Solving the incomplete markets model with aggregate uncertainty using parameterized cross-sectional distributions

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Abstract

This note describes how the incomplete markets model with aggregate uncertainty in Den Haan, Judd, and Juillard (2008) is solved using standard quadrature and projection methods. This is made possible by linking the aggregate state variables to a parameterized density that describes the cross-sectional distribution. A simulation procedure is used to obtain "reference" moments that can improve the shape of the density without adding additional state variables. This note compares several simulation procedures in which there is—as in the model—no cross-sectional sampling variation.

Key Words: Numerical solutions, projection methods, simulations
JEL Classification: C63, D52

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1 Introduction

This paper describes the algorithm used to solve the model with incomplete markets and aggregate risk (Model B) of Den Haan, Judd, and Juillard (2008). The algorithm solves for the policy functions using projection methods. We use a simulation procedure to reduce the dimension of the set of state variables, but the simulation step plays a minor role in the solution procedure and the algorithm can be run without it. In contrast, a simulation step is a key part to obtain the aggregate laws of motion in the most widely used algorithm to solve this type of model, i.e., the algorithm of Krusell and Smith (1998). Using simulated data to solve for aggregate policy rules has two disadvantages. First, by introducing sampling noise the policy functions themselves become stochastic. This effect can be reduced by using long samples, but sampling noise disappears at a slow rate. Second—and more importantly—the values of the state variables used to find the best fit for the aggregate law of motion are endogenous and are typically clustered around their means. But accuracy can be improved by using values that are more spread out.[1] In particular, the numerical literature advocates the use of Chebyshev nodes to ensure uniform convergence and the procedure used here allows for this efficient choice of grid points.

Using projection procedures to solve a model with a continuum of agents typically requires a parameterization of the cross-sectional distribution.[2] Here we improve on the procedure proposed in Den Haan (1997) by using reference moments, an idea originally proposed by Reiter (2008). These reference moments are used to improve the shape of the approximate distribution, but they are not state variables; using them reduces the dimensionality of the problem. If these moments are not state variables one needs information from outside the algorithm to pin down their values. This is why a simulation procedure is used, but this is the only role of simulated data in our algorithm. The philosophy that underlies our algorithm is similar to the one in Reiter (2008). The differences are mainly in terms of implementation where our's is less cumbersome.

Algan, Allais, and Den Haan (2008) (AAD hereafter) propose a new procedure to simulate cross-sections of a continuum of agents. The most common procedure to simulate models with a continuum of agents consists of using a finite number of agents and a random number generator to draw the idiosyncratic shocks. Consequently, the results are subject to cross-sectional sampling variation. Models with

[1] Recall that the standard errors of regression coefficients, $\sigma^2 X'X$, are lower when the x-values are more spread out.

[2] Den Haan and Rendahl (2008) show that aggregation without explicit distributional assumptions is possible when the individual policy functions are linear in the coefficients. They implicitly obtain information about the distribution by approximating some auxiliary policy rules.

a continuum of agents do not have this property and most solution procedures are based on this lack of sampling variation.[3] AAD show that sampling variation can be substantial and that properties of the laws of motion may be overlooked because of the presence of cross-sectional noise. In this note, we compare three procedures that all avoid cross-sectional sampling variation.

2 Algorithm

This section provides an overview of the key ingredients of the algorithm.[4]

Projection method The numerical solution of the incomplete markets model with aggregate uncertainty in Den Haan, Judd, and Juillard (2008) consists of a policy function $k'(\varepsilon, k, a, s; \Psi_k)$, where ε is the (exogenous) individual employment status, k the individual capital stock, a the exogenous aggregate state, s a set of variables that characterizes the cross-sectional joint distribution of capital and employment status, and Ψ_k the coefficients of the policy function. s refers to the beginning-of-period distribution after the new employment status has been observed.

A projection method consists of the following steps.

- Construct a grid of the state variables.
- At each grid point we can define an error term given values for ε, k, a, and s as

$$\begin{array}{c} \widetilde{u_E}(\varepsilon, k, a, s; \Psi_k) = \frac{1}{c} - \sum_{\varepsilon', a'} \left[\frac{\beta(r'+(1-\delta))}{c'} \right] p(\varepsilon', a'|\varepsilon, a) \\ = \\ \frac{1}{(r+(1-\delta))k + wl - k'} \\ - \sum_{\varepsilon', a'} \left[\frac{\beta(r'+(1-\delta))}{(r'+(1-\delta))k' + w'l' - k'(\varepsilon', k', a', s'; \Psi_k)} \right] p(\varepsilon', a'|\varepsilon, a) \end{array} \tag{1}$$

with

$$\begin{array}{c} k' = k'(\varepsilon, k, a, s; \Psi_k) \\ l = (1-\tau)\bar{l}\varepsilon + \mu(1-\varepsilon), \; l' = (1-\tau')\bar{l}\varepsilon' + \mu(1-\varepsilon') \\ r = \alpha a \left(\frac{K}{\bar{l}(1-u(a))} \right)^{\alpha-1}, \quad r' = \alpha a' \left(\frac{K'}{\bar{l}(1-u(a'))} \right)^{\alpha-1} \\ w = (1-\alpha) a \left(\frac{K}{\bar{l}(1-u(a))} \right)^{\alpha}, \quad w' = (1-\alpha) a' \left(\frac{K'}{\bar{l}(1-u(a'))} \right)^{\alpha} \\ \tau = \frac{\mu u(a)}{\bar{l}(1-u(a))}, \quad \tau' = \frac{\mu u(a')}{\bar{l}(1-u(a'))}. \end{array} \tag{2}$$

[3] For example, solution procedures typically specify that next period's distribution is fully determined by the current distribution and aggregate shocks.

[4] A more in depth discussion can be found in Algan, Allais, and Den Haan (2008).

