Markov perfect equilibria in differential games with regime switching

Ngo van Long, Fabien Prieur, Klarizze Puzon, Mabel Tidball

LAMETA

Laboratoire Montpelliérain d'Economie Théorique et Appliquée

UMR

Unité Mixte de Recherche

DOCUMENT de RECHERCHE

«Markov Perfect Equilibria in Differential Games with Regime Switching»

Ngo VAN LONG
Fabien PRIEUR
Klarizze PUZON
Mabel TIDBALL

DR n°2013-06

Unite de Formation et de Recherche d'Economie
Avenue Raymond DUGRAND C.S. 79606
3 4 9 6 0 MONTPELLIER Cedex 2

E-mail : lameta@lameta.univ-montp1.fr
web : www.lameta.univ-montp1.fr

Université Montpellier 1 — agro Montpellier Ecole Nationale Supérieure Agronomique — CNRS Centre National de la Recherche Scientifique — INRA

Markov Perfect Equilibria in Differential Games with Regime Switching

Ngo Van Long*, Fabien Prieur†, Klarizze Puzon‡ and Mabel Tidball§

February 15, 2013

Abstract

We propose a new methodology exploring Markov perfect equilibrium strategies in differential games with regime switching. Specifically, we develop a general game with two players having two kinds of strategies. Players choose an action that influences the evolution of a state variable, and decide on the switching time between alternative and consecutive regimes. Compared to the optimal control problem with regime switching, necessary optimality conditions are modified for the first-mover. When choosing her optimal switching strategy, this player considers her impact on the other player's actions and welfare, vice versa. In order to determine the optimal timing between regime changes, the notion of erroneous timing is introduced and necessary conditions for a particular timing to be erroneous are derived. We then apply this original material to an exhaustible resource extraction game. Sufficient conditions for the existence of an interior solution are compared to those characterizing an erroneous timing. The impact of feedback strategies for adoption time on the equilibrium depends on conflicting effects: the first mover incurs an indirect cost due to the future switching of her rival (incentive to delay the switch). But she is able to affect the other player's switching decision (incentive to switch more rapidly). In a particular case with no direct switching cost, the interplay between the two ensures that the first-mover adopts the new technology in finite time. Interestingly, this result differs from what is obtained in a non-game theoretic framework, i.e. immediate adoption.

Key words: differential games; regime switching; technology adoption; non-renewable resources

JEL classification: C61, C73, Q32.

*Department of Economics, McGill University, 855 Sherbrooke Street West, Montreal, QC H3A 2T7, Canada. E-mail: ngo.long@mcgill.ca

†LAMETA, University of Montpellier I and INRA, 2 place Viala 34060 Montpellier, France. E-mail: prieur@supagro.inra.fr

‡LAMETA, University of Montpellier I. E-mail: puzon@lameta.univ-montp1.fr

§INRA-LAMETA. E-mail: tidball@supagro.inra.fr

1 Introduction

Several optimal decision making problems in economics concern the timing of switching between alternative and consecutive regimes. Regimes may refer to technological and/or institutional states of the world. For instance, a firm with an initial level of technology may find it optimal to either adopt a new technology or to stick with the old one (Boucekkine, Saglam and Vallée, 2004). Another example is the decision to phase out existing capital controls in a given economy (Makris, 2001). In all these examples, the switching decision corresponds to a trade-off, i.e. adopting a new regime is associated with a cost. Given such, multi-stage optimization is generally used for the analysis of regime switching (Tomiyama, 1985). Switching instants between regimes are thus endogenously determined.

In this article, we consider regime switching strategies in differential games. Game theoretic settings involving optimal regime switching have been rarely considered in literature. The first set involves dynamic games of regime change with no state or stock variable. These models generally assume that the relevant state of the system is the identity of the players which have adopted new technology. An example is Reinganum (1981)'s study which considered technological adoption decisions of two ex ante identical firms. She assumed that firms adopt pre-commitment (open-loop) strategies. That is, it is as if a firm enters a binding commitment denoting its date of technology switch, knowing the adoption date of the other firm. Reinganum's primary finding is that, under open-loop strategy, one firm will innovate first and the other will innovate at a later date. Fudenberg and Tirole (1985) scrutinized Reinganum's study by using the concept of pre-emption equilibrium. Focusing on Markov perfect equilibrium as the solution concept, they noted that the second-mover may try to preempt its rival and become the first-mover (see Long, 2010, for a survey of the literature).

Meanwhile, the other strand of literature pertains to the strategic interaction of agents in relation to the dynamics of a given stock. For instance, Tornell (1997) presented a model relating economic growth and institutional change. Infinitely-lived agents solve a differential game over the choice of property-rights regime, e.g. common or private property, defined over a capital stock. It was shown that a potential equilibrium of the game involves multiple switching between regimes. But because only the symmetric equilibrium was considered, it was observed that players always choose to switch at the same instant. Consequently, the question of the timing between the switching strategies was not addressed. In addition, even if Tornell explicitly defined the Markov perfect equilibrium for the class of differential games with regime switching, a rigorous modelling of these strategies, for switching time, is missing in his analysis. A more recent example is the analysis by Boucekkine, Krawczyk and Valle (2011). They analyzed the trade-off between environmental quality and economic performance using a two-player differential game. Assuming that pollution results from the sum of consumption levels and there is no decay, they have proven the existence of an open-loop Nash equilibrium. They found out that each player chooses the technology without considering the choice made by the other player. No interior switching instant was derived. At the open-loop Nash equilibrium, either a player adopts technology immediately, or he sticks to the old one.

Nevertheless, to the authors' knowledge, there seems to be no existing study which explicitly discusses and models feedback Nash or Markov perfect equilibrium strategies in differential games with regime switching. This is where the first theoretical contribution of this paper lies. We develop a general differential game with two players having two kinds of strategies. First, players have to choose at each point in time an action that influences the evolution of a state variable. Second, they may decide on the timing of switching between alternative and consecutive regimes that differ both in terms of the payoff function and the state equation. For simplicity, we assume that each player can affect a regime change only once. At a feedback Nash equilibrium, we define the switching or timing strategy as a function of the state of the system. The relevant level of the state variable on which the strategy is based is the one corresponding to the instant when the switching problem arises. For instance, consider a timing where player i switches first, followed by player j. Then, player i's switching strategy is defined over the initial stock, whereas player j's strategy depends on the level of the state at which player i has decided to switch. For any of the two possible general timings, we characterize the necessary optimality conditions for switching times, both for interior and corner solutions. One interesting finding is that, compared to the standard optimal control problem with regime switching, necessary optimality conditions are modified only for the player who finds it optimal to move first. Indeed, when choosing the optimal date and level of the state variable for switching, this player must take into account that *(i)* her decision will influence the other player's switching strategy and *(ii)* the other player's switch will impact on her own welfare. Therefore, this player will adapt her strategy. Depending on the particular economic problem at hand, the interaction through switching times may be an incentive to either postpone or expedite regime switching. Another important issue is how to determine the optimal timing at the Markov perfect equilibrium. This issue is solved by providing necessary conditions for a particular timing to be erroneous. By erroneous we mean that at least one player would prefer lying in the other timing.

The second contribution of this paper is the application of this new game theoretic material to address the tragedy of the commons. A game of exhaustible resource extraction is considered. At a given cost, players have the option to have more efficient extraction technology. Not only do players choose their consumption levels, they also decide whether to adopt new technology and when. To date, only few have studied the relationship between natural resource exploitation and the timing of technology adoption. With a finite horizon two-stage optimal control problem, Amit (1986) explored the case of a petroleum producer who considers switching from primary to secondary recovery process. He observed that a technological switch occurs if the desired extraction rate is larger than can be obtained by the natural drive, or when the desired final output is more than can be obtained using the primary process. In a more recent paper, Valente (2011) analyzed a two-phase endogenous growth model which concerns a switch from an exhaustible resource input into a backstop technology. Adoption of new technology implies sudden decline in consumption, but an increase in the growth rate. Finally, Boucekkine, Pommeret, and Prieur (2012) explored a general control problem with both technological and ecological regime switch. They applied it to address the issue of optimal resource extraction under ecological irreversibility, and with the possibility to adopt backstop technology. It was observed that the opportunity to switch to a backstop technology may lead to an irreversible ecological regime. Overall, while the above-mentioned studies have explored resource management and regime switching, they only do so using optimal single-agent optimization programs. None have conducted an analysis using a differential game theoretic approach. Indeed, Section 4's resource extraction game tries to fill this gap in literature. It is assumed that heterogenous players start with a less efficient extraction technology and have to decide: *(i)* whether to switch to a more efficient technology, and *(ii)* when, given that switching involves a direct cost that depends on both the switching date and the level of the state variable.

Our main findings can be summarized as follows. We first identify a meaningful sufficient condition for a particular timing to be erroneous. This condition contrasts the difference of players' switching costs with the difference in technological gains from switching. Indeed, it is possible that both players find the timing erroneous. This happens when the player who is supposed to be the first to adopt has a relative disadvantage in adoption costs that is not compensated by any relative technological advantage. This notably encompasses the obvious situation in which the first mover incurs the higher switching cost and, at the same time, is the one who benefits the less from adoption. The opposite of this condition is necessary for the timing to be optimal. We then provide sufficient conditions for the existence of an interior solution where both players adopt the new technology in finite time and investigate the impact of feedback strategies for switching time on the first-mover own switching strategy (compared to the single-agent case). We emphasize the interplay between two opposite effects. First, in our application, the switch of the second mover is costly for the first mover because it implies a drop in her consumption of the resource. The switching cost of the latter is thus augmented by this term which is an incentive, other things equal, to delay the switch. At the same time, however, it turns out that the length of time between the two switches is increasing in the level of the state variable. From the point of view of the first mover, who controls this level, switching at a relatively abundant stock of resource is a means to postpone the switch of the other. Because of discounting, delaying the switch of the other player will allow the first-mover to incur a lower cost. This is an incentive to switch at an earlier date. In the particular case where the first player does not bear a direct switching cost, we show that she finds it worthwhile to adopt the new technology at finite and positive date. Indeed, this result differs from what one would obtain in the absence of interaction between players, *i.e.*, immediate adoption.

The plan of the paper is as follows. Section 2 describes the main assumptions of the general differential game with regime switching. Section 3 analyzes the optimality conditions that characterize a Markov perfect equilibrium. Section 4 applies these theoretical findings to a game of exhaustible resource extraction. Section 5 provides a brief discussion of the results, and Section 6 concludes.

2 The general problem

We consider a two-player differential game in which the instantaneous payoff of each player and the differential equation describing the stock dynamics depend on what regime the system is in. There are a finite number of regimes, and we assume that under certain conditions, the players are able to take action (at some cost) to affect a change of regime. Let X_i be the set of regimes that can be changed by player i only, Y be the set of regimes that both players can change, and Z the set of regimes that neither can change.

For simplicity, we assume that each player can affect a regime change only once. In particular, this implies that regime changes are irreversible.

Consider a simple model where there are four possible regimes, denoted by α, β, γ and δ. (An alternative notation is 11 for regime α, 21 for regime β, 12 for regime γ, and 22 for regime δ.)

We assume that the system is initially in regime α. Player 1 (which we will refer to as HE) can take a "regime change action" to switch the system from regime α to regime β, if player 2 (which we will refer to as SHE) has not taken her regime change action before him. Once the system is in regime β, only player 2 can take a regime change action, and this switches the system to regime δ. From regime α, player 2 can switch to regime γ (if player 1 has not taken his regime change before her). From regime γ, only player 1 can make a regime change, and this switches the system to regime δ. If the system is in regime α and players 1 and 2 take regime change action simultaneously, the regime will be switched to δ. (In this example, $X_1 = \{\gamma\}$, $X_2 = \{\beta\}$, $Y = \{\alpha\}$, and $Z = \{\delta\}$).

More generally, we can write a "transition matrix" and specify in each box what action is required to affect a change from one regime to another:

$$\begin{bmatrix} & \alpha & \beta & \gamma & \delta \\ \alpha & (0,0) & (1,0) & (0,1) & (1,1) \\ \beta & * & (0,0) & * & (0,1) \\ \gamma & * & * & (0,0) & (1,0) \\ \delta & * & * & * & (0,0) \end{bmatrix}$$

In this matrix, an entry $*$ means the indicated change is not possible: *e.g.* from regime β, it is not possible to switch to α. An entry $(0,0)$ means neither agent takes a regime change action. $(1,0)$ means player 1 takes a regime change action while player 2 does not. An entry $(1,1)$ means both players take their regime change action at the same point of time (simultaneously).

At each instant, each player also chooses an action C_i, for instance a consumption level, that affects the evolution of the state variable K. The instantaneous payoff to player i at time t when the system is in regime r where $r = \alpha, \beta, \gamma$ or δ is

$$F_i^r(C_i(t), C_{-i}(t), K(t))$$

If player i takes a regime change action at time T_i, he/she incurs a lumpy cost $\Omega_i(K(T_i), T_i)$.

Then, if $T_2 > T_1$, the total payoff for player 1 is

$$\int_0^{T_1} F_1^\alpha(C_1, C_2, K)e^{-\rho t}dt + \int_{T_1}^{T_2} F_1^\beta(C_1, C_2, K)e^{-\rho t}dt$$
$$+ \int_{T_2}^{\infty} F_1^\delta(C_1, C_2, K)e^{-\rho t}dt - \Omega_1(K(T_1), T_1)$$

with ρ the discount rate.

If $T_2 < T_1$, the total payoff for player 1 is

$$\int_0^{T_2} F_1^\alpha(C_1, C_2, K)e^{-\rho t}dt + \int_{T_2}^{T_1} F_1^\gamma(C_1, C_2, K)e^{-\rho t}dt$$
$$+ \int_{T_1}^{\infty} F_1^\delta(C_1, C_2, K)e^{-\rho t}dt - \Omega_1(K(T_1), T_1)$$

And if $T_1 = T_2 = T$, the total payoff for player 1 is

$$\int_0^{T} F_1^\alpha(C_1, C_2, K)e^{-\rho t}dt + \int_{T}^{\infty} F_1^\delta(C_1, C_2, K)e^{-\rho t}dt - \Omega_1(K(T), T)$$

The differential equation describing the evolution of the state variable K in regime r (where $r = \alpha, \beta, \gamma$ or δ) is

$$\dot{K} = f^r(C_1, C_2, K)$$

In the subsequent analysis, we use Markov perfect equilibrium (MPE) as the solution concept. As illustrated by the decomposition above, the game, that corresponds for instance to timing $0 \leqq T_1 \leqq T_2 \leqq \infty$, can be divided into three sub-games, each being associated with a particular regime. Indeed, for the timing considered, the sequence of regimes is: α, β and δ (or 11, 21 and 22). A natural way to proceed, for determining a MPE of this game, is to solve the problem recursively, starting from the regime arising after the last regime switching, here δ or 22. This is a natural extension of the method originally developed by Tomiyama (1985) and Amit (1986) to solve their two-stage optimal control problems.

The next assumption ensures that our problem, seen as a sequence of three sub-games, is well-behaved.

Assumption 1
- *The functions $F_i^r(.)$ and $f^r(.)$, for any regime $r = lk$, $l, k = 1, 2$, belong to the class C^1.*
- *The sub-game obtained by restricting the general problem to any regime r, satisfies the Arrow-Kurz's sufficiency conditions.*

These conditions will allow us to use some envelope properties that requires the differentiability of the value function (see Boucekkine, Pommeret and Prieur, 2012, for a detailed discussion).

