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# A Multiple Equilibria Model with Intrafirm Bargaining and Matching Frictions

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## Abstract

In this paper, we combine a matching model derived from Pissarides (2000) in the case of large firms with monopolistic competition on the product market and the model of intrafirm bargaining à la Stole and Zwiebel (1996). Moreover, we allow for increasing returns to scale in the aggregate production function leading to multiple equilibria. Then, we study the dynamics of such a framework for various size of returns to scale and propose numerical simulations. Finally, we show how the dynamical properties are altered in the case of multiple equilibria compared to that of a unique equilibrium and illustrate the issues of economic policy design in presence of multiple equilibria.

*Keywords:* matching frictions, monopolistic competition, intrafirm bargaining, multiple equilibria, economic policy

*JEL Classification:* C61, E24, E61, J41

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## Introduction

The understanding of labor market and its reactions facing economic policies is a widespread subject in economic literature. Since the 1960's, the sharp variations of unemployment rates in European countries have induced economists to consider the possibility of multiple unemployment equilibria. Today, this view is widely accepted. However, the possibility of multiple equilibria involves reconsideration in policy design. Thus, papers on economic policies design of labor market in the case of multiple equilibria have emerged.

Actually, as Cooper (1999) explains, the existence of multiple equilibria involves the possibility of differentiated policy implications according to the equilibrium achieved by the economy as well as the possibility by an appropriate economic policy to coordinate the economy on a Pareto superior equilibrium in the case of coordination failure<sup>1</sup> or to eliminate an inefficient one.

The search and matching models of Pissarides (2000) have permitted to take into account the fact that the employment cannot be increased without time and cost. The firms must post vacancies, what is costly, and commit on wage with potential workers, what is time-consuming, before any match is done and production takes place. In his book, Pissarides develops essentially the case of one worker-one firm which is more tractable, but his results can be generalized to a multi-workers firm. However, assuming a standard bargaining process between the firm and the worker, its approach has the main drawback to exclude the strategic interactions within the firm.

Stole and Zwiebel's (1996) paper studies such interactions. Indeed, they provide an intrafirm bargaining model in which contracts cannot commit the firms and its workers to wages and employment. The central assumption of this wage setting is to consider that the wage is renegotiated with all workers after a new hiring or laid off, so that each worker is considered as a marginal worker<sup>2</sup>. This bargaining process takes place within a multi-workers firm and, in this way it fills the lack of analysis about the effects of an additional worker on wage negotiation in Pissarides' model. As Cahuc and Wasmer (2001b, 2004) proved, these two approaches are complementary due to their analysis of different mechanisms at work in the labor market. Whereas the Pissarides model is explicitly dynamic and analyzes the labor market equilibrium in the presence of search and matching frictions but without strategic interactions within the firm, Stole and Zwiebel's approach analyzes the latter but in a static framework. Consequently, a model including both approaches will be more adequate to understand the labor market workings.

Furthermore, the size of competition degree on the product market has been also advanced in the understanding of labor market performances. Blanchard and Giovazzi (2003) show how the product market deregulation, involving lower

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<sup>1</sup>In the case where multiple Pareto ranked equilibria exist, there exists coordination failure if the economy reaches a Pareto inferior equilibrium (Cooper, 1999).

<sup>2</sup>Indeed, allowing renegotiation with all remaining workers, a worker obtains the same share of surplus than the one prior him and the one after him. As a consequence, order is irrelevant and all workers are considered as a marginal worker.

entry cost and higher competition degree, leads to higher real wages and lower unemployment. Nonetheless, they assume a standard Nash bargaining process. Ebell and Haefke (2003) analyze a similar issue but consider an intrafirm bargaining and show quantitatively that in such a framework the impact of product market competition on equilibrium unemployment is surprisingly weak.

In this paper, we study a matching model derived from that of Pissarides (2000) to which we add intrafirm bargaining à la Stole and Zwiebel, in large firms and monopolistic competition on the product market. In this way, we use a similar framework of that developed by Ebell and Haefke (2003). Our main contribution is to investigate the dynamical behaviour of such an economy for different kind of returns to scale. Indeed, all papers listed above consider either decreasing returns to scale or constant returns to scale and so the case of unique equilibrium. Our paper follows Mortensen (1999)'s paper and investigates also the case of increasing returns to scale in production involving the existence of multiple equilibria<sup>3</sup>. However, while Mortensen (1999) studies the global dynamics of Pissarides' model only, we extent the study to the case of intrafirm bargaining with monopolistic competition on the product market. Then, numerical simulations allow us to show how, in the case of multiple equilibria, the dynamical properties of the model as well as some comparative statics results are altered compared to the case of unique equilibrium, but also how an appropriate economic policy can eliminate an inefficient equilibrium. Thus, we illustrate all the distinct policy design issues in presence of multiple equilibria announced previously.

The paper is organized as follows. In section 1, we develop the mathematical model. After explaining the different assumptions on firms, labor and matching frictions, we analyze the firms' behavior and the wage determination under the intrafirm bargaining mechanism. Then, we deduce the general equilibrium both in and out of steady state. In section 2, the possible equilibrium cases according to the size of aggregate returns to scale and their dynamical behavior are studied. Numerical simulations are run and discussed in section 3.

## 1 The model

In this section, we present and solve the mathematical model both in and out of steady state in order to obtain its dynamical features. After a general presentation of our economy, we present the firm's program and the wage bargaining from which we deduce the general symmetric equilibrium of the economy.

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<sup>3</sup>Multiple equilibria can be also induced by increasing returns in the matching function, transactions costs, menu costs and others. See Cooper (1999) for a detailed report on this subject.

## 1.1 Hypothesis

### 1.1.1 Economy

We consider a continuous time model of an economy made up of agents which have the same discount rate  $r$ . The output is produced by multi-workers firms (or large firms). The labor is supplied by workers and each worker supplies one unit of labor. The labor is the only production factor used in this economy. The existence of matching frictions implies that firms need time and resources to hire workers.

### 1.1.2 Firms

We assume a monopolistic competition on the product market. We have a continuum of identical firms uniformly distributed on the interval  $[0, 1]$ . Each firm produces an imperfectly substitutable good with others thanks to a production technology with constant returns to scale given by  $y_i = An_i$ , where  $y_i$  represents the output of the firm  $i$  and  $n_i$  its labor employment. The efficiency parameter  $A$  is function of the average employment level in the economy  $\bar{n}$  in such way that the more the economy employs labor, the more the production technology of the firm is efficient (positive aggregate externality).  $A$  is defined as  $A = \bar{n}^\alpha$  where  $\alpha \geq 0$  gives the extent of the aggregate externality. We note that each firm is too small to have any influence on the aggregate state of the economy what entails the exogeneity of  $A$  at the firm level. The demand for the firm  $i$ 's output is given by  $y_i^d = Y(p_i/P)^{-\sigma}$ , with  $p_i$  the price of the firm  $i$ ,  $P$  the general price level,  $Y$  the aggregate output and  $\sigma > 1$  the demand elasticity for the good supplied by the firm  $i$ <sup>4</sup>.

### 1.1.3 Labor and matching frictions

Labor is supplied by a continuum of infinitely lived and identical workers of size normalized to one. At each time, a worker can be employed or unemployed. All unemployed workers are assumed to have the same search effort which is also normalized to one.

Given the presence of matching frictions in the labor market, the employment cannot be increased instantaneously. Indeed, firms must post vacancies in order to recruit which incurs a real cost  $c$  per unit of time and per unit of vacancy. Furthermore, we assume that the firm can post as many vacancies as necessary and without delay, so vacancies are "jump" variables<sup>5</sup>.

Vacancies are matched to the pool of unemployed workers according to the matching technology:  $m(u, v) = u^\eta v^{1-\eta}$ , where  $v$  represents the mass of vacancies,  $u$  the unemployment rate and  $\eta \in [0, 1]$  the matching elasticity with respect to the unemployment. This matching function is increasing and concave in each argument, and homogenous of degree one.

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<sup>4</sup>See appendix A for detailed calculations on the determination of the demand function.

<sup>5</sup>This assumption makes practicable the investigation of dynamics out-of-steady state hereafter.

Let the labor market tightness be  $\theta = \frac{v}{u}$ . A tight labor market features an economic environment where the new job offers are great compared to the pool of unemployed workers and is translated by a high value of  $\theta$  here. We note that  $\theta$  is exogenous to the firms' decision<sup>6</sup>. The firm meets unemployed workers at rate  $q(\theta) = \frac{m(u, v)}{v} = \theta^{-\eta}$ , with  $q'(\theta) < 0$ , while an unemployed worker meets vacancies at rate  $\theta q(\theta) = \frac{m(u, v)}{u} = \theta^{1-\eta}$ , with  $\frac{d[\theta q(\theta)]}{d\theta} > 0$ . Thus, when the labor market is tight ( $\theta$  high), it is more difficult for a firm to fill a vacant job ( $q(\theta)$  low) than for an unemployed worker to find a job ( $\theta q(\theta)$  high).

At each unit of time, a rate  $s$  of existing jobs is destroyed. This rate of job destruction is exogenous in our model. Thus, at the firm level, the employment evolves following the law of motion:  $\dot{n}_i = q(\theta)v_i - sn_i$ . Indeed, at each time, the employment of the firm  $i$  increases with the vacancies which are filled and decreases with the existing jobs which are destroyed.

The real wage  $w_i(n_i)$  is continuously and instantaneously negotiated soon after new informations arrive and is function of firm's employment.

#### 1.1.4 Sequence of events

The time schedule of the model can be illustrated by the following diagram:

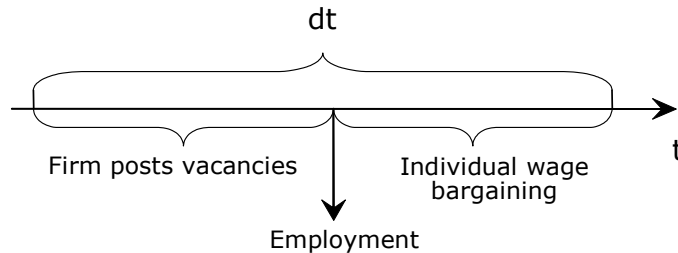


Figure 1: Sequence of events

The firm posts as many vacancies as necessary to hire in expectation the desired number of workers. It takes its decision while considering the real wage given by the incoming wage bargaining and expecting the fact that its employment level will have an effect on this. Then, once the employment level is determined, the individual wage bargaining takes place. The real wages are negotiated between the firm and each worker (individual bargaining) even its incumbents, so that each worker is treated as a marginal worker<sup>7</sup>.

<sup>6</sup>As stated previously, the firms are assumed to be too small to have any influence on the aggregate state of the economy.

<sup>7</sup>See Stole and Zwiebel (1996) for more details about the timing of a bargaining session.

## 1.2 Firm's behavior

The firm  $i$  maximizes the discounted value of future real profits  $\pi_i$ ; its state variable is its employment level  $n_i$  and its control variable is its amount of posted vacancies  $v_i$ . Thus, it opens as many vacancies as necessary<sup>8</sup>  $v_i \in [0; v_{i \max}]$  and without delay to have the desired employment level leading to the maximization of the discounted value of future real profits. Its discount rate is  $r$ .

Its problem is to solve<sup>9</sup>:

$$\begin{aligned} V(n_{i0}) &= \max_{v_i} \int_0^{+\infty} e^{-rt} \pi_i(n_i, v_i) dt \\ \text{s.t.} \quad \pi_i(n_i, v_i) &= \frac{p_i}{P}(y_i) y_i(n_i) - w_i(n_i) n_i - c v_i \end{aligned} \quad (1)$$

$$\dot{n}_i = q(\theta) v_i - s n_i, \quad n_i > 0, \quad n_i(0) = n_{i0} \quad (2)$$

$$y_i(n_i) = A n_i \quad (3)$$

$$\frac{p_i}{P}(y_i^d) = (y_i^d/Y)^{-1/\sigma}, \quad y_i^d = y_i(n_i) \quad (4)$$

$$0 \leq v_i \leq v_{i \max} \quad (5)$$

$$w_i(n_i) \quad \text{given} \quad (6)$$

We assume that the firm produces exactly the demanded output, so that the product market is clear (condition 4). Condition (5) expresses the fact that the firm expects an effect of its employment level on the bargaining outcome.