Here K stands for the aggregate capital stock, u the unemployment rate, which is determined by the aggregate exogenous state, and p for the probability of next period's exogenous random variables being equal to ε' and a' given that today's values are ε and a. Because of the non-negativity constraint on k' the relevant Euler-equation error is defined as

$$u_E(\varepsilon, k, a, s; \Psi_k) = \begin{cases} \widetilde{u_E}(\varepsilon, k, a, s; \Psi_k) \text{ if } k' > 0 \\ 0 \text{ if } k' = 0 \text{ and } \widetilde{u_E} \geq 0 \end{cases}$$

- Ψ_k is found by minimizing some objective criterion on the values of u_E at the nodes of the grid.

Two things are needed to be able to evaluate u_E. First, s and a must pin down K. If K would be an element of s then this would be trivial. Second, it must be possible to obtain the values of s' as a function of a, a', and s. This can be done if s implies an actual cross-sectional distribution. The cross-sectional distribution of the current period together with the individual policy function can then determine the characteristics of next period's distribution (and thus s') using standard quadrature techniques. Next, we explain how this can be done.

Linking s to a cross-sectional distribution Let the first $N_{\overline{M}}$ moments of the strictly positive capital holdings of agents with employment status ω be given by $\overleftarrow{m^{\omega,j}}$, with $j \in \{1, \cdots, N_{\overline{M}}\}$ and suppose that these are elements of s.[5] To link this set of moments with a density we approximate the density of individual capital holdings with a flexible functional form $P(k, \rho^\omega)$ and choose the parameters ρ^ω such that the moments coincide with those specified. The following functional form is used.

$$P(k, \rho^\omega) = \rho_0^\omega \exp \begin{pmatrix} \rho_1^\omega \left[k - \overleftarrow{m^{\omega,1}} \right] + \\ \rho_2^\omega \left[\left(k - \overleftarrow{m^{\omega,1}} \right)^2 - \overleftarrow{m^{\omega,2}} \right] + \cdots + \\ \rho_{N_{\overline{M}}}^\omega \left[\left(k - \overleftarrow{m^{\omega,1}} \right)^{N_{\overline{M}}} - \overleftarrow{m^{\omega,N_{\overline{M}}}} \right] \end{pmatrix}. \tag{3}$$

To completely characterize the cross-sectional distribution one would also need to include the fraction of agents at the constraint in s. The advantage of this particular functional form is that the coefficients $\rho_1^\omega, \cdots, \rho_{N_{\overline{M}}}^\omega$ can be found with the following minimization routine.

$$\min_{\rho_1^\omega, \rho_2^\omega, \cdots, \rho_{N_{\overline{M}}}^\omega} \int_0^\infty P(k, \rho^\omega) dk. \tag{4}$$

[5] An arrow pointing left (right) denotes beginning (end) of period.

The reason is that the first-order conditions of this minimization problem are exactly the conditions that the first $N_{\overline{M}}$ moments are equal to the set of specified moments.

$$\begin{aligned}
\int_0^\infty \left[k - \overleftarrow{m^{\omega,1}}\right] \; P(k;\rho^\omega)dk = 0 \\
\int_0^\infty \left[(k - \overleftarrow{m^{\omega,1}})^2 - \overleftarrow{m^{\omega,2}}\right] \; P(k;\rho^\omega)dk = 0 \\
\cdots \\
\int_0^\infty \left[(k - \overleftarrow{m^{\omega,1}})^{N_{\overline{M}}} - \overleftarrow{m^{\omega,N_{\overline{M}}}}\right] \; P(k;\rho^\omega)dk = 0
\end{aligned} \tag{5}$$

AAD show that the minimization problem is convex, which means that the first-order conditions are monotone and thus easy to solve.[6] The coefficient ρ_0^w is determined by the condition that the density integrates to one. By increasing the number of moments one increases the order of the approximating polynomial and the accuracy of the approximation.

The approximating densities are used to determine s' and are not necessarily of interest to the researcher. In fact, it may very well be the case that accurate predictions of s' can be obtained with approximating densities that are not accurate in all aspects. We document this in Section 3 by showing that an approximating density with continuous support (for strictly postive capital levels) can accurately predict next period's moments even though the true cross-sectional density has points with positive point mass, i.e., the CDF is discountinous.

Reference moments The elements of s fulfill two roles. First, they are a state variable and second they provide information about the shape of the distribution. For example, it is possible that the second-order moment is not important as a state variable because it doesn't vary very much but is still important to shape the distribution. If this happens one can reduce the cost of the algorithm by using reference moments. These moments are not state variables but are used to determine the cross-sectional density. The question arises how to get information on the reference moments. One possibility would be to use the solution of the model without aggregate uncertainty. Another possibility, and the one that is chosen here, would

[6] For alternative specifications of the functional form one would have to solve the coefficients from a system like (5) which is not guaranteed to be a well defined problem.

be to start with a guess on the reference moments, solve the model using the algorithm described above, and then simulate the economy to update the information on the reference moments. The reference moments could, without any additional complication, be a function of the included aggregate state variables.

Discussion of choices made Several choices were motivated by convenience. For example, similarity to choices made in other numerical work. Here we discuss two choices that the reader should be aware off. In the description of the algorithm given above, we assumed that the policy function for capital is approximated. One can also approximate the consumption choice or the conditional expectation and we chose the latter.[7] We approximate the conditional expectation using Chebyshev polynomials. This and a grid constructed using Chebyshev nodes leads to several desirable convergence properties.[8] But there are also disadvantages. First, the conditional expectation displays a sharp non-differentiability at the lowest level of k at which the agents chooses a zero capital stock, $\overline{k}$. For $k \leq \overline{k}$ the conditional expectation does not have to be approximated so we simply approximate the conditional expectation on those grid points at which the constraint is not binding. But this means not using the full set of Chebyshev nodes and some of the optimality properties may be lost. Moreover, the conditional expectation has other—less pronounced—non-differentiabilities due to the interaction of the constraint and the discrete support of ε.[9] In this particular problem these disadvantages are minor because the constaint only binds at very low levels of k, but when the constraint plays a more important role the reader should seriously consider using splines.

