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Abstract: *In the real investments literature, the investigated cash flow is assumed to follow some known stochastic process (e.g. Brownian motion) and the criterion to decide between investments is the discounted utility of their cash flows. However, for most new investments the investor may be ambiguous about the representation of uncertainty. In order to take such ambiguity into account, we refer to a discounted Choquet expected utility in our model. In such a setting some problems are to dealt with: dynamical consistency, here it is obtained in a recursive model by a weakened version of the axiom. Mimicking the Brownian motion as the limit of a random walk for the investment payoff process, we describe the latter as a binomial tree with capacities instead of exact probabilities on its branches and show what are its properties at the limit. We show that most results in the real investments literature are tractable in this enlarged setting but leave more room to ambiguity as both the mean and the variance of the underlying stochastic process are modified in our ambiguous model.*

1. Introduction

New investments are often decided in situations of uncertainty about the future states of the economy and future information arrivals. The real investments (or real options) literature (Pindyck and Dixit (1994) or Trigeorgis (1996)) assumes that uncertainty is described by known stochastic processes (e.g. Brownian motions), as does most of the literature in finance (derivative asset pricing, optimal portfolio choice, CAPM, etc.). In order to enlarge the scope of applications to uncertainty situations described by controversial probability distributions, or ambiguity about them, it is necessary to refer to new results in decision theory such as the multi-prior model of Gilboa and Schmeidler (1989) or the Choquet expected utility of Schmeidler (1989) that yield non-linear criteria. Non additivity of the criteria, or of the measure that express it, is the way to express ambiguity of the decision maker about what is the relevant representation of uncertainty. For instance, a convex measure, e.g. a capacity v such that $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, for all events A and B, expresses aversion to ambiguity. However many problems with non-linear models are still open. Notably, there is not a unique way to update non-additive measures such as Choquet capacities (see e.g. Kast,

Lapied and Toquebeuf, 2009). Furthermore, most models collapse to a linear one when dynamic consistency is required (Sarin and Wakker, 1998). Dynamics is central to investment problems, and the consistency of decisions with future decisions based on information arrivals is at the core of the real options theory. Epstein and Schmeidler (2003) propose a recursive multi-prior model that opens the way to applications: First, the multi-priors model is very close to the linear ones, so that the usual mathematics are (relatively) easy to adapt; Second, a recursive model, although not really dynamical (it only looks one period ahead), is sufficient to address most issues. However, because of its similarity with classical models it may be the case that this recursive multi-prior model restricts the kind of ambiguity that one wants to address. In this paper, we propose another approach where a particular family of probability distributions is considered and is summarized by a non-additive measure: a Choquet capacity. Our approach is axiomatic and subjective (the measure derives from the decision maker's preferences), there is no reference to an objective probability distribution that would be subjectively distorted (although it could be an interpretation). Furthermore we consider a discrete time dynamic model that we make converge toward a continuous time model, instead of the reverse. We address both the optimal portfolio choice of traded assets and the investment choice in a new non-traded asset (real investment).

Valuing flexibilities and option values in consistence with information arrivals is dealt with linear valuation models derived from No Arbitrage conditions[1] on equilibrium prices. Results are extended from the derivative assets valuation models when there exists an underlying traded asset. For situations where there are no such underlying assets, the real options literature uses an expected utility model representing the decision maker's preferences. In both cases, the underlying probability distribution is unambiguously known and the criterion is a linear form of future random cash flows (a Legesgue integral). Instead, we consider preferences could be represented by a non additive criterion: a Choquet integral with respect to a capacity, the core of which is a set of possible probability distributions over future states. In this setting, ambiguity about the underlying payoff process is taken into account, both at the descriptive and at the normative levels of the model. We keep close to the simplest models in the real options literature (Binomial processes and convergence to Brownian motions) so

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[1] Most of this literature is based on no arbitrage conditions, however these do not necessarily imply that prices are defined by additive integrals if bid-ask spreads prevail : see Jouini and Kallal (1995) or Chateauneuf et al. (1996), and De Waegenaere et al. (2003) for an equilibrium model with bid-ask spreads where prices define a Choquet integral.

that we can concentrate on extending their well-known results to ambiguity and a Choquet valuation criterion.

Similarly, in optimal portfolio choice it may be the case that one wants to consider some ambiguity about the assets underlying probability distribution. For instance, one may want to add an ambiguity premium to the risk premium to answer the equity premium puzzle. Or, if one considers that foreign investments distributions are more ambiguous than domestic ones, the home-bias puzzle may be addressed by a Choquet capacity, the core of which contains all possible ambiguous foreign asset underlying distributions.

Such approaches have already been developed in finance and decision theory, they face a difficult problem: The models' dynamic consistency. It is well known in dynamic decision theory (Sarin and Wakker 1998) that dynamic consistency is an axiom that implies criteria to be linear most of the time. However, another axiom called consequentialism[2] interacts with dynamic consistency. So, a number of results have been obtained to show how dynamic consistency can still hold in a non-linear model, if one or both axioms are weakened. On the basis of the Gilboa and Schmeidler (1989) Min-Epected-Utility static model, Epstein and Schneider (2003) developed a dynamic (recursive) model in discrete time. It satisfies both dynamic consistency and consequentialism and doesn't collapse into a linear model under the (non light) condition that the set of priors is rectangular. Such a model has been extended to continuous time by Chen and Epstein (2002) on the basis of the Duffie and Epstein (1992) stochastic differential utility model. In the model, uncertainty is described by a vectorial Brownian process defined on a space (Ω, F, P). Ambiguity is introduced by a set of density generators θ, that define a probability Q^{θ} on (Ω, F) equivalent to P. The set Θ of density generators defines the set of priors: $P^{\Theta} = \{Q^{\theta}: \theta \in \Theta\}$ and under the condition that Θ is the product of its projections, P^{Θ} is rectangular.

This model has been applied to finance: Chen et Epstein (2002) study the portfolio choice and the problem of the continuous time CAPM; Asano (2005), Miao and Wang (2007), Trojanowska and Kort (2006), Nishimura and Ozaki (2007) consider the real options valuation and decision problem. In all these cases, ambiguity deforms the objective probability distribution by changing the drifts of the stochastic processes, it leaves however the standard deviation constant.

In this paper, we address the same type of applications, however our model is constructed the other way around. First uncertainty is measured by the decision maker's subjective

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[2] (Hammond, 1989). Consequentialism expresses that events that couldn't occur given future information should not be taken into account by the decision maker.

representation of preferences. Dynamics is described by a discrete time Brownian motion in which probability ½ is replaced by a constant c (the ambiguous weight that the decision maker is putting both on the event « up » and the event « down » instead of the unambiguous ½). At the limit, we obtain a deformed Brownian motion where both the drift and the volatility are changed, in contrast with the Chen and Epstein (2002) model.

In section 2, preferences on payoffs processes of an investor (or decision maker) are represented by a discounted Choquet expectation, under an axiomatic that avoids utilities on payoffs. Ambiguity is taken into account by the Core of the Choquet capacity. In order to take flexibilities (and options) into account, preferences must satisfy a weakened axiom of dynamic consistency (Karni and Schmeidler 1991, Nishimura and Osaki 2003) and of model consistency (Sarin and Wakker 1998) that rely future conditional valuations to the present criterion.