Let us now define what is a MPE strategy in our model. A MPE strategy consists of a consumption policy and a switching rule describing the actions undertaken by each player at every possible state of the system.

To formulate player 1's maximization problem, we must then specify what knowledge he has about player 2's behavior. We focus on the case where each player has a consumption strategy which specifies consumption at time t as dependent only on *(i)* the current stock level, $K(t)$, and *(ii)* the current regime. Thus player 1 thinks that player 2's consumption strategy as a list of four functions : $\Phi_2^{\alpha}(K)$, $\Phi_2^{\beta}(K)$, $\Phi_2^{\gamma}(K)$ and $\Phi_2^{\delta}(K)$.

What does player 1 know about player 2's switching strategy?

Suppose player 1 thinks that if player 2 finds herself in regime β at date t (which implies that he switched at an earlier date $T_1 < t$), she will make a switch at a date $T_2 \geq t$. Then player 1 should think that the interval of time between the two switches, $T_2 - T_1$, is a function of the state of the system. The state of the system is defined in terms of the regime and the level of the state variable that are relevant for the switching problem of player 2. The relevant regime is the one that holds after T_1. Moreover, it is clear that $T_2 - T_1$ will not depend on any state level K because, in contrast to the consumption decision, it is a discrete choice. Because, player 2's switching problem arises once player 1 has switched, the level of the state that matters is $K(T_1) = K_1$. Thus, player 1 thinks that player 2's switching strategy takes the following form: $T_2 - T_1 = \theta_2^{\beta}(K_1)$. In a more general formulation, we could admit the case where player 1 thinks that if player 2 finds herself in regime α, she will make a switch at $T_2 = T_0 + \theta_2^{\alpha}(K_0)$.

Then we say that

Definition 1

- *Player 2's strategy (as guessed by player 1) is a $t-$uple $\psi_2 \equiv \left(\Phi_2^{\alpha}, \Phi_2^{\beta}, \Phi_2^{\gamma}, \Phi_2^{\delta}, \theta_2^{\beta}, \theta_2^{\alpha}\right)$.*
- *Player 1's strategy (as guessed by player 2) is represented by a $t-$uple, $\psi_1 \equiv \left(\Phi_1^{\alpha}, \Phi_1^{\beta}, \Phi_1^{\gamma}, \Phi_1^{\delta}, \theta_1^{\gamma}, \theta_1^{\alpha}\right)$.*
- *A Markov perfect equilibrium is a pair (ψ_1, ψ_2) such that ψ_1 is a best reply to ψ_2, for each possible initial condition and initial time.*

The next section presents the set of necessary optimality conditions that characterize a MPE of our differential game with regime switching.

3 Necessary Conditions

In the following analysis, player i's present value Hamiltonian and co-state variable in any regime lk are denoted respectively by H_i^{lk} and λ_i^{lk}. The results are presented for a particular timing: $0 \leqq T_1 \leqq T_2 \leqq \infty$, which allows us to dispense with some notations: we do not have to make explicit the dependance of switching strategies on the regime. Necessary optimality conditions for the other general timing, $0 \leqq T_2 \leqq T_1 \leqq \infty$, can easily be derived by symmetry. Finally note that in the theorem below, attention is paid only to the necessary optimality conditions related to the switching problem. Actually, deriving the optimality conditions for the consumption policies is pretty standard whereas the novelty of our analysis refers to switching rules.

Theorem 1 *1. The necessary optimality conditions for the existence of a MPE featuring the timing* $0 < T_1^* < T_2^* < \infty$ *are:*

- *For player 2:*

$$\begin{aligned} H_2^{21*}(T_2^*) - \tfrac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} &= H_2^{22*}(T_2^*) \\ \lambda_2^{21*}(T_2^*) + \tfrac{\partial \Omega_2(K_2^*, T_2^*)}{\partial K_2} &= \lambda_2^{22*}(T_2^*). \end{aligned} \tag{1}$$

- *For player 1:*

$$\begin{aligned} H_1^{11*}(T_1^*) - \tfrac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} &= H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \\ \lambda_1^{11*}(T_1^*) + \tfrac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} &= \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*), \end{aligned} \tag{2}$$

2. *Suppose player 1's switching problem has a solution* (T_1^*, K_1^*).

- *A necessary condition for a corner solution with immediate switching* $T_1^* = T_2^*$ *is*

$$H_2^{21*}(T_2^*) - \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} \leq H_2^{22*}(T_2^*) \text{ if } T_1^* = T_2^* < \infty \tag{3}$$

- *A necessary condition for a corner solution of the never switching type* $T_2^* = \infty$ *is*

$$H_2^{21*}(T_2^*) - \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} \geq H_2^{22*}(T_2^*) \text{ for any } T_2^* \geq T_1^* \tag{4}$$

3. *Suppose player 2's switching problem has a solution* (T_2^*, K_2^*).

- *A necessary condition for a corner solution with immediate switching* $0 = T_1^*$ *is*

$$H_1^{11*}(T_1^*) - \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} \leq H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \text{ if } 0 = T_1^* < T_2^* \tag{5}$$

- *A necessary condition for a corner solution of the never switching type* $T_1^* = T_2^*$ *is*

$$H_1^{11*}(T_1^*) - \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} \geq H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \text{ if } 0 < T_1^* = T_2^* \tag{6}$$

4. *Suppose that it is optimal for the two players to switch at the same date $T^* = T_1 = T_2$, for the same level of the state $K^* = K_1 = K_2$, then the following conditions must hold:*

$$\begin{aligned} H_2^{11*}(T^*) - \tfrac{\partial \Omega_2(K^*,T^*)}{\partial T_2} &= H_2^{22*}(T^*) \\ H_1^{11*}(T^*) - \tfrac{\partial \Omega_1(K^*,T^*)}{\partial T_1} &= H_1^{22*}(T^*) \\ \lambda_2^{11*}(T^*) + \tfrac{\partial \Omega_2(K^*,T^*)}{\partial K_2} &= \lambda_2^{22*}(T^*) \\ \lambda_1^{11*}(T^*) + \tfrac{\partial \Omega_1(K^*,T^*)}{\partial K_1} &= \lambda_1^{22*}(T^*) \end{aligned} \tag{7}$$

Proof. See the appendix A. ■

Theorem 1 exhibits the necessary conditions for the existence of potential interior and corner solutions. Let us first analyze the switching conditions for an interior solution. Of particular importance is the difference between the optimality conditions of the first-mover (player 1) and the second-mover (player 2). Player 2's conditions (1) are similar to the ones derived in multi-stage optimal control literature (Tomiyama (1985) and Amit (1986)). The first condition states that it is optimal to switch from the penultimate to the final regime when the marginal gain of delaying the switch, given by the difference $H_2^{21*}(.) - H_2^{22*}(.)$, is equal to the marginal cost of switching, $\frac{\partial \Omega_2(K_2^*,T_2^*)}{\partial T_2}$. The second condition equalizes the marginal benefit from an extra unit of the state variable K_2 with the corresponding marginal cost. It basically says that the value of the co-state, when approached from the intermediate regime, plus the incremental switching cost must just equal the value of the co-state, approached from the last regime. Hence, as long as a player finds it optimal to be the second mover, her optimality conditions are similar to the standard switching conditions of an optimal control problem.

The original part of the analysis stems from the problem faced by the player who opts to adopt first. Indeed, player 1's optimality conditions are modified. The first condition in (2) implies that player 1 also takes into account how her situation changes as a consequence of a switch of player 2. Player 1 decides on its optimal switching time by equalizing the marginal gain of delaying the switch, which is given by the difference $H_1^{11*}(.) - H_1^{21*}(.)$ to the marginal switching cost, $\frac{\partial \Omega_1(K_1^*,T_1^*)}{\partial T_1} - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)]$. The extra-term $[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)]$ is the marginal impact of player 2's switch on player 1. Depending on the nature of the problem, it can either be positive or negative. The second optimality condition is also modified. The cost of a marginal increase in K_1 now includes an extra-term: $\theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)]$. This term reflects the fact that player 1 takes into account the impact of the choice of his switching level K_1^* on player 2's timing strategy. Put differently, player 1 knows that modifying K_1^* is a means to delay or accelerate player 2's regime switching.

Indeed, it can be inferred that adopting first may entail a leadership role for player 1. Being the first-mover allows her to influence the player 2's strategy. Furthermore, through the anticipation of the second-mover's reaction, she can adapt her own strategy.

We now turn to the necessary conditions for the corner solutions. In doing so, a distinction should be made between different cases. The corner situations $T_1^* = 0$ and $T_2^* = \infty$ have been studied in literature (Boucekkine, Krawczyk, Vallée, 2011). At $T_1^* = 0$, player 1 disregards the first optimization phase, *i.e.* the one when she lies in regime 11. This is because a delay in switching yields a marginal gain that is not greater than the marginal loss of foregoing for an instant the benefit of the new regime, for player 1. Similarly, if player 2 were to adopt a never switching strategy then it would mean that for all $T_2 \geq T_1^*$, sticking to regime 21 offers a (marginal) reward, $H_2^{21*}(.) - H_2^{22*}(.)$, which is lower than or equal to $\frac{\partial \Omega_2(.,.)}{\partial T_2}$, the switching cost.

Of further interest is the interpretation of the corner solutions $T_1^* = T_2^*$, mentioned in the items 2 and 3 of Theorem 1. The switching conditions are of the same meaning as before. If player 2 chooses the corner $T_2^* = T_1^*$ it must mean that at T_1^* delaying the switch is associated with a marginal gain that is not greater than the marginal loss of foregoing for an instant the benefit of the new regime, $H_2^{22}(T_1^*)$. Condition in (6) is the corner optimum for player 1: if he chooses the corner $T_1^* = T_2^*$, it must be true that at T_2^* a delay in switching by player 1 yields a marginal gain that is at least as high as the marginal loss of foregoing for an instant the benefit of new regime, $H_1^{22}(T_2^*)$.

To analyze player 1's switching problem, we have assumed that player 1 is subject to the constraint $T_1 \leq T_2^*$, with T_2^* fixed. Then, using the tools originally developed by Tomiyama (1985) and Amit (1986), the corresponding finite horizon switching problem was solved. However, because the current analysis pertains to a differential game, Condition (6) cannot simply be interpreted as a necessary condition for having a corner solution $T_1^* = T_2^*$. Rather, this condition is necessary for the timing $0 \leqq T_1 \leqq T_2 \leqq \infty$ to be erroneous. Indeed, under (6), Player 1 would prefer switching at a later date than T_2^*. This is feasible because T_2^* is not fixed. Thus, (6) is also a sufficient condition for the optimal timing to be $0 \leqq T_2 \leqq T_1 \leqq \infty$.

The analysis of these erroneous timing situations will be crucial in the following section, an application of the theory to an exhaustible resource problem. It points out the conditions under which one player will optimally accept to adopt first, while the other will choose to be the second-mover. Such would allow the reduction of the set of MPE candidates. In our problem, there are a priori fifteen possible timings corresponding to the set of possible combinations between T_1 and T_2.[1] But, it is highly unlikely that heterogenous players decide on the same switching time. So, the timings $0 \leqq T_1 = T_2 < \infty$ should not give MPE candidates. Logically, one could expect that several cases are mutually exclusive. Analyzing the erroneous timing conditions, (5)-(6) and the ones obtained when analyzing the other timing, should be a means to understand which timing, between $0 < T_1 < T_2 \leqq \infty$ and $0 < T_2 < T_1 \leqq \infty$, contains the MPEs. Nonetheless, at this stage, we cannot rule out the existence of multiple MPEs.

[1] Of course, the same kind of switching conditions can be derived, by symmetry, for the other general timing $0 \leqq T_2 \leqq T_1 \leqq \infty$.

To conclude, we now consider the particular cases where the state variable follows a monotonic trajectory. This analysis is of particular use to applications involving the management of exhaustible resources. In this case, the path followed by the state variable is monotone non-increasing. Additional constraints on the switching level must be taken into account: $K_0 \geq K_1 \geq K_2 (\geq 0)$.

Corollary 1 • *Sufficient conditions for the corner $T_1^* = T_2^*$ are (3) and*

$$\lambda_2^{21*}(T_2^*) + \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial K_2} \leq \lambda_2^{22*}(T_2^*). \tag{8}$$

- *Sufficient conditions for the corner $0 = T_1^*$ are (5) and*

$$\lambda_1^{11*}(T_1^*) + \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} \leq \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*) \tag{9}$$

- *Sufficient conditions for the corner $T_1^* = T_2^*$ are (6) and*

$$\lambda_1^{11*}(T_1^*) + \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} \geq \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*) \tag{10}$$

We observe that introducing these constraints, we are now only able to provide two conditions that are only sufficient for the necessary condition characterizing a corner solution to hold.

The next section applies these theoretical findings to a game of exhaustible resource extraction.

4 Application

The concept of the tragedy of the commons has been used to explain a variety of economic phenomena. Pertinent examples include dynamic games involving the exploitation of a common property resource (Lane and Tornell, 1996, Ploeg, 2010 and Alvarez-Cuadrado and Long, 2011).[2] In general, it has been observed that the presence of rivalry among multiple agents tend to result to inefficient outcomes, e.g. overextraction of the natural resource.

Another common feature shared by the above-mentioned models is the assumption that players cannot adopt new technology that will improve their extraction efficiency. It is usually assumed that consumption is a fixed fraction of the extraction level. In this section, we relax this assumption and consider the possibility of technological adoption among players. That is, players not only choose their consumption. They also decide when to adopt the more efficient extraction technology. This puts forth another innovative contribution of this paper.

Indeed, the results of the theoretical analysis in Section 3 are used to study an exhaustible resource game. As will be discussed in this section, of particular relevance is the decision of competing agents to adopt new technology.

[2]For extensive surveys on dynamic games in resource economics, refer to Long (2010, 2011).

4.1 A resource extraction game

Hereafter we describe the economic environment. There are $n = 2$ players. Let $C_i(t)$ denote the consumption level of player i, $i = 1, 2$, at time $t \geq 0$. Meanwhile, let $E_i(t)$ be player i's extraction rate from the resource at time $t \geq 0$. Extraction is converted into consumption according to the following technology: $C_i(t) = \frac{E_i(t)}{\gamma_i}$, where $\frac{1}{\gamma_i}$ is a non-negative number that may reflect a player's degree of efficiency in transforming the extracted natural resource into a consumption good.

Two production technologies, described only by the parameter γ_i, are available to player i from $t = 0$. Because players' technological menus may differ, one needs to introduce another index for the regime. It is assumed that player 1 starts with technology $l = 1$ and has to decide: *(i)* whether she switches to technology $l = 2$, and *(ii)* when.[3] The ranking between the parameters satisfies: $\gamma_1^1 > \gamma_2^2$, which means that the second new technology is more efficient than the old one. Technological gain from adoption is thus measured by the ratio $\frac{\gamma_1^2}{\gamma_1^1} \in (0, 1)$.