The other choice that the reader should be aware of is that we parameterize the law of motion relating s' to the current-period aggregate state variables. Conditional on this law we then solve for the individual policy rules and then update the aggregate law of motion by projecting the calculated values of s' on the grid on the approximating functional form. Given the accuracy of the obtained fit there is no harm in this procedure, but there really is no reason for this two-step procedure. Even if one would like an approximating aggregate law of motion then it would be better to solve the model using the algorithm outlined above and then simply get an approximation for the aggregate law after one has obtained the solution of the model. Further information on the choices made can be found in the appendix and in AAD.

[7] Some motivation for choosing the conditional expectation is given in Christiano and Fischer (2000).

[8] See Judd (1998).

[9] See Den Haan (1997) and especially Figure 2.

3 Simulating a continuum cross-section of agents

Simulation procedures fulfill an important role in the numerical analysis of models with heterogeneous agent models. The popular procedure of Krusell and Smith (1998) uses simulated cross-sectional moments to determine the aggregate law of motion. Even in our algorithm—that is designed to avoid simulation procedures—we still use a simulation procedure to reduce the dimension of the set of state variables while keeping an accurate shape of the cross-sectional density. And even if the algorithm does not rely on a simulation procedure at all, then many characteristics of the solution can only be determined using a simulation procedure.

Given the importance of simulation procedured it is important to compare alternatives. The most popular procedure is to use a finite set of agents and to use a random number generator to determine the realizations of the idiosyncratic shocks. But this means that the outcome is subject to cross-sectional sampling variation, whereas both the model and the algorithm typically rely on there being none. AAD show that this sampling variation can be substantial especially for the smaller group in the population, i.e., the unemployed.

There are, however, procedures that avoid cross-sectional sampling variation, but to the best of our knowledge these have not been compared. Section 3.1 outlines three different simulation procedures, Section 3.2 compares the three simulation procedures for the model discussed here and for a model in which the CDF displays substantial discontinuities.

3.1 Three simulation procedures

To simplify the exposition we explain how to simulate across time a cross-sectional distribution of capital holdings when there are no shocks and no constraint, that is, when the policy function for k' is given by $k(k)$.[10]

[10]For most sensible choices of $k_{+1}(k)$ the distribution would then converge towards a single point. Adding stochastic elements that would prevent this is easy, but would make the exposition somewhat more tedious.

3.1.1 Simulation procedure of AAD

Let $f_t(k)$ be the distribution of capital holdings in period t and let f_1 be given.[11,12] Calculate the first $N_{\overline{M}}$ moments of the distribution of k' using quadrature methods and $k'(k)$ and f_1.[13] Using the procedure discussed in Section 2 one can then obtain the density $f_2(k)$ that corresponds to these moments.[14] Iteration on this procedure gives a time series $f_t(k)$. Given $f_t(k)$ any characteristic of the cross-sectional distribution can be calculated.

3.1.2 Grid-based procedure of Young[15]

Construct a grid of capital holdings, κ_i, $\bar{j} = 0, \cdots, N$, and let $p_t^{\bar{j}}$ be equal to the mass of agents with a capital stock equal to $\kappa_{\bar{j}}$. We have

$$\sum_{\bar{j}=0}^{N} p_t^{\bar{j}} = 1.$$

Calculate the values for p_{t+1}^{j} using the following algorithm.

- Initialize by setting $p_{t+1}^{j} = 0$ for all j.
- Calculate the values of p_{t+1}^{j} using the following procedure for $\bar{j} = 0, \cdots, N$.
  - Calculate $k'(\kappa_{\bar{j}})$. Let j be such that $\kappa_j \leq k'(\kappa_{\bar{j}}) < \kappa_{j+1}$.
  - The mass at the $\bar{j}^{\text{th}}$ grid point, $p_t^{\bar{j}}$, is allocated to the two grid points that enclose the choice $k'(\kappa_{\bar{j}})$—i.e., the j^{th} and the $(j+1)^{\text{th}}$ grid point—using the distance of $k'(\kappa_{\bar{j}})$ to the two grid points to determine the fractions. Thus,

$$p_{t+1}^{j} = p_{t+1}^{j} + \frac{\kappa_{j+1} - k'(\kappa_{\bar{j}})}{\kappa_{j+1} - \kappa_j} p_t^{\bar{j}}$$

[11] Alternatively, one can start the procedure with $N_{\overline{M}}$ moments. The density $f_1(k)$ can then be determined using the procedure of Section 2.

[12] It is easy to modify the procedure to include a constraint. $f_t(k)$ would in that case be the density of the strictly positive capital holdings and one would in addition keep track of the mass of agents at the constraint.

[13] To deal with the lower bound on capital, AAD explicitly calculate the fraction of agents at the constraint and then calculate the $N_{\overline{M}}$ moments for those agents with strictly positive capital holdings.

[14] If there is are no constraints on the range of k, then one has choose a lower and an upperbound for k that are outside the ergodic set or at least such that the mass below and above these two values is very small.

[15] Proposed inYoung (2008).

and

$$p_{t+1}^{j+1} = p_{t+1}^{j+1} + \frac{k'(\kappa_{\bar{j}}) - \kappa_j}{\kappa_{j+1} - \kappa_j} p_t^{\bar{j}}$$

- The sum of all the p_{t+1}^js is by construction equal to 1.

If the agent's employment status changes stochastically then one would have to change the end-of-period distribution calculated here into next period's beginning-of-period distribution taking into account the transition laws of the idiosyncratic shock. But this is straightforward given that these laws of motion are exogenous.

In the model without aggregate uncertainty this procedure can be expressed as a linear linear system that can be used to solve for the stationary distribution (and thus the equilibrium aggregate capital stock) by solving for the eigen factor corresponding to the unit eigen value.

