Cyclical versus Non-Cyclical Harvesting Policies in Renewable Resource Management

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Abstract

In this paper, we explore the link between cyclical and non-cyclical resource exploitation. We define an impulse control framework and show that the optimal harvesting behavior derived from this model can either be cyclical or a Most Rapid Approach Path. For the cyclical solution, we establish a link between our impulse control framework and the discrete time model by Dawid and Kopel [7]. For the Most Rapid Approach Path solution, we show the relation to Clark's [3] continuous control model. Our model is general enough to admit convex and concave revenue functions and to allow for the integration of different stock dependent cost functions. Using different harvest cost functions proposed in the literature, we show that particular functional forms of harvest costs may be at the origin of cyclical or non-cyclical solutions.

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Key words: optimal control, impulse control, renewable resource economics, harvest cost function.

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1 Introduction

The harvesting behavior in renewable resource economics may be seen as a continuous action, for example when describing the fishery or forestry sector as a whole, or as an action at a point in time: a fisherman driving out his gear once a week or a forester deciding to cut a stand once in a decade. In the first case, the observer is using a small scale and conceives resource harvesting as a smoothed movement of a more complex environment. In the second case, he is using a large scale and is focusing on an individual harvester. From a technical point of view, this implies different ways of modeling the harvest action. We make the assumption that the most coherent translation of this scale effect is operated by a change in the nature of the harvest variable, which in turn entails a change in the harvest cost function. We show in this paper that this change is sufficient to induce different optimal harvesting policies, in particular cyclical and non-cyclical solutions.

Non-cyclical solutions are characterized by a steady state equilibrium stock, the endogenously determined bioeconomic equilibrium - at least in infinite time horizon models, upon which we focus in the following. Cyclical solutions are defined by a recurring sequence of events: harvesting and waiting. Associated resource stocks in cyclical solutions may be exogeneous (as in the Faustmann model) or endogeneous (as in the following impulse control model).[1]

The standard reference situation is the continuous control model proposed by Clark [3, 4], with a unique state variable representing the stock of the natural resource. The gain function represents the net present value derived from resource harvesting and depends on resource stocks. Harvesting is represented by a harvest *rate*. The solution of the problem typically leads to a non-cyclical harvesting behavior.[2]

In deterministic models,[3] cycles may occur because of convex gain functions or increasing returns to scale, because of the presence of several state variables or different age-classes, or when stock effects and amenities are considered. Many models combine several of these elements.

In the literature on continuous infinite time models, Lewis and Schmalensee [18, 19] have shown that cycles can be optimal in presence of increasing returns to scale, stock effects and modest re-entry costs. They challenged the long-standing knowledge that cycles are not possible in continuous-time models with one state variable (see for example Léonard and Long [16] for an overview). Wirl [27] has shown that cycles are possible in a model with two state variables and stock effects. He based his analysis on the continuous control model by Clark, Clarke and Munro [5], which considers resource and capital stock dynamics.[4] As Wirl demonstrated, the sole introduction of a non-linear (quadratic) cost function leads to cyclical optimal harvesting behavior.

In the literature on discrete infinite time models, Dawid and Kopel [7, 8] have shown that strictly convex gain functions may lead to optimal cyclical solutions (or to fixed point solutions), in the absence of stock effects. Liski *et al.* [17] have demonstrated

[1]We do not consider optimal cyclical solutions as described by Faustmann's optimal forest rotation and further developments of this concept as rotations in these models are imposed by construction.

[2]Note that Clark has also analysed a discrete version of his model. The optimal solution consists in a steady state-like solution similar to the continuous control solution [3], chapter 7.

[3]We do not consider stochastic models in the following but see for example Jacquette [14, 15], Reed [20] or Singh *et al.* [23].

[4]In this model the harvest-gain function is concave, which, according to Wirl, renders cyclical solutions more difficult to occur.

the occurrence of cycles in a discrete time model with increasing returns to scale and missing adjustment costs, also in absence of stock effects. Finally, in early applications on fisheries economics, Hannesson [12] has pointed at so-called pulse-fishing solutions,[5] which he explained by increasing returns to scale and the presence of age classes in the population.

The present paper is directly motivated by the findings of Termansen [24] who shows that an impulse control model may generate different types of extreme harvesting regimes: those similar to Faustmann-like rotations and those similar to a steady state solution (see also Touza-Montero and Termansen [25]). In particular, Termansen explores the link between the optimal cyclical behavior and Clark-like solutions in the special case where harvest costs are independent of stock levels.

Our principal purpose is to discuss this relationship more systematically. We revisit this impulse control model, and we solve it completely by reducing it to two coupled optimization problems with two variables each. This allows us to formulate conditions under which the optimal harvesting behavior is cyclical or non-cyclical. These conditions involve an intimate combination of the growth function and the cost function, thereby emphasizing that convexity of the cost function, or its dependence on the stock level, are not the only issues worth considering.

We then discuss what happens in the situation of Clark's model, and how the steady-state solution emerges as a limit of small and frequent harvest operations. We also show that we can reproduce and generalize Dawid and Kopel's results, although these were obtained with a discrete-time model and time is continuous in our model. This allows us to claim that the model we propose is actually a "meta-model" encompassing discrete and continuous models previously described.

Part of this paper is devoted to a closer examination of when the particular functional form of the harvest cost function is at the origin of the cyclical solution. In contrast to Wirl [27], we deal with a control model in which only the resource stock evolves over time and in which the harvesting strategy is discrete. As Wirl, and in contrast to most other models with optimal cyclical harvesting policies (see for example Dawid and Kopel [7], Liski *et al.* [17], Lewis and Schmalensee [18, 19]), we suppose the harvest cost functions to be stock-dependent, such as the costs proposed by Clark [3]. The empirical literature in fisheries economics confirms our assumptions on the cost function in two ways: first, costs do most often depend on both, harvesting and resource stocks.[6] Second, harvest costs may be modeled through discrete or continuous functions.[7] Singh *et al.* [23] give an interesting example of a fishery industry where costs are stock-dependent and strictly convex in harvesting, as the functions we will consider below. Based on the theoretical literature, we remind two ways to derive the exact form of the cost function : from a production function based on the harvest amount on the one hand, from the profit function based on the harvest rate on the other hand. Introducing these cost functions in our impulse control framework allows us to explore the consequences of the scaling effect. We show that the optimal harvesting behavior depends on the cost function which

[5] Pulse fishing and chattering strategies imply very small jumps in the state variable.

[6] They may also depend on capital invested, but in many estimations investment and long-term decisions are neglected.

[7] See for example Grafton *et al.* [11], Arnason *et al.* [1] for models using the continuous harvest rate, $h(t)$, Homans and Wilen [13] for models using harvesting effort and Singh *et al.* [23] for a model describing harvesting as a discrete action, with h being the harvest amount.

is used, and that the optimal behavior can either be cyclical or converge to a steady state solution, like in a Clark-Munro continuous control model.

The rest of the article is organised as follows. We present the impulse control problem in section 2, we characterise the type of the solution in section 3 and treat the special case of optimal extinction in section 4. We then establish a link to Clark's continuous control solution in section 5 and to Dawid and Kopel's discrete control model in section 6. In section 7 we derive and study two particular types of cost functions: one which leads to a cyclical solution and one which leads to a non-cyclical solution. The last section is devoted to the conclusion. An extended appendix provides several proofs.

2 Preliminaries and Notation

We present in this section the impulse control model we are interested in, and propose an approach for characterizing its solutions (For the discussion of impulse control models, see for example Léonard and Long [16], Seierstaed and Sydsaeter [22]).

2.1 The Model

The resource dynamics

We consider a renewable resource, the dynamics of which, absent any harvest is given by:

$$\dot{x}(t) \;=\; F(x(t)) \;, \quad t \geq 0, \tag{1}$$

where $x(t)$ is the size of the population at any time t and F, stationary through time, is the growth rate function. The function F is assumed to satisfy the following conditions.

Assumption 1 There exist real numbers x_{sup} and x_{ns} such that $0 < x_{ns} < x_{sup} < +\infty$. The function $F : (0, x_{sup}) \to \mathbb{R}$ is positive over the interval $(0, x_{ns})$ and negative over the interval (x_{ns}, x_{sup}), with $F(0) = F(x_{ns}) = 0$, where $\lim_{x \downarrow 0} F(x) = F(0)$. The function F is bounded above. It is assumed that the differential equation (1) admits a unique solution for every initial stock $x_0 \in (0, x_{sup})$.

The quantity x_{sup} is the supremum of the carrying capacity of the environment. The long-run maximum sustainable level is x_{ns}, the level to which the population is converging for any x_0 such that $0 < x_0 < x_{sup}$.

The harvesting action

We are interested in the optimal economic exploitation of this resource by a discrete harvest action, i.e. in the framework of impulse control theory. Accordingly, we define an impulse exploitation policy IP $:= \{(t_i, I_i), i = 1, 2, \ldots\}$ as a sequence of catching dates t_i and instantaneous catches I_i, one for each date. The sequence of dates may be empty, finite or infinite. It is such that $0 \leq t_1$, and $t_i \leq t_{i+1}$ for each $i = 1, 2, \ldots$ and $\lim_{i \to +\infty} t_i = +\infty$. By convention, we shall assume that if the sequence is finite with $n \geq 0$ values, then $t_i = +\infty$ for all $i > n$.

The sequence of catches must be such that:

$$I_i \geq 0 \quad \text{and} \quad x_i - I_i \geq 0 \;, \tag{2}$$

where by notation:

$$x_i \;=\; \lim_{t \uparrow t_i} x(t)\;, \quad \text{with } x_1 = x_0 \text{ given if } t_1 = 0, \tag{3}$$

and such that the following constraints hold:

$$\dot{x}(t) = F(x(t)) \text{ for } t_i < t < t_{i+1} \text{ with } x(t_i) = x_i - I_i, \quad i = 1, 2, \ldots \tag{4}$$

$$\dot{x}(t) = F(x(t)) \text{ for } 0 < t < t_1 \text{ with } x(0) = x_0 \quad \text{if } t_1 > 0. \tag{5}$$

In other words: x_i is the size of the population just before the catching date t_i, and $x_i - I_i$ its size just after that same date. If $t_1 = 0$, the population x_1 is supposed to be inherited from the past, and denoted by x_0. Catches can not be negative nor exceed total population size. The conditions (2)–(5) define the set of *feasible* IPs, denoted by $\mathcal{F}_{x_0}$.

The harvester's profits

Monetary profits generated by any catch are depending upon the size of the catch and the size of the population at the catching time. We assume that the profit function is stationary through time so that whatever t_i, I_i and x_i, the current profits at time t_i amount to $\pi(x_i, I_i)$. The profit function is assumed to have the following standard properties.

Assumption 2 The function $\pi(x, I)$ is defined on the domain $\mathcal{D} := \{(x, I), x \in (0, x_{sup}), I \in [0, x]\}$. It is of class C^1 and bounded above by $\bar{\pi} < +\infty$, and such that $\pi(x, 0) = 0$, $\forall x \in (0, x_{sup})$. The derivative $\pi_I(x, I) := (\partial\pi/\partial I)(x, I)$ admits a limit when $I \downarrow 0$ for all $x \in (0, x_{sup})$.

Profits are discounted with the instantaneous interest rate, denoted by r. Indeed, the capital market is perfectly competitive and, hence, the instantaneous interest rate is assumed to be a strictly positive constant.