In section 3, we derive the joint capacity on the payoffs process in the case where uncertainty is described by a binomial tree. Then we investigate the limit behaviour of the joint capacity when the time intervals converge towards zero in order to obtain a "kind" of Brownian motion as the limit of the Choquet binomial random walk, in the following sections. We present some applications to real investments and portfolio choice in section 6. Section 7 relates our results to similar ones in the literature and concludes.

2. Valuing uncertain cash payoffs processes

In this part of the paper, we assume the DM oversees the future (i.e uncertain states at some future dates) as a finite set Ω. A project, or a new investment, is formalised by its future cash flow, i.e. payoffs contingent on states $\omega \in \Omega$, say $X(\omega)$ and the project is then represented by a measurable function $X: (\Omega, 2^{\Omega}) \rightarrow R$.

2.1 The basic model

The DM has preferences on a given project payoffs, hence she is able to compare measurable functions. Under our simplifying assumptions that Ω is a finite set, measurable functions are vectors in R^{Ω}. This set is endowed with a complete pre-order (of the DM's preferences) and, under some axioms that'll we present later on, this pre-order is represented by a real valued function $V: R^{\Omega} \rightarrow R$ such that $V(X)$ is the net present certainty equivalent of the uncertain payoffs X for the DM's preferences (i.e. in this case the decision criterion).

4

Given that, in the uncertain future, the project may be influenced by information arrivals, it is defined as a strategy contingent on information sets.

Information is formalised by a measurable function on a finite set I, $Y: (\Omega, 2^{\Omega}) \rightarrow (I, 2^{I})$ so that, for any $i \in I$, $[Y = i]$ defines a sub-algebra of 2^{Ω}, $\sigma([Y = i]) = \{A \cap [Y = i] / A \in 2^{\Omega}\}$.

When information $[Y = i]$ is obtained, the DM has preferences on uncertain cash payoffs that are measurable functions from $(\Omega, \sigma([Y = i]))$ into R that are, or may be, different from her ex-ante preferences. These preferences are represented by a function $V^{[Y = i]}: R^{\Omega} \rightarrow R$.

Taking flexibilities into account amounts to integrate the value of the options to modify the project into its present value, in accordance with information arrivals of the type $[Y = i]$, $i \in I$. Indeed, when information $[Y = i]$ obtains, the DM may modify her preferences over the project's payoffs and hence its valuation. For instance, her aversion to uncertainty may be reduced, or increased depending on the type of information ("good" or "bad" news). Another example may be that, the DM learning she has more wealth available, her marginal utility on wealth may decrease with respect to what it was before information arrived[3].

In order to model future dates and uncertain states we can set: $\Omega = S \times T$, where $S = \{s_1,\ldots,s_N\}$ represents the set of uncertain states to whom the payoffs are contingent and $T = \{0,\ldots,T\}$ is the set of dates at which the states may occur. A cash flow $X: \Omega \rightarrow R$ can be projected on both components and be considered whether as a random process or as a family of trajectories: $X = (X_t)_{t \in T} = (X_s)_{s \in S}$.

A DM is usually aware of her preferences over payoffs contingent on uncertain states, i.e. measurable functions from S to R, i.e. vectors in R^S here. For instance, these preferences may satisfy Diecidue and Wakker (2002)[4] axioms and be represented by a value function on the set of uncertain payoffs:

Axiom1: Preferences define a complete pre-order on the set of measurable functions.

Axiom 2: For any measurable function X, there exists a constant number (constant equivalent) for which the DM is indifferent to the function: $X \approx E(X)$ (where $\approx$ represents indifference).

Axiom 3: Preferences allow no comonotonic Dutch Books.

Let's recall:

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[3] This is excluded by the usual setting of Expected Utility theory where the utility of wealth is defined ex-ante. Here it is excluded because the subjective capacity captures the whole behaviour towards uncertainty.
[4] Or its extension in Kast and Lapied (2003) where uncertainty aversion is more detailed.

Definition 2.1.1: Two uncertain payoffs X and X' are comonotonic if they satisfy:

$$\forall \omega, \omega' \in \Omega, \ [X(\omega) - X(\omega')][X'(\omega) - X'(\omega')] \geq 0.$$

Definition 2.1.2: A comonotonic Dutch Book is a double sequence of comonotonic uncertain payoffs $(X_j, X'_j)_{j \in J}$ such that, for any j in J, X_j is weakly preferred to X'_j but

$$\forall \omega \in \Omega, \sum_{j \in J} X'_j(\omega) \geq \sum_{j \in J} X_j(\omega) \text{ and } \exists \omega \in \Omega, \sum_{j \in J} X'_j(\omega) > \sum_{j \in J} X_j(\omega).$$

Theorem 2.1.1 (Diecidue and Wakker, 2002):

For a preference relation on R^S satisfying axioms 1 and 2, for all X in R^S there exists a constant equivalent $E(X) \in R$ such that the following three statements are equivalent:

(i) $E(.): R^S \rightarrow R$ is strictly monotonic, additive on comonotonic vectors (but non necessarily additive on non comonotonic vectors).

(ii)   There exists a unique capacity such that $E(X)$ is the integral of X with respect to this measure.

(iii) $E(.)$ is such that axiom 3 is satisfied.

Otherwise stated: $\exists \nu$ a unique capacity on $(S, 2^S)$, $X_t \approx E(X_t) = \int_S X_t d\nu$. Notice that, in this representation theorem, the (subjective) capacity aggregates all of the DM's behaviour: there is no need for a utility on payoffs, and the certainty equivalent is merely an expected (subjective) value.

Similarly, a DM has preferences for present over future consumption (payoffs, here), i.e. preferences over payoffs contingent on dates. In this paper we assume that the DM's preferences over certain future payoffs satisfy Koopman's (1972) axioms and are represented by a linear form:

Axiom1': Preferences define a complete continuous pre-order on the set of measurable functions.

Axiom 2': Preferences are strictly monotonic.

Axiom 3': Preferences satisfy separability over dates.

Theorem 2.1.2 (Koopman 1972):

For a preference relation on R^T satisfying axioms 1' to 3' for all X in R^T there exists a present equivalent $D(X) \in R$ such that:

$$\exists \pi \text{ a unique bounded additive measure on } (T, 2^T), \ X_s \approx D(X_s) = \int_T X_s d\pi.$$

As usual in the real options pricing literature, we shall define the decision maker's criterion by the payoffs' discounted expectation (here a Choquet expectation):

X is preferred to X' iff $V(X) = DE(X) = \int_T [\int_S X_t dv] d\pi \geq \int_T [\int_S X'_t dv] d\pi = DE(X').$

Notice however that this criterion is somewhat arbitrary as it introduces a hierarchy into the treatment of time and of uncertainty contingency. Another criterion would be ED(X) if the DM considered payoffs on a each trajectories and then integrated the discounted results. Because the Choquet expectation is not additive, it is not the case that $ED(X) = DE(X)$ in general (for a general treatment of the double integral and a justification of one hierarchy over the other, see Kast and Lapied 2010). The hierarchy DE is the usual one and we choose it to keep our model close to the standard ones, this choice corresponds to a seventh axiom of the representation of preferences.

