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# Cyclical versus non-cyclical harvesting policies in renewable resource economics

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**Abstract:** In this paper, we explore the link between cyclical and non-cyclical resource exploitation. As already shown by Wirl [23] in the context of a two state-variable continuous control model, the form of the cost function is essential to explain the cyclical or non-cyclical nature of optimal solutions. We base our analysis on a one-state variable case. We present an impulse control model and study the impact of different cost functions on the type of solution. We show that the optimal harvesting behaviour derived from this model can either be cyclical or converge to a Most Rapid Approach Path solution, like the one in Clark's continuous control model [3]. We finally discuss the origin of the cost functions we studied and show that there are empirical underpinnings for the occurrence of cyclical optimal harvesting solutions.

**JEL classification:** C61, Q2.

**Key words:** optimal control, impulse control, renewable resource economics, harvest cost function.

# 1 Introduction

The harvesting behaviour in renewable resource economics is grounded on both, economic and biological forces. In bioeconomic models, the decision to harvest, to fish or to log depends on economic variables, such as prices, capital investment or harvest-costs, and on biological parameters, for example the resource stock and the resource growth. The harvesting behaviour itself may be seen as a continuous process, for example when describing local fishermen's activity, or as an action at a point in time: fishermen driving out their gear once a week or a forester deciding to cut a stand once in a decade. Different types of mathematical models are used for this purpose, such as continuous control models, discrete control models or impulse control models. Each type emphasises a different aspect of the harvesting behaviour and leads to different types of results.

For the purpose of this paper, we will distinguish two broad categories: models which lead to optimal cyclical harvesting policies and models which lead to optimal non-cyclical policies, where cycles are defined as a recurring sequence of events: harvesting and waiting. We will establish a link between cyclical and non-cyclical harvesting policies by the study of the harvest-cost function. In particular and in contrast to earlier studies (especially Wirl [23]), we propose to deal with the well known one-state control model proposed by Clark [3] and look for a more general model, which may either lead to a Clark like non-cyclical solution, or to a cyclical solution, depending on the harvesting cost function.

Before turning to the analysis of these two models, we will briefly review the different reasons why optimal resource exploitation may be cyclical: the non-concavity of the objective function may play a role, as well as stock effects and amenities, the presence of age classes in the population, the existence of increasing returns to scale and the form of the harvest cost function. Many well known models combine several of these elements. In the following, we will primarily be concerned with infinite time horizon models, such as the two models we will analyse in depth.

The first optimal cyclical behaviour described in resource economics was certainly by Faustmann's optimal forest rotation. In these forest models (see for example Reed [16], Kuuluvainen and Tahvonen [12] or Snyder and Bhattacharyya [19]) rotations are imposed by construction, because rotation time is the control variable of the problem. In addition, costs in these models are not stock dependent but represented either as fixed harvest- and replanting costs [16], [12] or time dependent maintenance costs [19]. Hence, cyclical solutions are not favoured.

A second type of cyclical harvesting behaviour occurs in models with increasing returns to scale. Indeed, Wirl [23] mentions the possibility that cyclical solutions may be due to the non-concavity of the harvesting gain function. There is no formal proof of this assumption but the intuition is that increasing returns to scale make it profitable to delay the harvesting decision. One example for a model with increasing returns to scale is the impulse control model of forest economics studied by Termansen [20] (see also Touza-Montero and Perrings [21]). Conditions on the existence of impulse control models are discussed by Seierstaed and Sydsaeter [17], Léonard and van Long [13] or Vind [22], but in finite time horizon only. Thus, in the forestry model proposed by Termansen, the optimal harvesting behaviour is rotational (as in classical forest models), but the cycles are not regular, because of the finite time horizon.

A third type of cyclical behaviour occurs in models with age-classes. One example is the so-called pulse fishing, (see for example Hannesson [10]), where fishermen drive out their gears in unregular cycles. He explains the cyclical behaviour by two facts: increasing returns to scale and imperfect selectivity of fishing gears. Indeed, it may be optimal for the fisherman to wait until the next age class has reached a significant population size before fishing again with higher efficiency.