The Euler first order condition entails that the optimal solution of the problem of firm  $i$ ,  $n_i^*(t)$ , is such that<sup>10</sup>:

$$\frac{\partial \frac{p_i}{P}(y_i)}{\partial y_i} \frac{\partial y_i(n_i)}{\partial n_i} y_i(n_i) + \frac{p_i}{P}(y_i) \frac{\partial y_i(n_i)}{\partial n_i} - w_i(n_i) - \frac{\partial w_i(n_i)}{\partial n_i} n_i = \frac{(s+r)c}{q(\theta)} - \frac{\eta c \dot{\theta}}{\theta q(\theta)}.$$

As  $\frac{\partial \frac{p_i}{P}(y_i)}{\partial y_i} \frac{y_i(n_i)}{\frac{p_i}{P}(y_i)} = -\frac{1}{\sigma}$ , the previous equality becomes:

$$\frac{p_i}{P}(y_i) = \frac{\sigma}{\sigma-1} \left\{ \left[ w_i(n_i) + \frac{\partial w_i(n_i)}{\partial n_i} n_i + (r+s) \frac{c}{q(\theta)} - \frac{\eta c \dot{\theta}}{\theta q(\theta)} \right] \left( \frac{\partial y_i(n_i)}{\partial n_i} \right)^{-1} \right\}.$$

The firm fix its price by taking a mark-up equal to  $\frac{\sigma}{\sigma-1}$  on its marginal cost of labor. This equation can be also rewritten as an expression relating the firm's employment and real wage:

$$w_i^D(n_i) = \frac{\sigma-1}{\sigma} \frac{p_i}{P}(y_i) \frac{\partial y_i(n_i)}{\partial n_i} - \frac{\partial w_i^S(n_i)}{\partial n_i} n_i - (r+s) \frac{c}{q(\theta)} + \frac{\eta c \dot{\theta}}{\theta q(\theta)}. \quad (7)$$

<sup>8</sup>We assume that the opened vacancies by the firm  $i$  cannot be greater than its total capacity noted  $v_{i \max}$ .

<sup>9</sup>The  $t$  index is removed for more convenient notations.

<sup>10</sup>See appendix B about the existence and uniqueness of solutions to the firm's problem.



This expression corresponds to the optimal "pseudo" labor demand of the firm  $i$ . In (7) and hereafter, we note  $w_i^D(n_i)$  the real wage of the "pseudo" labor demand and  $w_i^S(n_i)$  the real wage of the "pseudo" labor supply which will be derived later, even though both of them result from the bargaining process and represent the same thing. The derivative  $\frac{\partial w_i^S(n_i)}{\partial n_i}$  shows us that the firm expects that the bargaining result is influenced by its employment level. We note through  $\frac{\eta c \dot{\theta}}{\theta q(\theta)}$  that the labor demand is also driven by the state of the labor market  $\theta$ , the matching elasticity  $\eta$  and the expected evolution of the labor market tightness  $\dot{\theta}$ . Thus, the real wage that the firm  $i$  is willing to pay increases when the firm expects an increase in the labor market tightness  $\dot{\theta} > 0$  and decreases in the reverse case. Indeed, a tighter labor market involves a lower rate at which the firm meets unemployed workers ( $q'(\theta) < 0$ ) and so the firm consents to pay higher wage to obtain the desired employment level. The magnitude of the impact of firm's expectation is function of  $\eta$ ,  $c$  and  $\theta$ . When the labor market is tight,  $\theta q(\theta)$  is high, so that  $\frac{\eta c}{\theta q(\theta)}$  is weak; whereas when the labor market is less tight,  $\theta q(\theta)$  is weaker, so that  $\frac{\eta c}{\theta q(\theta)}$  is high. Thus, the effect of firm's expectation on labor market tightness will be greater in the case of weak labor market tightness than in the case of strong labor market tightness.

Now, we need to model the wage bargaining in order to get the "pseudo" supply of labor  $w_i^S(n_i)$  and complete the determination of  $w_i^D(n_i)$ <sup>11</sup>.

### 1.3 Intrafirm bargaining

The wage bargaining takes place between the worker and the firm (individual bargaining). The real wage is continuously and instantaneously negotiated with all workers after a new hiring or laid off (after any variation of the employment level). During the bargaining all workers are treated as marginal workers, so that incumbent workers have no insider power during wage negotiation. Moreover, all workers are assumed identical regarding their skill level. Given these assumptions, the negotiated wage is the same for all workers of the firm.

The firm opens vacancies which are matched to the pool of unemployed workers and lead to employment. The actual employment is fixed at the same time that the negotiation. Thus, the firm determines its employment level heeding its effect on the wage negotiation outcome. That's why we have the derivative  $\frac{\partial w_i^S(n_i)}{\partial n_i}$  in (7).

Let  $J_i$  and  $V_i$  the present discounted value of expected profit from an additional filled job<sup>12</sup> and a vacant one, and let  $E_i$  and  $U_i$  the present discounted

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<sup>11</sup>In order to obtain also the specification of  $\dot{\theta}$ , which can be derived from (7) only, we need to specify  $\frac{\partial w_i^S(n_i)}{\partial n_i}$  from to the wage bargaining.

<sup>12</sup>We must keep in mind that we have large firms here and not one job for each firm as in Pissarides (2000).

value of expected income stream of an employed and unemployed worker. Thus, the surplus of the firm and the worker from a match are equal to  $J_i - V_i$  and  $E_i - U_i$  respectively<sup>13</sup>.

Since the firms can open as many vacancies as necessary to obtain their optimal employment level and exploit all the new job profit opportunities, the free entry condition drives the value of  $V_i$  to zero and  $J_i$  is equal to the expected recruitment cost<sup>14</sup>:

$$J_i = \frac{\partial V(n_{i0})}{\partial n_{i0}} = \frac{c}{q(\theta)} \quad (8)$$

which entails:

$$J_i = \frac{\eta c \theta}{\theta q(\theta)}. \quad (9)$$

According to the first order conditions of the firm's program, we have also:

$$J_i = \left[ \frac{\sigma - 1}{\sigma} \frac{p_i}{P}(y_i) \frac{\partial y_i}{\partial n_i} - w_i^S(n_i) - \frac{\partial w_i^S(n_i)}{\partial n_i} n_i + \frac{\eta c \theta}{\theta q(\theta)} \right] / (r + s). \quad (10)$$

The present discounted utility of an employed worker  $E_i$  satisfies the Bellman equation:

$$rE_i = w_i^S(n_i) - s[E_i - U_i] + \dot{E}_i. \quad (11)$$

which is equal to the real wage  $w_i^S(n_i)$  minus the expected loss of income in the case of change of state (become unemployed) plus the expected gain from changes in job value  $\dot{E}_i$ .

In a similar way, the present discounted utility of an unemployed worker  $U_i$  satisfies the Bellman equation:

$$rU_i = b + \theta q(\theta)[E_i - U_i] + \dot{U}_i. \quad (12)$$

This expression represents the reservation wage of a worker which is equal to the sum of the unemployment benefits  $b$ , the expected gain in the case of change of state (get a job) and the expected gain from a change in unemployment value  $\dot{U}_i$ . Furthermore, we assume that workers stay in their job as long as  $E_i - U_i > 0$ <sup>15</sup>.

During the bargaining, the total surplus of the matching  $S_i = J_i + E_i - U_i$  is divided between the firm and the worker according to their respective bargaining power. As a result, following the usual Nash sharing rule, the negotiated real wage solves:

$$E_i - U_i = \frac{\gamma}{1 - \gamma} J_i \quad (13)$$

where  $\gamma \in [0, 1]$  represents the bargaining power of workers. With (13) and (8), we can rewrite the expression  $rU_i$  as follows:

$$rU_i = b + \frac{\gamma}{1 - \gamma} \theta c + \dot{U}_i. \quad (14)$$

<sup>13</sup>Note that both of them depend on  $w_i(n_i)$ .

<sup>14</sup>See Appendix B on the firm problem for further details.

<sup>15</sup>We show later that this inequality holds if and only if  $b < \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)$ .

Note that this expression is given at the firm level. Indeed,  $rU_i$  depends on parameters on which the firm and workers cannot have any influence. After differentiation of (13)<sup>16</sup> we can also write:

$$\dot{E}_i - \dot{U}_i = \frac{\gamma}{1-\gamma} \dot{J}_i = \frac{\gamma}{1-\gamma} \frac{\eta c \dot{\theta}}{\theta q(\theta)}. \quad (15)$$

The expressions (11), (13), (14) and (15) lead to the following first order differential equation:

$$w_i^S(n_i) = (1-\gamma)b + \gamma \left[ \frac{\sigma-1}{\sigma} \frac{p_i}{P}(n_i) \frac{\partial y_i}{\partial n_i} - \frac{\partial w_i^S(n_i)}{\partial n_i} n_i + \theta c \right] \quad (16)$$

the solution<sup>17</sup> of which gives us the negotiated wage:

$$w_i^S(n_i) = (1-\gamma)b + \gamma \left[ \frac{\sigma-1}{\sigma-\gamma} \frac{p_i}{P}(y_i) \frac{\partial y_i}{\partial n_i} + \theta c \right]. \quad (17)$$

The negotiated wage is independent from any dynamics what corroborates one of Pissarides' result according to which the negotiated wage holds both in and out of steady state<sup>18</sup>. This expression represents the "pseudo" labor supply at the firm level.

Now, we compute the derivative of equation (17) to insert it in the equation (7):

$$\frac{\partial w_i^S(n_i)}{\partial n_i} = \gamma \frac{\sigma-1}{\sigma-\gamma} \left( -\frac{1}{\sigma} \right) \frac{p_i}{P}(n_i) \frac{\partial y_i}{\partial n_i} n_i^{-1}. \quad (18)$$

This equation represents both the hiring externality due to the intrafirm bargaining framework and the slope of the wage curve (17). This hiring externality is always negative what translates that an additionnal worker will decrease the wages of all others. The substitution of (18) in (7) gives finally the "pseudo" labor demand including the wage negotiation outcome:

$$w_i^D(n_i) = \frac{\sigma-1}{\sigma-\gamma} \frac{p_i}{P}(y_i) \frac{\partial y_i}{\partial n_i} - (r+s) \frac{c}{q(\theta)} + \frac{\eta c \dot{\theta}}{\theta q(\theta)}. \quad (19)$$

## 1.4 General equilibrium and steady state

### 1.4.1 General equilibrium

At the general symmetric equilibrium all the firms and workers are identical. As a consequence, given the assumption on the labor force size and firms distribution, we have  $\frac{P_i}{P} = 1$ ,  $n_i = n = \bar{n}$ ,  $y_i = y = Y$ ,  $v_i = v$ ,  $J_i = J$ ,  $E_i = E$ ,

<sup>16</sup> Given that the negotiated wage is continuously renegotiated, this sharing rule holds also in rates of change.

<sup>17</sup> See appendix C for the detailed resolution.

<sup>18</sup> Indeed, the wage curve is independent of the law of motion of  $n_i$  and  $\theta$ .

$U_i = U$  and  $u = 1 - n$  which entails  $v = \theta(1 - n)$ . The aggregate production function becomes:  $Y = n^{1+\alpha}$ , with  $1 + \alpha \geq 1$ . As a result, in the presence of positive aggregate externality the returns to scale of the aggregate production technology are increasing. Otherwise, they are constant over time.