The problem of the manager is to choose some policy maximizing the sum of the discounted profits, that is, to solve the problem (P):

$$\text{(P)} \qquad \sup_{\text{IP} \;\; F_{x_0}} \Pi(\text{IP}) \;:=\; \sum_{i=1}^{\infty} e^{-rt_i}\, \pi(x_i, I_i)\;.$$

It is assumed here that the function Π is well defined over the whole set $\mathcal{F}_{x_0}$.[8][9]

2.2 The Dynamic Programming Principle

We deliberately omit to specify a profit function at this stage. One essential question of the following sections will thus be: what is the impact of the profit function on the type of solution? We will see the answer to this question in Section 7.

We use the Dynamic Programming approach to solve our problem. The following theorem insures the existence of a unique value for the problem.

[8] We argue in Appendix A.1 that this is the case under reasonable assumptions.

[9] Observe that we formulate our problem with a "sup" and not a "max" because we are interested in the possibility that the maximum is not reached inside the set $\mathcal{F}_{x_0}$.

Theorem 1 *The value function*

$$v(x) = \sup_{IP \in \mathcal{F}_x} \Pi(IP) \tag{6}$$

is the unique solution of the following variational equation:

$$v(x) = \sup_{\substack{y\in[0,x_{sup})\\ t\geq 0}} e^{-rt}\left[\pi(\varphi(t,x),\varphi(t,x)-y)+v(y)\right] , \tag{7}$$

where $\varphi(t,x)$ is the trajectory of the system, solution of the dynamics (1) *with $x(0) = x$.*

Proof. For this standard proof of dynamic programming see Davis [6, Theorem (54.19), page 236], González [10], Bensoussan-Lions [2]. ■

3 The Fundamental Equivalence Theorem between Optimal Impulse Policies and Optimal Cyclical Policies

In the following, we characterize the optimal solution of our problem (P). To do so, we first give a definition of cyclical policies and introduce an auxiliary problem (AP). We then present two properties of the optimal impulse control policy. We finally show the fundamental relations between problems (AP) and (P).

3.1 Cyclical Policies and the Auxiliary Problem

To proceed with the precise definition of cyclical policies, and with the evaluation of the corresponding total profits, we need some preliminary results. Define the function $\tau(x,y)$ as the time necessary for the dynamics to go from value x to y, $x \leq y$. It turns out that: for all $0 < x \leq y < x_{ns}$:

$$\tau(x,y) = \int_x^y \frac{1}{F(u)}\, du. \tag{8}$$

Since, by Assumption 1, $F(x_{ns}) = 0$, the integral defining $\tau(x,y)$ is singular when $y = x_{ns}$. Two cases may occur:

Slow growth $\lim_{y\uparrow x_{ns}} \tau(x,y) = +\infty$ for $0 < x < x_{ns}$: the population of the resource never actually reaches the level x_{ns};

Fast growth $\lim_{y\uparrow x_{ns}} \tau(x,y) < +\infty$ for $0 < x < x_{ns}$: the population of the resource does reach the level x_{ns} in finite time, whatever the starting value. In that case, we assume that the resource, starting at level x and reaching the level x_{ns}, stays at that level $x(t) = x_{ns}$ for $t \geq \tau(x, x_{ns})$, in accordance with the dynamics $\dot{x}(t) = F(x_{ns}) = 0$.

Finally, another feature of Assumption 1 is that $F(0) = 0$. As a consequence, if $x(0) = 0$, a solution to the dynamics (1) is $x(t) = 0$ for all $t \geq 0$. This implies the convention that $\tau(0,y) = +\infty$ if $y > 0$, and $\tau(0,0) = 0$.[10]

[10] This convention does not mean that $\lim_{x\downarrow 0} \tau(x,y) = +\infty$ in every situation. See more on this in Section 4.

Cyclical policies

We are now in position to define *cyclical policies*. A cyclical policy has two components: a cycle which is characterized by two values $\underline{x}$ and $\bar{x}$; and a transitory part which describes how the trajectory goes from the initial stock to the cycle. The transitory part consists in at most one harvest, such that the remaining population is less than $\bar{x}$. We first concentrate on the cycle.

A cycle has then two main parameters, which are such that $0 \leq \underline{x} < \bar{x} \leq x_{ns}$.[11] When in its cyclic part, a policy acts as follows: a), let the population grow up to $\bar{x}$; b), harvest down to $\underline{x}$; and repeat. Such a policy applies only to initial populations $x_0 \leq \bar{x}$. In other word, the transitory part can be dispensed with only for such initial population.

We will denote by $G(\underline{x}, \bar{x}, x_0)$ the value of discounted profits in a policy without the transitory part, applied to an initial population of x_0. The complete definition of the function G involves several cases, corresponding to the limit cases for $\bar{x}$ and $\underline{x}$.

i) If $0 \leq \underline{x} < \bar{x} < x_{ns}$: let $t_1 = \tau(x_0, \bar{x})$, $t_i = t_1 + (i-1)\tau(\underline{x}, \bar{x})$, for $i = 2...$, $I_i = \bar{x} - \underline{x}$, for all i. The total profit for this policy is:

$$G(\underline{x}, \bar{x}, x_0) := \pi(\bar{x}, \bar{x} - \underline{x}) \frac{e^{-r\tau(x_0,\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}} . \tag{9}$$

ii) If $0 \leq \underline{x} < \bar{x} = x_{ns}$:

For a Slow Growth, let $t_1 = +\infty$. In that case, $G(\underline{x}, x_{ns}, x_0) := 0$.

For a Fast Growth, let $t_1 = \tau(x_0, x_{ns})$, and $t_i = t_1 + (i-1)\tau(\underline{x}, x_{ns})$, for $i = 2...$, $I_i = x_{ns} - \underline{x}$, for all i.[12]

The total profit for this policy is given by (9) with $\bar{x} = x_{ns}$.

iii) If $\underline{x} = 0$: the value for the function G is obtained from the expressions above by replacing the term $\exp(-r\tau(\underline{x}, \bar{x}))$ by 0, since $\tau(0, y) = +\infty$.

At this point, the function G is defined for all $\underline{x} < \bar{x}$, where it corresponds to a specific impulse control policy, and for $x_0 \leq \bar{x}$. We proceed with completing the definition for the cases $\underline{x} = \bar{x}$ and for arbitrary x_0.

Note that when $\underline{x} = \bar{x}$, the denominator in (9) vanishes, as well as the nominator, by Assumption 2. Under Assumption 2, G may be defined by continuity when $\bar{x} \to \underline{x} = x$ and $x_0 \leq x$ in the following way:

$$G(x, x, x_0) \quad = \quad \pi_I(x, 0) \, \frac{F(x)}{r} \, e^{-r\tau(x_0,x)} . \tag{10}$$

Finally, using the fact that $\tau(x, y)$ defined in (8) is also defined for $y \leq x$, the expression (9) provides values for the function G when $x_0 > \bar{x}$ as well. Of course, these situations do not correspond to an implementable harvesting policy, and the function loses its economic meaning.

[11] Since $\bar{x}$ represents the population level until which the resource grows before harvesting, there is no meaning in considering $\bar{x} > x_{ns}$ since the population cannot grow to such a level.

[12] In the Fast Growth case, a more general cyclical policy could be defined with $t_1 = \tau(x_0, x_{ns}) + d$, and $t_i = t_1 + (i-1)(\tau(\underline{x}, x_{ns}) + d)$, for $i = 2...$, for some $d \geq 0$. However, it is easily seen that the associated profit is a decreasing function of d if $\pi() > 0$. If $\pi() < 0$, the best value of d would be $+\infty$; this case is not economically relevant however.

The auxiliary problem

Having defined the function $G(\underline{x}, \bar{x}, x_0)$ for all $0 \le \underline{x} \le \bar{x} \le x_{ns}$ and all $0 \le x_0 \le x_{ns}$, we define now the auxiliary problem (AP):

$$(\mathrm{AP}): \qquad \max_{\underline{x},\, \bar{x};\; 0 \le \underline{x} \le \bar{x} \le x_{ns}} G(\underline{x}, \bar{x}, x_0).$$

3.2 Submodularity and Optimal Trajectories

In this paragraph, we introduce a submodularity assumption on the profit function π and discuss its consequences on the shape of optimal trajectories for Problem (P). Other results on optimal trajectories are presented in Appendix A.3.

Consider now the following assumption.

Assumption 3 The function π is such that:

$$\pi(a, a-c) + \pi(b, b-d) \ \le\ \pi(a, a-d) + \pi(b, b-c) \tag{11}$$

for every $a \ge b \ge c \ge d$.

The following properties are well-known or easy to check.

Lemma 1 *Assume that π satisfies Assumption 3. Then:*

i) Let $g(x, y) = \pi(x, x-y)$ be defined for $x_{sup} > x \ge y \ge 0$. Then g is submodular, that is, for all a, b, c, d such that $\min(a, b) \ge \max(c, d)$:

$$\begin{aligned} & g(\min(a, b), \min(c, d)) + g(\max(a, b), \max(c, d)) \\ & \quad \le\ g(\min(a, b), \max(c, d)) + g(\min(a, b), \max(c, d))\ . \end{aligned}$$

ii) If π has second-order derivatives, then:

$$\frac{\partial^2 \pi}{\partial x \partial I} + \frac{\partial^2 \pi}{\partial x^2} \ \ge\ 0\ .$$

iii) If Assumption 2 holds in addition, then following inequality holds:

$$\pi(x, x-z) \ \ge\ \pi(x, x-y) + \pi(y, y-z)\ . \tag{12}$$

for all $x \ge y \ge z$.

iv) If $\pi(x, I) = R(I)$, then R is convex. Conversely, if R is convex, Assumption 3 holds.

Remark 1 Condition (12) (with strict inequality) is classically required to insure the existence of optimal impulse control policies. See for instance Davis [6]. Given Lemma 1 iii), Assumption 3 is therefore natural.

Assumption 3 implies interesting comparison properties on trajectories, which are independent from the dynamics $F(\cdot)$ or the discount rate r.

Lemma 2 *Consider an impulse control policy P which is such that there exists i and j with $i < j$ and: $x_i \geq x_j \geq x_i - l_i \geq x_j - l_j$. Then:*

i) If Assumption 3 holds with strict inequality in (11), then P cannot be optimal.

ii) If Assumption 3 holds with equality in (11) and if P is optimal, then there exist two other optimal policies: one P without the two harvests (x_i, l_i) and (x_j, l_j), and one P with one additional harvest (x_k, l_k) with $x_k = x_i - l_i$ and $l_k = x_j - x_i + l_j$.

The proof of this result is given in Appendix A.4.

3.3 The Fundamental Equivalence Theorem

Now we are going to prove the main relation between problems (P) and (AP). The results of this section are partly based on the next assumption. The solutions to Problem (AP) turn out *not* to depend on x_0, as stated in Lemma 9, see Appendix A.5. As a consequence, we can talk of solutions $(\underline{x}\ , \bar{x}\ )$ to the auxiliary problem (AP) independently of x_0.

Assumption 4 The problem (AP) has a unique solution, denoted with $(\underline{x}\ , \bar{x}\ )$, which is such that $\underline{x}\ < \bar{x}\ $.

Under assumption 4, let us define the following optimization problem (TP), which formalizes the "Transitory Problem". The transitory part of a cyclical policy consists in a) letting grow the stock until some value x; b) harvesting from x down to y; c) applying the cycle with harvesting interval $[\underline{x}, \bar{x}]$ from then on. The question is how to choose the quantities x and y.

$$\text{(TP)}: \quad \max_{\substack{x,y;\\ 0\leq y\leq x\leq x_{ns}\\ x_0\leq x;\ y\leq \bar{x}^*}} e^{-r\tau(x_0,x)}\left[\pi(x, x-y) + G(\underline{x}\ , \bar{x}\ , y)\right] .$$

The following theorem characterizes the solutions to the problem (P).