Hierarchy axiom: Preferences of the DM over payoffs contingent on $\Omega = S \times T$, are represented by $V = DE$. The subjective product measure $v \square \times \pi$ on $S \times T$ captures the DM's behaviour both on uncertainty and on time.

Taking into account future information arrivals in the present valuation will require some consistency between the DM's expected future behaviours and her present preferences, we shall express them by two more axioms.

2.2 Consistency with information arrivals

Dynamic consistency is supposed to answer the following question: Can we have X' preferred to X ex-ante if, for any state of information i in I, $V^{[Y=i]}(X) \geq V^{[Y=i]}(X')$?

The answer is yes, there are cases where we could. For instance, assume the set $[Y=i]$ excludes the set on which $X < X'$, so that on any set in $\sigma([Y=i])$, $X \geq X'$. Then, if preferences are monotonic we could have a contradiction between unconditional and conditional

7

valuations[5]. However we need not have one because all the information values in *I* are possible ex-ante (and not only those in [*Y*=*i*]) and the decision maker may still take into account payoffs for which $X < X'$ and then not prefer *X* to *X'*. But if, for all *i*'s, we had $X = X'$ on $[Y=i]^c$, then, indeed consistency with information arrivals would imply that:

$$\forall i \in I, V^{[Y=i]}(X) \geq V^{[Y=i]}(X') \Leftrightarrow V(X) \geq V(X').$$

This equivalence is the way Karni and Schmeidler (1991)[6], for instance, expressed Dynamic Consistency (they did it in terms of preferences instead of their representation by a value function as we did, and they limited information to the case where it could take one value only).

We'll require a similar but weaker condition, as expressed for example by Nishimura and Osaki (2003), so that the axiom of Dynamic Consistency will be stated as:

$$\forall X, X', [\forall i \in I, V^{[Y=i]}(X) \geq V^{[Y=i]}(X')] \Rightarrow V(X) \geq V(X').$$

Or, in terms of preferences:

Dynamic consistency axiom:

$\forall t = 1,\ldots, T-1, \forall X, X'$ such that: $X_\tau(s) = X'_\tau(s), \forall \tau = 0,\ldots, t, \forall s \in S,$

$[\forall i_t \in I_t, X \underset{\approx_{i_t}}{\succ} X'] \Rightarrow X \underset{\approx}{\succ} X'.$

In order to explicit the notations and to figure out the results, some requirements about the future DM's behaviour are usually imposed, namely that the future preferences (after information is released) satisfy the same axioms as the present ones (Sarin and Wakker 1998).

Model consistency axiom : Preferences conditional on information satisfy axioms 1, 2, 3 and 1', 2', 3', and the hierarchy axiom between time and uncertainty.

We can now state the following condition expressing consistency between conditional and unconditional expectations.

Proposition 2.2.1: Under the representation of preferences satisfying our nine axioms, for any $X \in R^{S \times T}, \forall \tau \in \{0,\ldots, T\}, \forall i \in I, \forall [Y_\tau=i] \subset F_\tau, \forall t, \tau \leq t \leq T, E_\nu(X_t) = E_\nu[E_\nu^{[Y_\tau=i]}(X_t)].$

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[5] For instance it would be the case if the decision maker's preferences satisfied consequentialism.
[6] But see also : Sarin and Wakker (1998), Machina (1998) and Ghirardato (2002).

Proof: Under the nine axioms, preferences are represented by value functions $V = DE$ and $V^{[Y=i]} = D^{[Y=i]}E^{[Y=i]}$ such that for any $X \in R^{S \times T}$ we can write:

$$V(X) = DE(X) = \sum_{t=0}^{T} \pi(t) E_\nu(X_t) \text{ and}$$

$$\forall \tau \in \{0,\ldots,T\}, \forall i \in I, \forall [Y_\tau = i] \subset F_\tau, V^{[Y_\tau=i]}(X) = \sum_{t=0}^{T} \pi^{[Y_\tau=i]}(t) E_\nu^{[Y_\tau=i]}(X_t)$$

with: $\forall t \in \{0,\ldots,\tau-1\}, \pi^{[Y_\tau=i]}(t) = 0, \pi^{[Y_\tau=i]}(\tau) = 1, \forall t \in \{\tau,\ldots,T\}, \pi(t) = \pi(\tau)\pi^{[Y_\tau=i]}(t).$

Let the certainty equivalent payoffs process be : $EC(X) = (E_\nu(X_0),\ldots, E_\nu(X_T))$ and

$EC^{[Y_\tau=i]}(X) = (X_0,\ldots,X_{\tau-1}, E_\nu^{[Y_\tau=i]}(X_\tau),\ldots,E_\nu^{[Y_\tau=i]}(X_T))$, we have by definition:

$V(X) = V(EC(X))$ and, $\forall i \in I, V^{[Y_\tau=i]}(X) = V^{[Y_\tau=i]}(EC^{[Y_\tau=i]}(X))$.

Under the dynamic consistency axiom the last equality implies:

$\forall i \in I, V(X) = V(EC^{[Y_\tau=i]}(X))$.

Thanks the definition of $\pi^{[Y_\tau=i]}$, this equality simplifies and yields[7]:

$\forall \tau \in \{0,\ldots, T\}, \forall i \in I, \forall [Y_\tau=i] \subset F_\tau, \forall t, \tau \le t \le T, E_\nu(X_t) = E_\nu[E_\nu^{[Y_\tau=i]}(X_t)].$ QED

Notice that this proposition says that dynamic consistency (together with all the other axioms) implies an implicit definition of conditional expectations (similar to its definition in the linear case): $E_\nu(X_t) = E_\nu[E_\nu^{[Y_\tau=i]}(X_t)]$. In the latter case, however, the definition implies that updating probabilities follows Bayes' rule, whereas several updating rules are compatible with such an implicitly defined conditional Choquet expectation (Kast, Lapied and Toquebeuf, 2009).

In this paper, we shall not rely on a particular updating rule, nor derive one, as if the decision maker had made up her mind, using for instance the Bayes' rule, or the Dempster-Shafer's one, or the Full Bayesian Updating rule, or any combination of these classical rules, or any more general one. As we shall see, our results do not rely on this choice.

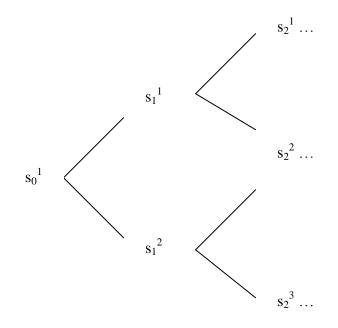## 3. Dynamically Consistent Choquet Random Walk (DCCRW)

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[7] Notice that this relies on preferences on time contingent payoffs are represented by a linear form. If D were another Choquet expectation (as in Kast and Lapied 2009) with the discount factor a non-additve measure instead of an additive one as in here, the result wouldn't hold.

Consider an investment with payoffs contingent on future states according to a binomial type tree (no probabilities for up and down movements are given).

3.1 Binomial tree

Time is defined by: $t = 0, 1,\ldots, T$.