Let $K(t)$ be the stock of the exhaustible resource, with the initial stock K_0 given. As in section 2, T_1 and T_2 are the switching times. Suppose $0 < T_1 < T_2$, then the evolution of the stock is given by the following differential equation:[4]

$$\dot{K} = \begin{cases} -\gamma_1^1 C_1 - \gamma_2^1 C_2 \text{ if } t \in [0, T_1] \\ -\gamma_1^2 C_1 - \gamma_2^1 C_2 \text{ if } t \in [T_1, T_2] \\ -\gamma_1^2 C_1 - \gamma_2^2 C_2 \text{ if } t \in [T_2, \infty) \end{cases} \tag{11}$$

At the switching time, if any, player i incurs a cost that is defined in terms of the level of the state variable at which the cost occurs. Let $S_i(K(T_i))$ be this cost, with $S_i'(.) \geq 0$. The direct switching cost is discounted at rate ρ. As seen from the initial period, if a switch occurs at T_i, the discounted cost amounts to $e^{-\rho T_i} S_i[K(T_i)]$. It takes the following form: $S_i[K(T_i)] = \chi_i + \beta_i K(T_i)$, $\chi_i > 0$ and $\beta_i \geq 0$. χ_i is the fixed cost related to technology investment. These may include initial outlay for machinery, etc. On the other hand, β_i represents the sensitivity of adoption cost on the level of the exhaustible resource at the instant of switch. It is assumed that the cost of adopting new technology is increasing in $K(T_i) = K_i$ and decreasing in T_i. This assumption can be attributed to learning.

Finally, each player's gross utility function $U(C_i)$ is increasing in her own consumption level, C_i. It is continuous and strictly concave. In the next subsection, we assume that it takes the logarithmic form: $U(C_i) = \ln(C_i)$.

Attention is first paid to the interior solution. Analysis of the corner solutions is postponed to the next subsection.

[3]The state of technology of the other player, 2, is labelled as k.

[4]In what follows, we will omit the time index for simplicity of notation. In addition, the regime index (l, k) will only be used when characterizing the solution valid in this particular regime.

4.2 Interior solution

We now analyze the above-mentioned differential game with two players. To allow for different switching instants, we consider the case where the players are heterogeneous in terms of both switching costs and technological menus.

From now on, consider the particular timing: $0 < T_1 < T_2 < \infty$. In other words, attention is paid to the interior solution where player 1 is supposed to adopt before player 2. Each player solves:

$$max_{\{C_i\},T_i} \int_0^\infty e^{-\rho t} \ln(C_i)dt - e^{-\rho T_i} S_i(K_i)$$

subject to (11), K_0 being given. This problem is solved by backward induction, starting from the game valid after the last switch.

4.2.1 Last period problem

The optimization program is identical for each player. For instance, player 1's problem is to maximize its present value of consumption flow given the dynamics of the resource, and a guess about player 2's consumption. Let us assume that $C_2(K) = b_2^2 K$. Then,

$$\max_{\{C_2\}} \int_{T_2}^\infty e^{-\rho t} \ln(C_1)dt$$

subject to

$$\dot{K} = -\gamma_1^2 C_1 - \gamma_2^2 b_2^2 K$$

with $K(T_2) = K_2$ given.

Writing the first order conditions and making the same guess, for player 1, we obtain:

$$\gamma_1^2 C_1^{22}(K) = \gamma_2^2 C_2^{22}(K) = \rho K \text{ and } K(t) = K_2 e^{-2\rho(t-T_2)} \tag{12}$$

and the value function is

$$V_i^{22}(K_2, T_2) = \frac{e^{-\rho T_2}}{\rho}\left[\ln(K_2) + \ln(\rho) - \ln(\gamma_i^2) - 2\right] = e^{-\rho T_2} v_i^{22}(K_2) \tag{13}$$

The following subsections analyze the two sub-games where players also have to decide on the instant to adopt the new technology. This introduces another source of interaction between players. We have now to characterize the timing strategies, together with the consumption strategies, at a MPE. Solving recursively for the timing $0 < T_1 < T_2 < \infty$, we first study the switching problem of player 2, *i.e.* of the last adopter.

4.2.2 Second period problem

Guessing linear feedback strategies, player 2's optimization program in this regime is:

$$\max_{\{C_2\},T_2} V_2^{21}(.) = \int_{T_1}^{T_2} e^{-\rho t} \ln(C_2)dt - e^{-\rho T_2} S_2(K_2) + V_2^{22}(K_2, T_2)$$

subject to

$$\dot{K} = -\gamma_1^2(a_1 + b_1 K) - \gamma_2^1 C_2$$

with $K(T_1) = K_1$ given, but free K_2.

During this period, player 1 has to make a guess for the switching date of player 2. But he takes both the date and the resource level at which player 2's switching problem starts as given. When going back to the first period problem, these two elements will be endogenized. At the moment, player 1 solves

$$\max_{\{C_1^{21}\}} V_1^{21}(.) = \int_{T_1}^{T_1+\theta_2[K(T_1)]} e^{-\rho t} \ln(C_1^{21}) dt + V_1^{22}(K_2, T_2)$$

subject to

$$\dot{K} = -\gamma_1^2 C_1^{21} - \gamma_2^1(a_2 + b_2 K)$$

with $K(T_1) = K_1$, T_1 and K_2 given.

We can determine the MPE in consumption strategies valid in regime $(2,1)$. We also characterize the instant and the level of resource at which it is optimal for Player 2 to switch. The results are summarized in the proposition below. Recall that we work with T_1 and K_1 fixed.

Proposition 1 • *In regime $(2,1)$ the consumptions strategies are given by*

$$\gamma_1^2 C_1^{21}(K) = \gamma_2^1 C_2^{21}(K) = \frac{\rho^2 \beta_2 (K_2^*)^2}{1 - \beta_2 \rho K_2^*} + \rho K. \tag{14}$$

- *The optimal level of the resource stock for switching, K_2^*, is defined by*

$$\rho S_2(K_2^*) + \ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] = \ln(1 - \beta_2 \rho K_2^*). \tag{15}$$

If the switching level of player 1 (defined in the next section) satisfies $K_1 \geq (\rho\beta_2)^{-1}$, then a sufficient condition for the existence of a unique K_2^* is:

$$\ln\left[\frac{\gamma_2^1}{\gamma_2^2}\right] > \rho\chi_2. \tag{16}$$

Otherwise, another sufficient existence condition is

$$\rho S_2(K_1) + \ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] > \ln(1 - \beta_2 \rho K_1) \tag{17}$$

- *The optimal switching date is $T_2^* = T_1 + \theta_2(K_1)$ with*

$$\theta_2^*(K_1) = \frac{1}{2\rho} \ln\left[(1 - \rho\beta_2 K_2^*)\frac{K_1}{K_2^*} + \rho\beta_2 K_2^*\right] = \frac{1}{2\rho} \ln\left[\frac{C_i^{21}(K_1)}{C_i^{21}(K_2^*)}\right]. \tag{18}$$

Proof. See the appendix B.2. ■

Based on Proposition 3, several remarks can be made. First, player 2's optimal K for switching is independent of player 1's decisions. Recall that by definition we have $\gamma_2^1 > \gamma_2^2$: switching to the new technology translates into a more efficient extraction of the resource. Then, the switching level exists and is unique as long as technology differential (gain from switching) is large enough compared to the fixed cost of switching.

Second, from equations (12) and (14), one can observe that $\gamma_2^1 C_2^{21}(T_2^-) = \gamma_2^2 C_2^{22}(T_2^+)$ iff $\beta_2 = 0$. Thus, players' resource extraction is not continuous at the switching date of player 2. This is due to the fact that the direct switching cost is a function of the level of state variable at the switching date.

Third, the optimal length between two switches, as defined by (18), or the optimal switching time for player 2, is defined in terms of K_1 (the level of state at the switching date of player 1), the discount rate and some parameters characterizing regime $(2,1)$, that players leave, and regime $(2,2)$, that players reach. Hence, player 1 is able to affect player 2 switching time T_2 and will take this influence into account in the first period problem. Note also that the optimal switching date of player 2 is increasing in K_1. The larger the resource stock at which player 1 decides to switch, the later the adoption of player 2. In other words, switching rapidly for player 1 tends to delay the adoption time of player 2.

Before analyzing the first period problem, we need to retrieve the value functions of player 1:

$$v_1^{21}(K_1,\theta_2) = \int_{T_1}^{T_2} e^{-\rho(t-T_1)} \ln(C_1^{21})dt + e^{-\rho\theta_2} v_1^{22}(K_2^*)$$

After some computations, one has

$$v_1^{21}(K_1,\theta_2) = \left[\left(-\frac{\ln C_1^{21}(K_1)e^{-\rho\theta_2}}{\rho} + 2e^{-\rho\theta_2}(\theta_2 + \frac{1}{\rho})\right) - \left(-\frac{\ln C_1^{21}(K_1)}{\rho} + \frac{2}{\rho}\right)\right] + e^{-\rho\theta_2} v_1^{22}(K_2)$$

Using $C_1^{21}(K_2) = \frac{\gamma_2^1}{\gamma_1^2} C_2^{21}(K_2)$, $v_1^{22}(K_2) = v_2^{22}(K_2) + \frac{1}{\rho}\ln\left(\frac{\gamma_2^2}{\gamma_1^2}\right)$ and (14), we obtain

$$v_1^{21}(K_1,\theta_2) = \frac{1}{\rho}\left[\ln(C_1^{21}(K_1)) - 2 + e^{-\rho\theta_2}\left(\ln\left(\frac{C_1^{21}(K_2)}{C_1^{21}(K_1)}\right) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + 2\rho\theta_2 + \rho S_2(K_2)\right)\right] \tag{19}$$

Note that using (18), we have $\ln\left(\frac{C_1^{21}(K_2)}{C_1^{21}(K_1)}\right) = -2\rho\theta_2$. Using this, equation (19) simply becomes

$$v_1^{21}(K_1,\theta_2) = \frac{1}{\rho}\left[\ln(C_1^{21}(K_1)) - 2 + e^{-\rho\theta_2}\left(\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho S_2(K_2)\right)\right] \tag{20}$$

4.2.3 First period problem

Player 1's problem deserves further attention because the player who adopts first has to consider the impact of her switching decision on the other player's switching strategy. This is the original part of our analysis and for this reason most of the resolution appears in the main text. Hereafter, we also focus on the Markov perfect switching strategy for the latest adopter. This notably implies that player 1 has to make a guess not only for the consumption strategy of player 2, but also for its timing strategy.

Assume that player 1 guesses *(i)* player 2 has in mind an optimal switch level K_2 (not necessarily the same as K_2^* that we found above from the analysis of player 2's switching conditions), and that *(ii)* player 2 has a switching time strategy $\theta_2(K_1)$ (not necessarily $\theta_2^*(K_1)$ defined by 18). We have to determine player 1's MPE strategy for the switching time given these guesses. The relevant level of the state, in the definition of the switching strategy of player 1, corresponds to the instant when the switching problem arises that is, K_0. Note that his switching time is also dependent on the technological state of the economy, described by the pair (γ_1^l, γ_2^h), $l, h = 1, 2$, but we do not need to make this dependence explicit.

Player 1's optimization program in this first period, *i.e.* the period that holds before any switch, is

$$\max_{\{C_1\},T_1} V_1^{11}(.) = \int_0^{T_1} e^{-\rho t}\ln(C_1)dt - e^{-\rho T_1}S_1(K_1) + e^{-\rho T_1}v_1^{21}[K_1, \theta_2(K_1)]$$

subject to,

$$\dot{K} = -\gamma_1^1 C_1 - \gamma_2^1(a_2 + b_2 K)$$

with $K(0) = K_0$ given, and free K_1.

Following the same methodology as in the second period, first we can write the switching conditions. Taking the derivative of the value function with respect to K_1, one obtains (the necessary condition for an interior $K_1^* < K_0$)

$$-\lambda_1(T_1^-) - e^{-\rho T_1}S_1'(K_1) + e^{-\rho T_1}\left[\frac{\partial v_1^{21}}{\partial K_1} + \frac{\partial v_1^{21}}{\partial \theta_2}\theta_2'(K_1)\right] = 0 \qquad (21)$$

This optimality condition emphasizes the impact of MPE strategies for switching times on players' behavior. When choosing the level of the state corresponding to the switching time, the player who moves first (Player 1), takes into account the impact of her decision on Player 2's switching strategy. This condition will be discussed further at the end of the section.

At the MPE, using (20) and the value for $\theta_2(K_1)$ and $\theta_2'(K_1)$, given in (18), the term inside the square brackets in equation (21) can be rewritten as:

$$\frac{\partial v_1^{21}}{\partial K_1} + \frac{\partial v_1^{21}}{\partial \theta_2}\theta_2'(K_1) = \frac{C_1^{21\prime}(K_1)}{\rho C_1^{21}(K_1)}\left[1 - \frac{e^{-\rho\theta_2(K_1)}}{2}\left(\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho S_2(K_2)\right)\right]$$

Thus, equation (21) is equivalent to

$$-\frac{1}{\gamma_1^1 C_1^{11}(T_1^-)} - \beta_1 + \frac{C_1^{21\prime}(K_1)}{\rho C_1^{21}(K_1)}\left[1 - \frac{1}{2e^{\rho\theta_2(K_1)}}\left(\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho S_2(K_2)\right)\right] = 0,$$

which reduces to

$$\gamma_1^1 C_1^{11}(T_1^-) = \frac{\Gamma + \rho K_1}{Z(K_1) - \beta_1\left(\Gamma + \rho_2 K_1\right)} \tag{22}$$

where $\Gamma = \frac{\rho^2\beta_2(K_2^*)^2}{1-\beta_2\rho K_2^*}$ and

$$\begin{aligned} Z(K_1) &\equiv \left[1 - \frac{1}{2e^{\rho\theta(K_1)}}\left(\ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \rho S_2(K_2^*)\right)\right] \\ &= 1 - \frac{\ln\left(1 - \rho\beta_2 K_2^*\right)}{2\sqrt{\left(1-\beta_2\rho K_2^*\right)\left(K_1/K_2^*\right) + \rho\beta_2 K_2^*}} > 1 \end{aligned}$$

where we have made use of the equation (15) and of the fact that $K_2^* \in (0, (\rho\beta_2)^{-1})$.

Define the function $F(K_1)$ by

$$F(K_1) = Z(K_1) - \beta_1(\Gamma + \rho K_1). \tag{23}$$

this function, that is defined over the interval (K_2^*, K_0), is decreasing in K_1. Thus, a necessary condition for the consumption level in (22) to be defined for some $K_1 \in (K_2^*, K_0)$ is $F(K_2^*) > 0$, which can be restated as

$$\frac{1 - \rho(\beta_1+\beta_2)K_2^*}{1 - \rho\beta_2 K_2^*} + \frac{\left[-\ln\left(1 - \rho\beta_2 K_2^*\right)\right]}{2} > 0. \tag{24}$$

From now on, we will assume that this technical condition holds. It notably imposes that β_1 is small enough. (A sufficient condition for consumption to be non negative for all $K_1 \in (K_2^*, K_0)$ is $Z(K_0) - \beta_1(\Gamma + \rho K_0) > 0$.)

Remark. The inequality (24) is not satisfied for all $K_2^* \in [0, (\rho\beta_2)^{-1})$. It is pretty easy to show that $\exists! \bar{K}_2^* \in (0, (\rho\beta_2)^{-1})$ such that $F(K_2^*) > 0$ for all $K_2^* < \bar{K}_2^*$.