Theorem 2 *Assume that Assumptions 1–4 hold. Let $(x\ (x_0), y\ (x_0))$ solve the maximization problem (TP). Then the value function of (P) is:*

$$v(x_0) = \begin{cases} G(\underline{x}\ , \bar{x}\ , x_0) & if \quad x_0 < \bar{x} \\ e^{-r\tau(x_0,x^*(x_0))}\left[\pi(x\ (x_0), x\ (x_0) - y\ (x_0)) + G(\underline{x}\ , \bar{x}\ , y\ (x_0))\right] & if \quad x_{ns} \geq x_0 \geq \bar{x} \ . \end{cases} \tag{13}$$

Moreover there exists a solution of (P) which is cyclical: it is given by:

$$t_1 = \tau(x_0, \bar{x}\ ), \quad t_i = t_1 + (i-1)\tau(\underline{x}\ , \bar{x}\ ), \quad i \geq 1, \qquad x_i = \bar{x}\ , \quad l_i = \bar{x}\ - \underline{x}\ , \quad i = 1, 2, \ldots,$$

if $x_0 < \bar{x}\ $, *and*

$$t_1 = \tau(x_0, x\ (x_0)), \quad t_2 = \tau(y\ (x_0), \bar{x}\ ), \quad t_i = t_2 + (i-2)\tau(\underline{x}\ , \bar{x}\ ), \quad i \geq 2,$$

$$x_1 = x\ (x_0), \quad l_1 = x\ (x_0) - y\ (x_0), \quad x_i = \bar{x}\ , \quad l_i = \bar{x}\ - \underline{x}\ , \quad i = 2, \ldots,$$

if $x_0 \geq \bar{x}\ $.

The proof of this result is given in Appendix A.5. The theorem states that any optimal cyclical policy has a cycle part with an harvesting interval $[\underline{x}\,, \bar{x}\,]$. It also describes the nature of the transitory part of optimal cyclical policies. In the case $x_0 < \bar{x}$, there is no transitory part, and the cycle is joined from the start. In the case $x_0 \geq \bar{x}$. The transitory part consists in letting the stock grow until $x\ (x_0)$, harvest it down to $y\ (x_0)$, then join the cycle.

We can now state the following relation between problems (P) and (AP), the proof of which is provided in Appendix A.6.

Theorem 3 *Let Assumptions 1–3 hold. Then:*

i) *If Assumption 4 holds in addition, then (P) has a solution which is cyclical;*

ii) *If (P) has a solution, then (P) has a solution which is cyclical, and there exists a solution to problem (AP) with* $0 \leq \underline{x} < \bar{x} \leq x_{ns}$.

We have therefore shown that there exists a cyclical solution to our problem (P) if and only if the solution to the auxiliary problem (AP) is interior. We can illustrate this point by interpreting the first order conditions of the auxiliary problem, for the case of an interior solution.

Lemma 3 *If* $(\underline{x}, \bar{x})$ *is a solution to the auxiliary problem (AP) with* $0 < \underline{x} < \bar{x} < x_{ns}$, $x_0 < \bar{x}$ *(interior solution), then the first order conditions are given by:*

$$\frac{\partial \pi}{\partial I} = \frac{r}{F(\underline{x})} \frac{e^{-r\tau(\underline{x},\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}}\ \pi(\bar{x}, \bar{x} - \underline{x})\ , \tag{14}$$

$$\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} = \frac{r}{F(\bar{x})} \frac{1}{1 - e^{-r\tau(\underline{x},\bar{x})}}\ \pi(\bar{x}, \bar{x} - \underline{x})\ . \tag{15}$$

Rearranging, we have:

$$\frac{\partial \pi}{\partial I} \frac{F(\underline{x})}{r} = \frac{e^{-r\tau(\underline{x},\bar{x})}}{1 - e^{-r\tau(\underline{x},\bar{x})}}\ \pi(\bar{x}, \bar{x} - \underline{x})\ , \tag{16}$$

$$\frac{d\pi}{dx} = \frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} = \frac{\partial \pi}{\partial I} \frac{F(\underline{x})}{F(\bar{x}) e^{-r\tau(\underline{x},\bar{x})}}\ . \tag{17}$$

The first condition states that, at the optimum, the marginal gain from harvesting the resource, weighted with the growth potential at the new resource stock as compared to the interest rate, should equal the value of standing timber, outcome of a maximized rotational harvest stream. The second condition states that the marginal gain derived from the stock effect is equal to the marginal gain from harvesting augmented by a correcting factor, which depends on the growth differential at the lower and upper limit of the rotational cycles, the latter being discounted over time. More precisely, the higher this growth differential, the greater the marginal gain due to the resource stock.

4 The Exhaustion Problem

This section is devoted to the question of knowing whether or not it can be optimal to exhaust the resource. The summary of the analysis is the following proposition. We say that the system has the "ELB" (Exhaustion locally better) property if exhausting the resource is better than leaving an arbitrarily small quantity $\underline{x}$. We shall not address here the issue of *global* optimality of exhaustion, the characterization of which seems to require stronger assumptions on the problem.

Proposition 1 *Assume that the gain function* $\pi(\bar{x}, I)$ *is positive. According to the form of the growth function* $F(x)$ *as* $x \to 0$*, we have:*

i) If $F(x) \sim \alpha x^{\beta}$ *with* $\alpha > 0$ *and* $\beta > 1$*, then the ELB property holds.*

ii) If $F(x) = \alpha x + O(x^2)$*, and if* $a = r/\alpha$*, then:*

ii.1) if $a > 1$*, then ELB holds.*

ii.2) if $a = 1$*, a necessary condition for ELB is that there exists an* x_m *that solves Equation* (50)*, and that:*

$$\pi(x_m, x_m) \ \lim_{k \to 0} \frac{\exp(-r\tau(k, x_m))}{k} \ \leq \ \pi_I(x_m, x_m)\ . \tag{18}$$

ii.3) if $a < 1$*, then ELB does not hold.*

iii) If $F(x) \sim \alpha x^{\beta}$ *with* $\alpha > 0$ *and* $0 \leq \beta < 1$*, then ELB does not hold.*

In other words, extinction is determined by the initial growh capacity of the resource and by the relative importance of the growth rate and the discount rate. The proof is provided in Appendix A.7.

5 Links between Impulse Control Models and Continuous Control Models

5.1 Comparison with Clark's Model

We may now establish a first link between the general impulse control model and the continuous control model, as proposed by Clark [3].

Consider a solution of problem (AP) on the boundary $\underline{x} = \bar{x}$. The maximization problem becomes:

$$\max_{0 \leq x \leq x_{ns}} G(x, x, x_0),$$

where G is given by (10). The first order condition is:

$$\pi_{Ix}(x, 0)F(x) \ + \ \pi_I(x, 0)[F\ (x) - r] \ = \ 0\ . \tag{19}$$

Let us denote by x a solution of (19). We can see that this condition coincides with the well known marginal productivity rule of resource exploitation when $\pi_I(x, 0)$ is the instantaneous profit function (see for example Clark [3] or Clark and Munro [4]). In fact, Equation (19) coincides with the equation which determines the steady state of the following Clark-like singular optimal control problem:

$$\text{(CP)} \qquad \max_{h(\cdot)} \int_0^\infty e^{-rt} \pi_I(x(t), 0) h(t)\, dt\ , \tag{20}$$

$$\dot{x} = F(x) - h, \tag{21}$$

for x_0 given and $0 \leq h(t) \leq h_{\max}$ for all t. This means that the conditions of a Clark-like steady state solution can also be triggered by the impulse control model that we propose, as shown in the next section.

5.2 Relationships between (P), (AP) and (CP)

We now establish the relationship between the gains in Clark's solution, the solution of (AP) on the boundary $\underline{x} = \bar{x}$, and the solution of Problem (P).

Theorem 4 *If the solution of (AP) is on the boundary* $\underline{x} = \bar{x} = x$ *, there does not exist a solution to (P).*

Theorem 5 *If the solution of (AP) is on the boundary* $\underline{x} = \bar{x} = x$ *, there exists a sequence of cyclical impulse controls with* $\bar{x} - \underline{x} =$ *approaching the value* $G(x\ , x\ , x_0)$, $\forall x_0 \leq x$ *, in the sense that: there exists a sequence* $\underline{x}_\epsilon$ *such that:*

$$\lim_{\epsilon \to 0} G(\underline{x}_\epsilon, \underline{x}_\epsilon + \ , x_0) = G(x\ , x\ , x_0).$$

For a direct proof, see Erdlenbruch [9]. This result is a consequence of general results on convergence of -approximations in compact sets (see Rockafellar and Wets [21]).

Remark 2 It turns out that the value $G(x\ , x\ , x_0)$ is the value of the solution to problem (CP) defined by (20)–(21), if $x_0 \leq x$.

If the solution of (AP) is such that $\underline{x} < \bar{x}$, the optimal outcome is cyclical. In this case, we know:

Theorem 6 *If (P) has a cyclical solution (the solution to the problem (AP) is such that* $\underline{x} < \bar{x}$*), then the gain from the impulse control model* $v(x_0)$ *is greater than the gain of problem (CP) defined by* (20)–(21), $G(x\ , x\ , x_0)$, *for all* $x_0 \leq x$ *.*

6 Links between Impulse Control Models and Discrete Models

In this section, we show that Dawid and Kopel's model [7] can be embedded in ours, through a proper choice of the dynamics, the cost function and the discount rate. Then, we explain the correspondence between the results of [7] and ours.

6.1 Equivalent Definitions with David and Kopel's model

6.1.1 Growth function and time span associated to the growth

The model of Dawid and Kopel is in discrete time. The population dynamics has the form:

$$x_{t+1} = f(x_t) - u_t = min[M, (1+\lambda)x_t] - u_t$$

with $x_t, u_t \geq 0$ $\forall t \geq 0$ and $M = 1$. We proceed with reproducing this behavior for our model. When no harvesting takes place, we must have: $\dot{x}(t) = F(x(t))$. Suppose:

$$F(x) = Ax \quad \text{if } x < x_{ns} = 1 \quad \text{and} \quad F(x) = 1 - x \quad \text{if } x \geq 1.$$

It can be checked that this function satisfies Assumption 1. Integrating the differential equation, we find that the stock evolves according to the following function:

$$x(t) = \varphi(t, x_0) = \min(x_0 e^{At}, 1) .$$

In order to reproduce the dynamics of Dawid and Kopel's discrete-time model, we fix a time duration Δ, and set:

$$x_{t+1} = \varphi(\Delta, x_t).$$

The dynamics are equivalent when $f(x_t) = \varphi(\Delta, x_t)$ for all x_t, which is the case when:

$$(1+\lambda)x_t = x_t e^{A\Delta}.$$

We deduce how the marginal growth factor A must be defined in terms of Dawid and Kopel's factor $1+\lambda$:

$$A = \frac{\log(1+\lambda)}{\Delta}. \tag{22}$$

Let us compute the time span necessary for the dynamics to get from x to y in terms of the new notation: for every $x \leq y \leq 1$,

$$\tau(x, y) = \int_x^y \frac{1}{F(u)} du = \int_x^y \frac{1}{Au} du = \frac{1}{A} \log \frac{y}{x} .$$

6.1.2 Discounted benefits

For the undiscounted gains, g, the correspondence with Dawid and Kopel's model is done by setting:

$$\pi(\bar{x}, I) = R(I).$$

Note that for this particular form of the gain function, Condition (11) is equivalent to the convexity of R, according to Lemma 1 iv).