3.1.3 Grid-based procedure of Rios-Rull[16]

Again construct a grid of capital holdings, κ_j, $j = 0, \cdots, N$. Let $\bar{p}_t^0$ be the mass of agents at κ_0 and let $\bar{p}_t^j$ be equal to the mass of agents with a capital stock bigger than κ_{j-1} and less than or equal to κ_j, for $j > 0$.[17] This mass is assumed to be distributed uniformly between grid points. We have

$$\sum_{j=0}^{N} \bar{p}_t^j = 1.$$

Let x^j be equal to the capital level at which an agent chooses κ_j.[18] Thus,

$$k'(x^j) = \kappa_j. \tag{6}$$

Now compute the distribution function of next period's capital at the grid points as

$$\bar{P}_{t+1}^j = \int_0^{x^j} d\bar{P}_{t+1}(k) = \sum_{j=0}^{\bar{j}} \bar{p}_t^j + \frac{x^j - \kappa_{\bar{j}}}{\kappa_{\bar{j}+1} - \kappa_{\bar{j}}} \bar{p}_t^{\bar{j}+1}, \tag{7}$$

[16] This procedure is used in Rios-Rull (1997), Heathcote (2005), Reiter (2006), and Den Haan (2008).

[17] Note that $p_t^j \neq \bar{p}_j$ except for $j = 0$. p_t^j is the mass *at* a grid point and $\bar{p}^j$ is the mass *in between* grid poitns.

[18] Note that if the capital choice would depend on aggregate state variables then x would be time varying.

where $\overline{j} = \overline{j}(x^j)$ is the largest value of j such that $\kappa_j \leq x^j$. The second equality follows from the assumption that mass is distributed uniformly between grid points. Note that $\bar{p}^0_{t+1} = \bar{P}^0_{t+1}$ and $\bar{p}^j_{t+1} = \bar{P}^j_{t+1} - \bar{P}^{j-1}_{t+1}$ for $j > 0$. Modifying the distribution to take into account unemployment risk is—as for the procedure of Young—straightforward.

3.2 Comparison and discussion

3.2.1 Experiment #1

In this experiment, we use our numerical solution for the individual policy functions of the model outlined above to simulate the cross-sectional distribution across time with the three simulation procedures.

The initial distribution is identical to the one used in Den Haan (2008). Figures 1 through 3 display the mean capital stock, the 1^{st} percentile, and the 10^{th} percentile for both the employed and the unemployed agents. The figures document that the difference between the generated series are small at least relative to the observed fluctuations. The largest percentage differences are observed for the 1^{st} percentile. For the employed (unemployed) we find that the differences are 1.35% (2.48%), 1.60% (1.78%), and 0.76% (1.60%) for AAD versus Rios-Rull, AAD versus Young, and Young versus Rios-Rull, respectively.[19] The differences are, thus, somewhat smaller when the Rios-Rull procedure is compared with the Young procedure, a finding that is also true for the mean and the 10^{th} percentile.

3.2.2 Experiment #2

Generating an accurate simulated panel for the model of the computational suite is relatively easy, because there are very few constrained agents, which means that any subsequent jumps in the CDF for higher levels of capital are very small. Moreover, the marginal propensity to save is almost constant and only varies with capital at low levels of capital.

Therefore, we also consider an example in which the marginal propensity to save varies strongly with capital and jumps in the CDF are important. Both features may give difficulties for the procedure of AAD. The continuous approximating density used in the AAD procedure, of course, misses the jumps of the CDF. Missing these jumps is not important as long as the marginal propensity to save is (locally) constant, but may matter if the savings function is nonlinear.

[19] We ignore the first one hundred observations. A careful look at the graph makes clear that the initial distribution is somewhat odd and there is some transition dynamics at the beginning of the sample.

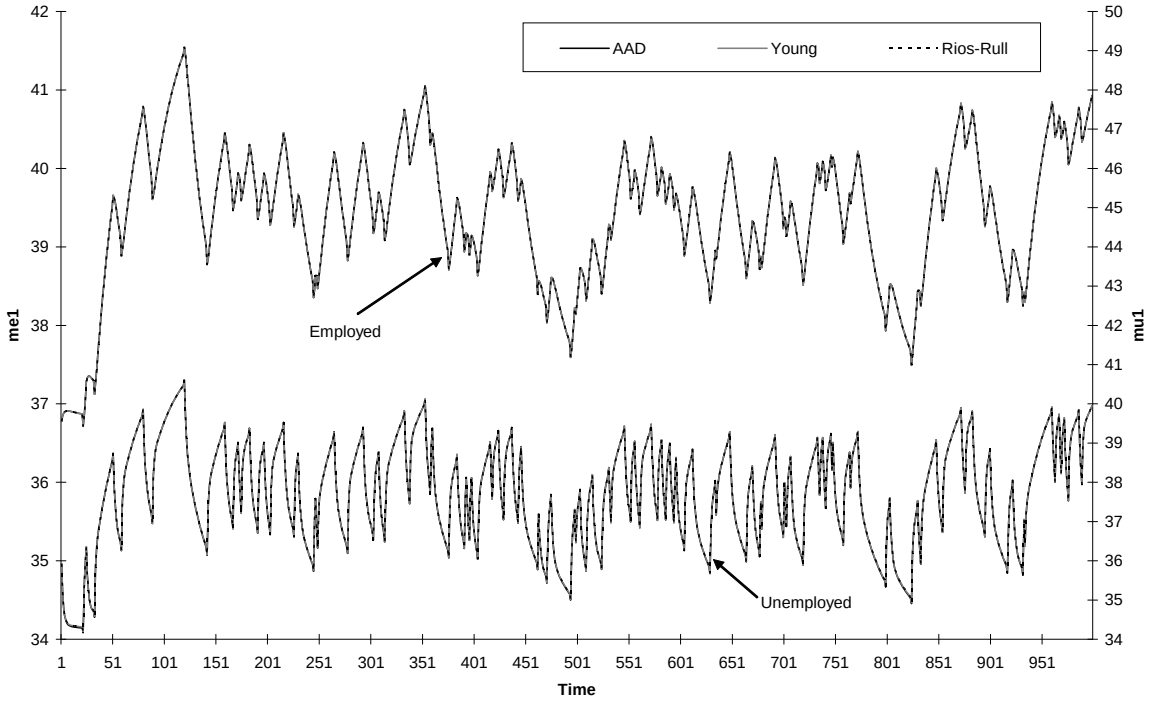


Figure 1: The means of the employed and the unemployed using AAD, Rios-Rull, and Young procedures