At the general equilibrium, the labor demand and labor supply are given by:

$$w^D(n) = \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)n^\alpha - (r + s)\frac{c}{q(\theta)} + \frac{\eta c \dot{\theta}}{\theta q(\theta)} \quad (20)$$

$$w^S(n) = (1 - \gamma)b + \gamma \left[ \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)n^\alpha + \theta c \right]. \quad (21)$$

Given that  $w^D(n) = w^S(n)$ , we obtain the following dynamical system characterizing the law of motion of  $\theta$  and  $n$  in our economy:

$$\begin{cases} \dot{\theta} = \frac{\theta q(\theta)}{\eta c} \left[ (1 - \gamma)b + \gamma \theta c + (r + s)\frac{c}{q(\theta)} - (1 - \gamma)\frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)n^\alpha \right] \\ \dot{n} = q(\theta)(1 - n)\theta - sn \\ \theta(0) = \theta_0, \quad n(0) = n_0. \end{cases} \quad (22)$$

The phase diagram which illustrates the dynamical behavior of our economy will be plotted in the space  $(n, \theta)$  (see figures 2 and 3).

#### 1.4.2 Steady state

At the steady state, the employment  $n$  and labor market tightness  $\theta$  are constant  $\dot{n} = \dot{\theta} = 0$ <sup>19</sup>. Hence, at the steady state we have:

$$w^D(n) = \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)n^\alpha - (r + s)\frac{c}{q(\theta)} \quad (23)$$

$$w^S(n) = (1 - \gamma)b + \gamma \left[ \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)n^\alpha + \theta c \right] \quad (24)$$

which are similar than those found in Cahuc and Wasmer (2004) and Ebell and Haefke (2003).

Since  $u = 1 - n$ , the tightness of labor market can be written:  $\theta = \frac{v}{u} = \frac{v}{1 - n}$  and given the constancy of employment rate at the steady state, we have also:

$$\dot{n} = q(\theta)v - sn = 0 \Leftrightarrow v = \frac{sn}{q(\theta)} = sn\theta^\eta. \quad (25)$$

<sup>19</sup>The curve  $\dot{n} = 0$  corresponds to the flow equilibrium condition which implies a pseudo Beveridge Curve, that is to say, a positive relation between vacancies and employment or labor market tightness and employment. The curve  $\dot{\theta} = 0$  leads to the vacancy curve which represents a relation between the labor market thigness and the employment, the sign of which depends on the size of returns to scale.

Using the previous expressions, we deduce the expression of  $\theta$  and  $q(\theta)$  as function of  $n$  only:

$$\theta(n) = \left[ \frac{sn}{1-n} \right]^{1/1-\eta}, \quad \text{with } \frac{\partial \theta(n)}{\partial n} > 0 \quad (26)$$

$$q(\theta(n)) = \left[ \frac{sn}{1-n} \right]^{-\eta/1-\eta}, \quad \text{with } \frac{\partial q(\theta(n))}{\partial n} < 0 \quad (27)$$

and we can also rewrite  $w^D(n)$  and  $w^S(n)$  as function of employment rate only:

$$w^D(n) = \frac{\sigma-1}{\sigma-\gamma} (1+\alpha)n^\alpha - (r+s)c \left[ \frac{sn}{1-n} \right]^{\eta/1-\eta} \quad (28)$$

$$w^S(n) = (1-\gamma)b + \gamma \left[ \frac{\sigma-1}{\sigma-\gamma} (1+\alpha)n^\alpha + c \left( \frac{sn}{1-n} \right)^{1/1-\eta} \right]. \quad (29)$$

## 2 Equilibrium

In this section, we study the equilibria of our economy and their properties. In our model, an equilibrium corresponds to any couple of variables  $(n^*, w^*)$  for which both firms and workers behave optimally ( $w^D(n) = w^S(n)$ ) at the steady state ( $\dot{\theta} = \dot{n} = 0$ ). As a consequence, it corresponds to the intersections between the "pseudo" demand of labor and the "pseudo" supply of labor, given by (28) and (29) respectively, in the space  $(n, w)$ . We note that the equilibrium values of employment rate and of labor market tightness can be also deduced from the intersections between the curves  $\dot{\theta} = 0$  and  $\dot{n} = 0$  in the space  $(n, \theta)$ .

Before the analysis of equilibrium cases, we compute the partial derivatives of equations (28) and (29):

$$\begin{aligned} \frac{\partial w^D(n)}{\partial n} &= \frac{\sigma-1}{\sigma-\gamma} \alpha (1+\alpha) n^{\alpha-1} - (r+s)c \left[ -\frac{\partial q(\theta(n))}{\partial n} q(\theta(n))^{-2} \right] \\ &= \frac{\sigma-1}{\sigma-\gamma} \alpha (1+\alpha) n^{\alpha-1} - (r+s)c \left[ \left( \frac{\eta}{1-\eta} \right) \frac{s}{(1-n)^2} \left( \frac{sn}{1-n} \right)^{\frac{2\eta-1}{1-\eta}} \right] \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial w^S(n)}{\partial n} &= \gamma \left[ \frac{\sigma-1}{\sigma-\gamma} \alpha (\alpha+1) n^{\alpha-1} + c \left( \frac{\partial \theta(n)}{\partial n} \right) \right] \\ &= \gamma \left[ \frac{\sigma-1}{\sigma-\gamma} \alpha (\alpha+1) n^{\alpha-1} + c \left( \frac{\eta}{1-\eta} \right) \frac{s}{(1-n)^2} \left( \frac{sn}{1-n} \right)^{\frac{\eta}{1-\eta}} \right]. \end{aligned} \quad (31)$$

We note that the curve  $w^S(n)$  is always increasing in the space  $(n, w)$  whatever the size of the aggregate externality, whereas the sign of the slope of the curve  $w^D(n)$  can change.

## 2.1 Constant returns to scale and unique equilibrium

**Proposition 1** *When the aggregate returns to scale are constant ( $\alpha = 0$ ), there is an unique equilibrium on the interval  $n \in ]0, 1[$  when  $b < \frac{\sigma - 1}{\sigma - \gamma}$  and none otherwise.*

**Proof.** In the space  $(n, w)$ , the expression  $w^D(n)$  is always decreasing  $\left(\frac{\partial w^D(n)}{\partial n} < 0\right)$

and  $w^S(n)$  is always increasing  $\left(\frac{\partial w^S(n)}{\partial n} > 0\right)$  when  $\alpha = 0$ . Now, to show the existence of an unique interior equilibrium<sup>20</sup> and so of one intersection between the curves in the space  $(n, w)$ , we must demonstrate that the curve  $w^D(n)$  is above the curve  $w^S(n)$  when  $n$  tends toward zero. We can easily show that  $w^D(0) = \frac{\sigma - 1}{\sigma - \gamma} > w^S(0) = (1 - \gamma)b + \gamma \frac{\sigma - 1}{\sigma - \gamma}$  holds if and only if  $b < \frac{\sigma - 1}{\sigma - \gamma}$  which is imposed<sup>21</sup>. ■

The study of the dynamical system given by (22)<sup>22</sup> allows us to establish the following proposition:

**Proposition 2** *When  $\alpha = 0$ , the unique interior equilibrium, when it exists, is a saddle point, the saddle path of which is given by  $\theta = 0$ .*

**Proof.** See appendix D. ■

We find thus the same equilibrium properties than in Pissarides (2000), that is to say the unique equilibrium is a saddle point which is reached for  $\theta = 0$ . The dynamics of the economy is presented in the figure 2 where the dashed line represents the saddle path.

<sup>20</sup>Note that we will always name an equilibrium on the interval  $]0, 1[$  as an interior equilibrium in the paper.

<sup>21</sup>This condition represents the sufficient condition for which we have always  $E_i > U_i$  in the case of constant aggregate returns to scale. When  $\alpha > 0$ , this condition becomes  $b < \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)$ .

<sup>22</sup>See appendix D for details about the study of the dynamical system (22).

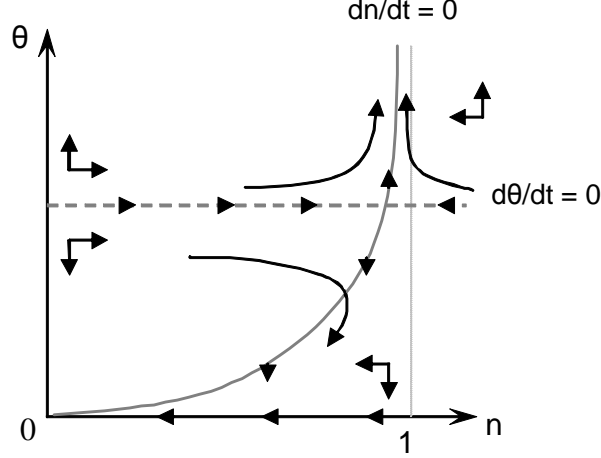


Figure 2: Phase diagram when  $\alpha = 0$

## 2.2 Increasing returns to scale and multiple equilibria

**Proposition 3** *When the aggregate returns to scale are increasing ( $\alpha > 0$ ), there are multiple distinct equilibria or none on the interval  $n \in ]0, 1[$ .*

**Proof.** To show the existence of multiple equilibria, we need to show the multiplicity of intersections between the curves  $w^D(n)$  and  $w^S(n)$  in the space  $(n, w)$ . The curve  $w^S(n)$  is strictly increasing in the space  $(n, w)$  when the aggregate returns to scale are increasing. The curve  $w^D(n)$  is increasing and decreasing when  $n$  becomes high (near to one). Indeed, its partial derivative (30) becomes negative for values of  $n > \tilde{n}$ <sup>23</sup>. Truly, when  $n > \tilde{n}$ ,  $\frac{\sigma - 1}{\sigma - \gamma} \alpha(\alpha + 1)n^{\alpha - 1}$

becomes inferior to  $(r + s)c \left[ \left( \frac{\eta}{1 - \eta} \right) \frac{s}{(1 - n)^2} \left( \frac{sn}{1 - n} \right)^{\frac{2\eta - 1}{1 - \eta}} \right]$  and (30) becomes negative. Moreover, when  $\alpha > 0$ , we have always  $w^D(0) = 0 < w^S(0) = (1 - \gamma)b$ . As a consequence, if the curves intersect, they must do it at least twice. ■

Although we cannot determine the exact number of interior equilibria, the study of dynamical system (22) shows that their behavior is the same than in the case where there exists two interior equilibria only. Furthermore, we will show later thanks to numerical simulations that we have always two equilibria

<sup>23</sup>This feature will be illustrated in the numerical simulations. The threshold  $\tilde{n}$  is such that  $\frac{\sigma - 1}{\sigma - \gamma} \alpha(\alpha + 1)n^{\alpha - 1} = (r + s)c \left[ \left( \frac{\eta}{1 - \eta} \right) \frac{s}{(1 - n)^2} \left( \frac{sn}{1 - n} \right)^{\frac{2\eta - 1}{1 - \eta}} \right]$

with realistic values for the parameters. Thus, we can deduce the following proposition concerning equilibria's properties:

**Proposition 4** *When  $\alpha > 0$ , the high interior equilibrium  $(n_H^*, w_H^*)$ , which is the one with high level of both employment and real wage, is a saddle point with a saddle path given by  $\theta \neq 0$ . The low interior equilibrium  $(n_L^*, w_L^*)$ , which is the one with low level of both employment and real wage, can be either attractive (a sink) or repulsive (a source) according to the parameters values<sup>24</sup>.*

**Proof.** See appendix D. ■

As a consequence, we don't find Pissarides's result anymore and note similarities with Mortensen (1999) concerning the properties of the low equilibrium. Indeed, Mortensen (1999) gives numerical examples for which the low equilibrium in his model is a sink and a source. As in this paper, the properties of his low equilibrium depends on some parameters values. According to appendix D, the dynamics of the economy can be represented by the following phase diagram where the dashed line represents the saddle path:

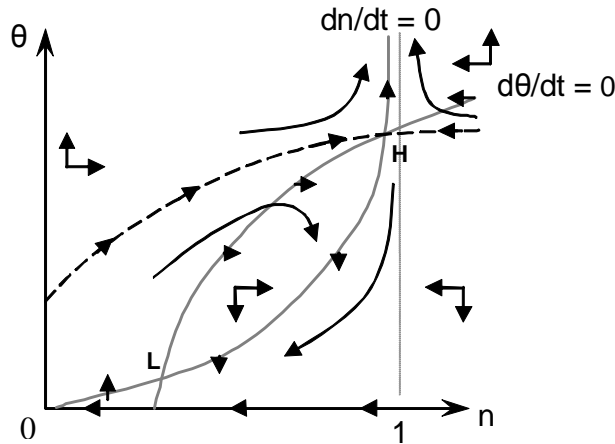


Figure 3: Phase diagram when  $\alpha > 0$

As indicated by the two phase diagrams, the dynamical system which is studied in details in appendix D (without resource constraint) has also two corner steady states which have identical properties whatever the value of  $\alpha$ . One of them represents a no employment state with  $(n, \theta) = (0, 0)$  where the economy could converge for some initial conditions below the saddle path<sup>25</sup>. The

<sup>24</sup>We will give examples where the low equilibrium is a sink in the numerical simulation part.