The second switching condition (wrt T_1) has the same form as the one of player 2:

$$e^{-\rho T_1}\ln(C_1^{11}(T_1)) - \lambda_1^{11}(T_1)\left[\gamma_2^1 C_2^{11}(T_1) + \gamma_1^1 C_1^{11}(T_1)\right] + \rho e^{-\rho T_1}(S_1(K_1) - v_1^{21}(K_1, \theta_2(K_1))) = 0$$

Since the relationship $\gamma_1^{11} C_1(t) = \gamma_2^{11} C_2(t)$ is valid in the first regime, *i.e.* for all $t \in [0, T_1]$, this condition can be restated as

$$\ln\left(C_1^{11}(T_1)\right) = 2 + \rho[v_1^{21}(K_1, \theta_2(K_1)) - S_1(K_1)] \tag{25}$$

Using the value of $C_1^{11}(T_1)$ and $C_1^{21}(T_1)$ respectively given by (22) and (14) (for the case of $T_1 > 0$), condition (25), that defines the optimal switching level K_1^*, reads:

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2(K_1)}\left[\rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[F(K_1)] \qquad (26)$$

Denote the LHS of (26) by $G(K_1)$ and the RHS by $H(K_1)$. Now we characterize the MPE in consumption strategies in regime 11 and provide sufficient conditions for having a unique solution to the first player's switching problem.

Proposition 2

- *If the following conditions hold:*

$$G(K_0) > H(K_0), \rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) < \rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \qquad (27)$$

  and,

$$\beta_1 < \frac{1}{2\rho K_2^*}\left[(1-\beta_2\rho K_2^*)\left(-\ln\left(1-\beta_2\rho K_2^*\right)\right)\right], \qquad (28)$$

  then there exists a unique $K_1^* \in (K_2^*, K_0)$ that solves (26).

- *In regime $(1,1)$ consumption strategies at the MPE are*

$$\gamma_1^2 C_1^{11}(K) = \gamma_2^1 C_2^{11}(K) = \Lambda + \rho K. \qquad (29)$$

  with,

$$\Lambda = \frac{\Gamma + \rho K_1^*(1 - F(K_1^*))}{F(K_1^*)} \qquad (30)$$

- *Taking the pair (K_1^*, K_2^*) as given, the optimal switching time is*

$$T_1 = \frac{1}{2\rho}\ln\left[\frac{K_0 + \frac{\Lambda}{\rho}}{K_1^* + \frac{\Lambda}{\rho}}\right], \qquad (31)$$

  with K_0 the initial stock.

Proof. See the appendix B.3. ■

The series of sufficient conditions in proposition (2) are more likely to be satisfied when:

(i) The initial resource stock is high enough (first inequality in (27)),

(ii) Player 1 is the one for whom adopting the new technology is relatively less costly, $S_1(K) < S_2(K)$ for all $K \in [0, K_0]$. She also the one who earns the highest benefit from adoption: $\frac{\gamma_1^2}{\gamma_1^1} < \frac{\gamma_2^2}{\gamma_2^1} < 1$.

(iii) Player 1's direct switching cost is not very sensitive to the resource stock i.e. β_1 is small enough.

We will see, in the next section, that these sufficient conditions (for the necessary conditions of an interior solution to hold) are intimately linked with the necessary conditions characterizing the corner regimes.

In the remaining part of this section, we will further address the impact of MPE strategies for switching times on the solution. Indeed, given that (player 2) switching strategy is based on the state of the system and player 1 is able to affect this state, it is crucial to understand how does player 1 adapt her strategy to player 2's switching decision. This also requires the solution to the following related issue: what is the impact of Player 2's future switch on player 1?

Recall that player 1's switching conditions are:[5]

$$\begin{array}{l} H_1^{11*}(T_1^*) - H_1^{21^*}(T_1^*) = \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \\ \lambda_1^{21*}(T_1^*) - \lambda_1^{11*}(T_1^*) = \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} - \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)]. \end{array}$$

In our application, they can be written as:

$$\begin{array}{l} \ln\left[\frac{C_1^{11}(T_1^*)}{C_1^{21}(T_1^*)}\right] = -\rho S_1(K_1^*) + e^{-\rho\theta(K_1^*)} \ln\left[\frac{C_1^{22}(T_2^*)}{C_1^{21}(T_2^*)}\right] \\ \left[\gamma_1^2 C_1^{21}(T_1^*)\right]^{-1} - \left[\gamma_1^1 C_1^{11}(T_1^*)\right]^{-1} = S_1'(K_1^*) + \theta_2'(K_1^*) e^{-\rho\theta(K_1^*)} \ln\left[\frac{C_1^{22}(T_2^*)}{C_1^{21}(T_2^*)}\right] \end{array} \tag{32}$$

Compared to the single-agent problem, both conditions are modified. Thus, we would like to understand how these modifications affect player 1's switching problem.

The LHS of the first condition in (32) reflects the marginal gain from extending the horizon of the first regime. If there exists $0 < T_1^* < T_2^*$ then this marginal gain must be equal to the marginal cost of switching at T_1^*. Now, the marginal switching cost (RHS) is augmented (in absolute magnitude) by the extra-term $e^{-\rho\theta(K_1^*)} \ln[\frac{C_1^{22}(T_2^*)}{C_1^{21}(T_2^*)}]$. Player 1 anticipates that her switching decision will be followed by the switch (in finite time too) of the second player and that this switch will be costly, given that $C_1^{22}(T_2^*) < C_1^{21}(T_2^*)$. She incurs a indirect (marginal) cost that corresponds to $e^{-\rho\theta_2(K_1^*)} \ln[\frac{C_1^{22}(T_2^*)}{C_1^{21}(T_2^*)}]$. Why is it so? Adopting a new technology translates into a decrease in the extraction rate. In particular, when player 2 switches, we have $\gamma_2^1 C_2^{21}(T_2^*) > \gamma_2^2 C_2^{22}(T_2^*)$. Intuitively, with the new technology, one needs less resource to produce a given amount of the consumption good. The impact of player 2's adoption on her own consumption is unclear because it depends on the size of the productivity differential $\frac{\gamma_2^2}{\gamma_2^1}$. However, it is clear that player 1 is worse off after player 2's switch because she bears the decrease in extraction, both players share the same extraction rate, and is not able to compensate this loss by an adaptation of her technology. By construction, she already switched at an earlier date and now sticks to her second technology.

So, it means that the marginal cost of switching is higher than it would be in the absence of player 2. Other things equal (K_1 constant), it implies that the

[5] At the MPE, the guess of player 1 must be consistent with the switching strategy actually adopted by player 2.

switch should occur at a later date *i.e.* player 1, when interacting with player 2, has an incentive to postpone adoption.

The second condition equalizes the marginal benefit from an extra unit of the state variable K_1 (LHS) with the corresponding marginal cost (RHS). The marginal cost is lower in the game than in the control problem because $\theta_2'(K_1^*)e^{-\rho\theta_2(K_1^*)}\ln[\frac{C_1^{22}(T_2^*)}{C_1^{21}(T_2^*)}] < 0$ ($\theta_2'(K_1^*) > 0$). Indeed, other things equal (T_1 constant), by increasing K_1, player 1 induces player 2 (because player 1 controls K_1 on which is based player 2's decision) to delay the instant of her switch that is, the instant when player 1 will incur the additional indirect (marginal) cost. The impact of player 2's switch will then be felt less acutely because of discounting. This in turn implies that player 1's adoption should occur for a higher K_1^*. This second effect makes it worthwhile for player 1 to adopt at an earlier date (because the trajectory of K is monotone non increasing).

In summary, the following conclusion can be drawn. At first glance, as a result of the interaction with player 2, player 1 will delay the adoption of the new technology (first-order effect corresponding to the first condition in (32)). It does not mean however that he will not adopt before player 2. According to the second condition in (32), the sooner the adoption of player 1, the lower the negative impact of player 2's adoption on his welfare (second-order effect).

Suppose now that $S_1(K_1) = 0$: player 1 does not bear any (direct) cost when she switches. Then, we know that the solution of the optimal control problem (single-agent problem) is $T_1^* = 0$: one adopts instantaneously because the new technology is more efficient than the old one.

But, it is clear that if the equations in (32) have a solution then conclusions will be very different in the switching game, with $S_1(K_1) = S_1'(K_1) = 0$. What are the features of this solution?

Player 1 incurs a indirect (marginal) cost when player 2 adopts. Then, it is optimal for player 1 to switch at a $0 < T_1^* < T_2^*$ because it allows her to compensate for the loss by increasing her extraction (which implies that consumption increases too) at her switching time *i.e.* one must have $\left[\gamma_1^1 C_1^{11}(T_1^*)\right]^{-1} > [\gamma_1^2 C_1^{21}(T_1^*)]^{-1}$. Interestingly enough, the interaction through switching time offsets the previous effect of adoption (identified for player 2): switching to the new, more efficient, technology translates into an increase of the extraction rate.

Under which conditions does a solution exist? Combining the two equations in (32), one obtains:

$$\ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] = 2(1 - Z(K_1)) + \ln[Z(K_1)] \tag{33}$$

with $Z(K_1)$ defined in the text. Let the RHS be denoted by $h(K_1)$. One has $h'(K_1) > 0$ for all $K_1 \in [K_2^*, K_0]$. In addition, evaluated at its lower bound, $h(.)$ "degenerates" in $\tilde{h}(K_2^*)$ with $\tilde{h}(K_2^*) = \ln(1 - \rho\beta_2 K_2^*) + \ln\left(1 - \frac{\ln(1-\rho\beta_2 K_2^*)}{2}\right)$. A necessary existence condition is $\tilde{h}(K_2^*) \leq \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right]$. Given that $\tilde{h}'(K_2^*) < 0$ for all $K_2^* \in [0, (\rho\beta_2)^{-1}]$, $\tilde{h}(0) = 0$ and $\tilde{h}[\rho\beta_2)^{-1}] = -\infty$, there exists a unique $\tilde{K}_2$ such that: $K_2^* \geq \tilde{K}_2 \Leftrightarrow \tilde{h}(K_2^*) \leq \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right]$.

So, if K_2^* is high enough i.e. if player 2 adopts rapidly (this is all the more likely when, for player 2, the switching cost is relatively low and/or the technology differential is low), then a sufficient existence condition is $h(K_0) > \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right]$, which is satisfied if the initial stock is high enough.

By continuity, all this reasoning is also correct when β_1 and $S_1(K_1)$ are not too high.

4.3 Corner solutions

In this section, we analyze the corner solutions and the erroneous timing situations. In what follows, we will extensively use one feature of the MPE in consumption strategies: $\gamma_1^l C_1^{lk} = \gamma_2^k C_2^{lk}$ in any regime lk. This in turn implies that $\lambda_1^{lk} = \lambda_2^{lk}$. Attention is paid to the first timing $0 \leqq T_1 \leqq T_2$. Results for the other timing are obtained by symmetry.

Let us first determine the conditions under which the timing analyzed so far is erroneous *i.e.* at least one player would prefer the opposite timing.

4.3.1 Erroneous timing

Take the (interior) solution of player 1's switching problem (T_1^*, K_1^*) as given. Suppose there is no interior solution to player 2's switching problem. it means that either player 2 would like to switch at or before T_1^* or she prefers adopting a never switching strategy. The latter case will be analyzed in the next subsection.

In the proposition below, we provide some sufficient conditions to be in the former case. It should be clear that this particular option taken by player 2 will in turn influence player 1's strategy. Here we need to introduce some notations. Let $f(K_1)$ be defined as follows:

$$f(K_1) = \frac{1 - \rho(\beta_1 + \beta_2)K_1}{1 - \rho\beta_2 K_1} - \frac{\ln(1 - \rho\beta_2 K_1)}{2}$$

Proposition 3 *Immediate switching $T_1^* = T_2$: assume that $2\beta_1 > \beta_2$ and*

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_1^{**}) \leq \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] + \rho S_1(K_1^{**}) \tag{34}$$

where K_1^{**} is the unique solution of[6]

$$\rho S_1(K_1) + \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] = \ln(1 - \rho\beta_2 K_1) + \ln\left[f(K_1)\right], \tag{35}$$

then it is optimal for player 2 to adopt at a date no later than T_1^*.

[6] K_1^{**} is the interior solution of player 1's switching problem when he anticipates that player 2 will stick to his strategy.

Proof. See the appendix C.1. ∎

Condition (34) characterizes a situation that is more than a simple corner solution. In order to analyze this case, we must proceed as follows: let $0 < T_1 < \infty$ be given, we want to determine under which conditions it is "optimal" for player 2, who maximizes the discounted value between T_1 and ∞, to switch immediately. This means that the necessary conditions are similar to the usual conditions of the multi-stage optimal control theory for immediate switching (see Tomiyama, 1985, Amit, 1986, Makris, 2001 and Boucekkine, Saglam and Vallée, 2004). However, it is worth noting that this particular situation cannot be interpreted as usual corner regime precisely because we are in a differential game. The consequence of this is that T_1 (the beginning of the planning period for player 2) is not fixed. This degenerated corner solution actually corresponds to a situation in which it is not optimal for player 2 to adopt after player 1. Put differently, as long as condition (34) holds, the correct timing, at the MPE, if a MPE exists, should be $0 \leqq T_2 \leqq T_1 \leqq \infty$.

In the same vein, it is possible that player 1, who is supposed to be the first mover, might prefer adopting the new technology at a date no earlier than T_2^*. In order to find sufficient conditions for this situation to occur, we analyze the corner $T_1^* = T_2^*$. Assuming now that we have found an interior solution (T_2^*, K_2^*) to player 2's adoption problem, our results are as follows.

Proposition 4 *Never switching $T_1 = T_2^*$: if $F(K_2^*) \leq \frac{1}{2}$ and*

$$\rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \tag{36}$$

then player 1 never finds it optimal to switch from the old to the new technology before T_2^*.

Proof. See the appendix C.1. ∎

The link between the two situations is apparent from propositions (3) and (4). Actually, it turns that the sufficient conditions (34) and (36) for an erroneous timing are identical for the two players, except that they do not have the same reference point. One way of ensuring that they are satisfied is to impose:

$$\rho[S_1(K) - S_2(K)] \geq -\left[\ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) - \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] \text{ for any } K \in [0, K_0], \tag{37}$$

where $-\ln\left(\frac{\gamma_i^2}{\gamma_i^1}\right)$, for $i = 1, 2$, can be understood as a measure of the gain from switching.

Condition (37) can be easily interpreted in economic terms. It basically states that the relative advantage of adoption (RHS), measured in terms of the differential of gains, is lower the relative advantage in terms of adoption costs (LHS), for player 1. Of course, this inequality is satisfied when player 2 incurs a lower direct switching cost and, at the same time, withdraws the higher benefit of adoption. But, it might also hold in intermediate situations where player 2's adoption cost is higher provided that the differential in technological gains is largely favorable to player 2.

4.3.2 Corner timings and simultaneous switches

In this sub-section, we briefly give an overview of the conditions under which the MPE may either be associated with corner timings or features simultaneous switches.

Proposition 5 • *Never switching* $0 < T_1 < T_2 = \infty$*: if*

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(0) \geq 0 \tag{38}$$

then player 2 never finds it optimal to switch from the old to the new technology.