Let's consider discounting. In the continuous case, any value $v(t)$ evolves as follows with discounting:

$$v(t) = v_0 e^{-rt}.$$

In discrete time, discounted values evolve as:

$$v_t = \delta v_{t-1} = \delta^t v_0.$$

Since we have chosen the correspondence $v_t = v(\Delta t)$, we should have:

$$\delta^t = e^{-rt\Delta}$$

that is:

$$\log \delta = -r\Delta \ .$$

We may now express Dawid and Kopel's threshold a in the new notation:

$$a \ = \ -\frac{\log \delta}{\log(1+\lambda)} \ = \ \frac{r\Delta}{A\Delta} \ = \ \frac{r}{A} \ .$$

6.1.3 The maximization problem

We proceed with the definition of the function G which is the basis of the auxiliary problem (AP). Two cases must be considered:

- The general problem with $\underline{x} < \bar{x}$. In this case,

$$G(\underline{x}, \bar{x}, x_0) \ = \ \frac{R(\bar{x} - \underline{x})(\frac{x_0}{\bar{x}})^{\frac{r}{A}}}{1 - (\frac{\underline{x}}{\bar{x}})^{\frac{r}{A}}} \ = \ \frac{R(\bar{x} - \underline{x})(\frac{x_0}{\bar{x}})^{a}}{1 - (\frac{\underline{x}}{\bar{x}})^{a}}. \tag{23}$$

  This expression holds including when $\bar{x} = x_{ns} = 1$ and $\underline{x} = 0$.

- The boundary problem. Remember that

$$\pi(\bar{x}, I) \ = \ R(I)$$

  furthermore

$$\lim_{I \to 0} \frac{\partial}{\partial I}\pi(\bar{x}, I) \ = \ R\ (0).$$

  Hence:

$$G(x, x, x_0) \ = \ R\ (0)\ \frac{Ax}{r}\ \left(\frac{x_0}{x}\right)^{\frac{r}{A}} \ = \ R\ (0)\ \frac{x_0^a}{a}\ x^{1-a}. \tag{24}$$

Dawid and Kopel define the elasticity of gains as the function:

$$\varepsilon(x) \ = \ \frac{R\ (x)}{xR(x)}\ . \tag{25}$$

We have:

Lemma 4 *The following results hold for all* $0 \leq \underline{x} \leq \bar{x} \leq 1$*:*

i) *If* $a < 1$*, then:*

$$0 \ < \ \varepsilon(\bar{x} - \underline{x}) - 1 \ < \ \frac{\partial G}{\partial \bar{x}}(\underline{x}, \bar{x}, x_0) \ < \ \varepsilon(\bar{x} - \underline{x}) - a\ . \tag{26}$$

ii) *If* $a > 1$*, then:*

$$\varepsilon(\bar{x} - \underline{x}) - a \ < \ \frac{\partial G}{\partial \bar{x}}(\underline{x}, \bar{x}, x_0) \ < \ \varepsilon(\bar{x} - \underline{x}) - 1\ . \tag{27}$$

iii) *If* $a = 1$, *then:*

$$\frac{1}{G(\underline{x}, \bar{x}, x_0)} \frac{\partial G}{\partial \bar{x}}(\underline{x}, \bar{x}, x_0) = \frac{1}{\bar{x} - \underline{x}} \left(\varepsilon(\bar{x} - \underline{x}) - 1 \right) > 0 . \tag{28}$$

Lemma 5 *If* $a > 1$, *then*

$$\frac{\partial G}{\partial \underline{x}} < 0 .$$

The proof of these lemmas follows from standard calculations. The fact that $\varepsilon > 1$ follows from the convexity of R and the fact that $R(0) = 0$.

As a consequence of Lemmas 4 and 5, we have the following optimization results:

Lemma 6 *The function* G *given by* (23) *and* (24) *has the following properties:*

i) *If* $a < 1$, *then there exists a unique* $(\underline{x}, \bar{x})$, *with* $0 < \underline{x} \leq \bar{x} = 1$, *solution to:*

$$\max_{0 \leq \underline{x} \leq \bar{x} \leq 1} G(\underline{x}, \bar{x}; x_0) . \tag{29}$$

ii) *If* $a > 1$, *then for all* $\bar{x}$,

$$\arg \max_{0 \leq x \leq 1} G(x, \bar{x}; x_0) = 0. \tag{30}$$

iii) *If* $a = 1$, *then*

$$\arg \max_{0 \leq \underline{x} \leq \bar{x} \leq 1} G(\underline{x}, \bar{x}; x_0) \in (0, 1) . \tag{31}$$

6.2 Relations with the Results by Dawid and Kopel

Dawid and Kopel have introduced a threshold value, a, for which the decision maker is indifferent between harvesting immediately or harvesting in the next period. Based on this definition, they state the following fundamental results, which we can also find in our framework:

As long the elasticity of gains ε, averaged over the harvest of one period, is larger than the threshold a, the optimal policy is to wait and defer the exploitation. Inversely, when the average elasticity of gains ε is smaller than a, immediate exploitation is optimal(see [7, Lemma 5]).

In the supra-marginal case, $a < 1$, Dawid and Kopel show that immediate extinction is never optimal. In our framework we can deduce this result from Proposition 1 *ii*.3), since the growth function is precisely of the form $F(x) = Ax$, and $a = r/A$. In the supra-marginal case, the optimal program is eventually cyclic, as Dawid and Kopel show in their Proposition 1. Optimal policies in our framework are also cyclical, under the conditions given in Theorem 3.

In the sub-marginal case, $a > 1$, Dawid and Kopel show that immediate extinction can be optimal. We can deduce this result from Proposition 1 *ii*.1) and from Lemma 5 *ii*). Moreover, Lemma 4 *i*) corresponds to Proposition 3, p. 290 of [7].

7 Cost Functions and Optimal Harvestin Policies

Clearly, the occurrence of an interior or a boundary solution in our model depends on the profit function that applies. The cost function is the most discriminating element of the different profit functions studied in the literature. Costs may depend on the capital investment in the sector (for example the number and type of vessels or the number of fishermen), the abundance of the resource stock, the harvesting rate or the harvest amount. In this paper, we study stock-dependent harvest cost functions, such as the one proposed by Clark [3]. We now need to define those functions which can be applied to our impulse control model. Several different types of cost functions have been proposed in the literature, that can be transformed into impulse cost functions.[13] We analyse these cost functions and their associated profit functions in the following and study the type of solution they trigger.

7.1 A Separable Profit Function Proposed in the Literature

Clark [3] derives the objective function for the continuous control model from a continuous production function. The harvest rate $h(t)$ is determined by the stock size $x = x(t)$ and the rate of harvesting effort $E = E(t)$ such that $h = Q(E, x)$, where the function $Q(E, x)$ is the production function of a given resource industry. Clark assumes this function to be a Cobb-Douglas function of the form

$$Q(E, x) = aE^{\alpha}x^{\beta},$$

where a,α and β are positive constants and especially $\alpha = 1$. The function ax^{β} can be replaced with an arbitrary, nondecreasing search efficiency function $S(x)$, such that $h = Q(E, x) = S(x)E$ and $E = \frac{h}{S(x)}$. The cost of a unit of effort is also a constant: $c = CE$. Then, the continuous control cost function can be written as:

$$c_c(x, h) = CE = \frac{C}{S(x)}h$$

and, for $a = 1$ and $\beta = 1$, we have:

$$c_c(x, h) = \frac{C}{x}h,$$

and the associated instantaneous profit is of the form:

$$\pi_c(x, h) \ = \ (p - \frac{C}{x})\ h,$$

where p is the resource price.

7.2 The Separable Profit Function Adapted to our Problem

In order to represent harvesting as a discrete process, we may follow the reasoning proposed by Clark (just using harvest amount and effort at a point in time instead of the continuous variables). The resulting profit function is given by:

$$\pi(x, I) \ = \ a(x)\ d(I)\ .$$

[13] By impulse cost function we mean the cost function such as it appears in an impulse control problem.

Note that this is a separable profit function: variable harvest costs, linked to the amount of harvesting, are independent of search costs, which depend on the stock availability. This is a usual assumption in renewable resource economics which seems reasonable to the extend to which search equipment and harvesting equipment are separate investments.

Proposition 2 *Assume that Problem (AP) for this profit function has a unique solution. Let* x *be solution of (19). If* $a(x\ ) > 0$ *and* $d\ (0) + d\ (0)\frac{a'(x^*)}{a(x^*)} > 0$ *then there exists a cyclical solution to (P).*

Proof. It is sufficient to prove that there exists an interior solution to the auxiliary problem (AP). Using the uniqueness of the solution, this would imply Assumption 4, and by Theorem 3, that a cyclical solution to (P) exists.

When $\pi(x, I) = a(x)d(I)$, equation (10) is given by:

$$G(x, x, x_0) \ = \ a(x)d\ (0)\ \frac{F(x)}{r}\ e^{r\tau(x_0,x)}\ .$$

From equation (19), we know that first order conditions hold for

$$a\ (x)F(x) + a(x)(F\ (x) - r) = 0. \tag{32}$$

We can develop G in a neigbourhood of the frontier $\underline{x} = \bar{x}$ using a Taylor series

$$G(x+h, x+k, x_0) \cong G(x, x, x_0) + \frac{F(x)}{r} e^{r\tau(x_0,x)} B(x, h, k),$$

where

$$B(x, h, k) = \left\{ \frac{\ }{2} a(x) \left[d\ (0) + d\ (0) \frac{r - F\ (x)}{F(x)} \right] + hd\ (0) \left[a\ (x) - a(x) \frac{r - F\ (x)}{F(x)} \right] \right\}, \qquad = h - k.$$

If there exists h and $\ > 0$ such that $B(x\ , h, h - \ ) > 0$, then there exists a direction pointing to the interior of the domain (where $\underline{x} < \bar{x}$) with greater profit than $G(x\ , x\ , x_0)$. This means that in this case the solution is not on the boundary $\underline{x} = \bar{x}$. Given that x verifies (32) then sufficient conditions for obtaining $B(x\ , h, k)$ greater than zero are:

$$a(x\ ) > 0$$

and

$$d\ (0) + d\ (0) \frac{a\ (x\ )}{a(x\ )} > 0.$$

■

We thus have identified a class of profit functions which leads to a cyclical optimal harvesting strategy in our impulse control problem.

7.3 The Termansen Case

Termansen [24] studies a profit function that belongs to this class of separable functions.