<sup>25</sup>For any initial conditions below the saddle path, the economy can converge only to this steady state in the case of unique equilibrium; whereas in the case of multiple equilibria it can also converge to the low equilibrium when the latter is attractive.



firms' behavior which leads to this no employment steady states is the absence of opened vacancies. Indeed, in this case, the employment level decreases due to the destruction of existing jobs and, without opened vacancies, converges to zero. The other corner steady state is a full employment state with  $(n, \theta) = (1, +\infty)$ . where the economy could converge for some initial conditions above the saddle path. However, given the resource constraint, this steady state is not workable. Needing an infinity of opened vacancies, an employment rate equal to one is incompatible with the constraint of a finite production level, because the firms do not open more vacancies than their capacity constraint<sup>26</sup>. Consequently, these two corner steady state are ruled out of the analysis in what follows. According to both phase diagrams, we also note that the saddle path represents the frontier between different dynamical behaviors of our economy.

### 2.3 Discussion

The previous propositions and figures show us how the dynamical behavior of the economy is altered as soon as a positive externality exists. We can state that:

- A given behavior of agents with given initial conditions will lead to a different outcome according to the size of aggregate returns to scale.
- When  $\alpha > 0$ , a given behavior of agents with given initial conditions can lead to a different outcome according to some parameters values.

The employment rate is sticky and stable; whereas the vacancies are forward looking and unstable. Thus, the economy converges to one equilibrium by means of opened vacancies which represent the firms' control variable. The tightness of labor market  $\theta$  determines the rate at which the firms meet unemployed workers  $q(\theta)$ . Thus, an expected future change in the labor market tightness leads the firms to expect a future change in the rate at which they meet unemployed workers and, as a result, influences immediately their supply of vacancies.

In the case of  $\alpha = 0$ , the saddle path is the  $\theta$ -stationary because the law of motion of  $\theta$  is independent from employment  $n$ . As a consequence, for any initial condition  $n_0$  and in the absence of anticipated future change in  $\theta$ , the economy is located on the saddle path and the employment level varies until there is convergence to equilibrium (Pissarides, 2000, p. 30). However, if the firms expect a future decrease in the labor market tightness ( $\dot{\theta} < 0$ ) they expect a future increase in the rate at which they meet unemployed. As a result, they are incited to open fewer vacancies because they will become easier to fill in the future. An expected fall in the labor market tightness leads thus to an immediate fall in opened vacancies which lead to an immediate fall in  $\theta$ , and the expectations are self-fulfilling. In the end, the economy can be caught in the no employment steady state (see figure 2).

In the case of  $\alpha > 0$ , the law of motion of  $\theta$  is function of  $n$  and the result of Pissarides does not hold. In this case the saddle path has the property that

<sup>26</sup>Note that in the firm's program the opened vacancies  $v_i$  are such that  $v_i \in [0; v_{i \max}]$ .

$\dot{\theta} > 0$  if  $n_0 < n_H^*$  and  $\dot{\theta} < 0$  if  $n_0 > n_H^*$ . When  $n_0 < n_H^*$ , if the firms expect an increase in the tightness of labor market  $\dot{\theta} > 0$  (optimistic expectations<sup>27</sup>), they will immediately open up more vacancies to get the desired employment because they expect a gradual fall in the rate at which they will meet unemployed later. As a result, the mass of opened vacancies, the employment and the tightness of labor market increase. However, the high equilibrium can be reached if and only if the firms have sufficient optimistic expectations. Actually, if their expectations are less optimistic (inferior to the saddle path), the employment level turns down before it reaches the high equilibrium (see figure 3) and the economy is caught to the low equilibrium (if it is attractive) or to the no employment steady state. Thus, a coordination failure appears.

Consequently, the policy implications are very different between the case of unique equilibrium and multiple equilibria. In the case of unique equilibrium, the government should inspire agents in the no change in  $\theta$  to reach the interior steady state; whereas it should induce agents to expect a given variation of the labor market tightness in the other case. In other words, while it should just maintain a stable economic environment in the first case, the government must act to coordinate agents' expectations in the latter. For example, by hiring subsidies or recruitment campaign, the government can induce the firms to expect a future tightening of labor market and it can induce the reverse by operating governmental budget cuts or tax wages.

### 3 Numerical simulations

In this section, we run numerical simulations in order to give support to the previous propositions and to assess the effects of some economic policies on labor market performances. Then, we give evidences on reverse policy implications according to the equilibrium reached by the economy as well as on the possibility for an economic policy to eliminate some unwanted equilibrium.

The model period is one year. We don't calibrate the model according to empirical studies on a particular economy, but our parametrization choices are based on previous works (Ebell and Haefke, 2003). However, some difficulties appear regarding some parameters. The parametrization is given in the following table:

*Table 1: Parameters values*

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<sup>27</sup>We consider this expectation as optimistic because a tight labor market is due to an excess of supply of job compared to the unemployed worker translating a good economic environment.

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$\gamma = 0.5$	Bargaining power of workers
$\eta = 0.5$	Elasticity of matching function
$s = 0.1$	Job destruction rate
$c = 0.2$	Vacancy cost
$r = 0.04$	Discount rate
$b = 0.3$	Unemployment benefit
$\sigma = 2$	Competition degree on good market

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In a first time, we set  $\gamma = \eta = 0.5$  which doesn't correspond to the Hosios (1990) condition here due to the presence of intrafirm bargaining (Cahuc and Wasmer, 2001a)<sup>28</sup>. Later, we will change the value of  $\gamma$  to show how this parameter influences the properties of the low equilibrium in the case of multiple equilibria. The job destruction rate value corresponds to a destruction of 10% of existing jobs each year and the discounted rate one to a 4% annual interest rate. The parameter  $b$  is commonly interpreted as the monetary compensation for the unemployed. Usually, it is easier to consider the replacement rate, i.e. the ratio of benefits to wage, during calibration. According to OECD reports, the average replacement rate can be very different from a country to another and from a family type to another but in average it ranges between 20% and 80%. In our model, the wage is determined endogenously and so when we fix a value for  $b$  we can give the replacement rate value after its determination. Here we fix the parameter  $b$  to 0.3<sup>29</sup>. The same problem appears for the parameter of vacancy cost  $c$ , we fix its value at 0.2 here. The parameter  $\sigma > 1$ , which represents the demand elasticity of the good supplied by the firms, translates the degree of competition on the good market in our economy<sup>30</sup>. When the competition is strongly monopolistic and firms have a high market power  $\sigma$  tends to 1 and, inversely, when the competition comes near to perfect competition,  $\sigma$  tends to  $+\infty$ . This parameter is fixed to 2 (monopolistic competition). Later, we will increase its value in order to check the effects of the good market competition.

In all following graphs, the black curves represent  $w^D(n)$  and the grey curves  $w^S(n)$  given by the expressions (28) and (29) respectively.

### 3.1 Equilibrium

In this part, we vary the extent of the externality in order to illustrate propositions 1 and 3. When there is no aggregate externality  $\alpha = 0$ , the aggregate

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<sup>28</sup>Hosios (1990) identified a general condition under which all the externalities of search process are internalised and all decisions are efficient: the matching elasticity with respect to unemployment must be equal to the worker's share of the match surplus in the case of constant returns to scale in the production technology. However, Cahuc and Wasmer (2001a) have showed that this condition does not hold anymore in the case of intrafirm bargaining.

<sup>29</sup>This value of  $b$  verifies always the condition  $b < \frac{\sigma - 1}{\sigma - \gamma}(1 + \alpha)$  specified in the subsection 2.1.

<sup>30</sup>This feature comes from the assumption on firms distribution on the interval  $[0, 1]$ .

returns to scale in the economy are constant; whereas they are increasing in the case of positive aggregate externality  $\alpha > 0$ .

Next, according to our parameterization and equilibrium outcomes, we compute the reservation wage of workers  $rU$ <sup>31</sup> given by expression (14) to compare our findings with those of Stole and Zwiebel (1996) and Ebell and Haefke (2003). In Stole and Zwiebel (1996), the negotiated wage is equal to the reservation wage and workers get no rent from employment. Indeed, they earn what they would expect to earn if they were unemployed. This result comes from the hiring externality which depresses wages when an additional worker is hired. Thus, the firms overemploy in order to moderate the workers' wages. In their paper, Ebell and Haefke (2003) show that the overemployment incentive induced by the hiring externality is reduced by the monopolistic competition on the good market. Indeed, monopolistic competition incites the firms to lower their output level in order to preserve their power on the prices. Thus, this underemployment distortion offsets partially the overemployment incentive due to intrafirm bargaining and workers get rents.

### 3.1.1 Constant aggregate returns to scale

We illustrate that we have an unique equilibrium in the case where the aggregate returns to scale are constant.

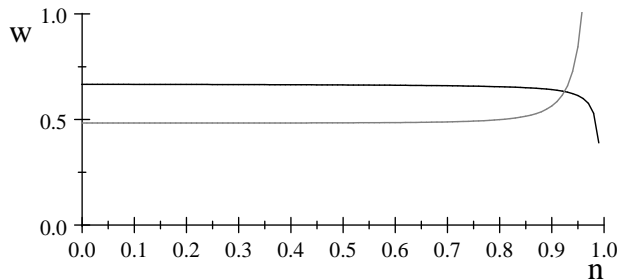


Figure 4: No externality  $\alpha = 0$

Thus, we find the usual results that the strategic complementarity introduced by the monopolistic competition is not strong enough to involve the presence of multiple equilibria. Concerning the workers' rent, we find the same result than Ebell and Haefke (2003). The workers get rents and their wages are inferior to the marginal productivity due to the presence of hiring cost. Thus, in spite of the presence of the hiring externality, the negotiated wage is greater than the reservation wage because of the underemployment distortion involved by the monopolistic competition.

<sup>31</sup>At the each equilibrium, the reservation wage can be deduced from the equilibrium values of  $n$  and equation (14).

### 3.1.2 Increasing aggregate returns to scale

Here, we run numerical simulations with two distinct values for  $\alpha$ :  $\alpha = 0.5$  and  $\alpha = 2$ . Doing so, we give support to the presence of multiple equilibria (proposition 3) and investigate the effects of the size of aggregate returns to scale on the equilibrium outcomes.