Uncertainty is described by a binomial tree so that the uncertain states, $s_1, \ldots, s_n$ in S are trajectories, i.e. sequences of nodes in the tree: for $i = 1,\ldots, N$, $s_i = (s_0, s_1^{i_1}, \ldots, s_T^{i_T})$ with $i_t = 1,\ldots, t+1$:

A each t, the possible nodes are in $S_t = \{s_t^1, \ldots, s_t^{t+1}\}$. The information process is such that the DM knows, at time t, the state that is realised at this date. The set of parts of S_t is A_t.

The preferences of the DM over payoff processes such that: $X = (X_0, \ldots, X_T)$ are represented by a discounted (by a discount factor π) Choquet expectation with respect to a capacity v on $(S, 2^S)$, as in section 2, so that the certainty equivalent of the process is:

$$DE(X) = \sum_{t=0}^{T} \pi(t) E_v(X_t),$$

where: $E_\nu(X_t) = \sum\limits_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t)$, with the usual notation for a Choquet integral for which, if,

for instance, $X_t(s_1) \leq \ldots \leq X_t(s_N)$, $\Delta\nu(s_n) = \nu(\{s_n,\ldots, s_N\}) - \nu(\{s_{n+1}, \ldots, s_N\})$, with $\{s_{N+1}\} = \varnothing$, for notational convenience.

In order to characterise a Choquet Random Walk, we impose that, for any node s_t at date t $(0 \leq t < T)$, if s^u_{t+1} and s^d_{t+1} are the two possible successors of s_t at date $t+1$ (for, respectively, an "up" or a "down" movement in the binomial tree), the conditional capacity is a constant: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$, with $0 < c < 1$. The common value c expresses the DM's ambiguity about the likelihood of the states to come.

The conditional capacities are normalized in the following way:

$$\nu(\varnothing/s_t) = 0, \nu(\{s^u_{t+1}, s^d_{t+1}\}/s_t) = 1, \forall B \in A_{t+1}, \nu(B/s_t) = \nu(B \cap \{s^u_{t+1}, s^d_{t+1}\}/s_t).$$

From proposition 2.2.1, dynamic consistency implies:

$$\forall\ \tau = 1,\ldots,T-1,\ \forall\ t = \tau,\ldots,T,\ \sum_{s_\tau \in S_\tau}\ [\ \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t/s_\tau)]\Delta\nu(s_\tau) = \sum_{s_t \in S_t} X_t(s_t)\Delta\nu(s_t) \qquad (3.1)$$

3.2 Characterization of the subjective capacity

Now that a Choquet random walk is characterised, we show that preferences that satisfy the dynamic consistency axiom don't leave much choice for the subjective capacity that represents them.

Proposition 3.2.1: A Dynamically Consistent Choquet Random Walk satisfying relation (3.1) is completely defined by a unique capacity ν satisfying: $\nu(s^u_{t+1}/s_t) = \nu(s^d_{t+1}/s_t) = c$.

Proof: In the proof, we can concentrate wlog on the characteristic functions $X_{\tau+n}$ of the sets in $A_{\tau+n}$, because any random variable has a unique decomposition into a non-negative linear combination of the characteristic functions in a cone containing $X_{\tau+n}$ and Choquet expectation is linear on this cone (Kast and Lapied, 1997).

Three cases are to be considered.

(i) If $t = \tau$, relation (3.1) is trivially satisfied.

(iii) If $t = \tau+1$, relation (3.1) becomes:

$$\forall\, t = 1,...,T{-}1, \quad \sum_{s_t \in S_t} [\,\sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1})\Delta v(s_{t+1}/s_t)]\Delta v(s_t) = \sum_{s_{t+1} \in S_{t+1}} X_{t+1}(s_{t+1})\Delta v(s_{t+1}) \quad (3.2)$$

For any $t = 1,...,T{-}1$, and any $B \in A_{t+1}$, the conditional capacity $v(B/s_t)$ can only take three different values, because, if s^u_{t+1} and s^d_{t+1} are the two possible successors of s_t at date $t+1$, we have:

$(s^u_{t+1} \notin B$ and $s^d_{t+1} \notin B) \Rightarrow v(B/s_t) = 0$.

$[(s^u_{t+1} \notin B$ and $s^d_{t+1} \in B)$ or $(s^u_{t+1} \in B$ and $s^d_{t+1} \notin B)] \Rightarrow v(B/s_t) = c$.

$(s^u_{t+1} \in B$ and $s^d_{t+1} \in B) \Rightarrow v(B/s_t) = 1$.

For $X_{t+1} = 1_B$, relation (3.2) can be written as:

$$v(B) = c\,v(\{s_t : [s^u_{t+1} \in B] \vee [s^d_{t+1} \in B]\}) + (1-c)v(\{s_t : [s^u_{t+1} \in B] \wedge [s^d_{t+1} \in B]\}) \quad (3.3)$$

All the capacities at date $t+1$ are then uniquely determined by capacities at date t. Going backward until date 1 where $v(s^1_1) = v(s^2_1) = c$, the set function v is completely defined and hence unique.

It remains to prove that set function v is a capacity, i.e. is an increasing measure:

$$B \subset D \Rightarrow \{s_t : [s^u_{t+1} \in B] \vee [s^d_{t+1} \in B]\} \subset \{s_t : [s^u_{t+1} \in D] \vee [s^d_{t+1} \in D]\},$$

then, from (3.3): $B \subset D \Rightarrow v(B) \leq v(D)$.

(ii) Finally, if $t = \tau+n$, $n > 1$, relation (3.1) partially characterizes the conditional capacities $v(B/s_\tau)$, where $B \in A_{\tau+n}$. We have, indeed $2^{\tau+n+1}$ equations (the number of characteristic functions of the sets in $A_{\tau+n}$) for $(\tau+1)\times 2^{\tau+n+1}$ conditional capacities (one for each B in $A_{\tau+n}$ and one for each node at date τ). Then, these relations cannot constrain the capacity v defined by case (ii). QED

Proposition 3.2.2: In a Dynamically Consistent Choquet Random Walk the capacity v is sub-linear[8] if and only if $c \leq \tfrac{1}{2}$. Moreover it does not reduce to a probability if and only if $c \neq \tfrac{1}{2}$.

Proof: We only have to prove that: $[\forall\, t = 1,...,T,\ \forall\, B \in A_t,\ v(B) + v(B^C) \leq 1] \Leftrightarrow c \leq \tfrac{1}{2}$, because non-additivity results from the same reasoning.

First, if the capacity is sub-linear, at date 1, for $B = s_1{}^1$ and $B^C = s_1{}^2$, $v(B) + v(B^C) = 2c \leq 1$, implies $c \leq \tfrac{1}{2}$.

The reciprocal obtains by induction and let's assume that $c \leq \tfrac{1}{2}$ in the sequel:

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[8] A sublinear (or convex) capacity characterises aversion to ambiguity (Gilboa and Schmeidler 1993).

At the first stage, $B \in A_1$. $B = \varnothing$ or $B = S_1$, implies $\nu(B) + \nu(B^C) = 1$, and $B = s_1^{\,1}$ or $B = s_1^{\,2}$ yields: $\nu(B) + \nu(B^C) = 2c$. The property is then established at date 1.