In her case, the growth function is of the form $F(x) = s_0 x(1 - x/K)$, the gain is of the form $g(x, y) = a(x)d(x - y)$ with

$$a(x) \ = \ p - \frac{bc}{x}\ , \qquad d(I) \ = \ I\ .$$

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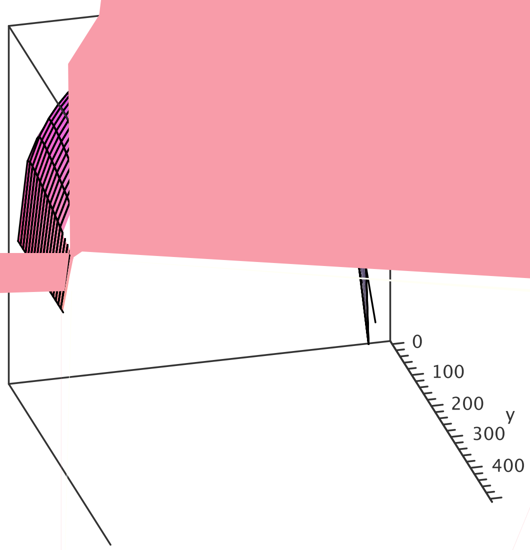

Let: $x(t) = u$, $x(t_0) = x_0$, $\dot{x}dt = du$, $x(t_1) = x_1$. We thus have:

$$\begin{aligned}\lim_{h_{\max}\to\infty} \int_{x_0}^{x_1} (p - c(u)) \frac{h_{\max}}{F(u) - h_{\max}} du &= \int_{x_0}^{x_1} (p - c(u)) \lim_{h_{\max}\to\infty} \frac{h_{\max}}{F(u) - h_{\max}} du \\ &= \int_{x_1}^{x_0} (p - c(u)) du \ .\end{aligned}$$

This leads us to define the profit function in our impulse control problem as:

$$\pi(\bar{x}, \bar{x} - \underline{x}) \ = \ \int_{\underline{x}}^{\bar{x}} [p - c(x)] \ dx \ . \tag{33}$$

Remark 3 For such gain functions, it is easy to check that:

i) for every $a \geq b > c \geq d$,

$$\begin{aligned}\pi(a, a - c) + \pi(b, b - d) &= \int_{d}^{a} [p - c(x)] \ dx \ + \ \int_{c}^{d} [p - c(x)] \ dx \\ &= \pi(a, a - d) + \pi(b, b - c) \ .\end{aligned}$$

ii) Assumption 3 holds therefore, but with equality in (11), and we have:

$$\pi_{xI} \ + \ \pi_{II} \ = \ 0 \ .$$

iii) As stated in Lemma 7 (see Appendix A.2), the function y (x_0) is constant in this case, with y $(x_0) = x$ solution of (19).

Let us suppose in addition that the cost function and dynamics of the problem are specifically given by:

$$c(x) = \frac{b}{x^{\alpha+1}}, \quad F(x) = g_0 x(K - x), \quad \alpha \geq 0 \ . \tag{34}$$

In this case we obtain the following result:

Proposition 3 *For $\pi(\cdot)$, $c(\cdot)$ and $F(\cdot)$ given by (33)–(34), x solution of (19) is a global maximum of problem (AP).*

Proof. We are going to prove that $\forall \underline{x} < \bar{x}$ we have $H(\underline{x}, \bar{x}) := \frac{\partial G}{\partial \underline{x}}(\underline{x}, \bar{x}) > 0$ hence $\underline{x} = \bar{x} = x$ is a global maximum. Consider:

$$H(\underline{x}, \bar{x}) = \left[-p + c(x) + [p(\bar{x} - \underline{x}) - \int_{\underline{x}}^{\bar{x}} c(u) du] \ e^{-r\tau} \frac{r}{F(\underline{x})(1 - e^{-r\tau})} \right] \frac{e^{-r\tau}}{1 - e^{-r\tau}}.$$

Where $\tau = \tau(\underline{x}, \bar{x})$; Let us study the sign of $H(\underline{x}, \bar{x})$.

$$H(\underline{x}, \bar{x}) = \frac{H_2(\underline{x}, \bar{x}) \ \underline{x} \ (p - c(\underline{x}))(K - \underline{x}) + \alpha \ \underline{x} \ g_0 \ (K - \underline{x}) + H_3(\underline{x}, \bar{x})}{H_1(\underline{x}, \bar{x})}$$

where:

$$H_1(\underline{x}, \bar{x}) = \alpha\, \underline{x}\, g_0\, (K - \underline{x}) \left[\left(\frac{\bar{x}(K - \underline{x})}{\underline{x}(K - \bar{x})} \right)^{\frac{r}{g_0 K}} - 1 \right] < 0,$$

$$H_2(\underline{x}, \bar{x}) = -\alpha\, g_0 \left(\frac{\bar{x}(K - \underline{x})}{\underline{x}(K - \bar{x})} \right)^{\frac{r}{g_0 K}} < 0,$$

and

$$H_3(\underline{x}, \bar{x}) = r\, p\, \alpha(\bar{x} - \underline{x}) + r\, b\, c\, (\bar{x}^{-\alpha} - \underline{x}^{-\alpha}).$$

We have that $H_3(\bar{x}, \bar{x}) = 0$ and $\dfrac{\partial H_3}{\partial \underline{x}}(\bar{x}, \bar{x}) = r\, \alpha(-p + c(\underline{x}))$. For all $\underline{x}$ such that $-p + c(\underline{x}) < 0$ we have $H_3(\underline{x}, \bar{x}) > 0$. All these inequalities imply that $H(\underline{x}, \bar{x}) > 0$. ■

The optimal harvesting strategy associated to the integral profit function leads to a non-cyclical optimal harvesting strategy in our impulse control problem. Note that Clark [3] studies this profit function in models where the harvesting process is supposed to be a discrete action. Gains associated to the solution from our impulse control model tend towards the gains associated to Clark's continuous case in which $c(x) = \frac{b}{x^{\alpha+1}}$. This is a consequence of Theorem 5 for initial populations $x_0 < x$. The case of initial populations larger than x remains to be investigated.

8 Conclusion

We have proposed an impulse control framework for the management of renewable resources which is general enough to include concave and convex benefit functions, including stock dependent cost functions. The optimal management of the resource is expressed as optimization problem (P), the solution of which is shown to satisfy the dynamic programming principle. Introducing the class of "cyclical policies", we have reduced the solution of Problem (P) to the sequential solution of two static optimization problems with two variables each.

We have demonstrated that this model can generate two types of optimal solutions: cyclical solutions and solutions which correspond to a Clark-like Most Rapid Approach Path. We have also confirmed that the functional form of the harvest cost function, next to the form of the growth function, is one important element in the explanation of these types of solutions.

With this analysis, we have thus validated our assumption that a change in the scale of analysis may lead to a change in the nature of the solution, via the adaptation of the harvest variable and the harvest cost function. Indeed, when the harvest process is considered at a small scale, the harvest cost function is likely to be an integral function, such as the one derived from the continuous profit function. The optimal harvest policy is non-cyclical and leads to average production gains that are equal to the continuous harvesting policy defined by Clark. This type of functions is most common in the theoretical and empirical fishery economics literature.

When the observer uses a large scale and considers harvest as being an individual action, the cost function is probable to be a separable function. The optimal harvest policy is cyclical and leads to potentially higher production gains than the continuous harvesting policy defined by Clark. This type of function is used in the theoretical and empirical literature in forestry and fishery economics.

Hence, using a different scale when analysing a renewable resource sector may lead to fundamentally different optimal stock levels and harvesting policies. This fact may have important implications for the definition of public policies, especially when threshold effects exist.

References

[1] Arnason, R., L. K. Sandal, S. I. Steinshamn and N. Vestergaard (2004). Optimal feedback controls: comparative evaluation of the cod fisheries in Denmark, Iceland, and Norway. American Journal of Agricultural Economics 86(2): 531 - 542.

[2] Bensoussan, A. J.L. Lions (1982). Contrôle impulsionnel et inéquations quasi-variationnelles, Méthodes Mathématiques de l'Informatique, Dunod.

[3] Clark, C.W. (1990). Mathematical Bioeconomics, The Optimal Management of Renewable Resources, John Wiley and Sons.

[4] Clark, C.W. and G.R. Munro (1975). The Economics of Fishing and Modern Capital Theory: A simplified Approach, Journal of Environmental Economics and Management 2: 92-106.

[5] Clark, C.W., Clarke, F.H. and G.R. Munro (1979). The optimal exploitation of renewable resource stocks: Problems of irreversible investment, Econometrica 47, 25-47.

[6] Davis, M.H.A. (1993). Markov Models and Optimization, Prentice Hall.

[7] Dawid, H. and M. Kopel (1997). On the Economically Optimal Exploitation of a Renewable Resource: The Case of a Convex Environment and a Convex Return Function. Journal of Economic Theory 76: 272-297.

[8] Dawid, H. and M. Kopel (1999). On optimal cycles in dynamic programming models with convex return function. Economic Theory 13: 309-327.

[9] Erdlenbruch, K. (2005). Essais en économie des ressources renouvelables. Thèse, Université Montpellier I, 172 p.

[10] Gonzalez, R. (1979); Thèse de 3ème cycle, Université de Paris Dauphine.

[11] Grafton, R. Q., Sandal, L.K. and S.I Steinshamn (2000). How to Improve the Management of Renewable Resources: The Case of Canada's Northern Cod Fishery. American Journal of Agricultural Economics 82(3): 570-580.

[12] Hannesson, R. (1975). Fishery Dynamics: A North Atlantic Cod Fishery. The Canadian Journal of Economics 8(2), 151-173.

[13] Homans, F. R. and J. E. Wilen (1997) A Model of Regulated Open Access Resource Use. Journal of Environmental Economics and Management 32: 1-21.

[14] Jacquette, D. L. (1972). A discrete-time population-control model. Mathematical Biosciences 15:231-252.

[15] Jacquette, D. L. (1974). A discrete-time population-control model with setup cost. Operations Research 22: 298-303.

[16] Léonard, D. and N. V. Long (1998). Optimal Control Theory and Static Optimization in Economics. Cambridge University Press, 353 p.

[17] Liski, M.,Kort, P.M. and A. Novak (2001). Increasing returns and cycles in fishing. Resource and Energy Economics 23, 241-258.

[18] Lewis,T.R. and R. Schmalensee (1977). Nonconvexity and Optimal Exhaustion of Renewable Resources. International Economic Review 18(3): 535-552.

[19] Lewis,T.R. and R. Schmalensee (1979). Non-Convexity and Optimal Harvesting Strategies for Renewable Resources. The Canadian Journal of Economics 12(4): 677-691.

[20] Reed, W.J. (1984). A stochastic model for economic management of a renewable animal resource. Mathematical Biosciences 22: 313-337.

[21] Rockafellar, R.T., J-B. Wets (1998). Variational Analysis. Springer, 733 p.

[22] Seierstad, A. and K. Sydsaeter (1987). Optimal Control Theory with Economic Applications. Amsterdam, Elsevier.

[23] Singh, R., Weninger, Q. and M. Doyle (2006). Fisheries management with stock growth uncertainty and costly capital adjustment. Journal of Environmental Economics and Management 52, 582-599.

[24] Termansen, M. (2007). Economies of scale and the optimality of rotational dynamics in forestry. Environmental and Resource Economics 37: 643-659.

[25] Touza-Montero, J. and M. Termansen (2002). The Faustman model as a special case. Unpublished working paper. Environment Department, University of York, UK.

[26] Vind, K. (1967) Control Systems with Jumps in the State Variables. Econometrica 35, 273-277.

[27] Wirl, F. (1995) The Cyclical Exploitation of Renewable Resource Stocks May Be Optimal. Journal of Environmental Economics and Management 29, 252-261.

A Appendix

A.1 Boundedness of the Profit Function

In this section, we discuss sufficient conditions for the existence of values for impulse control policies.

Let $F_{\max}$ be an upper bound on function F, which exists under Assumption 1. From the equation of the dynamics (1), we have for all $t_1 \leq t_2$:

$$x(t_2) \;\leq\; x(t_1) \;+\; F_{\max}(t_2 - t_1)\;. \tag{35}$$

Next, denote with π^+ the function: $\pi^+(x, I) = \max(\pi(x, I), 0)$. Assume that there exists a bounded function $\;\; : [0, x_{sup}) \to \mathbb{R}_+$ such that:

$$\pi^+(x, I) \;\leq\; \;(x) I\;.$$

We show below that these assumptions imply that the gain associated with any impulse control policy of the family $\mathcal{F}_{x_0}$ is bounded above.