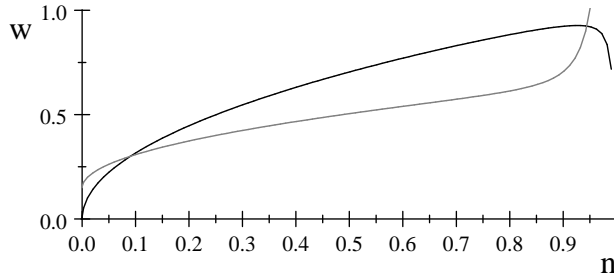


Figure 5: Externality  $\alpha = 0.5$

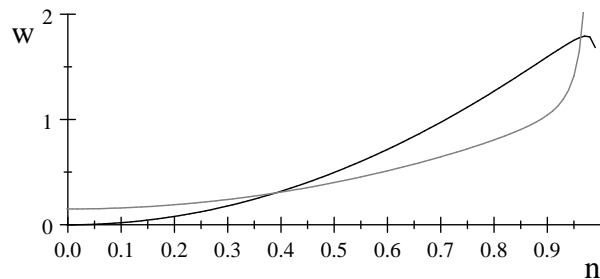


Figure 6: Externality  $\alpha = 2$

The previous graphs show the existence of two positive equilibria which are characterized as follows: the low equilibrium  $(n_L^*, w_L^*)$ , with a low employment rate and a low real wage, and the high equilibrium  $(n_H^*, w_H^*)$ , with a high employment rate and a high real wage. At the low equilibrium, we find Stole and Zwiebel's (1996) result. Indeed, the workers get no rents and the equilibrium wage is equal to the reservation wage which is itself approximately equal to the unemployment benefits  $b$ . However, we cannot state that this result comes from the overhiring incentive as in Stole and Zwiebel given the weakness of the employment level<sup>32</sup>. At the high equilibrium, our findings are similar to those of Ebell and Haefke (2003). As in the case of constant aggregate returns to scale, the underemployment incentive due to monopolistic competition offsets partially the overhiring incentive. Thus, workers get rents and the employment level is high. Finally, we note that the equilibrium employment rate and real wage raises at both equilibria when the aggregate externality increases.

<sup>32</sup>An explanation of this result is given in the next subsection where we show why this equilibrium is likely only in the case where the bargaining power of workers is very low.

### 3.2 Dynamical properties

The dynamical properties of the model in the case of unique equilibrium are always the same whatever the values of parameters. As a consequence, a carefully investigation of dynamics is necessary only when there exists multiple equilibria.

As mentioned previously, the high equilibrium is always a saddle point. Conversely, the low equilibrium can be attractive (a sink) or repulsive (a source) according to the values of parameters. Following the proof in appendix D and many numerical simulations, we can conclude that the low equilibrium is:

- always repulsive when  $\gamma \geq \eta$
- repulsive or attractive when  $\gamma < \eta$ <sup>33</sup>.

We have run several numerical simulations with various parameters values from which we can conclude that we must have a really low bargaining power of workers  $\gamma$  compared to the matching elasticity with respect to unemployment  $\eta$  in order to obtain an attractive low equilibrium. An example of values is  $\eta = 0.5$  and  $\gamma = 0.1$ , the other parameters values being given in table 1. According to the numerical simulations, we can also conclude that the attractiveness of the low equilibrium is not influenced by the degree of competition on the good market and the size of returns to scale. In figure 7, we illustrate this case of attractive low equilibrium where  $\gamma = 0.1$ .

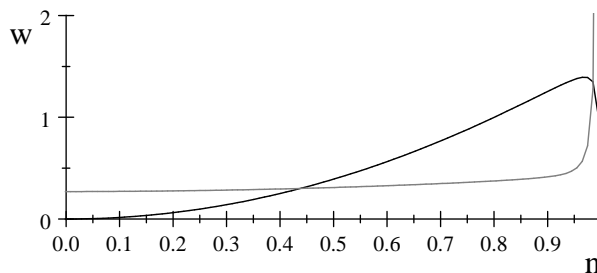


Figure 7: Attractive low equilibrium with  $\alpha = 2$

Thus, the economy can be caught at the low equilibrium only when the workers have a sufficient low bargaining power compared to the matching elasticity. This result can be explained as follows: When the bargaining power of workers is sufficiently low, the firms don't need to overemploy in order to moderate the workers' wage aspirations and the under-employment incentive due to the monopolistic competition on the good market leads to a low employment level. Consequently, workers get no rent at this equilibrium while the employment level is low. However, in the reverse case, the firms need to overemploy and the high equilibrium is the only stable equilibrium in the model. In the next

<sup>33</sup>This result is similar to this of Mortensen (1999) who shows how the low equilibrium is a sink or a source according to the value of  $\frac{\gamma}{1-\eta}$ .

subsections, we execute numerical simulations with  $\gamma = 0.1$  to obtain always the case of an attractive low equilibrium. Even if an economic policy shock can be unable to move the economy out the low equilibrium's basin of attraction, it is interesting to know how this low equilibrium changes. This is why we investigate comparative statics at the low equilibrium too.

### 3.3 Economic policy

Now, we compute the effects of a higher degree of competition and higher unemployment benefits. Next, we compare the comparative statics results between the unique equilibrium case and the multiple equilibria one.

#### 3.3.1 Higher degree of competition

Here we investigate the effects of a higher competition degree in the good market which can be due to market deregulation policies. To do it, we use two distinct values of  $\sigma$  translating the competition degree on the good market. The solid curves represent the case where  $\sigma = 2$  (the benchmark case), that is to say the case of a monopolistic competition. The dashed curves represent the case of a more competitive good market with  $\sigma = 100$ .

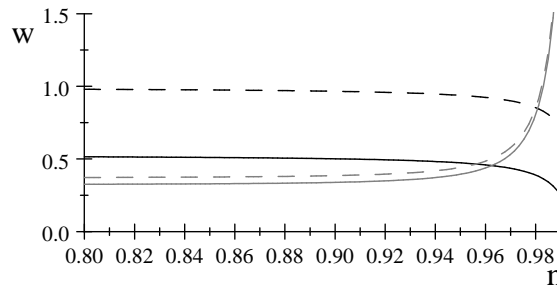


Figure 8: Stronger competition  
when  $\alpha = 0$

In the case of constant aggregate returns to scale, a higher competition on the product market improves both employment rate and real wage. These results are similar to those of Blanchard and Giavazzi (2003).

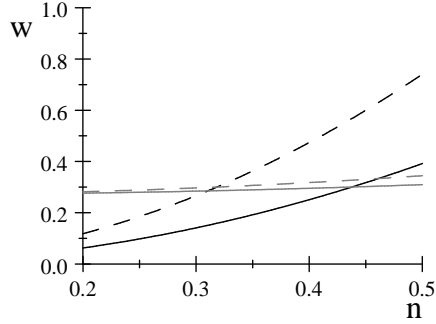


Figure 9: Stronger competition at the low equilibrium when  $\alpha = 2$

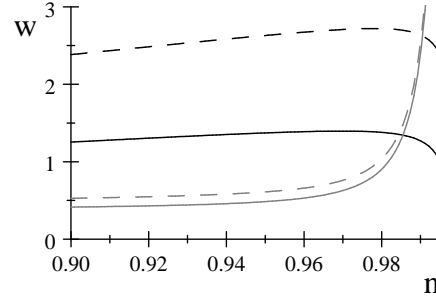


Figure 10: Stronger competition at the high equilibrium when  $\alpha = 2$

In the case where the aggregate returns to scale are increasing, we find results similar to Blanchard and Giavazzi (2003) at the high equilibrium only. Indeed, at the high equilibrium, a higher competition degree has a positive effect on real wage and a tiny but positive effect on the employment (figure 10). However, at the low equilibrium, a higher competition degree involves a lower employment rate and has a tiny positive effect on real wage (figure 9). Thus, the comparative statics result on employment has the reverse sign.

### 3.3.2 Unemployment benefits increase

Here, we investigate the effects of more generous unemployment benefits. We still use the parameters values of table 1 with  $\gamma = 0.1$  and  $b = 0.3$  in the benchmark case (solid curves) and compare the equilibrium outcomes with the case where unemployment benefits are higher,  $b = 0.4$  (dashed curves).

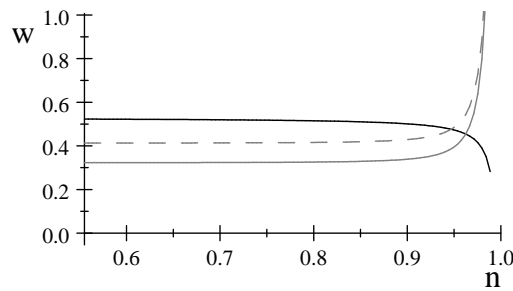


Figure 11: Higher unemployment benefits when  $\alpha = 0$

In the case of constant aggregate returns to scale we find the usual results that raising unemployment benefits involves a reduction in the employment rate and an increase in real wage. Actually, higher unemployment benefits imply



higher reservation wage for workers and reduce incentives to work. As a result, the real wage is higher; while the employment rate is lower.

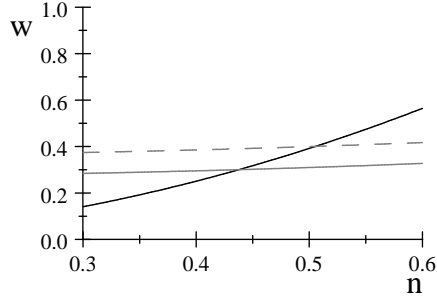


Figure 12: Higher unemployment benefits at the low equilibrium when  $\alpha = 2$

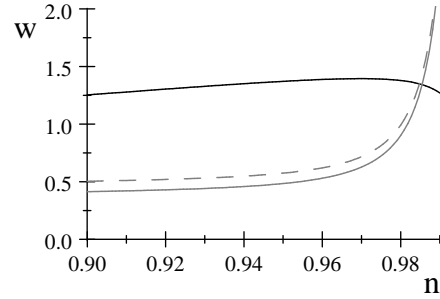


Figure 13: Higher unemployment benefits at the high equilibrium when  $\alpha = 2$

In the case of increasing returns to scale, an increase in unemployment benefits has:

- a tiny negative effect on employment and tiny positive effect on real wage at the high equilibrium (Figure 13)<sup>34</sup>
- a positive effect both on the employment rate and on the real wage at the low equilibrium (Figure 12).

Thus, the comparative statics results are still reversed for the employment rate.

### 3.4 Discussion

In this part, we discuss the economic policy implications in our economy. Table 2 summarizes our findings.

Table 2: Comparative statics results

<sup>34</sup>The effects go in the same direction than in the case of unique equilibrium, but they are weaker.

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	$\alpha = 0$	$\alpha > 0$
Product market deregulation:	$\frac{\partial n^*}{\partial \sigma} > 0$ $\frac{\partial w^*}{\partial \sigma} > 0$	$\frac{\partial n_H^*}{\partial \sigma} \geq 0$ $\frac{\partial w_H^*}{\partial \sigma} > 0$
$\Delta \sigma > 0$		$\frac{\partial n_L^*}{\partial \sigma} < 0$ $\frac{\partial w_L^*}{\partial \sigma} \geq 0$
Unemployment benefits increase:	$\frac{\partial n^*}{\partial b} < 0$ $\frac{\partial w^*}{\partial b} > 0$	$\frac{\partial n_H^*}{\partial b} \leq 0$ $\frac{\partial w_H^*}{\partial b} \geq 0$
$\Delta b > 0$		$\frac{\partial n_L^*}{\partial b} > 0$ $\frac{\partial w_L^*}{\partial b} > 0$

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In the case of unique equilibrium we find always the usual results of the literature. When there exists multiple equilibria, all the results are also similar to those of the literature at the high equilibrium. However, at the low equilibrium, a market deregulation and more generous unemployment benefits have the reverse effect on the equilibrium employment rate and the expected one on the real wage. We note also that at the high equilibrium, even though the effects on the equilibrium employment rate go in the same direction than these of the unique equilibrium, they are always weaker. These findings show us how the economic policy implications will be different between the high equilibrium and the low equilibrium. We note also that even if the economy is on the saddle point the effects can be weaker than those expected in the case of multiple equilibria.