- *Immediate switching* $0 = T_1 < T_2 < \infty$*: if* $F(K_0) \geq 1$ *and,*

$$\rho S_1(K_0) + \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] \leq \ln(1 - \beta_2 \rho K_2^*), \tag{39}$$

with K_2^**, the unique solution of (15) and* $F(.)$*, defined in (23), then player 1 instantaneously adopts the new technology.*

- *Immediate and never switchings* $0 = T_1 < T_2 = \infty$*: if (38),*

$$\rho S_1(K_0) + \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] \leq 0, \tag{40}$$

and,

$$\rho K_0 \geq \frac{\gamma_1^1 - \gamma_1^2}{\beta_1 \gamma_1^1 \gamma_1^2} \tag{41}$$

hold, then players adopt strictly opposite strategies, one adopting immediately the new technology, the other sticking to the old technology.

Proof. See the appendix C.2. ■

Sufficient conditions for corner solutions have a very simple interpretation. For instance, according to condition (38), a player never finds it worthwhile to adopt the new technology when the fixed cost of adoption, weighted by the rate of time preference, is larger than the gain from switching. In the same vein, a player is willing to adopt the new technology immediately when the switching cost at the initial resource level is lower than the gain from adoption.

Finally, there are three remaining cases. Players might wish to adopt their new technology at the same date and for the same stock of resource. Or, they might prefer switching instantaneously or on the contrary adopting never switching strategies. The conditions for these cases are summarized in the proposition below.

Proposition 6 • *It cannot be optimal for players with different switching costs to adopt at the same positive and finite date.*

- *Immediate switching* $0 = T_1 = T_2$*: if*

$$\rho S_i(K_0) + \ln\left[\frac{\gamma_i^2}{\gamma_i^1}\right] \leq 0 \tag{42}$$

and,

$$\rho K_0 \geq \frac{\gamma_i^1 - \gamma_i^2}{\beta_i \gamma_i^1 \gamma_i^2} \tag{43}$$

for $i = 1, 2$*, then both players find it worthwhile to adopt instantaneously the new technology.*

- *Never switching* $T_1 = T_2 = \infty$*: if*

$$\ln\left[\frac{\gamma_i^2}{\gamma_i^1}\right] + \rho S_i(0) \geq 0 \tag{44}$$

for $i = 1, 2$*, then players will not to switch from the old to the new technology.*

Proof. See the appendix C.3. ■

Regarding the first item, using the feature that $\lambda_1^{lk} = \lambda_2^{lk}$ in any regime (l, k), it is clear that the switching conditions

$$\begin{array}{c} \lambda_2^{11*}(T^*) + \frac{\partial \Omega_2(K^*, T^*)}{\partial K_2} = \lambda_2^{22*}(T^*) \\ \lambda_1^{11*}(T^*) + \frac{\partial \Omega_1(K^*, T^*)}{\partial K_1} = \lambda_1^{22*}(T^*) \end{array}$$

with $K^* = K_1 = K_2$ the level of the state variable at the switching date $T^* = T_1 = T_2$, cannot be satisfied at the same time as long as players bear different direct switching costs. More precisely, if $S_1'(K) \neq S_2'(K)$ for all $K, \Leftrightarrow \beta_1 \neq \beta_2$, then this case cannot arise.

5 Discussion

In this section, we provide a synthesis of Section 4. Once all possible combinations of switching times, for $0 \leqq T_1 \leqq T_2 < \infty$, have been studied, it is possible to develop the general reasoning that allows us to conclude that a MPE will be associated with, for instance, the interior timing $0 < T_1 < T_2 < \infty$.

The first thing to do is check that neither player finds this timing erroneous. Necessary conditions for the timing not to be erroneous are simply the opposite of the sufficient conditions for an erroneous one. Thus, we must have (see the appendix devoted to the analysis of corner solutions): $f(K_1^{**}) > 1$ and

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_1^{**}) > \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] + \rho S_1(K_1^{**}) \tag{45}$$

from player 2's erroneous timing conditions. We also have to impose $F(K_2^*) > 1$ and

$$\rho S_1(K_2^*) + \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] \geq \rho S_2(K_2^*) + \ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right], \tag{46}$$

from player 1's problem.

Next, it must be the case that players do not find it optimal to be situated at their respective corner solutions. Again, necessary conditions for this statement to be true are the opposite of the sufficient for the corner $0 = T_1^*$ and $T_2^* = \infty$ to arise at the MPE. So, it is necessary that

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(0) < 0, \tag{47}$$

$F(K_0) < 1$ and,

$$\rho S_1(K_0) + \ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] > \ln(1 - \beta_2 \rho K_2^*), \tag{48}$$

Assume that there exists K_1^{**}, solution of (35) with $f(K_1^{**}) > 1$, $F(K_2^*) > 1$ and (45)-(48) hold. What are the remaining conditions needed to ensure the existence of an interior MPE? One can observe that $F(K_2^*) > 1$ and (46) correspond to part of the sufficient conditions presented in Proposition 2 for the existence of K_1^* at the interior solution. Moreover, it appears that (47) is also a sufficient condition for existence of K_2^* (see Proposition 1). Thus, we can establish the following.

Corollary 2 *Suppose that there exists K_1^{**}, solution of (35) with $f(K_1^{**}) > 1$, $F(K_2^*) > 1$ and (45)-(48) hold. A sufficient condition for the existence of K_1^* is:*

$$G(K_0) > H(K_0). \tag{49}$$

If this critical level satisfies $K_1^* \geq (\rho\beta_2)^{-1}$ then there exists a MPE featuring the timing $0 < T_1 < T_2$. Otherwise ($K_1^* < (\rho\beta_2)^{-1}$), there is a second sufficient condition that reads:

$$\rho S_2(K_1^*) + \ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] > \ln(1 - \beta_2 \rho K_1^*). \tag{50}$$

Remark 1. Recall that under (46)-(48), there may well exist other MPEs primarily because they are just the opposite of sufficient conditions for corner and erroneous timing. So, we cannot guarantee the uniqueness of the (interior) solution. If one seeks to determine a unique solution, it is enough to refer to the sufficient conditions for corner solutions stated in Propositions 5 and 6.

Remark 2. K_1^{**} is not related to a particular solution. It is just defined to study the hypothetical scenario in which the timing is erroneous for player 2. But, it is possible that there exists no K_1^{**}. In this case, one logically expects that player 1, given that player 2 would prefer sticking to her strategy, prefers switching immediately. So, K_1^{**} should be replaced with K_0. However, studying this hypothetical scenario, we can show that there exists no solution featuring $0 \leftarrow T_1 \leftarrow T_2$.

6 Conclusion

In this paper, we have developed a general two-player differential game with regime switching. The interaction between players is assumed to be governed by two kinds of strategies. At each point in time, they have to choose an action that influences the evolution of a state variable. In addition, they may decide on the switching time between alternative and consecutive regimes. At a feedback Nash equilibrium, the switching strategy is defined as a function of the state of the system. Compared to the standard optimal control problem with regime switching, necessarily optimality conditions are modified only for the first-mover. When choosing the optimal date and level of state variable for switching, this player must take into account that *(i)* her decision will influence the other player's switching strategy, and *(ii)* the other player's switch will affect her welfare. Furthermore, we have exhibited the necessary conditions characterizing the timing at the Markov perfect equilibria. Erroneous timing strategies were eventually analyzed.

At the latter part of this paper, we applied this new theoretical framework to solve a game of exhaustible resource extraction with technological regime switching. It was assumed that, at a given cost, players have the option to adopt a more efficient extraction technology. We then obtained sufficient conditions guaranteeing that both players switch in finite time. Moreover, we investigated the impact of feedback strategies for switching time on the first-mover technology adoption strategy. There is an interplay between two conflicting effects. First, the switch of the second mover is costly for the first-mover because it implies a drop in her consumption. Thus, the first-mover may opt to delay adoption. Meanwhile, because of discounting, delaying the switch of the other play will allow the first-mover to incur a lower indirect cost. This is an incentive for the first-mover to adopt at an earlier date.

Overall, the methodology presented in this paper may pave the way to handle a wider class of problems in economics. Potential extensions include the analysis of cooperative outcomes, the consideration of ecological switching, and the like. These issues will be addressed in the authors' future research endeavors.

References

[1] Alvarez-Cuadrado, F. and Long, NV.: Relative consumption and renewable resource extraction under alternative property-rights regimes. Resource and Energy Economics **33**(4), 1028-1053 (2011).

[2] Amit, R.: Petroleum reservoir exploitation: Switching from primary to secondary recovery. Operations Research **34**, 534-549 (1986).

[3] Boucekkine, R., Saglam, C. and Vallée T.: Technology adoption under embodiment: a two-stage optimal control approach. Macroeconomic Dynamics **8**, 250-271 (2004).

[4] Boucekkine, R., Krawczyk, J. and Vallée T.: Environmental quality versus economic performance: A dynamic game approach. Optimal Control Applications and Methods **32**, 29-46 (2011).

[5] Boucekkine, R., Pommeret, A and Prieur F. Optimal regime switching and threshold effects: Theory and application to a resource extraction problem under irreversibility.AMSE working paper 1213 (2012).

[6] Boucekkine, R., Pommeret, A. and Prieur F.: Technological vs. ecological switch and the environmental Kuznets Curve. American Journal of Agricultural Economics **95**, 252-260 (2013).

[7] Fudenberg, D Tirole, J.: Preemption and rent equalization in the adoption of new technology. Review of Economic Studies, **52**(3), 383-401 (1985).

[8] Lane, P. and Tornell, A. 1996. Power, growth, and the voracity effect. Journal of Economic Growth **1**(2), 213-41 (1996).

[9] Long, NV.: A Survey Of Dynamic Games In Economics. World Scientific Publishing Co. Pte. Ltd. (2010).

[10] Long, NV.: Dynamic Games in the Economics of Natural Resources: A Survey. Dynamic Games Applications **1**, 115-148 (2011) DOI 10.1007/s13235-010-0003-2

[11] Makris, M.: Necessary conditions for infinite horizon discounted two-stage optimal control problems. Journal of Economic Dynamics and Control **25**,1935-1950 (2001).

[12] Ploeg, van der F.: Voracious transformation of a common natural resource into productive capital. International Economic Review **51**(2), 365-381 (2010).

[13] Reinganum, JF.: On the diffusion of new technology: A game theoretic approach. Review of Economic Studies **48**, 385-405 (1981).

[14] Tomiyama, K.: Two-stage optimal control problems and optimality conditions. Journal of Economic Dynamics and Control **9**, 317-337 (1985).

[15] Tornell, A.: Economic growth and decline with endogenous property rights. Journal of Economic Growth, **2**(3), 219-50, (1997).

[16] Valente, S.: Endogenous growth, backstop technology adoption, and optimal jumps. Macroeconomic Dynamics **15**, 293-325 (2011)

Appendix

A Proof of Theorem 1

Let the triplet $(C_1^*(t), C_2^*(t), K^*(t))$ be the path followed by each player's strategy and the state variable at a Markov perfect equilibrium (MPE), for every $t \in [0, +\infty)$. A restriction of this path to $[T_{j-1}, T_j]$, $j = 1, 2$, with $T_0 = 0$, continues to characterize the solution of the subgame with $K(T_j) = K_j^*$, T_{j-1} and T_j fixed and with the maximization of $\int_{T_{j-1}}^{T_j} F_i(K, C_1, C_2)e^{-\rho t}dt$ as player i's objective, $i = 1, 2$. In addition, a restriction of $(C_1^*(t), C_2^*(t), K^*(t))$ to $[T_2, +\infty)$ is a MPE of the infinite horizon game with $K(T_2) = K_2^*$, T_2 fixed and with the maximization of $\int_{T_2}^{\infty} F_i(K, C_1, C_2)e^{-\rho t}dt$ as player i's objective.

The proof uses standard calculus of variations techniques in a sequence of three subgames as explained in the main text. The problem is solved recursively, starting from the game arising after the last switch. The proof focuses on the timing $0 \leqq T_1 \leqq T_2 \leqq \infty$, i.e. on the case where player 1 is the first to switch, followed by player 2. The necessary optimality conditions for the other timing $0 \leqq T_1 \leqq T_2 \leqq \infty$ can be obtained by symmetry.

The index of the technology for player 1 is k, $k = 1, 2$, whereas the technology of the second player is indexed by l, $l = 1, 2$. The system is said to be in regime (k, l) when player 1 uses technology l and player 2 uses technology k. Note that both the payoff $F_i(K, C_1, C_2)$, the function of the state equation $f(K, C_1, C_2)$ and the strategies can be dependent of the particular regime in which the system lays. So, k, l will be used as a superscript.

In each subgame, we determine the Markov Perfect Equilibrium i.e. we restrict attention to strategies of the type: $C_j(t) = \Phi_j(K(t))$. Player's optimization problems are solved using the Pontryiagin method. This implies that when solving player 1's problem, in any regime, we have to introduce a guess about the other player's strategy, here $C_2(t) = \Phi_2(K(t))$.

For each subgame, except the last one, attention is mainly paid to the problem faced by the player who undertakes the switching decision. When required, we also present the optimality conditions of the other player.

- *Last regime* $(2, 2)$*, for* $t \geq T_2$:

In this regime, player j solves:

$$\max_{C_j} \int_{T_2}^{\infty} F_j^{22}(K, C_j, \Phi_{-j}(K))e^{-\rho t}dt, \tag{51}$$

subject to,

$$\dot{K} = f^{22}(K, C_j, \Phi_{-j}(K)). \tag{52}$$

where T_2 and the initial condition $K(T_2) = K_2$ are fixed. $K(T_2)$ will be made free in the next stage. The present value Hamiltonian of the problem, H_j^{22}, is given by $H_j^{22} = F_j^{22}(K, C_1, C_2)e^{-\rho t} + \lambda_j^{22} f^{22}(K, C_1, C_2)$, where λ_j^{22} is player j co-state variable associated with K in regime $(2, 2)$. This problem does not deserve further attention since it yields straightforward first-order necessary conditions (including the appropriate transversality conditions). Let us denote by superscript $*$ the paths identified by these conditions (abstracting here from existence and uniqueness issues). Let $V_j^{22*}(K_2, T_2)$ be the value function, we have the usual envelope conditions: for $j = 1, 2$

$$\begin{aligned} \frac{\partial V_j^{22*}}{\partial T_2} &= -H_j^{22}(T_2), \\ \frac{\partial V_j^{22*}}{\partial K_2} &= \lambda_j^{22}(T_2). \end{aligned} \tag{53}$$

- *Second regime* $(2, 1)$, *for* $t \in [T_1, T_2]$:

In this regime, player 2 (the one who decides on the switching time) solves:

$$\max_{C_2, T_2} \int_{T_1}^{T_2} F_2^{21}(K, C_2, \Phi_1(K)) e^{-\rho t} dt - \Omega_2(K_2, T_2) + V_2^{22*}(K_2, T_2) \tag{54}$$

subject to,

$$\dot{K} = f^{21}(K, C_2, \Phi_1(K)), \tag{55}$$

where T_1 and the initial condition $K(T_1) = K_1$ are fixed. But, $K(T_2) = K_2$ and T_2 are free.