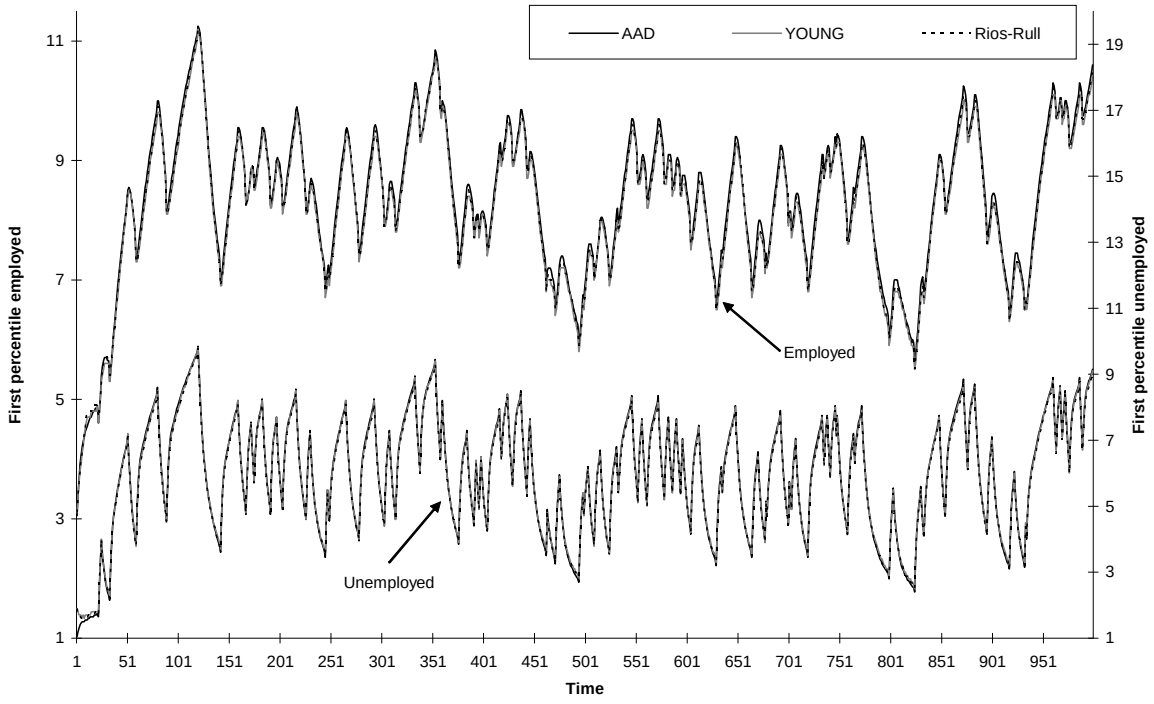


Figure 2: The first percentile for the employed and the unemployed using AAD, Rios-Rull, and Young procedure.

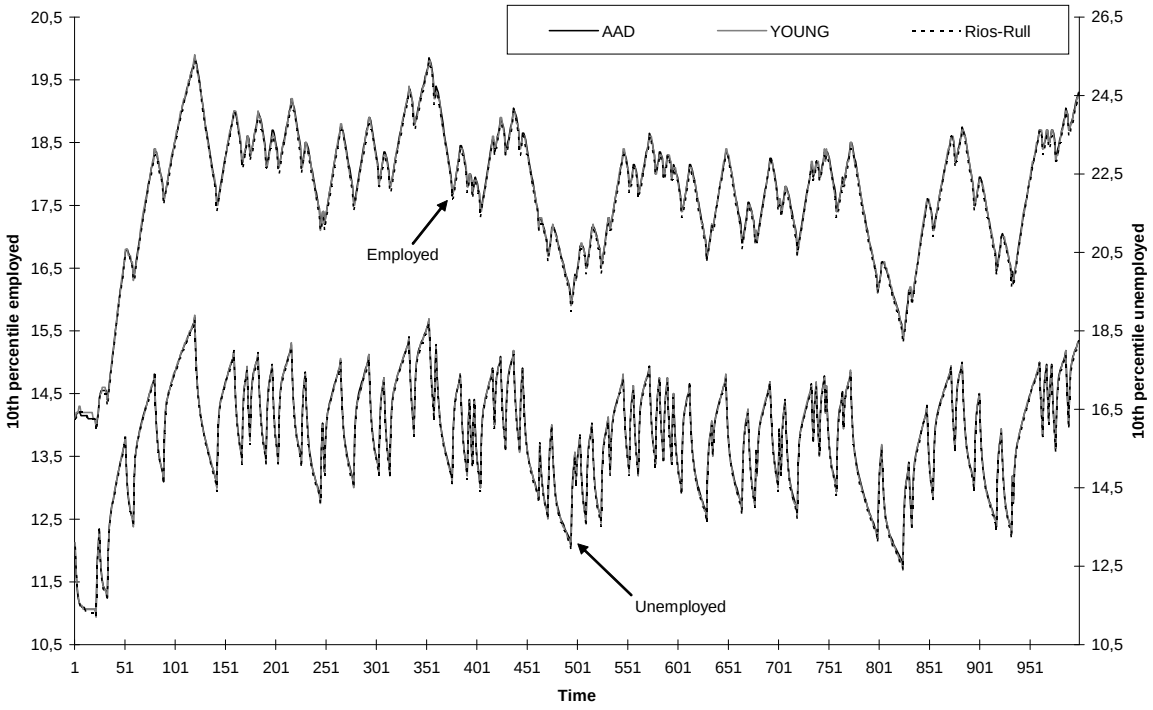


Figure 3: The 10th percentile for the employed and the unemployed using AAD, Rios-Rull, and Young procedure.