Let $\Delta t \leq (F_{\max})^{-1}$. We first show that the total quantity of resource collected in any interval of length less than Δt is bounded. Indeed, because policies of the class $\mathcal{F}_{x_0}$ are such that $\lim_{i\to+\infty} t_i = +\infty$, there exist finite numbers p and q such that:

$$\begin{aligned}
\sum_{i \mid a \leq t_i < a+\Delta t} I_i &= \sum_{i=p}^{q} x(t_i^-) - x(t_i) \\
&= x(t_p^-) - x(t_p) + \sum_{i=p+1}^{q} x(t_{i-1}) + [x(t_i^-) - x(t_{i-1})] - x(t_i) \\
&= x(t_p^-) - x(t_q) + \sum_{i=p+1}^{q} [x(t_i^-) - x(t_{i-1})] \\
&\leq x_{ns} + \sum_{i=p+1}^{q} F_{\max}(t_i - t_{i-1}) \;=\; x_{ns} + F_{\max}(t_q - t_p) \\
&\leq x_{ns} + F_{\max}\Delta t\;.
\end{aligned}$$

Next, we show that the total gain of any policy is bounded above. Indeed, decomposing the time horizon into intervals of duration Δt, we find:

$$\begin{aligned}
\Pi &= \sum_{m=0}^{\infty} \sum_{i \mid m\Delta t \leq t_i < (m+1)\Delta t} e^{-rt_i} \pi(x_i, I_i) \\
&\leq \sum_{m=0}^{\infty} \sum_{i \mid m\Delta t \leq t_i < (m+1)\Delta t} e^{-rm\Delta t} \;(x_i) I_i \\
&\leq \sup_x \;(x) \sum_{m=0}^{\infty} e^{-rm\Delta t} \sum_{i \mid m\Delta t \leq t_i < (m+1)\Delta t} I_i \\
&\leq \sup_x \;(x) \sum_{m=0}^{\infty} e^{-rm\Delta t} (x_{ns} + F_{\max}\Delta t) \\
&= (\sup_x \;(x)) \frac{x_{ns} + F_{\max}\Delta t}{1 - \exp(-r\Delta t)}\;.
\end{aligned}$$

A.2 Complements on the Auxiliary Problem

We provide here additional properties of the function $y(x)$, which solves Problem (TP) under the additional constraint that $x_0 = x$.

Lemma 7 *Let Assumption 4 holds, and assume that the function π is smooth enough, and that the solution to the problem:*

$$\max_{\substack{y; \\ 0 \le y \le x_0}} \left[\pi(x_0, x_0 - y) + G(\underline{x}, \bar{x}, y)\right] ,$$

denoted by $y(x_0)$, is interior. Then the function $y(x)$ is decreasing for $x \ge \bar{x}$. If equality holds in (11)*, the function $y(x)$ is constant and equal to $\underline{x}$ since $y(\bar{x}) = \underline{x}$.*

Proof. Let $H(x,y) = \pi(x, x-y) + G(\underline{x}, \bar{x}, y)$. If $y = y(x)$ maximizes $H(x,y)$ for x fixed as an interior solution, then for all x,

$$H_y(x, y(x)) \;=\; 0 , \qquad H_{yy}(x, y(x)) \;\le\; 0 .$$

Differentiating the identity $0 = H_y(x, y(x))$ with respect to x, we obtain:

$$y'(x) \;=\; -\frac{H_{xy}(x, y(x))}{H_{yy}(x, y(x))} \;=\; \frac{\pi_{xI} + \pi_{II}(x, x - y(x))}{H_{yy}(x, y(x))} .$$

The numerator of this expression is positive, by Lemma 1 *iii*). The denominator is negative according to the second-order condition. This expression is therefore negative.

If equality holds in (11), the numerator actually vanishes, and $y'(x) = 0$. The value of $y(x)$ is therefore constant. ■

A.3 Dynamic Programming and Optimal Trajectories

In this appendix, we propose a technical result which is useful in a variety of situations. This result is actually an application of the Dynamic Programming principle of Theorem 1, to the comparison of trajectories.

Before stating the result, we need some preliminaries. Assume that a policy P is such that $x_{i+1} \ge x_i$. Let $\delta t = \tau(x_i - I_i, x_i)$. Consider the following modifications of the reference policy P:

Policy A (remove the harvesting at t_i)

> **for** $k < i$, $t_k^A = t_k$, $I_k^A = I_k$;
> **for** $k \ge i$, $t_k^A = t_{k-1} - \delta t$, $I_k^A = I_{k-1}$.

Policy B (copy once the harvesting occurring at t_i)

> **for** $k \le i$, $t_k^B = t_k$, $I_k^B = I_k$;
> **for** $k > i$, $t_k^B = t_{k+1} + \delta t$, $I_k^B = I_{k+1}$.

Policy C (reproduce endlessly the harvesting occurring at t_i)

> **for** $k < i$, $t_k^C = t_k$, $I_k^C = I_k$;

for $k \geq i$, $t_k^C = t_i + (k-i)\delta t$, $I_k^C = I_i$.

Assume now that the policy P is such that $x_{i+1} \in (x(t_{i-1}^+), x_i]$, where by convention, $t_{-1} = 0$ in the case $i = 1$. In that case, there exists a time $T = t_i - \tau(x_{i+1}, x_i)$ such that $x(T) = x_i$. Let, as above, $\delta t = \tau(x_i - I_i, x_i)T$. and define the policies A, B and C just as above.

These policies are illustrated in Figure 2 a) in the first case, and b) in the second one.

We can now state the result:

Lemma 8 *Consider an impulse control policy P which is such that either $x_i \in (x(t_i^+), x_{i+1}]$ or $x_{i+1} \in (x(t_{i-1}^+), x_i]$ for some i. Then, the gain of policy P is smaller than that of policies A or C constructed above.*

Proof. Assume first that policy P is such that $x_i \in (x(t_i^+), x_{i+1}]$, which implies $x_{i+1} \geq x_i$. Let G_P, G_A, G_B and G_C be the total profits for policies P, A, B and C. Denote with V_0 the current-value gains associated with the part of the trajectory before t_i (which is common to all these policies) and let $G_\pi = V_0 + e^{-rt_i}\tilde{G}_\pi$ for policies $\pi \in \{$ P, A, B, C $\}$. It is easy to see that

$$
\begin{aligned}
\tilde{G}_P &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_B \\
\tilde{G}_A &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_P \\
\tilde{G}_C &= \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_C \ .
\end{aligned}
$$

As a consequence, we have the identity: $\tilde{G}_P - \tilde{G}_A = e^{-r\delta t}(\tilde{G}_B - \tilde{G}_P)$. This implies that $\tilde{G}_P \leq \max(\tilde{G}_A, \tilde{G}_B)$. Next, if we have $\tilde{G}_P \leq \tilde{G}_A$, then we have $\tilde{G}_A \leq \pi(x_i, x_i - I_i) + e^{-r\delta t}\tilde{G}_A$ so that:

$$
\tilde{G}_A \leq \frac{\pi(x_i, x_i - I_i)}{1 - e^{-r\delta t}} = \tilde{G}_C \ .
$$

This proves the statement.

Consider now the case $x_{\epsilon\delta}$

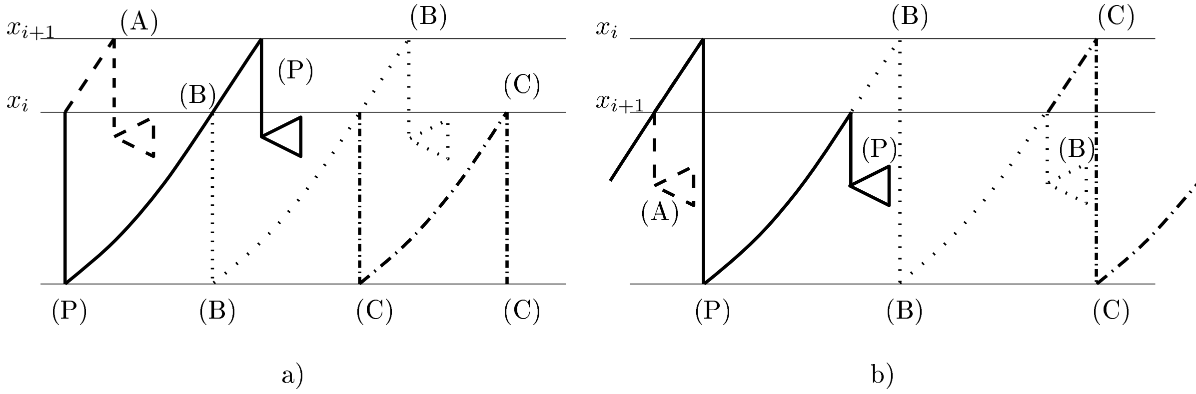

Figure 2: The original policy P and its modifications A, B and C. The triangle represents the remainder of the trajectory, which is common to P, A and B, up to a shift in time.

> **for** $k = j$**,** $t_j^A = t_j$, $I_j^A = b - c$;
> **for** $k > j$**,** $t_k^A = t_{k-\ell} + \delta t$, $I_k^A = I_{k-\ell}$.

Policy (B) (remove the piece of trajectory from b to c):

> **for** $k < i$**,** $t_k^B = t_k$, $I_k^B = I_k$;
> **for** $k = i$**,** $t_k^B = t_k$, $I_k^B = a - d$;
> **for** $k > i$**,** $t_k^B = t_{k+\ell} - \delta t$, $I_k^B = I_{k+\ell}$.

These policies can be visualized in Figure 3, which represents the evolution of the population under each of the three policies. The triangle represents the rest of the trajectory, which is the same for all three policies, except for a shift in time. The rectangle represents an arbitrary piece of trajectory, which can possibly exit the range $[b, c]$.[14]

The discounted profits G associated with the original policy P can be written as:

$$G \; = \; V_0 \; + \; R_i \; \pi(a, a-c) \; + \; R_i \; V_1 \; + \; R_j \; \pi(b, b-d) \; + \; R_j \; V_d$$

where R_i and R_j are the discounts:

$$R_i \; = \; e^{-rt_i} \qquad R_j \; = \; e^{-rt_j} \; ,$$

and where V_0, V_1 and V_d are the current-value gains associated with the first part of the trajectory, and the pieces of the trajectory, respectively, in the intervals (t_i, t_j) and $(t_j, +\infty)$:

$$V_0 \; = \; \sum_{k=1}^{i-1} e^{-rt_k} \; \pi(x_k, I_k) \qquad V_1 \; = \; \sum_{k=i+1}^{j-1} e^{-r(t_k - t_i)} \; \pi(x_k, I_k)$$

$$V_d \; = \; \sum_{k=j+2}^{\infty} e^{-r(t_k - t_j)} \; \pi(x_k, I_k) \; .$$

[14]The situation where $b = c$ is allowed, in which case the piece of trajectory may be empty. In that case, there is a double harvest at the same instant in time.