Previously, we have showed that the low equilibrium can be attractive when the bargaining power of workers  $\gamma$  is sufficiently weak compared to the matching elasticity  $\eta$ . Otherwise, the low equilibrium is always repulsive and it cannot be reached whatever the initial conditions and the behavior of agents. Thus, an appropriate economic policy increasing the bargaining power of workers<sup>35</sup> eliminate any possibility of occurrence of the low equilibrium.

## Conclusion

In this paper, we have used a model similar to that of Ebell and Haefke (2003) with monopolistic competition on the product market, matching frictions and intrafirm bargaining on the labor market. The main contributions of our paper are: to allow for increasing aggregate returns to scale in such a framework leading to multiple equilibria, to give a detailed study of the model dynamics for constant and increasing aggregate returns to scale, and to investigate the equilibria multiplicity consequences on economic policy implementation. Then,

<sup>35</sup>After numerical simulation of such an economic policy, we have found that its effects are those found in the literature in all cases ( i.e higher real wage and lower employment). This is why we haven't reported the result here.

we have shown how the global dynamics are altered in case of equilibria multiplicity compared to the case of unique equilibrium as well as the possibility of coordination failures in the former case. Numerical simulations and comparative statics study have also allowed us to conclude that a product market deregulation and an increase in unemployment benefits have opposite effects on employment following the equilibrium reached by the economy when two equilibria exist. Finally, we have showed why the occurrence of the low equilibrium can be likely only when the workers' bargaining power is very weak due to intrafirm bargaining and how it can be avoided thanks to an appropriate economic policy. Thus, we have exemplified all the issues of economic policy design in the presence of multiple equilibria: the necessity to coordinate agents' expectations to reach the high equilibrium, the differentiated effects according to the equilibrium achieved and the possibility to influence the stability properties of equilibria.

## References

- [1] Blanchard, O. and Giavazzi, F. (2003), "Macroeconomic Effects of Regulation and Deregulation in Goods and Labor Markets", *Quarterly Journal of Economics* 118, 879-907;
- [2] Cahuc, P. and Wasmer, E. (2001a), "Labor Market efficiency, Wages and Employment when Search Frictions Interact with Intrafirm Bargaining", *IZA Discussion Paper N° 304*, June 2001;
- [3] Cahuc, P. and Wasmer, E. (2001b), "Does intra-firm bargaining matter in the large firm's matching model?", *Macroeconomic Dynamics*, 5, 742-747;
- [4] Cahuc, P. and Wasmer, E. (2004), "A theory of wages and labour demand with intra-firm bargaining and matching frictions", *CEPR Discussion Paper N° 4605*, September 2004;
- [5] Cooper, R.W (1999), *Coordination games: Complementarities and Macroeconomics*, Cambridge University Press;
- [6] Ebell, M. and Haefke, C. (2003), "Product Market Deregulation and Labor Market Outcomes", *IZA Discussion Paper N°957*, December 2003;
- [7] Hart, R.F. and Feichtinger, G. (1987), A New Sufficient Condition for Most Rapid Approach Paths, *Journal of Optimization Theory and Applications*, Vol.54, 403-411;
- [8] Pissarides, C.A. (2000), *Equilibrium Unemployment Theory, Second Edition*, Cambridge, The MIT Press;
- [9] Mortensen, D.T. (1999), "Equilibrium Unemployment Dynamics", *International Economic Review*, 40(4), November 1999, 889-914;

- [10] Stole, L. A. and Zwiebel, J. (1996), "Intrafirm Bargaining under Non-binding Contracts", Review of Economic Studies, March 1996, 63, 375-410;

## Appendix

### Appendix A: Determination of the demand for the firm $i$ 's output

Households are both consumers and workers. They are risk neutral and have Dixit-Stiglitz preferences over a continuum of  $i$  differentiated goods uniformly distributed over the interval  $[0, 1]$ . A representative household  $j$  derives its demand in good  $i$  by solving:

$$\underset{c_i}{Max} \left( \int_0^1 c_{ij}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \quad (A1)$$

under the budget constraint  $\int_0^1 \frac{p_i}{P} c_{ij} di = I_j$ .

The parameter  $\sigma > 1$  represents the elasticity of substitution between goods,  $p_i$  the price of good  $i$ ,  $P$  the price index given by  $P = \left[ \int_0^1 p_i^{\frac{\sigma-1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}}$  and  $I_j$  the real income of the representative household  $j$ .

Given the absence of saving and normalization of the identical households population, we obtain the aggregate demand for the good  $i$ :

$$y_i^d = c_i = \left( \frac{p_i}{P} \right)^{-\sigma} I. \quad (A2)$$

Furthermore, assuming market clearing on the product market, we have  $PI = PY$  where  $I = \int_0^1 I_j dj$  represents the aggregate real income here and  $Y$  the aggregate output. Thus, we can write (A2) as follows:

$$y_i^d = \left( \frac{p_i}{P} \right)^{-\sigma} Y. \quad (A3)$$

The expression (A3) represents a standard monopolistic competition demand function with an elasticity of substitution among differentiated goods given by  $-\sigma$ .

## Appendix B: Existence and uniqueness of the firm's problem

The firm  $i$  solves:

$$\begin{aligned}
 V(n_{i0}) &= \max_{v_i} \int_0^{+\infty} e^{-rt} \pi_i dt \\
 s.t \quad \pi_i &= \frac{p_i}{P}(y_i)y_i(n_i) - w_i(n_i)n_i - cv_i \\
 \dot{n}_i &= q(\theta)v_i - sn_i, \quad n_i > 0, \quad n_i(0) = n_{i0} \\
 y_i(n_i) &= An_i \\
 \frac{p_i}{P}(y_i^d) &= (y_i^d/Y)^{-1/\sigma}, \quad y_i^d = y_i(n_i) \\
 0 &\leq v_i \leq v_{i\max} \\
 &w_i(n_i) \quad \text{given}
 \end{aligned}$$

which is equivalent to the following variational problem:

$$\max_{n_i} \int_0^{\infty} e^{-rt} \left( f(n_i) - c \frac{\dot{n}_i + sn_i}{q(\theta)} \right) dt \quad (\text{B1})$$

with  $n_i(0) = n_{i0}$  and  $0 \leq \frac{\dot{n}_i + sn_i}{q(\theta)} \leq v_{i\max}$ , and where  $f(n_i) = \frac{p_i}{P}(y_i)y_i(n_i) - w_i(n_i)n_i$

For convenience, we rewrite the variational problem as follows:

$$\max_n \int_0^{\infty} [G(t, n_i) + H(t, n_i)\dot{n}_i] dt \quad (\text{B2})$$

where  $G(t, n_i) = e^{-rt} \left[ f(n_i) - \frac{cs}{q(\theta)} n_i \right]$  and  $H(t, n_i) = -e^{-rt} \frac{c}{q(\theta)}$ .

The Euler first order condition entails that the optimal solution of the problem of firm  $i$ , (the turnpike solution)  $n_i^*(t)$ , is such that:

$$\frac{\partial G(t, n_i)}{\partial n_i} = \frac{\partial H(t, n_i)}{\partial t} \quad (\text{B3})$$

$$\begin{aligned}
 \Leftrightarrow \quad \frac{\partial f(n_i)}{\partial n_i} &= \frac{(s+r)c}{q(\theta)} - \frac{\eta c \dot{\theta}}{\theta q(\theta)} \\
 \Leftrightarrow \quad \frac{\partial \frac{p_i}{P}(y_i)}{\partial y_i} \frac{\partial y_i(n_i)}{\partial n_i} y_i(n_i) + \frac{p_i}{P}(y_i) \frac{\partial y_i(n_i)}{\partial n_i} - w_i(n_i) - \frac{\partial w_i(n_i)}{\partial n_i} n_i &= \frac{(s+r)c}{q(\theta)} - \frac{\eta c \dot{\theta}}{\theta q(\theta)}
 \end{aligned}$$

The following theorem, called the most rapid approach theorem or turnpike theorem (Hart and Feichtinger, 1987), gives the optimal solution for this problem:

**Theorem 5** *If*

$$G_n(t, n) > (<)H_t(t, n), \quad \text{when } n > (<)n^*(t),$$

where  $n_i^*(t)$  is the unique solution of (B3) and if for all admissible paths  $n_i(t)$ , the following condition holds:

$$\lim_{t \rightarrow \infty} e^{-rt} \int_{n(t)}^{n^*(t)} H(t, x) dx = \lim_{t \rightarrow \infty} e^{-rt} \frac{c}{q(\theta)} (n_i^*(t) - n_i(t)) \geq 0,$$

then, the optimal solution of problem (B1) is the most rapid approach to  $n_i^*(t)$  i.e.:

$$v_i^*(t) = \begin{cases} 0 & \text{si } n_i > n_i^*(t) \\ v_{im} & \text{si } n_i < n_i^*(t) \\ \frac{\dot{n}_i^* + sn_i^*}{q(\theta)} & \text{si } n_i = n_i^*(t). \end{cases} \quad (\text{B4})$$

We are going to check that the hypothesis of this theorem holds with the wage function  $w_i^S(n)$  given by (17). To begin with, let us compute

$$J_i = \frac{\partial V(n_{i0})}{\partial n_{i0}}$$

which is used in the wage negotiation.

Suppose  $n_{i0} > n_i^*(0)$ , in this case the optimal control is zero and the optimal path  $n_i^*(t)$  is the solution of:

$$\dot{n}_i = -sn_i, \quad n_i(0) = n_{i0}.$$

Let  $n_i^{opt}(t) = n_{i0}e^{-st}$  be this solution and

$$tm \quad \text{such that } n_i^*(tm) = n_i^{opt}(tm) \quad (\text{B5})$$

where  $tm$  is the time at which the optimal path meets  $n_i^*(t)$  and note that  $tm = tm(n_{i0})$ . Consequently, we have

$$V(n_{i0}) = \int_0^{tm} e^{-rt} f(n_i^{opt}(t)) dt + \int_{tm}^{\infty} e^{-rt} \left( f(n_i^*(t)) - c \frac{\dot{n}_i^*(t) + sn_i^*(t)}{q(\theta)} \right) dt \quad (\text{B6})$$

and differentiating with respect to  $n_{i0}$ , yields

$$\frac{\partial V(n_{i0})}{\partial n_{i0}} = \int_0^{tm} e^{-rt} \frac{\partial f(n_i^{opt}(t))}{\partial n_{i0}} dt + e^{-rtm} f(n_i^{opt}(tm)) \frac{\partial tm}{\partial n_{i0}} + \quad (\text{B7})$$

$$\int_{tm}^{\infty} e^{-rt} \frac{\partial \left( f(n_i^*(t)) - c \frac{\dot{n}_i^*(t) + sn_i^*(t)}{q(\theta)} \right)}{\partial n_0} dt - e^{-rtm} \left( f(n_i^*(tm)) - c \frac{\dot{n}_i^*(tm) + sn_i^*(tm)}{q(tm)} \right) \frac{\partial tm}{\partial n_0}.$$

As  $n_i^*$  does not depend on  $n_{i0}$ ,  $n_i^*(tm) = n_i^{opt}(tm) = n_{i0}e^{-stm}$  and by (B5):

$$\dot{n}_i^*(tm) \frac{\partial tm}{\partial n_0} = e^{-rtm} - n_{i0}se^{-rtm} \frac{\partial tm}{\partial n_0},$$

we obtain that

$$\frac{\partial tm}{\partial n_{i0}} = \frac{e^{-rtm}}{\dot{n}_i^*(tm) + n_{i0}se^{-rtm}},$$

so that (B6) becomes:

$$\frac{\partial V(n_{i0})}{\partial n_{i0}} = \int_0^{tm} e^{-(r+s)t} f'^{opt}(t) dt + e^{-(r+s)tm} \frac{c}{q(\theta)}. \quad (\text{B8})$$

When  $tm$  goes to zero with  $n_{i0} > n_i^*(0)$ , we have:

$$\frac{\partial V(n_i^*(0)^+)}{\partial n_{i0}} = \frac{c}{q(\theta)}. \quad (\text{B9})$$

The same argument is valid when  $n_{i0} < n_i^*(0)$ , and

$$\frac{\partial V(n_i^*(0)^-)}{\partial n_{i0}} = \frac{c}{q(\theta)}.$$

**Remark 6** Note that for this singular control problem,  $\frac{c}{q(\theta)}$  is the value of the co-state variable of the control problem only when  $n_{i0} = n_i^*(0)$  or, in other words, when starting on  $n_i^*(t)$ .

$$H(n_i, v_i, \lambda) = (f(n_i) - cv_i) + \lambda(q(\theta)v_i - sn_i),$$

actually:

$$\frac{\partial H(n_i, v_i, \lambda)}{\partial v_i} = 0 \iff \lambda = \frac{c}{q(\theta)}.$$

Now, we verify that the equation (B2) has a unique solution when the wage is given by the negotiated wage (equation (17)). The substitution of (17) in (B2) gives:

$$n^{-\frac{1}{\sigma}} = (\sigma - \gamma) \frac{(1 - \gamma)b + \gamma\theta c + c(r + s)/q(\theta) - \eta c\dot{\theta}/q(\theta)}{(1 - \gamma)(\sigma - 1)A^{1-1/\sigma}\alpha Y^{1/\sigma}} = K. \quad (\text{B10})$$

As a result, there exists a unique positive solution to this equation when  $K > 0$ . Since  $\sigma > 1$ , this statement always holds when  $\eta c\dot{\theta} < (1 - \gamma)bq(\theta) + \gamma\theta q(\theta)c + c(r + s)$ .