Finally, Wirl [23] has shown that the form of the cost function could change a non-cyclical into a cyclical solution. He based his analysis on the continuous control model by Clark, Clarke and Munro [5], a model with two state variables and infinite time horizon. In this model, the harvest-gain function is concave, which, according to Wirl, renders cyclical solutions more difficult to occur and harvest costs are stock dependent. As Wirl shows, the sole fact of introducing a non-linear (quadratic) cost function in this model triggers a cyclical optimal harvesting behaviour.

In this paper, we will establish a similar link between cyclical and non-cyclical harvesting behaviour, but based on a control model with only one state variable, such as the model proposed by Clark [3]. As in the above two state-variable model analyzed by Wirl [23], maximisation in the simple Clark model is realised over an infinite time horizon, the harvest-gain function is concave and harvest costs are stock dependent and linear in the harvesting rate. The optimal solution consists in approaching most rapidly the bioeconomic optimum, a steady state, and then maintaining the resource at this level.<sup>1</sup> We construct an infinite-time impulse control model based on the same resource stock and the same natural growth function. As above, the harvest-gain function is stock dependent and linear in the harvest amount. The two models only differ in the way the harvesting process is considered: whereas harvesting is represented by the harvest rate in the continuous control model, it is described by the harvest amount and the moments of harvest in the impulse control model. In addition, the impulse cost may take on different functional forms. We show that the optimal harvesting behaviour depends on the cost function which is used, and that the optimal behaviour can either be cyclical or converge to a Most Rapid Approach Path solution, like in a Clark-Munro continuous control model. To conclude, we explore different ways the impulse cost function may be derived.

The paper is organized as follows. In section 2 we present the impulse control problem and characterize the type of the solution. In section 3 we show under which conditions the optimal solution will be cyclical and establish a link to a Clark-like continuous control solution. In section 4 we study two particular cost functions: a separable cost function and an integral cost function. We also show the situations in which these cost functions may be applied and the consequences that their use may have on harvest gains and harvest strategies.

## 2 An impulse control problem of resource exploitation

Before introducing the impulse control model, we remind in the first section of the appendix (6.1) the continuous control model proposed by Clark [3]. Clark's model leads to

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<sup>1</sup>The discrete version leads to similar types of solutions [3].

the non-cyclical solution. We want to relate this type of behaviour with the following impulse control model, presented as our problem (P).

## 2.1 The problem

We analyse the following problem (P). Maximize over  $t_i, x_i, I_i$  for  $i = 1 \dots \infty$ :

$$\mathcal{G}(\{t_i, x_i - I_i, x_i\}_{i=1}^{\infty}, x) = \sum_{i=1}^{\infty} e^{-rt_i} g(x_i, x_i - I_i) \quad (1)$$

s.t.

$$\begin{aligned} \dot{x}(t) &= F(x(t)) \quad \text{if } t \geq 0, \quad t \notin t_i, \quad i = 1, 2, \dots \quad x(0) = x_0 \text{ given,} \\ \lim_{t \rightarrow t_i^+} x(t_i) &= \lim_{t \rightarrow t_i^-} x(t_i) - I_i, \quad x_i = \lim_{t \rightarrow t_i^-} x(t_i), \\ I_i &\leq x_i, \quad x(t) \in [0, K]. \end{aligned}$$

Where  $g$  stands for the gain from resource use,  $t_i$  the time of harvest,  $I_i$  the amount of harvest at that time,  $x_i$  is the resource stock just before harvesting and  $r$  is the discount rate. The initial resource stock,  $x_0$ , is known. The dynamic of the resource,  $x(t)$ , is determined by the natural growth function  $F(x)$  as long as there is no harvest. When harvest occurs, the difference in the resource stock just before harvest  $\lim_{t \rightarrow t_i^-} x(t_i)$  and the resource stock just after harvest  $\lim_{t \rightarrow t_i^+} x(t_i)$  is equal to the harvest amount itself:  $I_i$ . Finally, the resource stock is nonnegative and bounded by the carrying capacity,  $K$ . The control variables of our problem are:  $t_i$  the time of harvest, the resource stock at that time,  $x_i$  and the amount of harvest  $I_i$ . We suppose an infinite time horizon,  $i = 1 \dots \infty$ . In our problem (P), we deliberately do not specify any profit function. One essential question of the following sections will thus be: what is the appropriate profit function that we should use (and under which conditions)? What are the impacts of the profit function on the type of solution? We will see the answer to this question in section 4.