Suppose that it is also true at date t and consider some $B \in A_{t+1}$. From relation (3.3):

$$\nu(B) + \nu(B^C) = c\,\nu(\{s_t : [s_{t+1}^u \in B] \vee [s_{t+1}^d \in B]\}) + (1-c)\,\nu(\{s_t : [s_{t+1}^u \in B] \wedge [s_{t+1}^d \in B]\})$$
$$+ c\,\nu(\{s_t : [s_{t+1}^u \in B^C] \vee [s_{t+1}^d \in B^C]\}) + (1-c)\,\nu(\{s_t : [s_{t+1}^u \in B^C] \wedge [s_{t+1}^d \in B^C]\}).$$

With the following notations:

$$\overline{D} = \{s_t : [s_{t+1}^u \in D] \vee [s_{t+1}^d \in D]\} \text{ and } \underline{D} = \{s_t : [s_{t+1}^u \in D] \wedge [s_{t+1}^d \in D]\}, \text{ it follows that:}$$

$$\nu(B) + \nu(B^C) = c[\nu(\overline{B}) + \nu(\overline{B^C})] + (1-c)[\nu(\underline{B}) + \nu(\underline{B^C})].$$

We have: $\overline{B^C} = \underline{B}^C$ and $\underline{B^C} = \overline{B}^C$ therefore:

$$\nu(B) + \nu(B^C) = c[\nu(\overline{B}) - \nu(\underline{B}) + \nu(\underline{B}^C) - \nu(\overline{B}^C)] + \nu(\underline{B}) + \nu(\overline{B}^C).$$

With $\underline{B} \subset \overline{B} \Rightarrow \nu(\overline{B}) - \nu(\underline{B}) \geq 0$, and $\overline{B}^C = \underline{B}^C \subset \overline{B^C} = \underline{B}^C \Rightarrow \nu(\underline{B}^C) - \nu(\overline{B}^C) \geq 0$, and because $c \leq \frac{1}{2}$, it follows that:
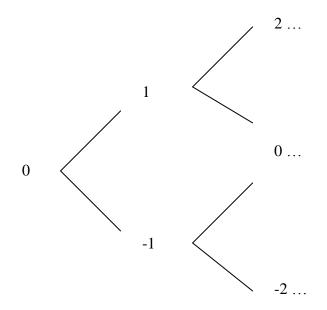
$$\nu(B) + \nu(B^C) \leq \frac{1}{2}[\nu(\overline{B}) - \nu(\underline{B}) + \nu(\underline{B}^C) - \nu(\overline{B}^C)] + \nu(\underline{B}) + \nu(\overline{B}^C), \text{ or:}$$

$$\nu(B) + \nu(B^C) \leq \frac{1}{2}[\nu(\overline{B}) + \nu(\overline{B}^C) + \nu(\underline{B}) + \nu(\underline{B}^C)].$$

As $D \in A_{t+1} \Rightarrow [\overline{D} \in A_t \wedge \underline{D} \in A_t]$, by hypothesis: $\nu(\overline{B}) + \nu(\overline{B}^C) \leq 1$, $\nu(\underline{B}) + \nu(\underline{B}^C) \leq 1$, and then: $\nu(B) + \nu(B^C) \leq 1$. QED

3.3 Symmetric Random Walk

We call Symmetric Random Walk a binomial process for which the "up" and the "down" movements correspond to the same magnitude. Without loss of generality, we take this increment to be the unity, and the departure point to be zero. In the probabilistic model, i.e. the case where $c = \frac{1}{2}$, this process is a discrete time Brownian motion.

To compute the Choquet expectation of such a process we need to characterize the decumulative distribution function of capacity ν.

Proposition 3.3.1: The decumulative function of capacity ν is obtained by iteration from:

$$\forall\ t = 2,...,T,\ \forall\ n = 1,...,\ t,\ \nu(s_t^1,...,s_t^n) = c\,\nu(s_{t-1}^1,...,s_{t-1}^n) + (1-c)\nu(s_{t-1}^1,...,s_{t-1}^{n-1}) \quad (3.4)$$

and $\nu(s_1^1) = c$.

The closed form of the decumulative function is:

$$\forall\ t = 1,...,\ T,\ \forall\ n = 1,...,\ t-1, \qquad \nu(s_t^1,...s_t^n) = c^{t-n+1} \sum_{j=0}^{n-1} \binom{j}{t-n+j} (1-c)^j \quad (3.5)$$

Proof: We have:

$$\forall i = 1,...,n-1, \nu(s_t^1,...,s_t^n / s_{t-1}^i) = 1, \nu(s_t^1,...,s_t^n / s_{t-1}^n) = c, \forall j = n+1,...,t, \nu(s_t^1,...,s_t^n / s_{t-1}^i) = 0$$

If we apply relation (3.2) to $X = 1_{s_t^1 \cup ... \cup s_t^n}$, it follows that:

$$\nu(s_t^1,...,s_t^n) = c\,\nu(s_{t-1}^1,...,s_{t-1}^n) + (1-c)\nu(s_{t-1}^1,...,s_{t-1}^{n-1}) \quad (3.4)$$

If we put the expression of the decumulative function given by relation (3.5) in the right hand side of relation (3.4), we have:

$$c\,\nu(s_{t-1}^1,...,s_{t-1}^n) + (1-c)\nu(s_{t-1}^1,...,s_{t-1}^{n-1})$$

$$= c[c^{t-n} \sum_{j=0}^{n-1} \binom{j}{t-n+j-1}(1-c)^j] + (1-c)[c^{t-n+1} \sum_{j=0}^{n-2} \binom{j}{t-n+j}(1-c)^j]$$

$$= c^{t-n+1}[\sum_{j=0}^{n-1} \binom{j}{t-n+j-1}(1-c)^j + \sum_{j=0}^{n-2} \binom{j}{t-n+j}(1-c)^{j+1}]$$

$$= c^{t-n+1}[1 + \sum_{j=1}^{n-1} \binom{j}{t-n+j-1}(1-c)^j + \sum_{j=1}^{n-1} \binom{j-1}{t-n+j-1}(1-c)^j]$$

$$= c^{t-n+1}\{1 + \sum_{j=1}^{n-1}[\frac{(t-n+j-1)!}{j!(t-n-1)!} + \frac{(t-n+j-1)!}{(j-1)!(t-n)!}](1-c)^j\}$$

$$= c^{t-n+1}[\binom{0}{t-n}(1-c)^0 + \sum_{j=1}^{n-1}\frac{(t-n+j)!}{j!(t-n)!}(1-c)^j]$$

$$= c^{t-n+1} \sum_{j=0}^{n-1} \binom{j}{t-n+j}(1-c)^j$$

$$= v(s_t^1,\ldots,s_t^n)$$

Therefore, relation (3.5) satisfies relation (3.4). QED

Proposition 3.3.2: The Choquet Expectation of the payoffs at date t of a Symmetrical Choquet Random Walk is: $\forall\ t = 0,\ldots, T,\quad E(X_t) = t\,(2c-1)$ (3.6).