After some some standard calculations, one obtains:

$$V_2^{21} = \int_{T_1}^{T_2} \left[H_2^{21} + \dot{\lambda}_2^{21} K \right] dt - \left\{ \lambda_2^{21}(T_2) K_2 - \lambda_2^{21}(T_1) K_1 \right\} - \Omega_2(K_2, T_2) + V_2^{22*}(K_2, T_2)$$

To find the necessary optimality conditions, we derive the first-order variation of V_2^{21} with respect to the state and control variables' paths, for fixed T_1, $K(T_1) = K_1$ and free T_2 and K_2. This yields, after rearranging terms:

$$\begin{aligned} \delta V_2^{21} = & \int_{T_1}^{T_2} [(\tfrac{\partial H_2^{21}}{\partial K} + \tfrac{\partial H_2^{21}}{\partial C_1} \Phi_1'(K) + \dot{\lambda}_2^{21}) \delta K + \tfrac{\partial H_2^{21}}{\partial C_2} \delta C_2] dt \\ & + (H_2^{21}(T_2) - \tfrac{\partial \Omega_2(K_2, T_2)}{\partial T_2} + \tfrac{\partial V_2^{22*}}{\partial T_2}) \delta T_2 - (\lambda_2^{21}(T_2) + \tfrac{\partial \Omega_2(K_2, T_2)}{\partial K_2} - \tfrac{\partial V_2^{22*}}{\partial K_2}) \delta K_2. \end{aligned}$$

A trajectory is optimal if any small departure from it decreases the value function, that is $\delta V_2^{21} \leq 0$ for any $\delta K(t)$, $t \in (T_1, T_2)$, for any $\delta C_2(t)$, $t \in [T_1, T_2]$, and for any δT_2 and δK_2, which gives the following necessary conditions for an interior maximizer, $T_1 < T_2 < \infty$:

$$\begin{cases} \frac{\partial H_2^{21}}{\partial C_2} = 0, \; \frac{\partial H_2^{21}}{\partial K} + \frac{\partial H_2^{21}}{\partial C_1} \Phi_1'(K) + \dot{\lambda}_2^{21} = 0, \\ H_2^{21}(T_2) - \frac{\partial \Omega_2(K_2, T_2)}{\partial T_2} + \frac{\partial V_2^{22*}}{\partial T_2} = 0, \; \lambda_2^{21}(T_2) + \frac{\partial \Omega_2(K_2, T_2)}{\partial K_2} - \frac{\partial V_2^{22*}}{\partial K_2} = 0. \end{cases} \tag{56}$$

The first two equations are the standard Pontryagin conditions, the last two are optimality conditions with respect to the switching time, T_2, and the free state value, K_2. Together with conditions in (53) obtained from the third subproblem, one gets conditions (1) of Theorem 1, that is:

$$\begin{aligned} H_2^{21*}(T_2^*) - \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} &= H_2^{22*}(T_2^*) \\ \lambda_2^{21*}(T_2^*) + \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial K_2} &= \lambda_2^{22*}(T_2^*). \end{aligned} \tag{57}$$

Regarding the necessary conditions for corner solutions:

Suppose $T_1 = T_2^*$ (and $\delta K_2 = 0$) then the only possible variations of T_2^* are such that $\delta T_2 \geq 0$. For $\delta V_2^{21} \leq 0$ it must be true that

$$H_2^{21*}(T_2^*) - \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} \leq H_2^{22*}(T_2^*) \tag{58}$$

Suppose in addition that the state variable follows a monotone trajectory, for instance a monotone non increasing one (which would be the case if K is an exhaustible resource). Then, we have to take into account an additional constraint: $K_1 \geq K_2^*$. If this constraint is binding (by definition, this holds in the corner situation $T_1 = T_2$), then the only possible variations of K_2^* are of the type $\delta K_2 \leq 0$. For $\delta V_2^{21} \leq 0$ (assuming $\delta T_2 = 0$), we must have

$$\lambda_2^{21*}(T_2^*) + \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial K_2} \leq \lambda_2^{22*}(T_2^*), \tag{59}$$

which gives another necessary condition for the corner case.

Finally, player 2 finds it optimal to stick to her first technology i.e. $T_2^* = \infty$, when:

$$H_2^{21*}(T_2^*) - \frac{\partial \Omega_2(K_2^*, T_2^*)}{\partial T_2} \geq H_2^{22*}(T_2^*) \text{ (for any } T_2^* > T_1) \tag{60}$$

Let us now have a look to player 1's problem. We also consider feedback strategies for the switching time. This implies that player 2 switching strategy is defined in terms of the level of state at which her switching problem starts that is, of K_1: $T_2 = T_1 + \theta_2(K_1)$. Player 1's does not take any switching decision in this regime but she makes the guess that player 2's switching time will be dependent on the level of the state variable K_1 at which player 1 switches. One has to incorporate the guess of player 1 about player 2 switching time in her value function. At the MPE, this guess will be consistent with the actual switching strategy of player 2. So, player1's value function in the same regime reads:

$$V_1^{21} = \int_{T_1}^{T_1+\theta_2(K_1)} \left[H_1^{21} + \dot{\lambda}_1^{21} K\right] dt - \left\{\lambda_1^{21}[T_1 + \theta_2(K_1)]K_2 - \lambda_1^{21}(T_1)K_1\right\} + V_1^{22*}[K_2, T_1 + \theta_2(K_1)]$$

Following the same steps as before, one can obtain the Pontryagin conditions for player 1:

$$\frac{\partial H_1^{21}}{\partial C_1} = 0, \ \frac{\partial H_1^{21}}{\partial K} + \frac{\partial H_1^{21}}{\partial C_2} \Phi_2'(K) + \dot{\lambda}_1^{21} = 0. \tag{61}$$

In addition, we the partial derivatives of the value function with respect to T_1 and K_1 (making use of (53)) are:

$$\begin{aligned} \frac{\partial V_1^{21*}}{\partial K_1} &= \theta_2'(K_1)[H_1^{21}(T_2) - H_1^{22}(T_2)] + \lambda_1^{21}(T_1) \\ \frac{\partial V_1^{21*}}{\partial T_1} &= H_1^{21}(T_2) - H_1^{21}(T_1) - H_1^{22}(T_2), \end{aligned} \tag{62}$$

we call these conditions the modified envelope conditions.

Note finally that by construction of the second sub-game, its value function V_2^{21} depends on the fixed initial condition and T_1: $V_2^{21*}(T_1, K_1)$. Again one can write the following envelope properties:

$$\frac{\partial V_2^{21*}}{\partial T_1} = -H_2^{21*}(T_1),\ \frac{\partial V_2^{21*}}{\partial K_1} = \lambda_2^{21*}$$

• *First regime* $(1,1)$*, for* $t \in [0, T_1]$:

In the initial regime, for the timing considered so far, player 1 has now to choose whether she switches and when. The optimization program is:

$$\max_{C_1, T_1} \int_0^{T_1} F_1^{11}(K, C_1, \Phi_2(K))dt - \Omega_1(K_1, T_1) + V_1^{21*}(K_1, T_1)$$

subject to,

$$\dot{K} = f^{11}(K, C_1, \Phi_2(K)),$$

where K_0 is given and $K(T_1) = K_1$ and T_1 are free.

The value function can written as:

$$V_1^{11} = \int_0^{T_1} \left[H_1^{11} + \dot{\lambda}_1^{11} K\right] dt - \left[\lambda_1^{11}(T_1)K_1 - \lambda_1^{11}(0)K_0\right] - \Omega_1(K_1) + V_1^{21*}(K_1, T_1)$$

Computing the first-order variation of V_1^{11} with respect to the state and control variables' paths, for free T_1 and K_1, one obtains:

$$\begin{array}{l}\delta V_1^{11} = \int\limits_0^{T_1} [(\frac{\partial H_1^{11}}{\partial K} + \frac{\partial H_1^{11}}{\partial C_2}\Phi_2'(K) + \dot{\lambda}_1^{11})\delta K + \frac{\partial H_1^{11}}{\partial C_1}\delta C_1]dt \\ +(H_1^{11}(T_1) - \frac{\partial \Omega_1(K_1,T_1)}{\partial T_1} + \frac{\partial V_1^{21*}}{\partial T_1})\delta T_1 - (\lambda_1^{11}(T_1) + \frac{\partial \Omega_1(K_1,T_1)}{\partial K_1} - \frac{\partial V_1^{21*}}{\partial K_1})\delta K_1.\end{array} \tag{63}$$

A trajectory is optimal if any small departure from it decreases the value function, that is $\delta V_1^{11} \leq 0$ for any $\delta K(t)$, $t \in (0, T_1)$, for any $\delta C_1(t)$, $t \in [0, T_1]$, and for any δT_1 and δK_1. Hence, the necessary conditions for an interior maximizer, $0 < T_1 < T_2$ are:

$$\left\{\begin{array}{l}\frac{\partial H_1^{11}}{\partial C_1} = 0,\ \frac{\partial H_1^{11}}{\partial K} + \frac{\partial H_1^{11}}{\partial C_2}\Phi_2'(K) + \dot{\lambda}_1^{11} = 0, \\ H_1^{11}(T_1) - \frac{\partial \Omega_1(K_1,T_1)}{\partial T_1} + \frac{\partial V_1^{21*}}{\partial T_1} = 0,\ \lambda_1^{11}(T_1) + \frac{\partial \Omega_1(K_1,T_1)}{\partial K_1} - \frac{\partial V_1^{21*}}{\partial K_1} = 0.\end{array}\right. \tag{64}$$

The (last) two switching conditions can be rewritten, using (62), as:

$$\begin{array}{l}H_1^{11*}(T_1^*) - \frac{\partial \Omega_1(K_1^*,T_1^*)}{\partial T_1} = H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \\ \lambda_1^{11*}(T_1^*) + \frac{\partial \Omega_1(K_1^*,T_1^*)}{\partial K_1} = \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*),\end{array} \tag{65}$$

they correspond to conditions (2) of Theorem 1. Thus, conditions in (57) and (65) give the necessary conditions for the optimal timing to be $0 < T_1 < T_2 < \infty$.

Optimality conditions for corner solutions can easily be deduced from the analysis above:

Suppose $0 = T_1^*$ (and $\delta K_1 = 0$) then the only possible variations of T_1^* are such that $\delta T_1 \geq 0$. For $\delta V_1^{11} \leq 0$ it must be true that

$$H_1^{11*}(T_1^*) - \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} \leq H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \text{ (for any } T_1^* < T_2^*) \tag{66}$$

In the opposite situation, $T_1^* = T_2$, the variations T_1^* are non positive: $\delta T_1 \leq 0$. For $\delta V_1^{11} \leq 0$, we must have

$$H_1^{11*}(T_1^*) - \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial T_1} \geq H_1^{21^*}(T_1^*) - [H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] \text{ (for any } T_1^* > 0) \tag{67}$$

Consider again that the state variable follows a monotone no increasing trajectory. Then, we have an addition constraint: $K_0 \geq K_1^* \geq K_2$. If $K_1 = K_0$ (which corresponds to $0 = T_1$), then the only possible variations of K_1^* are $\delta K_1 \leq 0$. Thus $\delta V_1^{11} \leq 0$ (assuming $\delta T_1 = 0$) is ensured only if

$$\lambda_1^{11*}(T_1^*) + \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} \leq \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*) \tag{68}$$

which gives another necessary condition for the corner case.

In the opposite situation, $K_1^* = K_2$, the following condition

$$\lambda_1^{11*}(T_1^*) + \frac{\partial \Omega_1(K_1^*, T_1^*)}{\partial K_1} \geq \theta_2'(K_1^*)[H_1^{21*}(T_2^*) - H_1^{22*}(T_2^*)] + \lambda_1^{21*}(T_1^*) \tag{69}$$

must be satisfied.

Another eventuality is that both players find it optimal to switch at the same instant $T_1 = T_2 = T$, and for the same resource stock, $K_1 = K_2 = K$. This is of course a knife-edge situation, which is highly unlikely at least when one assumes a sufficient degree of heterogeneity between players. In that case, the set of necessary conditions reduce to:

$$\begin{array}{l} H_2^{11*}(T^*) - \frac{\partial \Omega_2(K*, T^*)}{\partial T_2} = H_2^{22*}(T^*), \\ \lambda_2^{11*}(T^*) + \frac{\partial \Omega_2(K^*, T^*)}{\partial K_2} = \lambda_2^{22*}(T^*), \\ H_1^{11*}(T^*) - \frac{\partial \Omega_1(K^*, T^*)}{\partial T_1} = H_1^{22^*}(T^*), \\ \lambda_1^{11*}(T^*) + \frac{\partial \Omega_1(K^*, T^*)}{\partial K_1} = \lambda_1^{22*}(T^*). \end{array} \tag{70}$$

Finally, there may exist double corner solutions: $T_1 = T_2 = 0$ and $T_1 = T_2 = \infty$. To be completed

About the corner solutions when the state variable follows a monotone (non increasing path):

Consider again that the state variable follows a monotone no increasing trajectory. Then, we have an addition constraint: $K_0 \geq K_1^* \geq K_2$. If $K_1^* = K_0$, then the only possible variations of K_1^* are $\delta K_1 \leq 0$. But $K_1^* = K_0 \Leftrightarrow 0 = T_1^*$.

So, We necessarily have at the same time $\delta T_1 \geq 0$. Thus, from (63), $\delta V_1^{11} \leq 0$ is ensured only if

$$(H_1^{11}(T_1) - \frac{\partial \Omega_1(K_1, T_1)}{\partial T_1} + \frac{\partial V_1^{21*}}{\partial T_1})\delta T_1 - (\lambda_1^{11}(T_1) + \frac{\partial \Omega_1(K_1, T_1)}{\partial K_1} - \frac{\partial V_1^{21*}}{\partial K_1})\delta K_1 \leq 0 \tag{71}$$

which gives the necessary condition for the corner case that replaces condition (66) (holding when K_1^* can move in any direction). It is also clear that conditions (66) and (68) are just sufficient conditions for (71) to be satisfied.

In the same vein, when $K_1^* = K_2 \Leftrightarrow T_1^* = T_2$, the only possible variations of both variables are $\delta K_1 \geq 0$ and $\delta T_1 \leq 0$. It turns out that a the necessary condition for this corner case to be the solution is still given by (71). In this case, (67) and (69) are just some sufficient conditions.

B Analysis of interior solutions

B.1 Solution of the last period problem

To verify that (12) is correct, define

$$g_2 = \gamma_2^2 C_2^{22},\ g_1 = \gamma_1^2 C_1^{22}$$

then, with the guess that $g_2 = \rho K$, we have

$$\rho v_1^{22}(K) = \max_{g_1} \left\{ \ln g_1 - \ln \gamma_2^2 - \frac{d}{dK} v_1^{22}(K) \left[g_1 + \rho K \right] \right\}$$

Guessing $v_1^{22}(K) = A_1 + B_1 \ln(K)$

$$\rho \left[A_1 + B_1 \ln(K) \right] = \max_{g_1} \left\{ \ln g_1 - \ln \gamma_2^2 - \frac{B_1}{K} \left[g_1 + \rho K \right] \right\}$$

So $\frac{1}{g_1} = \frac{B_1}{K}$ and

$$\rho A_1 + \rho B_1 \ln(K) = \ln K - \ln B_1 - \ln \gamma_2^2 - 1 - B_1 \rho$$

Thus

$$B_1 = \frac{1}{\rho} \text{ and hence } \rho A_1 = \ln \rho - 2 - \ln \gamma_2^2$$

So

$$v_1^{22}(K) = \frac{1}{\rho} \left[\ln K + \ln \rho - 2 - \ln \gamma_2^2 \right]$$

and

$$\gamma_1^2 C_1^{22} = g_1 = \rho K$$

B.2 Proof of Proposition 1

Assume player 1 has switched at some $T_1^* \geq 0$ for some $K_1^* \leq K_0$. Taking these elements as given, player 2's value function in regime $(2,1)$ can be rewritten as:

$$V_2^{21}(.) = \int_{T_1}^{T_2} \left[H_2^{21}(.) + \dot{\lambda}_2^{21} K \right] dt - \left[\lambda_2^{21}(T_2)K_2 - \lambda_2^{21}(T_1)K_1 \right] - e^{-\rho T_2} S_2(K_2) + V_2^{22}(K_2, T_2) \tag{72}$$

where $H_2^{21}(.)$ is the Hamiltonian in present value and λ_2 is the co-state variable, also in present value value.