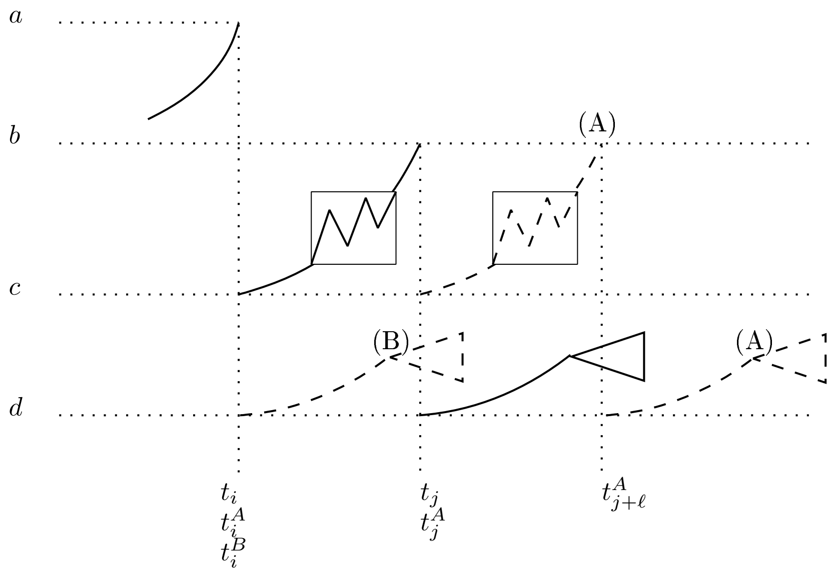

Figure 3: The original policies and its modifications (A) and (B)

The total discounted gains associated with policies (A) and (B) are:

$$\begin{aligned} G_A &= V_0 + R_i\,\pi(a, a-c) + R_i\,V_1 + R_j\,\pi(b, b-c) + R_j\,V_1 + R_j\,\rho\,\pi(b, b-d) + R_j\,\rho\,V_d \\ G_B &= V_0 + R_i\,\pi(a, a-d) + R_i\,V_d\,, \end{aligned}$$

with $\rho = R_j/R_i = \exp(-r(t_j - t_i))$. Accordingly, the modifications in the profit implied by switching from the original policy to either (A) or (B) are:

$$\begin{aligned} G - G_A &= R_j\ \left(\pi(b, b-d) - \pi(b, b-c) + V_d - \rho\pi(b, b-d) - V_1 - \rho V_d\right) \\ G - G_B &= R_i\ \left(\pi(a, a-c) - \pi(a, a-d) + V_1 + \rho\pi(b, b-d) + \rho V_d - V_d\right)\ . \end{aligned}$$

As a consequence, we have the following identity:

$$\pi(a, a-c) + \pi(b, b-d) - \pi(a, a-d) - \pi(b, b-c) = \frac{1}{R_j}(G - G_A) + \frac{1}{R_i}(G - G_B)\ .$$

Under Assumption 3, the left-hand side is negative. If the inequality in (11) is strict, it is even strictly negative. This implies that one at least of G_A or G_B is strictly larger than G. The policy cannot be optimal. This proves i).

If equality holds (11) and the policy P is assumed to be optimal, then $G_A = G_B = G$ and policies (A) and (B) are optimal as well. These policies have the properties claimed in ii).

A.5 Proof of Theorem 2

The proof is separated into two cases. If x_0 is "small enough", the proof is done by trajectory comparison arguments. For the case of "large" x_0, the proof consists in embedding the optimization problems (AP) and (TP) into a more general optimization problem, then solving this more general problem. The solution turns out to be provided by (AP) and (TP), and satisfy the dynamic programming equation.

Throughout the rest of this section, Assumption 4 is assumed to hold, so that the optimal values for (AP), $\underline{x}$ and $\bar{x}$, are well defined.

Let $w(\cdot)$ be defined, as in (13), as:

$$w(x) = \begin{cases} G(\underline{x}\ ,\bar{x}\ ,x) & \text{if} \quad x < \bar{x} \\ \\ e^{-r\tau(x,x^*(x))}\left[\pi(x\ (x), x\ (x) - y\ (x)) + G(\underline{x}\ ,\bar{x}\ ,y\ (x))\right] & \text{if} \quad x_{sup} \geq x \geq \bar{x}\ . \end{cases} \tag{36}$$

where $(x\ (x), y\ (x))$ is any solution of the problem (TP) with initial population $x_0 = x$.

The following result will be useful for the proof.

Consider problem (AP). Its solution does not depend on the initial stock value x_0, as long as this value is small enough:

Lemma 9 *Assume that* $(\underline{x}\ ,\bar{x}\ )$ *solves (AP) for some value of* $x_{ns} > x_0 > 0$. *Then it solves (AP) for every value of* x_0.

Proof. The result follows from the fact that for all x_0, x_1:

$$G(\underline{x}, \bar{x}, x_0) \; = \; e^{-r\tau(x_0,x_1)}\; G(\underline{x}, \bar{x}, x_1)\ .$$

Therefore the two functions are proportional, with a proportionality factor which is strictly positive if $0 < x_0 < x_{ns}$ and $0 < x_1 < x_{ns}$. The problems (AP) for x_0 and (AP) for x_1 have therefore the same solutions. If $x_1 = 0$, or if $x_1 = x_{ns}$, then $G = 0$ and any $(\underline{x}\ ,\bar{x}\ )$ maximizes it. If $x_1 = x_{ns}$ and the growth is fast, the reasoning above applies. ■

A.5.1 Proof for $x_0 < \bar{x}$

Lemma 10 *If Assumptions 3 and 4 hold, then the function* $w(x_0)$ *solves the dynamic programming equation* (7) *for all* $x_0 < \bar{x}$.

According to Theorem 1, the value function of problem (P) verifies:

$$v(x) \;=\; \max_{\substack{t\geq 0\\ 0\leq y\leq \phi(t,x)}} e^{-rt}\left[\pi(\varphi(t,x), \varphi(t,x) - y) + v(y)\right] \tag{37}$$

$$= \max\left\{ \max_{0\leq y\leq x}\left[\pi(x, x-y) + v(y)\right]\ , \right. \tag{38}$$

$$\left. \max_{\substack{x<\bar{x}\leq x_{ns}\\ 0\leq y\leq \bar{x}}} e^{-r\tau(x,\bar{x})}\left[\pi(\bar{x}, \bar{x}-y) + v(y)\right] \right\} . \tag{39}$$

This decomposition is obtained by separating the case $t = 0$ (expression (38)) from the case $t > 0$, and performing the change of variable $t = \tau(x, \bar{x})$ in (39). This change of variable maps the time interval $t \in (0, +\infty)$ to the interval on populations $\bar{x} \in (x, x_{ns})$ in the Slow Growth case, and $\bar{x} \in (x, x_{ns}]$ in the Fast Growth case, since $x \leq x_{ns}$.

We must show that the function $w(x)$, defined in (36), a solution of Equation (37).

By assumption, $x < \bar{x}$. Replacing $v(y)$ by its value in (37), the right-hand side can be written as:

$$M = \max \Bigg\{ \max_{0 \le y \le x} \left[\pi(x, x-y) + G(\underline{x} \; , \bar{x} \; , y) \right] \; , \tag{40}$$

$$\max_{\substack{x < \bar{x} \le x_{ns} \\ 0 \le y < \bar{x}^*}} e^{-r\tau(x,\bar{x})} \left[\pi(\bar{x}, \bar{x}-y) + G(\underline{x} \; , \bar{x} \; , y) \right] \; , \tag{41}$$

$$\max_{\substack{x < \bar{x} \le x_{ns} \\ \bar{x}^* \le y \le \bar{x}}} e^{-r\tau(x,\bar{x})} \Bigg[\pi(\bar{x}, \bar{x}-y) \tag{42}$$

$$+ e^{-r\tau(y,x^*(y))} \left[\pi(x \; (y), x \; (y) - y \; (y)) + G(\underline{x} \; , \bar{x} \; , y \; (y)) \right] \Bigg] \Bigg\} \; .$$

We recognize in the term (41) the problem (TP). We prove first that this is the largest of the three. Consider, for some $y = y_0$, the value in brackets in in (42). It corresponds to a policy P with two harvests $\bar{x} \to y_0$ and $x \; (x) \to y \; (x)$. Two cases may happen, according to which of $\bar{x}$ and $x \; (x)$ is the largest.

Case $\bar{x} \ge x \; (x)$: in this case, these two harvests are overlapping (since $y \; (x) < \bar{x} \; \le y_0$), in which case Lemma 2 applies. The policy P is dominated by at least one of two modifications. If the dominating policy is the one without the second cut, then its value is present in (41) when y has the value $y \; (y_0)$. If the dominating policy is the one with an additional cut, then it is easy to see (see for instance the proof of Lemma 8) that the policy with a cyclic harvesting with interval $[\bar{x} \; , y]$ is even better. But this policy provides a gain equal to $\pi(\bar{x}, \bar{x}-y) + e^{-\tau(y,\bar{x}^*)} G(y, \bar{x} \; , y) \le \pi(\bar{x}, \bar{x}-y) + e^{-\tau(y,\bar{x}^*)} G(\underline{x} \; , \bar{x} \; , y)$. Policy P is therefore again dominated by some policy represented in (41).

Case $\bar{x} < x \; (x)$: in this case, Lemma 8 applies, and policy P is dominated by at least one of two modifications. Either the dominating policy is the modification "A" without a second cut: its gain is one of the values in (41). Or the dominating policy is the one with a cyclic harvesting. The reasoning above then applies and there exists a value in (41) which dominates the value in (42). We have shown that (42) is smaller than (41).

Next, we show that (40) is dominated by (41). Each y in (40) corresponds to some policy P_y which two first harvests are $x \to y$ and $\bar{x} \; \to \underline{x}$. Since x is smaller than $\bar{x}$, we are in the situation of Lemma 8 again. The policy P_y is therefore dominated: either by the policy A which consists in applying directly the cycle with interval $[\underline{x} \; , \bar{x} \;]$, or by the cyclical policy with interval $[y, x]$. This one is in turn dominated by the cyclical policy A according to Assumption 4. In both cases, P_y is dominated by C. Since the gain associated with C is present in (41) (with $\bar{x} = \bar{x}$ and $y = \underline{x}$), the term in (40) is dominated by the term in (41).

At this stage, we have proved that (41) dominates the two other terms, so that:

$$M = \max_{\substack{x < \bar{x} \le x_{ns} \\ 0 \le y < \bar{x}^*}} e^{-r\tau(x,\bar{x})} \left[\pi(\bar{x}, \bar{x}-y) + G(\underline{x} \; , \bar{x} \; , y) \right] \; .$$

There remains to be proved that the maximum in the right-hand side is reached at $\bar{x} = \bar{x}$ and $y = \underline{x}$. Each value of the right-hand side is the gain of some policy P which two first cuts are $x_1 = \bar{x}$ and $x_2 = \bar{x}$. Whether $\bar{x} < \bar{x}$ or $\bar{x} > \bar{x}$, the application of Lemma 8 implies that P is dominated: either by policy "A" which has the value $G(\underline{x} \; , \bar{x} \; , x)$, or by policy "C" which is has the value $G(y, \bar{x}, x) < G(\underline{x} \; , \bar{x} \; , x)$ by Assumption 4 and Lemma 9.

The value of M is readily seen to be $e^{-r\tau(x,\bar{x}^*)}G(\underline{x}\ ,\bar{x}\ ,\bar{x}\ ) = G(\underline{x}\ ,\bar{x}\ ,x) = w(x)$. The function w solves the Bellman equation for $x < \bar{x}$.