Proof: The payoffs of X at date t are: $X(s_t^1) = t, X(s_t^2) = t-2,\ldots,X(s_t^t) = -t+2, X(s_t^{t+1}) = -t$.
Then, their Choquet Expectation is:

$$E(X_t) = -t[1 - v(s_t^1,\ldots,s_t^t)] - (t-2)[v(s_t^1,\ldots,s_t^t) - v(s_t^1,\ldots,s_t^{t-1})]$$

$$+\ldots + (t-2)[v(s_t^1,s_t^2) - v(s_t^1)] + tv(s_t^1)$$

$$= -t + 2[v(s_t^1,\ldots,s_t^t) + v(s_t^1,\ldots,s_t^{t-1}) +\ldots+ v(s_t^1)].$$

Relation (3.4) in proposition 3.3.1 implies:

$$v(s_t^1,\ldots,s_t^t) + v(s_t^1,\ldots,s_t^{t-1}) +\ldots+ v(s_t^1) = c[v(s_{t-1}^1,\ldots,s_{t-1}^t) - v(s_{t-1}^1,\ldots,s_{t-1}^{t-1})]$$

$$+ v(s_{t-1}^1,\ldots,s_{t-1}^{t-1}) - v(s_{t-1}^1,\ldots,s_{t-1}^{t-2}) +\ldots+ v(s_{t-1}^1,s_{t-1}^2) - v(s_{t-1}^1) + v(s_{t-1}^1)]$$

$$+ v(s_{t-1}^1,\ldots,s_{t-1}^{t-1}) +\ldots+ v(s_{t-1}^1)$$

$$= c + v(s_{t-1}^1,\ldots,s_{t-1}^{t-1}) +\ldots+ v(s_{t-1}^1).$$

It follows that:

$$E(X_t) = -t + 2[c + v(s_{t-1}^1,\ldots,s_{t-1}^{t-1}) +\ldots+ v(s_{t-1}^1)]$$

$$= -t + 2[c + c + v(s_{t-2}^1,\ldots,s_{t-2}^{t-2}) +\ldots+ v(s_{t-2}^1)]$$

…

$$= -t + 2[(t-1)c + v(s_1^1)] = t(2c-1). \text{ QED}$$

Remark 1: $c < \frac{1}{2} \Rightarrow E(X_t) < 0$. This is consistent with the DM's aversion to ambiguity that makes her give a negative value to a fair game.

Remark 2: Other Symmetric Random Walks can be obtained from this one by a positive affine transformation. The Choquet integral is linear with respect to this transformation.

For $\forall\ t = 0,\ldots, T$, $Y_t = a\ X_t + b$, $a > 0$, we have: $E(Y_t) = a\ t\ (2c-1) + b$.

As a result we can address the cases where the mean is non zero and we can make volatility vary.

4. Independence

We want to consider two "independent random variables" X_T and Y_T that represent the final payoffs of two Symmetric Choquet Random Walks. Obviously, "independent random variables" have to be defined in the context of non-additive measures. We do it in the following, using a slightly more general definition than the one usually adopted (e.g. Marinacci, 1997). In the sequel, we omit the time index, which is useless because we assumed dynamic consistency.

The possible values of Y are: $T, T-2, \ldots, -T+2, -T$. For a real interval $[a, b]$, which doesn't contain all these values but at least one of them, let D be the information of the decision maker: $D = \{s \in S\ /\ Y(s) \in [a, b]\}$. Similarly, let us define: $B = \{s \in S\ /\ X(s) \in [a, b]\}$.

In this framework, relation (3.1) becomes:

$$\sum_{i=D,D^C}[\ \sum_{s \in S}X(s)\Delta v^i(s)]\Delta v(i) = \sum_{s \in S}X(s)\Delta v(s) \qquad (4.1)$$

As in section 3.1, the conditional capacities are normalised in the following way:

$\forall\ D \subset S$, $v(\varnothing/D) = 0$, $v(S/D) = 1$, $\forall\ B \subset S$, $v(B/D) = v(B \cap D/D)$.

Whatever the updating rule used to define the conditional capacity, independence between two random variables expresses the idea that conditioning their joint distribution by any one of them yields the other's marginal measure:

Definition 4.1: The random variables X and Y are independent if and only if:

$\forall \, D = \{s \in S \, / \, Y(s) \in [a, b]\} \subset S, \, \forall \, B = \{s \in S \, / \, X(s) \in [a, b]\} \subset S : \nu(B/D) = \nu(B).$

Notice that, most papers following Marinacci (1997) give a definition that is a particular case of this one: $\nu(B \cap D) = \nu(B) \, \nu(D)$. Indeed, *whatever the updating rule* we have:

Proposition 4.1: Under relation (4.1), if the random variables X and Y are independent:

$\forall \, D \subset S, \, \forall \, B \subset S, \, \nu(B \cap D) = \nu(B) . \nu(D).$

Proof: $\displaystyle\sum_{s \in S} 1_B(s) \Delta \nu(s) = \nu(B) = \nu(B/D) = \nu(B \cap D/D) = \sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^D(s).$

On the one hand, we have:

$\displaystyle\sum_{s \in S} [\sum_{s \in S} 1_B(s) \Delta \nu(s)] . 1_D(s) \Delta \nu(s) = \nu(B) \sum_{s \in S} 1_D(s) \Delta \nu(s) = \nu(B).\nu(D),$

and on the other hand:

$\displaystyle\sum_{s \in S} [\sum_{s \in S} 1_B(s) \Delta \nu(s)] . 1_D(s) \Delta \nu(s) = \sum_{s \in S} [\sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^D(s)] . 1_D(s) \Delta \nu(s)$

$\displaystyle = [\sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^D(s)] . \nu(D) = \sum_{i = D, D^C} [\sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^i(s)] \Delta \nu(i),$

because:

$\displaystyle\sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^{D^C}(s) = \nu(B \cap D/D^C) = \nu(\varnothing/D^C) = 0.$

Relation (4.1), for $\forall \, s \in S$, $X(s) = 1_{B \cap D}(s)$, implies:

$\displaystyle\sum_{i = D, D^C} [\sum_{s \in S} 1_{B \cap D}(s) \Delta \nu^i(s)] \Delta \nu(i) = \sum_{s \in S} 1_{B \cap D}(s) \Delta \nu(s) = \nu(B \cap D),$

and then: $\nu(B \cap D) = \nu(B) . \nu(D)$. QED

5. Convergence towards a Brownian motion

We first characterise the variations of the decumulative function of the previous dynamically consistent Choquet random walk.