In the second experiment, the individual policy function of an unemployed agent is assumed to be equal to

$$k^u_{+1} = \max\{0, k - 25\}$$

and the policy function of an employed agent is equal to

$$k^e_{+1} = -\gamma_0 + k + \exp(\alpha_0 + \alpha_1 k + \alpha_2 k^2).$$

For the chosen parameter values,[20] the marginal propensity to save of an employed agent varies from 0.00 when $k = 0$ to almost 1 when $k = 99$. The laws of motion of the exogenous random variables are as in experiment #1.

Although these policy functions do not depend on the aggregate state, the choices still do because the employment status depends on the aggregate state. The chosen policy function may look strange, but is motivated by its ability to generate large jumps in the cross-sectional distribution. When the aggregate state randomly changes, then the distributions generated with different procedures look similar because of these random aggregate shocks. The differences between the solution procedures become more clear if we keep the economy in the same aggregate state (that is, the economy remains in either the good or the bad state) and those results are reported here. The fraction of agents at the constraint is now substantially higher than in the problem discussed above. This higher fraction of constrained agents leads to several substantial jumps in the CDF as is clear in Figure 4 that plots the CDF obtained with the AAD and Young procedure when the economy has been in the bad state for a long time period.

Figure 4 documents that the CDFs obtained with the different procedures display substantial differences. The time series of standard characteristics of the cross-sectional distribution, however, are much more similar. This is documented in figures 5 and 6, which plot the simulated mean capital stocks and the fraction of agents at the constraint when the economy is (and remains) in the bad aggregate state. Figures 7 and 8 report the results when the economy is (and remains) in the good aggregate state.

3.2.3 Discussion

Although the procedures are quite different, they generate very similar results in both experiments. Of course, our results may not carry over to all problems and one always should check whether the simulated data are accurate. But the results

[20] $\alpha_0 = 2.70805$, $\alpha_1 = -0.06667$, $\alpha_2 = 0.000326$, and $\gamma_0 = -0.6$.

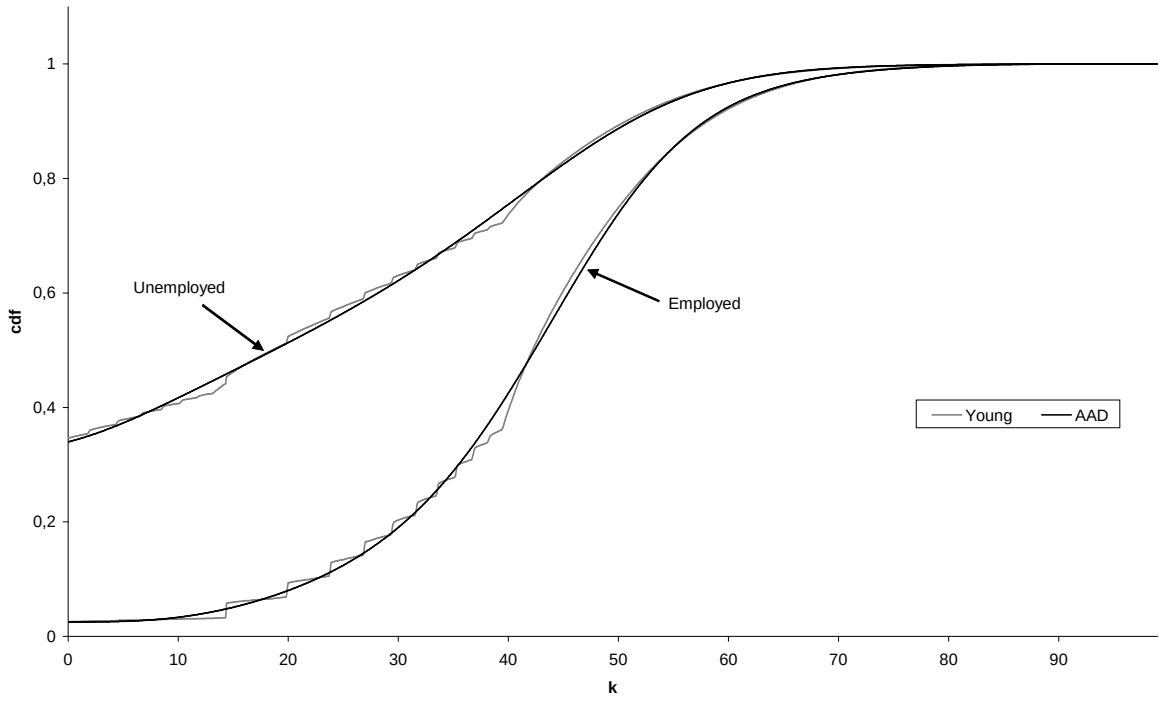


Figure 4: The cumulative distibution function for the employed and the unemployed when the economy has been in the bad state for a long time.