A.5.2 Proof for $x_0 \geq \bar{x}$

Lemma 11 *If Assumptions 3 and 4 hold, then the function $w(x_0)$ solves the dynamic programming equation for all $x_{ns} \geq x_0 \geq \bar{x}$.*

Proof. Replacing $v(y)$ by its value in (37), the right-hand side can be written as:

$$M = \max\Bigg\{ \max_{0 \leq y < \bar{x}^*} \left[\pi(x_0, x_0 - y) + G(\underline{x}\ ,\bar{x}\ ,y)\right] , \tag{43}$$

$$\max_{\bar{x}^* \leq y \leq x_0} \Big[\pi(x_0, x_0 - y) \tag{44}$$

$$+ e^{-r\tau(y,x^*(y))} \left[\pi(x\ (y), x\ (y) - y\ (y)) + G(\underline{x}\ ,\bar{x}\ ,y\ (y))\right]\Big] ,$$

$$\max_{\substack{x_0 < \bar{x} \leq x_{ns} \\ 0 \leq y < \bar{x}^*}} e^{-r\tau(x_0,\bar{x})} \left[\pi(\bar{x}, \bar{x} - y) + G(\underline{x}\ ,\bar{x}\ ,y)\right] , \tag{45}$$

$$\max_{\substack{x_0 < \bar{x} \leq x_{ns} \\ \bar{x}^* \leq y \leq \bar{x}}} e^{-r\tau(x_0,\bar{x})} \Big[\pi(\bar{x}, \bar{x} - y) \tag{46}$$

$$+ e^{-r\tau(y,x^*(y))} \left[\pi(x\ (y), x\ (y) - y\ (y)) + G(\underline{x}\ ,\bar{x}\ ,y\ (y))\right]\Big]\Bigg\} .$$

Following the reasoning in proof of Lemma 10, the terms (44) and (46) are respectively dominated by (43) and (45). There remains:

$$\begin{aligned} M &= \max\Bigg\{ \max_{0 \leq y \leq \bar{x}^*} \left[\pi(x_0, x_0 - y) + G(\underline{x}\ ,\bar{x}\ ,y)\right] , \\ &\qquad \max_{\substack{x_0 < \bar{x} \leq x_{ns} \\ 0 \leq y \leq \bar{x}^*}} e^{-r\tau(x_0,\bar{x})} \left[\pi(\bar{x}, \bar{x} - y) + G(\underline{x}\ ,\bar{x}\ ,y)\right] \Bigg\} \\ &= \max_{\substack{x_0 \leq \bar{x} \leq x_{ns} \\ 0 \leq y \leq \bar{x}^*}} e^{-r\tau(x_0,\bar{x})} \left[\pi(\bar{x}, \bar{x} - y)\ +\ G(\underline{x}\ ,\bar{x}\ ,y)\right] . \end{aligned}$$

This is the definition of Problem (TP). The solution is therefore $(x\ (x_0), y\ (x_0))$, which concludes the proof. ■

A.6 Proof of Theorem 3

The statement i) of Theorem 3 is a direct consequence of Theorem 2.

For statement ii), we need the following result, which is a corollary of Assumption 3 and Lemma 2.

Lemma 12 *If Assumption 3 holds, then for every solution to problem (P) which is not cyclical, there exists a cyclical solution with the same value.*

Proof. It is necessary to characterize first what a non-cyclical solution may be. From the definition of cyclical policies in Section 3.1, it can be seen by inspection that the set of possible values for the population $x(t)$ is an interval included in $[0, x_{ns}]$, and that every single value a) either is reached once only, b) or is reached an infinite number of times according to a periodic sequence $s_1, s_1 + T, s_1 + 2T, \ldots$ for some $T > 0$, c) or is 0. A solution which is not cyclical would therefore: i) either reach population values which are not an interval, ii) or reach some value $v \neq 0$ a number of times which is neither 1 nor infinity, iii) or reach some value $v \neq 0$ according to a sequence of instants which is not periodic.

The first step is to exclude non-cyclical solutions to (P) which are such that $x(s) = x(t)$ for some $s < t$. For such a policy (A), consider the smallest such t. Let (B) be the policy which consists in performing the same catches as (A) up to time t, next applying the optimal cyclical policy with initial population $x(t)$ but shifted in time by t units. The values reached by policy (B) are reached either once or an infinite number of times at periodic intervals. As a consequence of Theorem 1, the value function of policy (B) is the same as (A). Therefore, a policy which is such that ii) or iii) can be replaced by a cyclical one.

The second step is to eliminate policies of type i). For such policies, there exists some $i < j$ and a sequence of values $a > b \geq c > d$, such that for some i, $x_i = a$, $l_i = a - c$, and $x_j = b$, $l_j = b - d$. According to Lemma 2, such a policy cannot be optimal if Assumption 3 holds strictly. In the other case, the policy can be replaced with another policy with the same total profit but one harvest less. If this policy is not cyclical, an induction is applied to construct a cyclical policy which has the same profit as the original one. ■

According to this lemma, we know that we can restrict our attention to cyclical solutions of (P). Such solutions are characterized theorem 2. Their cyclic part is given by an harvesting interval $[\underline{x}\ , \bar{x}\ ]$ which is necessarily an interior solution of (AP).

A.7 Proof of Proposition 1

First of all, recall that according to the definition of the function G in Section 3.1, we have:

$$G(0, \bar{x}, x_0) \;=\; \pi(\bar{x}, \bar{x})\, e^{-r\tau(x_0, \bar{x})}\,,$$

and that this is the profit of a cyclical policy with harvesting interval $[0, \bar{x}]$, applied to the initial stock $x_0 \leq \bar{x}$. The ELB property holds if there exists a x_m such that, x_0 being given:

$$\arg\max_{\underline{x},\, \bar{x};\, \underline{x} \leq \bar{x}} G(\underline{x}, \bar{x}, x_0) \;=\; (0, x_m)\,.$$

A necessary condition for this to happen is that the pair $(0, x_m)$ be a local maximum of the function $G(\cdot)$. We therefore compute an expansion of the function in the vicinity of $(x_m, 0)$, starting with the expression:

$$G(k, x_m + h; x_0) \;=\; \pi(x_m + h, x_m + h - k)\, e^{-r\tau(x_0, x_m+h)}\, \frac{1}{1 - e^{-r\tau(k, x_m+h)}}\,. \qquad (47)$$

We shall limit the expansions to the first order term, in order to obtain necessary conditions for such an extremum to exist. Let $\eta = |h + k|$. Omitting the arguments of the functions when they are (x_m, x_m), we have:

$$\begin{aligned}
\pi(x_m+h, x_m+h-k) &= \pi \;+\; h\left(\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I}\right) \;-\; k\frac{\partial \pi}{\partial I} \;+\; O(\eta^2) \\
\tau(x_0, x_m+h) &= \tau(x_0, x_m) + \tau(x_m, x_m+h) \\
&= \tau(x_0, x_m) + \frac{h}{F(x_m)} + O(\eta^2)
\end{aligned}$$

from which:

$$\begin{aligned}
&\pi(x_m+h, x_m+h-k)\; e^{-r\tau(x_0,x_m+h)} \qquad (48)\\
&= \; e^{-r\tau(x_0,x_m)} \;\left(g + h(\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} - \frac{rg}{F(x_m)}) - k\frac{\partial \pi}{\partial I} + O(\eta^2)\right) \;.
\end{aligned}$$

For the development of $\tau(k, x_m+h) = \tau(k, x_m) + \tau(x_m, x_m+h)$, it is necessary to specify the behavior of the growth function in the vicinity of the origin. Indeed, we know that:

$$\tau(k, x_m) \;=\; \int_k^{x_m} \frac{1}{F(u)}\, du \;.$$

Proof of i). Assume first that $F(x) \sim \alpha x^\beta$ with $\alpha > 0$ and $\beta > 1$. In this case, the integral above diverges as $k \to 0$, and actually:

$$\tau(k, x_m) \;\sim\; \frac{k^{1-\beta}}{\alpha(\beta-1)} \;.$$

As a consequence, $\exp(-r\tau(k, x_m)) = o(k^m)$ for all $m > 0$. In other words, this function tends to 0 faster than any polynomial, as $k \to 0$. It follows that the third factor in the right-hand side of (47) is negligible in the expansion. The expansion for G coincides with the right-hand side of (48), and we have:

$$G(k, x_m+h; x_0) - G(0, x_m; x_0) \;=\; e^{-r\tau(x_0,x_m)} \;\left(h(\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} - \frac{rg}{F(x_m)}) - k\frac{\partial \pi}{\partial I} + O(\eta^2)\right) \;. \tag{49}$$

Under the assumption that $(\partial\pi/\partial I)(\bar{x}, \underline{x}) < 0$ for all $\bar{x}, \underline{x}$, the difference above can always be made negative for a positive choice of k. As a consequence, it is always better to exhaust the resource in this case. The best choice for x_m is to satisfy the condition:

$$\frac{\partial g}{\partial \bar{x}}(x_m, 0) \;=\; \pi(x_m, x_m)\; \frac{r}{F(x_m)} \;, \tag{50}$$

Proof of ii). Next, consider the case where $F(x) = \alpha x + O(x^2)$. In this case, there holds:

$$\tau(k, x_m) \;=\; \frac{1}{\alpha} \log \frac{x_m}{k} \;+\; O(1) \;.$$

Consequently, $\exp(-r\tau(k, x_m)) \sim C k^{r/\alpha}$ for some constant C. Denote with a the ratio r/α. The expansion of the third term in (47) is:

$$\begin{aligned}
\frac{1}{1 - e^{-r\tau(k,x_m+h)}} &= 1 \;+\; C\; k^a\; e^{-r\tau(x_m,x_m+h)} \;+\; o(k^a) \\
&= 1 \;+\; C\; k^a\; (1 + \frac{h}{F(x_m)} + o(h)) \;+\; o(\eta^a) \;.
\end{aligned}$$

If $a > 1$, then the term k^a is negligible before k, and (49) holds. The same conclusion as above entails: it is locally better to exhaust the resource in this case. If $a < 1$, the terms of order k are negligible with respect to k^a. The expansion of G is then:

$$G(k, x_m + h; x_0) \;=\; e^{-r\tau(x_0,x_m)} \left(\pi + h(\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} - \frac{r\pi}{F(x_m)}) + Ck^a \pi + o(\eta^a) \right) . \quad (51)$$

If x_m is chosen so as to satisfy the condition (50), the coefficient of h in the right-hand side of (51) is zero. The conclusion is that the difference

$$G(k, x_m + h; x_0) - G(0, x_m; x_0) \;\sim\; e^{-r\tau(x_0,x_m)}\, Ck^a g(x_m, 0)$$

is positive for all deviation k. It is therefore not optimal to exhaust the resource. Finally, if $a = 1$, the expansion of G is:

$$G(k, x_m + h; x_0) \;=\; e^{-r\tau(x_0,x_m)} \left(\pi + h(\frac{\partial \pi}{\partial x} + \frac{\partial \pi}{\partial I} - \frac{r\pi}{F(x_m)}) + k(C\pi + \frac{\partial \pi}{\partial I}) + o(\eta) \right) . \quad (52)$$

Again, if x_m solves (50), the coefficient of h vanishes. The conclusion depends on the sign of the coefficient

$$V \;=\; C\pi(x_m, x_m) \;-\; \frac{\partial \pi}{\partial I}(x_m, x_m) \, .$$

If $V > 0$ then exhausting the resource is not optimal. If $V < 0$, then it is locally optimal. If $V = 0$, a second-order expansion is necessary in order to conclude. Observe that in the present case, the constant C is such that $\exp(-r\tau(k, x_m)) \sim Ck$, so that $C = \lim_{k\downarrow 0} \tau(k, x_m)/k$.

Proof of iii). Finally, assume that $F(x) \sim \alpha x^\beta$ with $\alpha > 0$ and $0 \le \beta < 1$. In this case, the integral which defines $\tau(k, x_m)$ converges as $k \to 0$, to some value T. As a consequence, we have:

$$\lim_{k\downarrow 0} G(k, x_m, x_0) \;=\; e^{-r\tau(x_0,x_m)}\, \pi(x_m, x_m)\, \frac{1}{1 - e^{-rT}} \, .$$

Since $G(0, x_m, x_0) = \exp(-r\tau(x_0, x_m))\pi(x_m, x_m)$ and $\pi(x_m, x_m) \ge 0$, we have:

$$G(0, x_m, x_0) \;\le\; \lim_{k\downarrow 0} G(k, x_m, x_0) \, .$$

Therefore, exhausting the resource is not optimal.