Proposition 5.1: $\forall \, t = 1, ..., T, \, \forall \, n = 1, ..., t + 1,$

$\Delta \nu_t^n \equiv \nu(s_t^1, ... s_t^n) - \nu(s_t^1, ... s_t^{n-1}) = \binom{n-1}{t} c^{t-n+1} (1-c)^{n-1}$ (5.1) where we set: $\nu(s_t^0) = 0.$

Proof: Relation (5.1) is true for $t = 1$ and we suppose that it holds true for a given t $(t \leq T)$. With relation (3.4), we have:

$$\Delta v_{t+1}^n = c\, v(s_t^1, \ldots s_t^n) + (1-c)v(s_t^1, \ldots s_t^{n-1}) - c\, v(s_t^1, \ldots s_t^{n-1}) - (1-c)v(s_t^1, \ldots s_t^{n-2})$$

$$= c[v(s_t^1, \ldots s_t^n) - v(s_t^1, \ldots s_t^{n-1})] + (1-c)[v(s_t^1, \ldots s_t^{n-1}) - v(s_t^1, \ldots s_t^{n-2})]$$

$$= c\Delta v_t^n + (1-c)\Delta v_t^{n-1}$$

$$= c\binom{n-1}{t}c^{t-n+1}(1-c)^{n-1} + (1-c)\binom{n-2}{t}c^{t-n+2}(1-c)^{n-2}$$

$$= \left[\binom{n-1}{t} + \binom{n-2}{t}\right]c^{t-n+2}(1-c)^{n-1}$$

$$= \binom{n-1}{t+1}c^{t-n+2}(1-c)^{n-1}$$

Relation (5.1) is then satisfied for $t + 1$ and then for any t by induction. QED.

In order to interpret formula (5.1), recall that on a comonotonic cone, a capacity is represented by a particular probability distribution. Here it is a standard binomial distribution with parameters such that $B(T, p)$ is in the core of capacity v. Then, there's an easy link with the multi-priors approach.

For a convex capacity $(c < \frac{1}{2})$, the core is given by:

Core $v = \{\mu$, probability distribution $/ \mu \geq v\}$, and, over one period:

$$[p = \mu(s_{t+1}^u / s_t) \geq v(s_{t+1}^u / s_t) = c, 1 - p = \mu(s_{t+1}^d / s_t) \geq v(s_{t+1}^d / s_t) = c] \Rightarrow p \in [c, 1-c].$$

For a Symmetric Random Walk: $E_\mu[X_{t+1} - X_t / s_t] = 2p - 1$, so that the MaxMin criterion yields: $\underset{\mu \in core(v)}{ArgMin}\; E_\mu(X_{t+1} - X_t / s_t) = (c, 1-c)$. This probability distribution, applied at each period, yields the binomial distribution corresponding to formula (5.1).

Proposition 5.2: When the time interval converges toward 0, the Symmetric Random Walk defined by (5.1) converges towards a general Wiener process with mean $m = 2c-1$ and variance $s^2 = 4c(1-c)$.

Proof: Recall that X_t is independent from Y_t and the Y_t's are independent. In the discrete time probabilistic model generated by (5.1), we have: $X_{t+1} = X_t + Y_t$, in which Y_t takes the values: 1 with probability c and -1 with probability $1-c$.

Then: $E[Y_t] = 2c-1 = m$, $Var[Y_t] = 4c(1-c) = s^2$.

If we define a discrete time process W_t by: $W_t = m\,h + s\,h^{1/2}\,U_t$, where U_t takes values: 1 with probability ½ and −1 with probability ½, then $E[W_t] = m\,h$, $Var[W_t] = s^2\,h$.

It is a standard result for Brownian motions that W_t converges toward a general Wiener process in continuous time: $W(t) = m\,t + s\,B(t)$, where $B(t)$ is the Brownian motion. QED

Notice that if $c < ½$, then $m < 0$ and $s^2 < 1$: both the mean and the variance are lower than in the probabilistic model. Indeed, ambiguity aversion yields lower weights on the ups and downs ($c < ½$) for given values ($+1$, $−1$), hence the variance is lower.

6. Applications

We can find possible applications of this approach to any model with ambiguity, for example macroeconomics "uncertainty models" where central banks are not aware of the "true" economic model. Another famous application would be a generalisation of the portfolio optimal choice under risk (Merton 1969, 1971, 1973). Here, let us consider a generalisation of the so called real option theory, i.e. a basic problem of optimal investment (cf. Dixit and Pindyck (1994)).

A firm can use a patent to invest and, after that, develop a production. The investment is irreversible and the corresponding cost is totally sunk. Therefore, we face an optimal stopping problem: The firm has to choose the optimal date to exert its option to invest (if it is worth it). In the basic model, the profit obtained when the patent is used follows a geometric Brownian motion $(\pi_t)_{0 \leq t \leq T}$ (where T is the expiration date of the patent after which there is no profits) such that:

$$d\pi_t = \mu\,\pi_t\,dt + \sigma\,\pi_t\,dB_t, \qquad (6.1)$$

with $\pi_0 > 0$, where B_t is the standard Brownian motion and μ and σ are some real numbers, with $\sigma > 0$ and $\mu < \rho$, where ρ, $\rho > 0$, is the firm discount rate.

With the Choquet Random Walk, the profit is:

$$d\pi_t = \mu\,\pi_t\,dt + \sigma\,\pi_t\,dW_t, \qquad (6.2)$$

where W_t is given by Proposition 5.2:

$$dW_t = m\,dt + s\,dB_t, \qquad (6.3)$$

with: $m = 2\,c − 1$, and $s^2 = 4\,c\,(1 − c)$, and then:

$$d\pi_t = (\mu + m\,\sigma)\,\pi_t\,dt + s\,\sigma\,\pi_t\,dB_t \qquad (6.4)$$

This relation is of the same type as (6.1), with $\mu' = \mu + m\,\sigma$ and $\sigma' = s\,\sigma$ in place of μ and σ. We can see that $0 < c < \frac{1}{2}$ implies $-1 < m < 0$ and $0 < s < 1$, and then $\mu - \sigma < \mu' < \mu$ and $0 < \sigma' < \sigma$, a reduction of the instantaneous mean. But we also have the reduction of the volatility: an unexpected result!

If we suppose that the horizon T is infinite, the value of the utilized patent $V(\pi_t, t)$ does not depend on t directly and the model becomes stationary. The well known solution of the optimal stopping problem is then to invest at date t if and only if the value $V(\pi_t)$ is larger than a reservation value V^* such that:

$$V^* = \frac{\alpha' I}{\alpha' - 1} \qquad (6.5)$$

where I is the cost of the investment and the constant α' is given by:[9]

$$\alpha' = \frac{-(\mu' - \frac{1}{2}\sigma'^2) + \sqrt{(\mu' - \frac{1}{2}\sigma'^2)^2 + 2\rho\sigma'^2}}{\sigma'^2} \qquad (6.6)$$

The effect of the Choquet distortion on the standard solution is equivocal, because it reduces at the same time the instantaneous mean and the volatility. It follows that the comparison between α' and α (with parameters μ and σ) is a matter of empirical data.

Consider now a stationary version of the Intertemporal Capital Asset Pricing Model (Merton (1969, 1971, 1973).

Let $(k_t)_{0 \le t \le T}$, the capital of the investor, which has to be allocated between a riskless asset with rate of return r, $r > 0$, and a risky asset the price of whom follows a geometric Brownian motion:

$$dP_t = \mu\,P_t\,dt + \sigma\,P_t\,dB_t \qquad (6.7)$$

with $P_0 > 0$, where B_t is the standard Brownian motion and μ and σ are some real numbers, with $\mu > 0$ and $\sigma > 0$.

If (x_t) is the part of the capital invested in the risky asset at date t, the program of the agent for a time horizon T is:

$$\underset{(x_t)_{0 \le t \le T}}{Max}\ E_t[u(k_T)],\ k_0 > 0 \qquad (6.8)$$

where $u(.)$ is an increasing and concave utility function.