## Appendix C: Solving the differential equation

The differential equation to be solved is:

$$w_i(n_i) = (1 - \gamma)b + \gamma\theta c + \gamma \left[ \frac{\sigma - 1}{\sigma} \frac{p_i}{P}(n_i) \frac{\delta y_i}{\delta n_i} - \frac{\delta w_i(n_i)}{\delta n_i} n_i \right]. \quad (\text{C1})$$

The method of resolution is standard and follows Cahuc and Wasmer (2004). Initially, we can disregard the term which does not depend on  $n_i$  (the constant term) and add it back in later. Given that  $\frac{p_i}{P}(y_i) \frac{\delta y_i}{\delta n_i} = A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}} Y^{\frac{1}{\sigma}}$ , the equation (C1) becomes:

$$w_i(n_i) = \gamma \frac{\sigma - 1}{\sigma} A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}} Y^{\frac{1}{\sigma}} - \gamma \frac{\delta w_i(n_i)}{\delta n_i} n_i$$

And can be rewritten as follows:

$$\frac{w_i(n_i)}{\gamma n_i} + \frac{\delta w_i(n_i)}{\delta n_i} - \frac{\sigma - 1}{\sigma} A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}-1} Y^{\frac{1}{\sigma}} = 0 \quad (\text{C2})$$

The homogenous version of (C2) is:

$$\frac{w_i(n_i)}{\gamma n_i} + \frac{\delta w_i(n_i)}{\delta n_i} = 0$$

which has the solution:

$$w_i(n_i) = K n_i^{-\frac{1}{\gamma}}. \quad (\text{C3})$$

We take the derivative of (C3), using the fact that  $K$  may depend upon  $n_i$ :

$$\frac{\delta w_i(n_i)}{\delta n_i} = -K \frac{1}{\sigma} n_i^{-\frac{1}{\gamma}-1} + n_i^{-\frac{1}{\gamma}} \frac{\delta K}{\delta n_i}. \quad (\text{C4})$$

Now, we substitute (C3) and (C4) in (C2) and we obtain:

$$\begin{aligned} n_i^{-\frac{1}{\gamma}} \frac{\delta K}{\delta n_i} - \frac{\sigma - 1}{\sigma} A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}-1} Y^{\frac{1}{\sigma}} &= 0 \\ \Leftrightarrow \frac{\delta K}{\delta n_i} &= \frac{\sigma - 1}{\sigma} A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}-1+\frac{1}{\gamma}} Y^{\frac{1}{\sigma}}. \end{aligned} \quad (\text{C5})$$

Given that  $\frac{p_i}{P}(y_i) \frac{\delta y_i}{\delta n_i} = A^{1-\frac{1}{\sigma}} n_i^{-\frac{1}{\sigma}} Y^{\frac{1}{\sigma}}$ , the integral over both sides of (C5) gives:

$$K = \gamma \frac{\sigma - 1}{\sigma - \gamma} \frac{p_i}{P}(y_i) \frac{\delta y_i}{\delta n_i} n_i^{\frac{1}{\gamma}} + C \quad (\text{C6})$$

where  $C$  is a constant of integration. Now, we substitute (C6) in (C3) and we obtain:

$$w_i(n_i) = \gamma \frac{\sigma - 1}{\sigma - \gamma} \frac{p_i}{P}(y_i) \frac{\delta y_i}{\delta n_i} + C n_i^{-\frac{1}{\gamma}}. \quad (\text{C7})$$



Following Cahuc and Wasmer (2004), the terminal condition  $\lim_{n_i \rightarrow 0} n_i w_i = 0$ , which reports the fact that the firm-level bargained wage should not explode as firm-level employment  $n_i$  approaches zero, implies that  $C = 0$ . As a consequence, the constant of integration can be withdrawn and, after adding back the constant term, we obtain the following solution for the differential equation (C1):

$$w_i(n_i) = (1 - \gamma)b + \gamma\theta c + \gamma \left[ \frac{\sigma - 1}{\sigma - \gamma} \frac{p_i}{P}(y_i) \frac{\delta y_i}{\delta n_i} \right]. \quad (\text{C8})$$

## Appendix D: Study of dynamical system

$$\dot{\theta} = \frac{\theta^{1-\eta}}{\eta c} \left[ (1 - \gamma)b + \gamma\theta c + (r + s)c\theta^\eta - (1 - \gamma) \frac{\sigma - 1}{\sigma - \gamma} (1 + \alpha)n^\alpha \right] \quad (\text{D1})$$

$$\dot{n} = \theta^{1-\eta} - (\theta^{1-\eta} + s)n \quad (\text{D2})$$

We analyse the steady states of the dynamical system.

### D.1 The case $\alpha > 0$

We analyse first the case  $\alpha > 0$ . Equation (D2) gives:

$$\dot{n} = 0 \quad \Longleftrightarrow \quad n = \frac{\theta^{1-\eta}}{\theta^{1-\eta} + s} := f(\theta). \quad (\text{D3})$$

We have the following properties of  $f$ :

$$f(0) = 0, \quad \lim_{\theta \rightarrow \infty} f(\theta) = 1, \quad f'(\theta) = \infty, \quad f(\theta) \text{ increasing and concave.}$$

Equation (D1) gives

$$\dot{\theta} = 0 \quad \Longleftrightarrow \quad \theta = 0 \quad \text{or} \quad n = (a_1 + a_2\theta + a_3\theta^\eta)^{1/\alpha} := g(\theta) \quad (\text{D4})$$

where

$$a_1 = \frac{b(\sigma - \gamma)}{(\sigma - 1)(1 + \alpha)}, \quad a_2 = \frac{\gamma c(\sigma - \gamma)}{(1 - \gamma)(\sigma - 1)(1 + \alpha)}, \quad a_3 = \frac{(r + s)c(\sigma - \gamma)}{(1 - \gamma)(\sigma - 1)(1 + \alpha)}.$$

are such that  $a_i > 0$ ,  $i = 1, 2, 3$ . We have the following properties of  $g$ :

$$g(0) > 0, \quad \lim_{\theta \rightarrow \infty} g(\theta) = \infty, \quad g(\theta) \text{ increasing,} \quad g'(0) < \infty.$$

We can verify

**Theorem 7** *If*

$$g(0) = \left[ \frac{b(\sigma - \gamma)}{(\sigma - 1)(1 + \alpha)} \right]^{1/\alpha} > 1$$

*the steady states of the dynamical system given by (D1) and (D2) are*

$$(\theta = 0, n = 0), \quad (\theta = \infty, n = 1).$$

We impose the restriction  $g(0) > 1 \Leftrightarrow b < \frac{\sigma-1}{\sigma-\gamma}(1+\alpha)$  that ensures that the inequality  $E_i > U_i$  always holds in the bargaining process. As a consequence, we cannot determine the number of interior solution (different from  $(\theta = 0, n = 0)$ , or  $(\theta = \infty, n = 1)$ ), but their behaviour is the same as one of the cases when two interior steady states exist (that is also the case of numerical simulations).

We can prove

**Theorem 8** *For  $\alpha > 0$  and in the case of two interior steady states*

- i) The high interior steady state is a saddle point*
- ii) There exists initial conditions such that the dynamical system arrives to  $\theta = 0$  in finite time. For these initial conditions the dynamical system converges to  $(\theta = 0, n = 0)$ .*
- iii)  $(\theta = \infty, n = 1)$  is a stable steady state.*

**Proof.** Note that the system given by (D1) and (D2) can be rewritten in the following way using the definitions of  $f$  and  $g$  :

$$\dot{\theta} = \frac{\theta^{1-\eta}c'}{\eta c} [g(\theta)^\alpha - n^\alpha] =: \phi(\theta, n) \quad (\text{D5})$$

$$\dot{n} = (\theta^{1-\eta} + s)(f(\theta) - n) := \psi(\theta, n) \quad (\text{D6})$$

where  $c' = (1 - \gamma)\frac{\sigma-1}{\sigma-\gamma}(1 + \alpha)$ . We compute

$$\begin{aligned} a = \frac{\partial \phi}{\partial \theta} &= \frac{\theta^{1-\eta}c'}{\eta c} \alpha g(\theta)^{\alpha-1} g'(\theta) + \frac{(1-\eta)\theta^{-\eta}}{\eta c c'} [g(\theta)^\alpha - n^\alpha] \\ &= \frac{\theta^{1-\eta}c'}{\eta c} (a_2 + a_3 \eta \theta^{\eta-1}) + \frac{(1-\eta)\theta^{-\eta}}{\eta c c'} [g(\theta)^\alpha - n^\alpha] \\ b &= \frac{\partial \phi}{\partial n} = -\frac{\theta^{1-\eta}c'}{\eta c} \alpha n^{\alpha-1} < 0 \\ c &= \frac{\partial \psi}{\partial \theta} = (\theta^{1-\eta} + s) f'(\theta) > 0 \\ d &= \frac{\partial \psi}{\partial n} = -(\theta^{1-\eta} + s) < 0 \end{aligned}$$

evaluated at a steady state  $(\theta, n) = (\theta^*, n^*)$  such that  $n^* = f(\theta^*) = g(\theta^*)$ . We obtain:

$$a^* = \frac{\theta^{1-\eta}c'}{\eta c} (a_2 + a_3 \eta \theta^{\eta-1}) > 0$$

$$b^* = b < 0$$

$$c^* = (\theta^{1-\eta} + s) f'(\theta) > 0$$

$$d^* = d < 0.$$

Call

$$Jc = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

then

$$\chi_{Jc}(z) = \det(Jc - zI) = z^2 - (a^* + d^*)z + a^*d^* - b^*c^*$$

and the discriminant of this equation is

$$\Delta = (a^* + d^*)^2 - 4(a^*d^* - b^*c^*). \quad (\text{D7})$$

Note that

$$a^*d^* - b^*c^* = \frac{(\theta^*)^{1-\eta}}{\eta c c'} \alpha(n^*)^{\alpha-1} ((\theta^*)^{1-\eta} + s)(f'(\theta^*) - g'(\theta^*))$$

then the sign of  $a^*d^* - b^*c^*$  is given by the sign of

$$f'(\theta^*) - g'(\theta^*).$$

When there exists two steady states, as in figure D2, at  $\theta^*$  high  $f'(\theta^*) - g'(\theta^*) < 0$ , then  $\Delta > 0$ , then the two roots of the characteristic polynomial are reals and have different sign (because the determinant of  $Jc$ ,  $a^*d^* - b^*c^*$ , is negative). This implies that the high steady state is a saddle point, then i) is proved.