## 2.2 Characterization of the solution

In contrast to the solutions proposed by Seierstaed and Sydsaeter [17], Léonard and van Long [13] or Termansen [20], we use the Dynamic Programming approach to solve our problem. The following theorem insures the existence of a unique solution to our problem.

### 2.2.1 The Dynamic Programming approach

**Theorem 1** *The value function*

$$v(x) = \max_{\{t_i, x_i, I_i\}_{i=1}^{\infty}} \mathcal{G}(\{t_i, x_i - I_i, x_i\}_{i=1}^{\infty}, x) \quad (2)$$

*is the unique solution of the following variational equation:*

$$v(x) = \max_{y \in [0, K], t \geq 0} e^{-rt} [g(\phi(t, x), y) + v(y)], \quad (3)$$

*where  $\phi(t, x)$  is the dynamic of the system.*

**Proof.** For this standard proof of dynamic programming see Davis [6], González [8], Bensoussan-Lions [2]. ■

**Remark 2** *In finite time horizon, the same kind of problem can be solved by the aid of the Maximum Principle. Interpretation of the first order conditions helps to get some economic insights for the reason why cyclical optimal harvesting behaviour occurs. This is why we remind this solution here, although we will not use the Maximum Principle in the following (see also Termansen [20]). As shown by Léonard and van Long [13] or Seierstad and Sydsaeter [17], the Hamiltonian and the first order conditions are given by the following equations.*

$$H(x, \lambda) = \lambda(t)F(x(t)),$$

where  $\lambda(t)$  is the adjoint variable at time  $t$  and also represents the shadow price of cuttings. We note the profit derived from cuttings  $\pi(x, I, t) = e^{-rt}g(x, x - I)$ . Necessary conditions at the points without jumps ( $t \neq t_i$ ) are

$$\dot{\lambda}(t) = \frac{-\partial H}{\partial x}(x(t), \lambda(t)) = -\lambda(t)\frac{\partial F(x)}{\partial x(t)}, \quad (4)$$

$$\lambda(t) \geq \frac{\partial \pi(x(t), 0, t)}{\partial I}, \quad (5)$$

where (5) is associated to the fact that only downward jumps are possible. In the following, we write  $x_i$  for  $\lim_{t \rightarrow t_i^-} x(t_i)$ , the stock level just before cuttings, and  $x_i^+$  for  $\lim_{t \rightarrow t_i^+} x(t_i)$ , the stock level just after cuttings. Likewise, we call  $\lambda_i^-$  and  $\lambda_i^+$  the value of the shadow price just before and just after the jumps. Necessary conditions at the jump points are:

$$\lambda_i^+ = \frac{\partial \pi(x_i, I_i, t_i)}{\partial I}, \quad (6)$$

$$\lambda_i^+ - \lambda_i^- = -\frac{\partial \pi(x_i, I_i, t_i)}{\partial x}, \quad (7)$$

$$H(x_i^+, \lambda_i^+) - H(x_i, \lambda_i) - \frac{\partial \pi(x_i, I_i, t_i)}{\partial t} = 0. \quad (8)$$

These conditions show the economic intuition between the optimal harvesting strategy: at the optimum, the value of the first resource unit that is not harvested anymore is equal to the marginal profit derived from the last resource unit that was harvested (6). The variation in shadow prices due to the harvesting is equal to the marginal profit derived from the resource stock (7). Finally, the change in the value of the Hamiltonian due to the harvesting is equal to the marginal profit derived from a different timing of harvesting, and thus to the opportunity cost of capital (8). The main caveat of the solution path proposed for such models (for example by Termansen [20]) is the lack of information which renders an analytical solution difficult to obtain.

### 3 The optimal harvesting behaviour

In the following, we characterize the optimal solution to our problem (P). We first obtain the gain function of our problem (P) and then introduce an auxiliary problem (PA) in order to solve problem (P). We study the auxiliary problem taking into account the case of an interior solution and a solution on the boundary.

#### 3.1 Existence of an optimal cyclical solution of (P)

We first define the conditions on  $g$  and  $F$  under which the behaviour resulting from the optimal solution of problem (P) is a cyclical behaviour  $[\underline{x}, \bar{x}]$  consisting in:

- letting the resource  $x_t$  grow until  $\bar{x}$ , (with  $x_0 \leq \bar{x}$ ),
- harvest at  $\underline{x}$ .