In the iso-elastic case:

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[9] It is easy to check that $\mu < \mu' < \rho$ implies $\alpha > 1$ and then (6.5) is a solution of the problem.

$$u(k) = \frac{k^{1-\alpha}}{1-\alpha}, \; \alpha > 0, \; \alpha \neq 1,$$

the optimal solution is a constant:

$$x*(t,k) = x = \frac{1}{\alpha}\left(\frac{\mu - r}{\sigma^2}\right) \qquad (6.9)$$

With the Random Choquet Walk, this solution becomes:

$$x*(t,k) = x' = \frac{1}{\alpha}\left(\frac{\mu' - r}{\sigma'^2}\right) \qquad (6.10)$$

with $\mu' = \mu + m\,\sigma$, $\sigma' = s\,\sigma$, $m = 2\,c - 1$, and $s^2 = 4\,c\,(1-c)$.

If $0 < c < \frac{1}{2}$, we have $\mu - \sigma < \mu' < \mu$ and $0 < \sigma' < \sigma$. It is easy to check that:

$$x' < x \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < -\frac{m}{1-s^2} = \frac{1}{1-2c}.$$

The value of the market price of risk λ relatively to the ambiguity parameter c, gives the hierarchy between investment in the risky asset under ambiguity and under risk. Similarly:

$$\frac{\partial x'}{\partial c} > 0 \Leftrightarrow \lambda \equiv \frac{\mu - r}{\sigma} < \frac{1 - 2c + 2c^2}{1 - 2c}.$$

The fact that investment in the risky asset is increasing with the reduction of ambiguity (when c increase towards ½, the ambiguity decreases) depends on the value of the market price of risk.

In both applications, the effect of ambiguity is ambiguous! Our results differ from the ones obtained from applications of Epstein and Schneider (2003)'s recursive multi-priors model. As we shall see in the next section, results in both applications to real options (Nishimura and Osaki 2007) and to continuous time CAPM (Chen and Epstein 2002) are not modified because the effect of ambiguity is not straightforward. This is due to the deformations of both the mean and the variance in our model of Choquet expectations.

7. Related literature and conclusions

It may be useful to make more precise the link between Dynamically Consistent Choquet Random Walks and Recursive Multiple-priors in Epstein-Schneider (2003). In section 5, we have seen that, for a convex capacity, it was very simple to exhibit the possible conditional one-step-ahead priors, so that our model is recursive, indeed. But the Recursive Multiple-

priors is based on a fundamental hypothesis: Rectangularity, and our model can be consistent with this approach only if this axiom is satisfied. We shall examine this point.

Let us recall some notations briefly.

The states space is Ω, and the filtration is $\{F_t\}_{0 \leq t \leq T}$, ($F_t$ is a finite partition).

P is the set of priors, $P_t(\omega) = \{p_t(\omega): p \in P\}$ is the set of Bayesian updates, and $P_t^{+1}(\omega) = \{p_t^{+1}(\omega): p \in P\}$ is the set of conditional one-step-ahead measures.

Definition 7.1: P is $\{F_t\}$-rectangular if for all t and ω,

$$P_t(\omega) = \{\int_\Omega p_{t+1}(\omega')dm : p_{t+1}(\omega') \in P_{t+1}(\omega') \forall \omega', m \in P_t^{+1}(\omega)\},$$

$$\text{or} \qquad P_t(\omega) = \int P_{t+1} dP_t^{+1}(\omega).$$

For a Dynamically Consistent Choquet Random Walk, the right hand side of the last equality is:

$$\int P_{t+1} dP_t^{+1}(\omega) = \{ \sum_{s_t \in S_t} p(\omega/s_t)m(s_t) : p,m \in core\, v\} = \{p(\omega/s_t^i)m(s_t^i) : p,m \in core\, v\},$$

for the state s_t^i such that $\omega = \{s_0,\ldots, s_t^i,\ldots, s_T\}$.

Under the assumption of independent random variables, wlog, we consider the state s_t^{t+1}:

$$\int P_{t+1} dP_t^{+1}(\omega) = \{p^{T-t}m^t : p,m \in [c,1-c]\} = [c^T,(1-c)^T].$$

$p \in core\, v$ implies from relation (5.1) in which $t = T$, and $n = T + 1$: $p(\omega) \geq v(\omega) = c^T$ and $p(\omega^c) = 1 - p(\omega) \geq v(\omega^c) = (1 - c)^T$.

The left hand side in the definition of rectangularity is then: $P_t(\omega) = [c^T,(1-c)^T]$, and this property is satisfied. As a consequence, all results obtained by the Multi-priors apply to the Choquet approach when the capacity is convex ($c \leq \frac{1}{2}$). However, the models differ and applications may not coincide.

In order to emphasise the parallels and differences between the Dynamically Consistent Choquet Random Walk and the Recursive Multiple-priors, we can consider the continuous time model in Chen and Epstein (2002).

In this last approach, we start from an "objective" distribution P, and a Brownian motion (B_t) with respect to P. The possible priors are the measures Q^θ such that:

$$\frac{dQ^\theta}{dP} = z_T^\theta, \text{ where } z_T^\theta \text{ is a } P\text{-martingale.}$$

From the Girsanov theorem, it follows that a Brownian motion (B_t^θ) with respect to Q^θ satisfies: $dB_t^\theta = \theta_t dt + dB_t$, where (θ_t) is a stochastic process adapted to the filtration generated by (B_t). In this equation, the variance is not affected, this differs from the Dynamically Consistent Choquet Random Walk where, indeed: $dW_t = m\, dt + s\, dB_t$.

We can see how close our approach is to the ones obtained under Epstein and Schneider (2003) recursive multi-priors model, however there are some important differences in the results. For instance in our Choquet model, the continuous time Brownian motion is deformed, in contrast with what happens in Chen and Epstein (2002).

The literature on Dynamic risk measures is very close to this method. Risk is measured, at time t, by the maximum of the expectations of minus the sum of discounted cash-flows, with respect to probabilities in a close and convex set, conditionally to the information at time t. This result is obtained under some properties: coherence, dynamically consistency and relevancy[10]. The crucial one, consistency, is equivalent to the rectangularity of Epstein and Schneider (2003), as noted by Riedel (2004)[11].

Dynamically consistent Choquet random walks are a generalization of the classical approach when some ambiguity of the decision maker about the payoff processes is taken into account. It has been seen that it is manageable and yields results that can extend the usual results in the real option literature. The effects of ambiguity is perceived both in the deformation of the mean and of the variance at the limit, when time intervals converge toward zero, in contrast with what happens in the probabilistic case. Because the random walk is defined step by step, the dynamic valuation is, indeed recursive. It is tempting, in the case when the capacity is convex, to refer to the literature on the Gilboa and Schmeidler (1989) multi-priors approach, following works and applications proposed by Epstein and Schneider (2003) and Epstein and Chen (2002) for example. Their results rely on the property they dubbed "rectangularity" that is satisfied by a DCCRW. However the ambiguity on the limiting variance that appears in the DCCRW is absent in the multi-Prior, so that DCCRW offer some other practical perspectives in applications.

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[10] See Riedel (2004), Theorem 1.
[11] Note 2, p. 186.

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