + \frac{\zeta^2}{W}(\lambda_S^0 + \lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{1}{\rho+2\beta}(e^{(\rho+\beta)t} - e^{-\beta t}) \\ &\quad - \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{1}{\rho+\alpha+\beta}(e^{(\rho+\alpha)t} - e^{-\beta t}) \end{aligned} \tag{C.4.1}$$

$$\begin{aligned} Z^{(B)}(t) &= \zeta(\widetilde{x} - \zeta c_s/W)/\alpha \\ &\quad + e^{-\alpha t}\Big(\overline{Z} - \frac{\beta}{\alpha-\beta}S_m + \beta\zeta(\widetilde{x} - \zeta c_s/W)/\alpha/(\alpha-\beta) \end{aligned}$$

$$
\begin{aligned}
&\quad +\frac{\zeta^2}{W}(\lambda_S^0+\lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{(\rho+\alpha+\beta)(\alpha-\beta)}\\
&\quad -\frac{\zeta^2}{W}\frac{\beta^2}{(\alpha-\beta)^2}\lambda_Z^0/(\rho+2\alpha)\Big)\\
&+\frac{e^{-\beta t}}{\alpha-\beta}\Big(\beta S_m-\zeta(\widetilde{x}-\zeta c_s/W)\\
&\quad -\frac{\zeta^2}{W}(\lambda_S^0+\lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{\rho+2\beta}\\
&\quad +\frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{\beta}{\rho+\alpha+\beta}\Big)\\
&+\frac{\zeta^2}{W}(\lambda_S^0+\lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{(\rho+2\beta)(\rho+\alpha+\beta)}e^{(\rho+\beta)t}\\
&-\frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{\beta}{(\rho+\alpha+\beta)(\rho+2\alpha)}e^{(\rho+\alpha)t}\ .
\end{aligned}
\tag{C.4.2}
$$

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