• **Interior solution** $T_1^* < T_2 < \infty$ **and** $0 \leq K_2 < K_1^*$:
Player 2 chooses a switching time T_2 and a stock switching level K_2 to maximize (72). The FOC wrt K_2 is

$$-\lambda_2^{21}(T_2^-) - e^{-\rho T_2} S_2'(K_2) + e^{-\rho T_2} \frac{1}{\rho K_2} = 0 \tag{73}$$

Given that the FOC wrt C_2^{21} yields $\frac{e^{-\rho t}}{C_2^{21}} = \gamma_2^1 \lambda_2$, this is equivalent to

$$C_2^{21}(T_2^-) = \frac{\rho K_2}{\gamma_2^1(1 - \beta_2 \rho K_2)} \tag{74}$$

The second switching condition is obtained by taking the derivative of the value with respect to T_2 is (assuming an interior solution, i.e. $T_2 \in (T_1, \infty)$),

$$e^{-\rho T_2} \ln(C_2^{21}(T_2^-)) - \lambda_2^{21}(T_2^-) \left[\gamma_2^1 C_2^{21}(T_2^-) + \gamma_1^2 C_1^{21}(T_2^-) \right] + \rho e^{-\rho T_2} S_2(K_2) - \rho V_2^{22}(K_2, T_2) = 0 \tag{75}$$

Let us determine the consumption strategies. Hereafter, we assume that the feedback strategy, $C_i^{21}(K)$, is linear in K. Define

$$h_1 = \gamma_1^2 C_1 \text{ and } h_2 = \gamma_2^1 C_2$$

Then

$$\dot{K} = -h_1 - h_2$$

Player 1 takes K_2 and T_2 as given. Suppose player 1 also guesses that $h_2 = \mu_2 + \delta_2 K$. Then player 1's maximizes

$$\int_{T_1}^{T_2} e^{-\rho t} \left[\ln h_1 - \ln \gamma_1^2 \right] dt + V_1^{22}(K_2, T_2)$$

s.t.

$$\dot{K} = -h_1 - \mu_2 - \delta_2 K$$

His FOC are

$$\frac{\dot{h}_1}{h_1} = -\rho - \delta_2$$

Similarly, suppose player 2 guesses that $h_1 = \mu_1 + \delta_1 K$. Then his FOC gives

$$\frac{\dot{h}_2}{h_2} = -\rho - \delta_1$$

On the other hand, from the guesses that $h_i = \mu_i + \delta_i K$, we have

$$\frac{\dot{h}_1}{h_1} = \frac{\delta_1 \dot{K}}{\mu_1 + \delta_1 K} = \frac{-2\delta_1 \left[\frac{\mu_1+\mu_2}{2} + \frac{\delta_1+\delta_2}{2} K\right]}{\mu_1 + \delta_1 K}$$

$$\frac{\dot{h}_2}{h_2} = \frac{\delta_2 \dot{K}}{\mu_2 + \delta_2 K} = \frac{-2\delta_2 \left[\frac{\mu_1+\mu_2}{2} + \frac{\delta_1+\delta_2}{2} K\right]}{\mu_2 + \delta_2 K}$$

Therefore, for the FOCs and the guesses to be consistent, we require

$$\frac{-2\delta_1 \left[\frac{\mu_1+\mu_2}{2} + \frac{\delta_1+\delta_2}{2} K\right]}{\mu_1 + \delta_1 K} = -\rho - \delta_2$$

$$\frac{-2\delta_2 \left[\frac{\mu_1+\mu_2}{2} + \frac{\delta_1+\delta_2}{2} K\right]}{\mu_2 + \delta_2 K} = -\rho - \delta_1$$

These two equations must hold for all K, hence they imply that $\delta_1 = \delta_2 = \rho$ and $\mu_1 = \mu_2 = \mu$

To determine μ_2, use eq (74) and the fact that $\delta_2 = \rho$ to obtain

$$\mu + \rho K_2 = \frac{\rho K_2}{1 - \beta_2 \rho K_2}$$

Thus

$$\mu = \frac{\rho^2 \beta_2 (K_2)^2}{1 - \beta_2 \rho K_2}$$

and

$$h_i(K) = \frac{\rho^2 \beta_2 (K_2)^2}{1 - \beta_2 \rho K_2} + \rho K \tag{76}$$

In particular, at $K = K_2$

$$h_i(K_2) = \frac{\rho K_2}{1 - \beta_2 \rho K_2}$$

Next, we determine what is the optimal level for switching. Using (76), the switching condition (75) simplifies to

$$\ln\left(\frac{\rho K_2}{\gamma_2^1 (1 - \beta_2 \rho K_2)}\right) = 2 + \rho(v_2^{22}(K_2) - S_2(K_2)) \tag{77}$$

This equation defines the optimal level for switching, K_2^*. After some manipulations, (77) reduces to

$$-\ln(1 - \beta_2 \rho K_2) = \ln\left[\frac{\gamma_2^1}{\gamma_2^2}\right] - \rho\left[\chi_2 + \beta_2 K_2\right] \tag{78}$$

The left-hand side is defined for all $K_2 \in \left[0, \frac{1}{\rho\beta_2}\right)$. Assume that $K_1^* \geq \frac{1}{\rho\beta_2}$. Then, the LHS is an increasing function of K_2, varying from zero to ∞ as K goes from zero to $1/\rho\beta_2$. The right-hand side is strictly positive at $K_2 = 0$ iff $\ln\left[\frac{\gamma_2^1}{\gamma_2^2}\right] > \rho\chi_2$. Since $\beta_2 > 0$, right-hand side is strictly decreasing in K_2. Thus, if $\ln\left[\frac{\gamma_2^1}{\gamma_2^2}\right] > \rho\chi_2$, there exists a unique solution K_2^* in $\left[0, \frac{1}{\rho\beta_2}\right)$.

The last part of the proof consists in defining the optimal switching date. Replacing consumptions with the expressions given by (76) in the state equation, one obtains:

$$\dot{K} = -2\rho K - 2\Gamma$$

with $\Gamma = \gamma_1^2 a_1 = \gamma_2^1 a_2 = \mu$. The solution of this differential equation is

$$K^{21}(t) = \left[K_1 + \frac{\rho\beta_2 (K_2)^2}{1-\beta_2\rho K_2}\right] e^{-2\rho(t-T_1)} - \frac{\rho\beta_2 (K_2)^2}{1-\beta_2\rho K_2} \tag{79}$$

Evaluating (79) in T_2 and defining θ_2 as the optimal length between two switches: $\theta_2 = T_2 - T_1$, one has

$$K_2^* = \left[K_1 + \frac{\rho\beta_2 (K_2^*)^2}{1-\beta_2\rho K_2^*}\right] e^{-2\rho\theta_2} - \frac{\rho\beta_2 (K_2^*)^2}{1-\beta_2\rho K_2^*},$$

Then

$$\left[\frac{K_1}{K_{2^*}} + \frac{\rho\beta_2 K_2^*}{1-\beta_2\rho K_2^*}\right] e^{-2\rho\theta_2} = 1 + \frac{\rho\beta_2 K_2^*}{1-\beta_2\rho K_2^*} = \frac{1}{1-\beta_2\rho K_2^*}$$

$$(1-\beta_2\rho K_2^*)\,(K_1/K_2^*) + \rho\beta_2 K_2^* = e^{2\rho\theta_2} = \left(e^{\rho\theta_2}\right)^2$$

$$\sqrt{(1-\beta_2\rho K_2^*)\,(K_1/K_2^*) + \rho\beta_2 K_2^*} = e^{\rho\theta_2} \tag{80}$$

which gives the solution

$$\theta_2(K_1) = \frac{1}{2\rho}\ln\left[(1-\rho\beta_2 K_2^*)\frac{K_1}{K_2^*} + \rho\beta_2 K_2^*\right] = \frac{1}{2\rho}\ln\left[\frac{C_i^{21}(K_1)}{C_i^{21}(K_2^*)}\right]. \tag{81}$$

where the second equality comes from $C_i^{21}(T_1)e^{2\rho T_1} = C_i^{21}(T_2)e^{2\rho T_2}$.

B.3 Proof of Proposition 2

Using the fact that (for the case of $T_1 > 0$),

$$\begin{gathered}\gamma_1^1 C_1^{11}(T_1^-) = \tfrac{\Gamma+\rho K_1}{F(K_1)} \\ \gamma_1^2 C_1^{21}(T_1^+) = \Gamma + \rho K_1\end{gathered}$$

the optimality condition (25) can be rewritten as:

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = e^{-\rho\theta_2(K_1)}\left[\rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)\right] + \ln[F(K_1)] \tag{82}$$

with $F(K_1) = Z(K_1) - \beta_1(\Gamma + \rho K_1)$ defined in the main text. We're trying to solve this equation in K_1 for $K_1 \in [K_2^*, K_0]$.

Denote the LHS (respectively the RHS) of (82) by $G(K_1)$ (respectively $H(K_1)$).

Note that $G(K_1)$ is an increasing function of K_1 on the interval $[K_2^*, K_0]$ whereas $H(K_1)$ is non monotone.

It follows that if $G(K_2^*) < H(K_2^*)$ and K_0 is sufficiently large so that $G(K_0) > H(K_0)$, then there exists a unique $K_1^* \in (K_2^*, K_0)$ that satisfies (82).

Evaluating $G(.)$ at the **lower** bound K_2^* yields $G(K_2^*) = \rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right)$. By evaluating $H(.)$ at K_2^*, one obtains

$$H(K_2^*) = \rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) + \ln\left[1 - \frac{\ln(1-\beta_2\rho K_2^*)}{2} - \frac{\beta_1\rho K_2^*}{1-\beta_2\rho K_2^*}\right].$$

which is well-defined under (24).

So, a set of sufficient conditions for $G(K_2^*) < H(K_2^*)$ are (i): $\rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) < \rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right)$ and (ii): $-\frac{\ln(1-\beta_2\rho K_2^*)}{2} - \frac{\beta_1\rho K_2^*}{1-\beta_2\rho K_2^*} \geq 0$. The second inequality is satisfied iff β_1 is sufficiently small such that

$$\beta_1 < \frac{1}{2\rho K_2^*}\left[(1-\beta_2\rho K_2^*)\left(-\ln\left(1-\beta_2\rho K_2^*\right)\right)\right]$$

Next, we can solve for the MPE in consumption strategies, holding in the first period problem. Noticing that player 2 has no means to influence player 1's switching decision due to the particular timing we have considered ($0 < T_1 < T_2$), her problem is simply given by

$$\max_{\{C_2\}} V_2^{11}(.) = \int_0^{T_1} e^{-\rho t}\ln(C_2)dt + e^{-\rho T_2}v_2^{21}(K_1)$$

subject to,

$$\dot{K} = -\gamma_2^1 C_2 - \gamma_1^1(a_1 + b_1 K)$$

with $K(0) = K_0$ and K_1 given. Combining again the two players FOCs, it's easy to find that $\gamma_1^1 b_1 = \gamma_2^1 b_2 = \rho$ and $\gamma_1^1 a_1 = \gamma_2^1 a_2 = \Lambda$. In order to determine the parameter Λ, we evaluate the MPE strategy in K_1 and equalize to the resulting value for consumption to the one defined in condition (22). This yields,

$$\Lambda = \frac{\Gamma + \rho K_1^*(1 - F(K_1^*)}{F(K_1^*)}.$$

Finally, we have to find the almost explicit value of the switching time T_1. For that purpose, first note that at the MPE of the first regime, the resource stock is given by: $K^{11}(t) = \left(K_0 + \frac{\Lambda}{\rho}\right)e^{-2\rho t} - \frac{\Lambda}{\rho}$. Evaluating this expression in T_1 and equalizing with the optimal value K_1^*, one has:

$$T_1 = \frac{1}{2\rho}\ln\left[\frac{K_0 + \frac{\Lambda}{\rho}}{K_1^* + \frac{\Lambda}{\rho}}\right].$$

C Corner solutions

C.1 Proof of proposition 3

• **Corner solution** $0 < T_1^{**} = T_2 < \infty$ **and** $0 \leq K_2 = K_1^{**}$:

In this situation, assuming $0 < T_1 < \infty$ but considering $T_2 \to T_1$, the optimality conditions of player 1 are:

$$\begin{aligned}
&H_1^{11}(T_1) + \rho e^{-\rho T_1} S_1(K_1) = H_1^{22}(T_1) \\
&\lambda_1^{11}(T_1) + e^{-\rho T_1} S_1'(K_1) = \lambda_1^{21}(T_1) - \theta_2'(K_1)[H_1^{22}(T_1) - H_1^{21}(T_1)]
\end{aligned}$$

In our application, these conditions simplify to:

$$\gamma_1^1 C_1^{11}(T_1) = \frac{\rho K_1}{\gamma_1^1(1-\beta_2\rho K_1)} \frac{1}{\frac{1-\rho(\beta_1+\beta_2)K_1}{1-\rho\beta_2 K_1} - \frac{\ln(1-\rho\beta_2 K_1)}{2}}, \tag{83}$$

which gives the consumption level at the switching date here. Denote the denominator of the second term by $f(K_1)$. And,

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) = \ln(1-\rho\beta_2 K_1) + \ln\left[f(K_1)\right], \tag{84}$$

which defines an optimal switching level K_1^{**} as long as $f(K_1^{**}) > 0$.

From the general proof of theorem, two sufficient conditions for this case to occur are:

$$H_2^{21}(T_2) - \frac{\partial \Omega_2(K_2, T_2)}{\partial T_2} \leq H_2^{22}(T_2) \tag{85}$$

and,

$$\lambda_2^{21}(T_2) + \frac{\partial \Omega_2(K_2, T_2)}{\partial K_2} \leq \lambda_2^{22}(T_2) \tag{86}$$

for all $T_2 \geq T_1^{**}$ and $K_2 \leq K_1^{**}$.