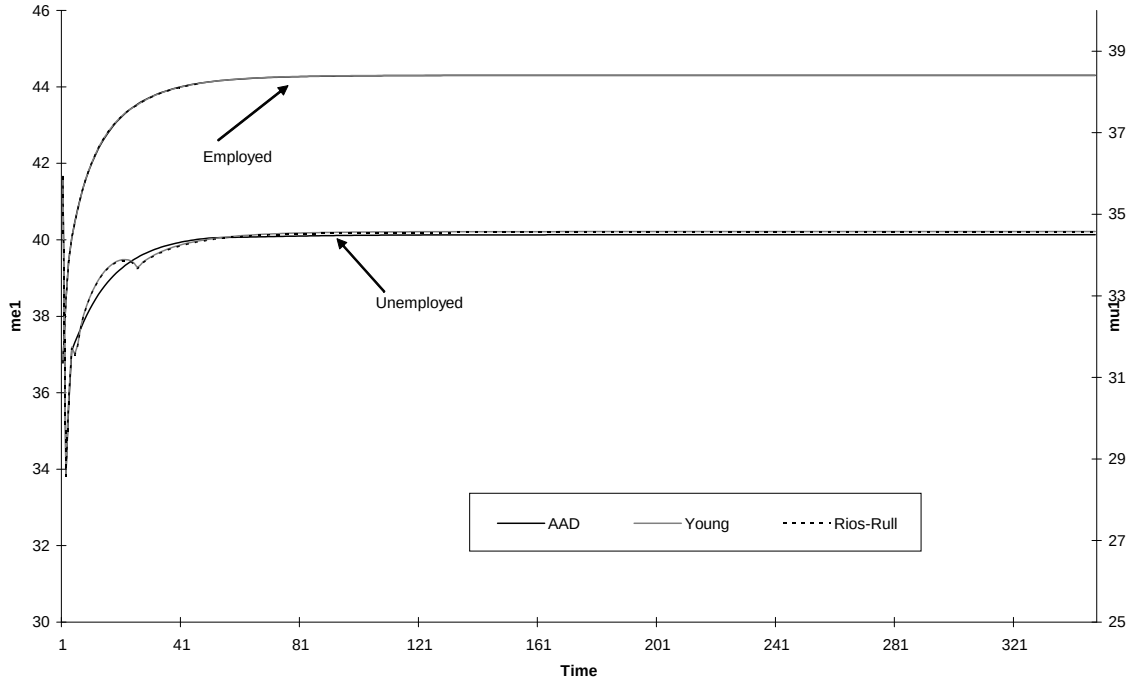


Figure 5: The means of the employed and unemployed when the economy is in the bad state

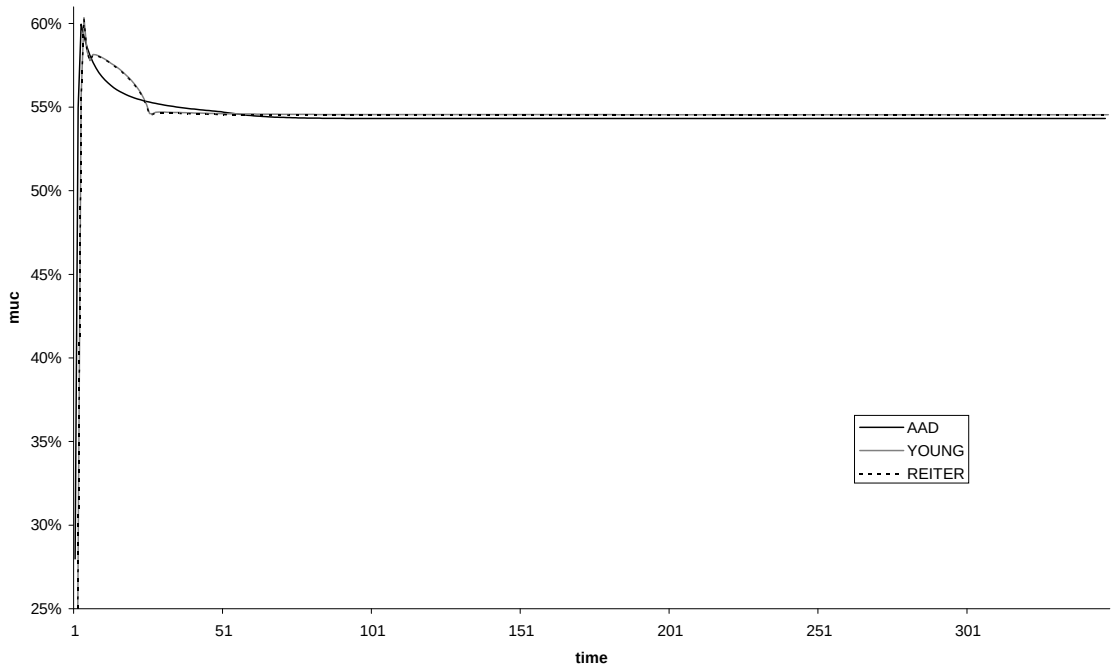


Figure 6: The fraction of agents at the constraint when the economy is in the bad state

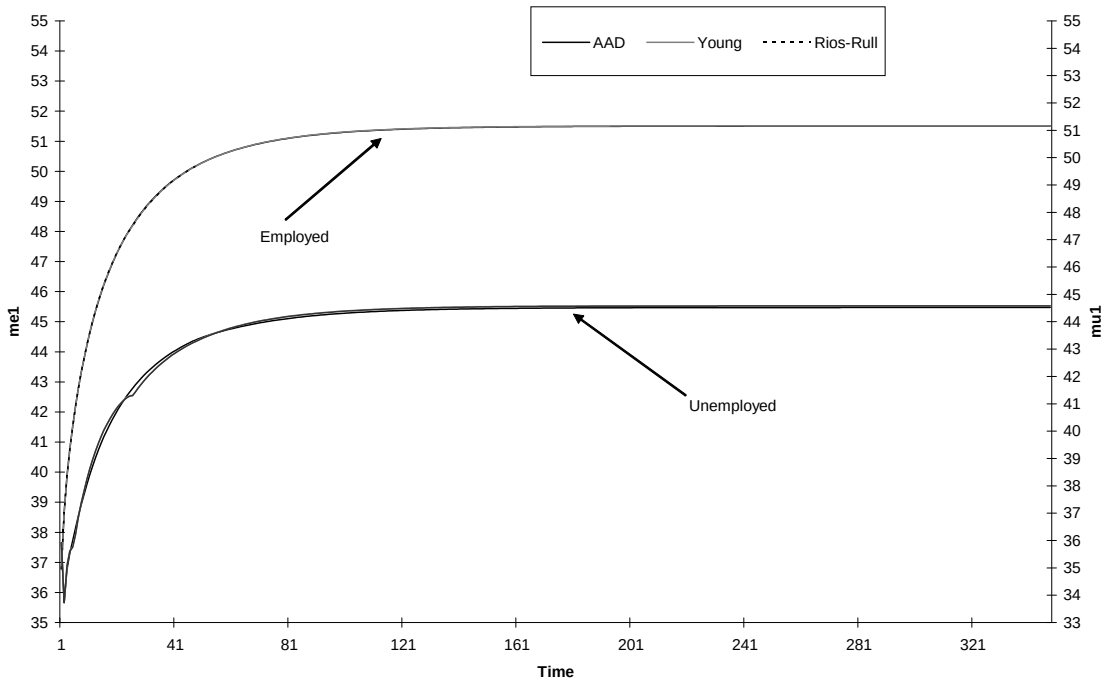


Figure 7: The means of the employed and unemployed when the economy is in the good state