Now we want to prove that for some values of initial conditions, the dynamical system arrives to  $\theta = 0$  in finite time. Note that for a fixed  $\bar{\theta}$  there exists  $\bar{n}$  and  $\bar{t}$  such that:

$$\forall t \in [0, \bar{t}], n(t) \geq \bar{n}, \quad \frac{c'}{\eta c} (g(\bar{\theta})^\alpha - \bar{n}^\alpha) := M(\bar{\theta}, \bar{n}) < 0.$$

Then for all  $t \in [0, \bar{t}]$ ,  $\theta(t)$  is a decreasing function. Moreover, calling  $M(\bar{\theta}, \bar{n}) = M$ ,

$$\dot{\theta}(t) < M\theta(t)^{1-\eta}.$$

As the solution of

$$\dot{\theta}(t) = M\theta(t)^{1-\eta}, \quad \theta(0) = \theta_0,$$

is  $\tilde{\theta}(t) = (\theta_0^{1-\eta} + \eta M t)^{1/\eta}$ , we have that

$$\theta(t) < \tilde{\theta}(t).$$

Note that

$$\tilde{\theta}(t_1) = 0 \quad \iff \quad t_1 = -\frac{\theta_0^\eta}{\eta M}.$$

If  $t_1 \leq \bar{t}$  then we have proved that  $\theta(t)$  arrives to  $\theta = 0$  in finite time. To see this note that as

$$\dot{n}(t) \geq sn(t), \quad \text{implies} \quad n(t) \geq n_0 e^{-st},$$

we need to prove that

$$n_0 e^{-st_1} > \bar{n}.$$

Taking  $\bar{\theta} = \theta_0$  a sufficient condition is to take  $\bar{\theta}$  and  $\bar{n}$  such that

$$1 \geq n_0 \geq \bar{n} e^{-s \frac{\bar{\theta} \eta}{\eta M}}.$$

Now we prove that  $(\theta = \infty, n = 1)$  is a stable steady state. If  $\theta$  is large enough and for all  $0 < n < 1 < g(\theta)$  we have that  $\dot{\theta} > 0$ ,  $\lim_{t \rightarrow \infty} \dot{\theta}(t) = \infty$  and  $\lim_{t \rightarrow \infty} \theta(t) = \infty$ . If  $n(t)$  converges to  $0 \leq n_{sup} < 1$  then from (D2)  $\lim_{t \rightarrow \infty} \dot{n}(t) = \infty$ , that is a contradiction, then  $\lim_{t \rightarrow \infty} n(t) = 1$ . Then iii) is proved. ■

We can compute the trace of  $J$ .

$$a^* + d^* = (\theta^*)^{1-\eta}(\gamma - \eta) + r. \quad (\text{D8})$$

When  $\Delta < 0$  (and this is the case in numerical simulations for the low interior steady state), the sign of  $a^* + d^*$  determines if the steady state is attractive or repulsive. Indeed, the steady state is repulsive when  $a^* + d^* > 0$  and attractive when  $a^* + d^* < 0$ . We can easily deduce that when  $\gamma \geq \eta$  (if  $\Delta < 0$ ) the trace of  $J$  is always positive and the steady state is always repulsive. An example with  $\Delta < 0$  and  $a^* + d^* > 0$  (repulsive steady state) is  $\sigma = 2, \alpha = 2, r = 0.04, s = 0.1, \gamma = 0.5, \eta = 0.5, c = 0.2$  and  $b = 0.3$  which is represented in figure D4. An example with  $\Delta < 0$  and  $a^* + d^* < 0$  (attractive steady state) is  $\sigma = 2, \alpha = 10, r = 0.04, s = 0.1, \gamma = 0.1, \eta = 0.5, c = 0.2$  and  $b = 0.3$  (see figure D3). After numerous numerical simulations, we note that the low interior steady state is always attractive when  $\gamma = 0.1 < \eta = 0.5$  for different values of  $\alpha \in ]0, 10[$  and  $\sigma \in [2, 100]$ .

Figure D1 shows us the behaviour of the high steady state (saddle point) for  $\sigma = 2, r = 0.04, s = 0.1, \gamma = 0.5, \eta = 0.5, c = 0.2, \alpha = 2$  and  $b = 0.3$ . This figure also shows some trajectories that converge to  $(0, 0)$  and  $(\infty, 1)$ .

## D.2 The case $\alpha = 0$

When  $\alpha = 0$  we have that

$$\dot{\theta} = 0 \iff \theta = 0 \quad \text{or} \quad (1 - \gamma)\left(b - \frac{\sigma - 1}{\sigma - \gamma}\right) + \gamma\theta c + (r + s)c\theta^\eta := h(\theta) = 0.$$

We can easily see that if  $b - \frac{\sigma - 1}{\sigma - \gamma} > 0$  there exists no solution of  $h(\theta) = 0$ . If  $b - \frac{\sigma - 1}{\sigma - \gamma} < 0$  there exists an unique solution of  $h(\theta) = 0$ .

**Theorem 9** *For  $\alpha = 0$  the interior steady state (when it exists) is a saddle point. The two corner steady state have the same properties those of the case  $\alpha > 0$ .*

See Figure D1.

### D.3 Phase diagrams

In the following figures, the tightness of labor market  $\theta$  is always on the Y-axis and the employment rate  $n$  on the X-axis. The dashed curves represent  $\dot{\theta} = 0$  and the solid curves represent  $\dot{n} = 0$ .

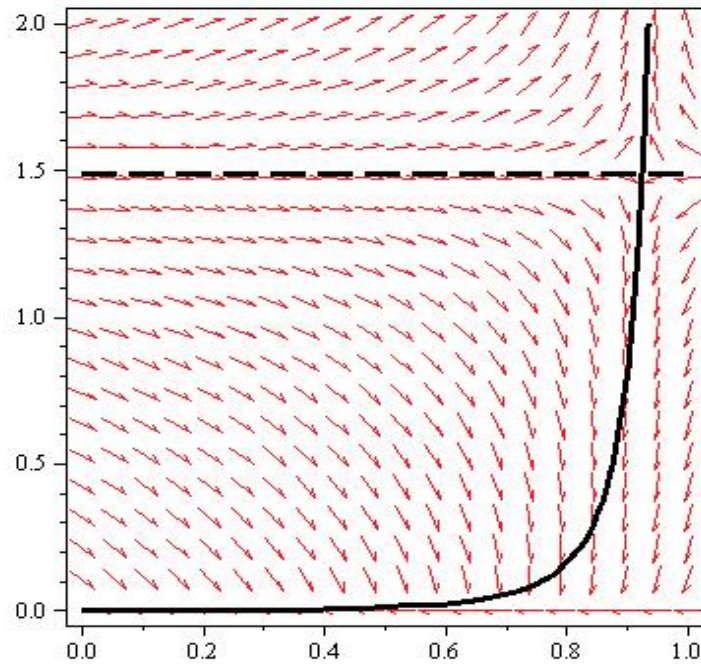


Figure D1: Equilibrium paths when  $\alpha = 0$

We note that the saddle path corresponds to  $\dot{\theta} = 0$  as in Pissarides (2000).

However, in the case where  $\alpha > 0$ , the saddle path has the property that  $\dot{\theta} > 0$  if  $n_0 < n_H^*$  and  $\dot{\theta} < 0$  if  $n_0 > n_H^*$ . The equilibrium paths illustrated in the figure D2 show us this statement.

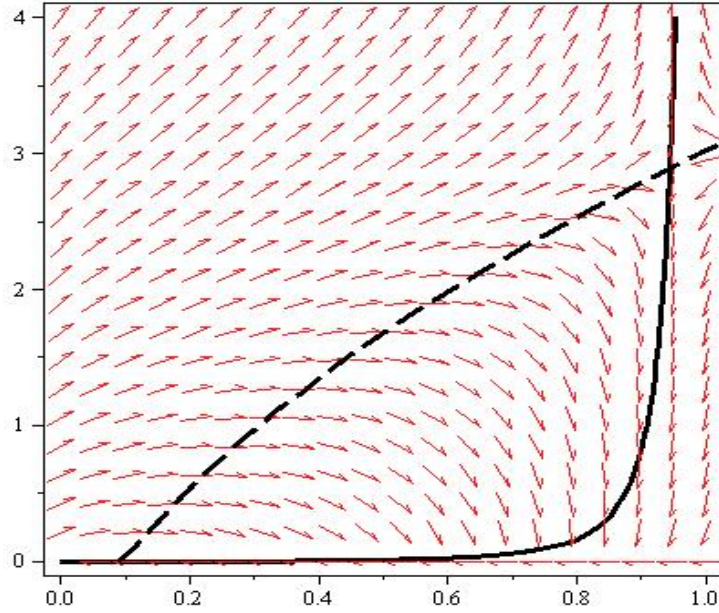


Figure D2: Equilibrium paths when  $\alpha = 0,5$

The Figures D3 and D4 show respectively how the economy converges to the low equilibrium in the case where this is attractive and diverges otherwise.

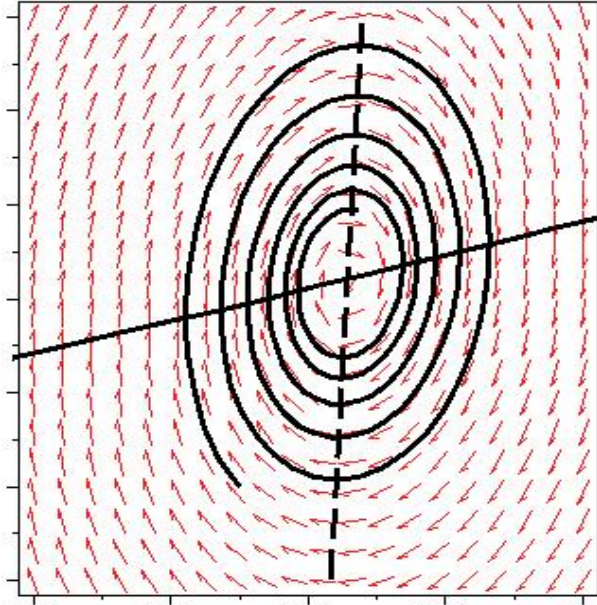


Figure D3: Attractive low equilibrium  
with  $\alpha = 10$  and  $\gamma = 0,1$

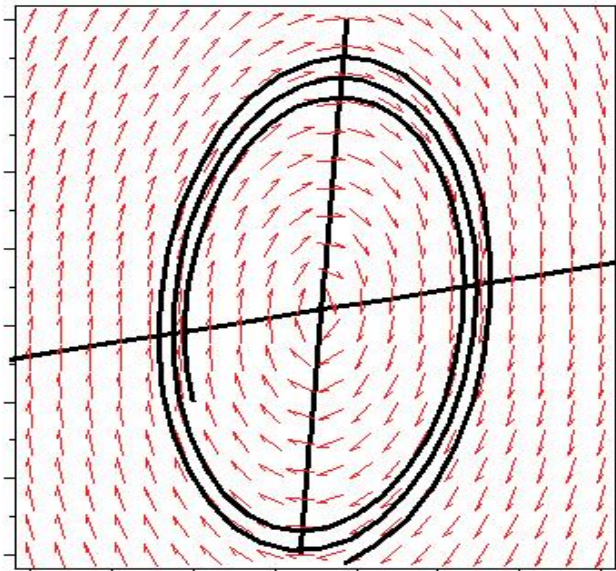


Figure D4: Repulsive low equilibrium  
with  $\alpha = 2$  and  $\gamma = 0,5$

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