For our example, these equations reduces to:

$$C_2^{21}(T_2^-) \geq \frac{\rho K_2}{\gamma_2^1(1-\beta_2\rho K_2)} \tag{87}$$

$$\ln(C_2^{21}(T_2)) \leq \ln(C_2^{22}(T_2)) - \rho S_2(K_2) \tag{88}$$

In the following analysis, we distinguish between two sub-cases. Let us first consider the opportunity to switch at some $T_2 > T_1$ and suppose that (87) holds with the equality. Then, the remaining condition (88) reduces to:

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_2) \leq \ln(1-\beta_2\rho K_2) \tag{89}$$

for all $K_2 > K_1$. Second, in the particular point in time $T_1^{**} = T_2$ (and $K_1^{**} = K_2$), we cannot use (87) with the equality because it contradicts the fact that in any regime lk, one has $\gamma_1^l C_1^{lk}(t) = \gamma_2^l C_2^{lk}(t)$. But, we can precisely use the latter relationship and the fact that regime $(2,1)$ actually vanishes into regime 11 in this situation, to conclude that: $\gamma_2^l C_2^{21}(T_1^-) = \gamma_1^l C_1^{11}(T_1^-)$ given by (83). Thus, conditions (87)-(88) can be rewritten as:

$$(0<) f(K_1^{**}) < 1 \tag{90}$$

and

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_1^{**}) \leq \ln(1-\beta_2\rho K_1^{**}) + \ln[f(K_1^{**})] \tag{91}$$

where K_1^{**} is the solution of (84).

Now, assume that $2\beta_1 > \beta_2$, then $f'(K_1^{**}) < 0$ for all $K_1^{**} \in [0, K_0]$. In addition, $f(0) = 1$. Thus, condition (90) is satisfied. The material is sufficient to show that condition (91), valid for $T_1^{**} = T_2$, implies condition (104), valid for $T_1^{**} < T_2$. Indeed, first note that because $S_2'(K_2) > 0$ we have

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_1^{**}) \geq \ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_2) \text{ for all } K_2 \leq K_1^{**}$$

Second, because $f(.)$ is decreasing, one has:

$$\ln(1-\beta_2\rho K_1^{**}) + \ln[f(K_1^{**})] \leq \ln(1-\beta_2\rho K_2) + \ln[f(K_2)] \text{ for all } K_2 \leq K_1^{**}$$

Third, use the fact that because $f(K_2) \in [0,1]$ for all $K_2 \leq K_1^{**}$ to obtain

$$\ln(1-\beta_2\rho K_2) + \ln[f(K_2)] \leq \ln(1-\beta_2\rho K_2) \text{ for all } K_2 \leq K_1^{**}$$

which completes the proof: if (84) has a solution then a sufficient condition to be at the corner $0 < T_1^{**} = T_2 < \infty$ is

$$\ln\left[\frac{\gamma_2^1}{\gamma_2^2}\right] + \rho S_2(K_1^{**}) \leq \ln\left[\frac{\gamma_1^1}{\gamma_1^2}\right] + \rho S_1(K_1^{**}) \tag{92}$$

- **Corner solution** $0 < T_1^* = T_2^{**} < \infty$ **and** $K_1 = K_2^*$:

We work by symmetry in this corner situation. The conditions now involves the following inequalities:

$$H_1^{11}(T_1) - \frac{\partial\Omega_1(K_1,T_1)}{\partial T_1} \geq H_1^{21}(T_1) - [H_1^{21}(T_2^*) - H_1^{22}(T_2^*)]$$

$$\lambda_1^{11}(T_1) + \frac{\partial\Omega_1(K_1,T_1)}{\partial K_1} \geq \theta_2'(K_1)[H_1^{21}(T_2^*) - H_1^{22}(T_2^*)] + \lambda_1^{21}(T_1)$$

for any $T_1 > 0$ and $K_1 \leq K_0$. Or, for our example,

$$\ln[C_1^{11}(T_1)] + \rho S_1(K_1) \geq \ln[C_1^{21}(T_1)] + e^{-\rho\theta_2(K_1)}\ln(1-\beta_2\rho K_2^*) \tag{93}$$

$$\gamma_1^1 C_1^{11}(T_1^-) \leq \frac{\Gamma + \rho K_1}{F(K_1)} \tag{94}$$

In the case where $T_1 < T_2^{**}$, (110) and (111), holding with an equality, are sufficient to be in the corner. They reduce to one inequality which is exactly the opposite of (107)

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq e^{-\rho\theta_2(K_1)} \ln(1-\beta_2\rho K_2^*) + \ln[F(K_1)] \tag{95}$$

In the case where $T_1 = T_2^{**} \leftrightarrow K_1 = K_2^*$ (definition of T_2^{**}), we can use the fact that $\gamma_1^1 C_1^{11}(T_2^{**}) = \gamma_2^1 C_2^{21}(T_2^{**}) = \Gamma + \rho K_2^*$. Then, conditions (110)-(111) are equivalent to:

$$(0 <) F(K_2^*) < 1 \tag{96}$$

$$\rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \ln(1-\beta_2\rho K_2^*) \tag{97}$$

Following the same approach as in the two previous proofs, it easy to show that (97) and a condition a bit stronger than (96) are sufficient to be at the corner $T_1 = T_2^{**}$. Let $H(K_1)$ be the RHS of (97). A sufficient condition for having $H'(K_1) < 0$ for all $K_1 \geq K_2^*$ is $F(K_1) < \frac{1}{2}$, which is satisfied if $F(K_2^*) \leq \frac{1}{2}$. Replace condition (96) with the latter inequality. Then, we have $H(K_1) \leq H(K_2^*)$ for all $K_1 \geq K_2^*$, which is equivalent to

$$\ln(1-\beta_2\rho K_2^*) > \ln(1-\beta_2\rho K_2^*) + \ln[F(K_2^*)] \geq e^{-\rho\theta_2(K_1)} \ln(1-\beta_2\rho K_2^*) + \ln[F(K_1)]$$

In addition, we know that

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \text{ for all } K_1 \geq K_2^*$$

thus, $F(K_2^*) \leq \frac{1}{2}$ and (97) imply (95). Finally, recall that K_2^* solves (104). Thus, (97) can be rewritten as:

$$\rho S_1(K_2^*) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho S_2(K_2^*) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) \tag{98}$$

C.2 Proof of proposition 5

- **Corner solution** $0 < T_1^{**} < T_2 = \infty$:

If player 2 adopts a never switching strategy, then it must hold that:

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_2) \geq \ln(1-\beta_2\rho K_2) \tag{99}$$

for all $(0 \leq) K_2 \leq K_1$, where we have made use of

$$C_2^{21}(T_2^-) \geq \frac{\rho K_2}{\gamma_2^1(1-\beta_2\rho K_2)} \tag{100}$$

Assuming that player 1's switch occurs in finite time, a sufficient condition for (99) is:

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(0) \geq 0 \tag{101}$$

Remark. When $T_2 = \infty$, player 1's problem reduces to a single agent's switching problem and it is pretty simple to determine the switching level (assuming there is an interior solution). It solves:

$$\ln\left[\frac{\gamma_1^2}{\gamma_1^1}\right] + \rho S_1(K_1) = \ln(1 - \beta_1 \rho K_1) \tag{102}$$

Now, if one assumes that (101) does not hold, then a sufficient condition for this corner case not to be optimal (for player 2) is:

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_1) \leq 0 \tag{103}$$

with K_1 solution of (102).

- **Corner solution** $0 = T_1 < T_2^* < \infty$ **and** $K_0 = K_1$:

Suppose that we have an interior solution for player 2. Then, the following must be satisfied: under the conditions of proposition 2, there exists a unique K_2^* that solves:

$$\ln\left[\frac{\gamma_2^2}{\gamma_2^1}\right] + \rho S_2(K_2) = \ln(1 - \beta_2 \rho K_2) \tag{104}$$

Remark. In this case the switching level of player 2, K_2^*, is still given by (104). The general expression of the switching time is obtained in (81) but of course this switching time is different from the one found at the interior solution (for $T_1^* > 0$).

If player 1 finds it optimal to switch instantaneously then:

$$H_1^{11}(T_1) - \frac{\partial \Omega_1(K_1, T_1)}{\partial T_1} \leq H_1^{21}(T_1) - [H_1^{21}(T_2^*) - H_1^{22}(T_2^*)]$$

$$\lambda_1^{11}(T_1) + \frac{\partial \Omega_1(K_1, T_1)}{\partial K_1} \leq \theta_2'(K_1)[H_1^{21}(T_2^*) - H_1^{22}(T_2^*)] + \lambda_1^{21}(T_1)$$

for any $T_1 < T_2^*$ and $K_1 \geq K_2^*$.

In our application, these conditions simplify to:

$$\gamma_1^1 C_1^{11}(T_1^-) \geq \frac{\Gamma + \rho K_1}{F(K_1)} \tag{105}$$

with $F(K_1) = Z(K_1) - \beta_1 (\Gamma + \rho K_1)$. And,

$$\ln[C_1^{11}(T_1)] + \rho S_1(K_1) \leq \ln[C_1^{21}(T_1)] + e^{-\rho\theta_2(K_1)} \ln(1 - \beta_2 \rho K_2^*) \tag{106}$$

Again, a distinction is made between two sub-cases. First consider that, for any $T_1 > 0$, condition (105) holds with equality and use the resulting expression for $C_1^{11}(T_1)$ and the fact that $\gamma_1^2 C_1^{21}(T_1) = \Gamma + \rho K_1$ to obtain a single condition:

$$\rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq e^{-\rho\theta_2(K_1)} \ln(1 - \beta_2 \rho K_2^*) + \ln[F(K_1)] \tag{107}$$

that must hold for any $0 < T_1 < T_2$ and $K_1 > K_2^*$.

A the particular date $0 = T_1$ (implying that $K_1 = K_0$), it must be that (105) holds with inequality and we obtain the level of the first period consumption by using the fact that (regime 11 vanishes in regime 21 and) $\gamma_1^1 C_1^{11}(0) = \gamma_2^1 C_2 21(0) = \Gamma + \rho K_0$. Using this expression, conditions (105) and (106) reduce to:

$$F(K_0) \geq 1 \tag{108}$$

and,

$$\rho S_1(K_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq \ln(1 - \beta_2 \rho K_2^*) \tag{109}$$

In order to show that (108)-(109) are sufficient to be in the corner case considered that is, imply (107), first note that under (108) and $F'(K_1) < 0$, one has $F(K_1) > 1$ for all $K_1 \leq K_0$. Then, we have

$$\ln(1-\beta_2\rho K_2^*) \leq e^{-\rho\theta_2(K_1)} \ln(1-\beta_2\rho K_2^*) \leq e^{-\rho\theta_2(K_1)} \ln(1-\beta_2\rho K_2^*) + \ln[F(K_1)],$$

for any $K_1 \in [K_2^*, K_0]$.

Moreover, we use the feature that:

$$\rho S_1(K_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \geq \rho S_1(K_1) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \text{ for any } K_1 \leq K_0$$

to reach the conclusion that (108)-(109) are sufficient to be in the corner $T_1 = 0$.

- **Immediate and never switching:** $0 = T_1 < T_2 = \infty$.

The analysis of the case follows quite easily from the one of the corner $0 < T_1^{**} < T_2 = \infty$ and $0 = T_1^{**} < T_2 < \infty$. (99) gives a sufficient condition for player 2 to be at the corner $T_2 = \infty$ and we have already mentioned that in such a situation player 1 faces a standard two-stage control problem. Sufficient conditions for having that there is no point at which player 1 wishes to adopt her new strategy are:

$$H_1^{11}(T_1) - \frac{\partial \Omega_1(K_1, T_1)}{\partial T_1} \leq H_1^{21}(T_1)$$

$$\lambda_1^{11}(T_1) + \frac{\partial \Omega_1(K_1, T_1)}{\partial K_1} \leq \lambda_1^{21}(T_1)$$

if $T_1 = 0$ and $K_1 = K_0$. Note that in this case, player 1 simply compares the (marginal) value she would obtain under the permanent regime 11 with the corresponding value she would get by switching directly to 21. In particular, what would differ is the initial consumption since if the same regime is valid for all t then $\gamma_1^l C_1(0) = \rho K_0$ for $l = 1, 2$ i.e. the rule that defines the extraction rate would be the same.

Therefore, in our application, the conditions above reduce to:

$$\rho K_0 \geq \frac{\gamma_1^1 - \gamma_1^2}{\beta_1 \gamma_1^1 \gamma_1^2} \tag{110}$$

and,

$$\rho S_1(K_0) + \ln\left(\frac{\gamma_1^2}{\gamma_1^1}\right) \leq 0 \tag{111}$$

C.3 Proof of proposition 6

• **Simultaneous interior switches:** $0 < T_1 = T_2 = T < \infty$.

Suppose that the two players want to adopt their new technology at the same date and for the same stock of resource. The optimality conditions corresponding to this case are, for an interior solution:

$$\begin{array}{l} H_2^{11*}(T^*) - \frac{\partial \Omega_2(K^*,T^*)}{\partial T_2} = H_2^{22*}(T^*) \\ H_1^{11*}(T^*) - \frac{\partial \Omega_1(K^*,T^*)}{\partial T_1} = H_1^{22*}(T^*) \\ \lambda_2^{11*}(T^*) + \frac{\partial \Omega_2(K^*,T^*)}{\partial K_2} = \lambda_2^{22*}(T^*) \\ \lambda_1^{11*}(T^*) + \frac{\partial \Omega_1(K^*,T^*)}{\partial K_1} = \lambda_1^{22*}(T^*) \end{array}$$

with $K = K_1 = K_2$ the level of the state variable at the switching date $T = T_1 = T_2$. Using the feature that $\lambda_1^{lk} = \lambda_2^{lk}$ in any regime lk, it is clear that the two last switching conditions cannot be satisfied at the same time as long as players bear different direct switching costs. More precisely, if $S_1'(K) \neq S_2'(K)$ for all K,$\Leftrightarrow$ $\beta_1 \neq \beta_2$, then this case cannot arise.

• **Simultaneous instantaneous switches:** $T_1 = T_2 = 0$.

It is straightforward that sufficient conditions for this case correspond to (110) and (111), that must be satisfied now for the two players.

• **Never switching for both players:** $T_1 = T_2 = \infty$.

The last situation can also be deduced from the corner case $T_1 < T_2 = \infty$. The sufficient condition is (99). It must hold for the two players.

Documents de Recherche parus en 2013

DR n°2013 - 01: Estelle MIDLER, Charles FIGUIÈRES, Marc WILLINGER
«Choice overload, coordination and inequality: threehurdles to the effectiveness of the compensationmechanism?»

DR n°2013 - 02: Charles FIGUIÈRES, Ngo Van LONGY, Mabel TIDBALL
«The MBR Intertemporal Choice Criterion and Rawls' Just Savings Principle»

DR n°2013 - 03: Agathe ROUAIX , Charles FIGUIÈRES, Marc WILLINGER
« The trade-off between welfare and equality in a public good experiment» Do the poor need Robin Hood or the Sheriff of Nottingham? »

DR n°2013 - 04: Mohamed ALI BCHIR, Marc WILLINGER
«Does the exposure to natural hazards affect risk and time preferences? Some insights from a field experiment in Perú. »

DR n°2013 - 05: Magali JAOUL-GRAMMARE, Brice MAGDALOU
«Opportunities in Higher Education : An Application in France »

DR n°2013 - 06: Ngo VAN LONG, Fabien PRIEUR, Klarizze PUZON, Mabel TIDBALL
«Markov Perfect Equilibria in Differential Games with Regime Switching»

Contact :

Stéphane MUSSARD : mussard@lameta.univ-montp1.fr