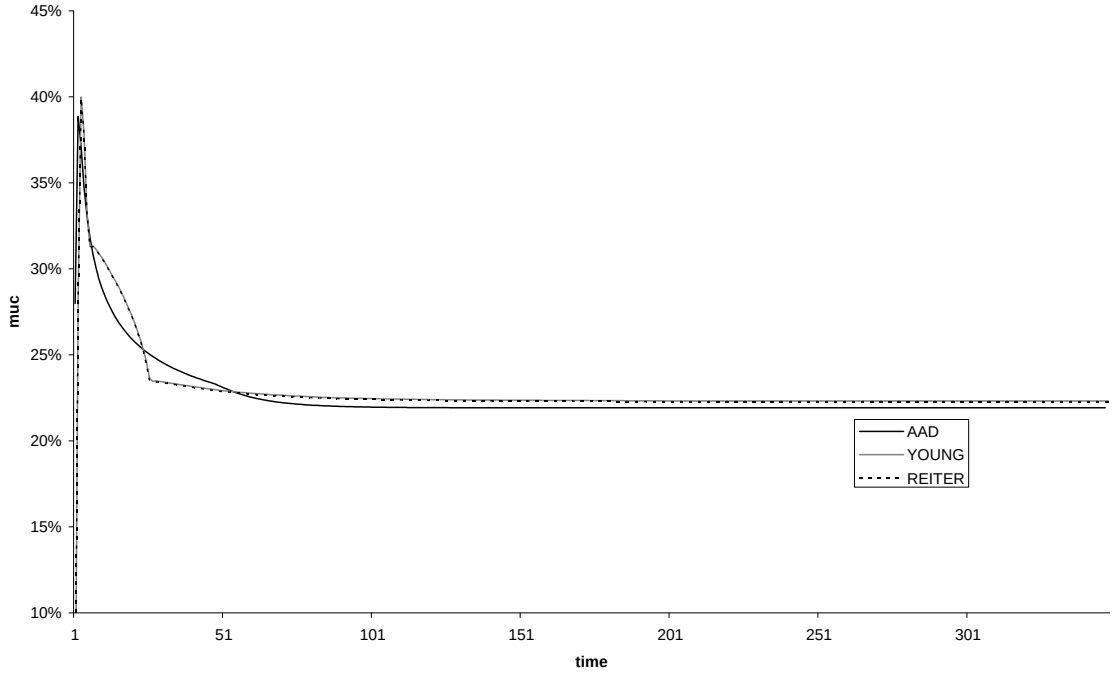


Figure 8: The fraction of agents at the constraint when the economy is in the good state

presented here indicate that convenience may be an important element in the choice made as well.

In terms of programming, the easiest procedure is the one proposed by Young (2008). In contrast to the grid-based procedure of Rios-Rull (1997), it does not require calculating the inverse, which can be a costly operation.[21] Both grid procedures allow quite naturally for discontinuities in the CDF. But the second example showed that the procedure of AAD also can lead to an accurate characterization of the movements across time of key characteristics of the cross-sectional distribution, even in the presence of substantial discontinuities. The main advantage of the procedure of AAD is that it characterizes the cross-sectional distribution with a much smaller number of parameters. For the procedures discussed here the simulation procedure of AAD uses ten parameters whereas the grid-based method uses one thousand. For some applications it may be extremely helpful to limit the number of parameters.

A Appendix

The state variables used are

$$s = \left[a_{-1}, a, \overrightarrow{m_{-1}^{u,c}}, \overleftarrow{m^{e,1}}, \overleftarrow{m^{u,1}}\right],$$

where $\overrightarrow{m_{-1}^{u,c}}$ stands for the fraction of constrained unemployed agents at the end of the last period, and $\overleftarrow{m^{\omega,1}}$ stands for the beginning-of-period mean capital holdings of agents with employment status ω and strictly positive capital holdings. Note that this set of state variables has enough information to determine $\overleftarrow{m^{u,c}}$ and $\overleftarrow{m^{e,c}}$.[22] In addition to these moments we use five higher-order moments (for both the employed and the unemployed) to determine the density of the cross-sectional distribution. In the simulation we use a total of 10 moments.

Parameter settings of the numerical procedure, such as the order of the polynomial and the number of grid points, are given in Table 1. For the exogenous random variables we use two grid points related to the two possible realizations.

[21] Reiter (2006) proposes several approximating steps to speed up the procedure.

[22] Alternatively, we could have used $s = [a, \overleftarrow{m^{u,c}}, \overleftarrow{m^{e,c}}, \overleftarrow{m^{e,1}}, \overleftarrow{m^{u,1}}]$. The advantage of our choice is that a_{-1} can take on only two values and is, therefore, "cheaper" as a state variable than an additional fraction of constrained agents.

Table 1: Construction of the grid

State variables	$k, \varepsilon, a_{-1}, a, \overrightarrow{m_{-1}^{u,c}}, m^{e,1}, m^{u,1}$		
range of values and number of grid points	k	$[0, 99]$	50 grid points
	$\overrightarrow{m_{-1}^{u,c}}$	$[0, 0.002]$	5 grid points
	$\overleftarrow{m^{e,1}}$	$[35, 42.4]$	5 grid points
	$\overleftarrow{m^{u,1}}$	$[33.5, 41.5]$	5 grid points
location of points	Chebyshev nodes		

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