We define the length of a cycle in the following way (cf. appendix 6.2):

$$\tau(x, y) = \int_x^y \frac{1}{F(u)} du.$$

For this cyclical behaviour, we define

$$G(\underline{x}, \bar{x}, x_0) := g(\bar{x}, \underline{x}) \frac{e^{-r\tau(x_0, \bar{x})}}{1 - e^{-r\tau(\underline{x}, \bar{x})}}. \quad (9)$$

$G$  corresponds to  $\mathcal{G}$  valued in the sequence:  $t_0 = \tau(x_1, \bar{x})$ ,  $t_i = t_0 + i\tau(\underline{x}, \bar{x})$ , for  $i = 2, \dots$ ,  $x_i = \bar{x}$ ,  $x_i - I_i = \underline{x}$ , for  $i = 1, \dots$ .

#### 3.2 The auxiliary problem (PA)

Let us introduce the auxiliary problem:

$$(PA) = \max_{\underline{x}, \bar{x}; \underline{x} \leq \bar{x}} G(\underline{x}, \bar{x}, x_0).$$

As illustrated in figure ??, we have now written our problem as only depending on the stock levels  $\bar{x}$  and  $\underline{x}$ . We first analyse the type of solutions of this auxiliary problem. In particular, we want to know whether the gain function is maximized in an interior solution (for  $\bar{x} \neq \underline{x}$ ) or in a boundary solution (for  $\bar{x} = \underline{x}$ ).

##### 3.2.1 An interior solution

**Lemma 1** *If  $(\underline{x}, \bar{x})$  is a solution to the auxiliary problem (PA) with  $0 < \underline{x} < \bar{x} < K$  (interior solution), then the first order conditions are given by:*

$$\frac{1}{g} \frac{\partial g}{\partial \underline{x}} = -r \frac{1}{F(\underline{x})} \frac{e^{-r\tau}}{1 - e^{-r\tau}}, \quad (10)$$

$$\frac{1}{g} \frac{\partial g}{\partial \bar{x}} = r \frac{1}{F(\bar{x})} \frac{e^{-r\tau}}{1 - e^{-r\tau}}. \quad (11)$$

(see appendix 6.3 for additional explanations on the Lemma).

**Theorem 3** Let  $(\underline{x}^*, \bar{x}^*)$  be an interior solution of (PA). Then the value function of (P) is:

$$v(x) = \begin{cases} G(\underline{x}^*, \bar{x}^*, x) & \text{if } x \leq \bar{x}^* \\ g(x, \underline{x}^*) + G(\underline{x}^*, \bar{x}^*, x) & \text{if } x > \bar{x}^*. \end{cases}$$

Moreover the optimal solution of (P) is given by

$$x_i = \bar{x}^*, \quad I_i = \bar{x}^* - \underline{x}^*, \quad t_0 = \tau(x_0, \bar{x}^*), \quad t_i = t_0 + i\tau(\underline{x}^*, \bar{x}^*), \quad i = 1 \dots$$

**Proof.** see appendix 6.4. ■

### 3.2.2 A solution on the boundary

Let us now turn to the case:  $\underline{x} = \bar{x} = x$  (the case  $\underline{x} = 0$  and/or  $\bar{x} = K$  will be excluded of our study). Note that in this case, (9) is not well defined since the denominator vanishes. Assume therefore  $\lim_{\underline{x} \rightarrow \bar{x}} g(\underline{x}, \bar{x}) = 0$ . Change the variables and write:

$$g(\underline{x}, \bar{x}) := \bar{g}(\bar{x}, I); \quad I = \bar{x} - \underline{x}.$$

Note we assume that the limit of  $\frac{\partial \bar{g}}{\partial I}$  is well defined:

$$\lim_{I \rightarrow 0} \frac{\partial \bar{g}}{\partial I}(\bar{x}, I) := \bar{g}_I(\bar{x}),$$

$G$  may be defined by continuity when  $\bar{x} \rightarrow \underline{x}$  in the following way:

$$G(x, x, x_0) = \bar{g}_I(\bar{x}) \frac{F(x)}{r} e^{-r\tau(x_0, x)}. \quad (12)$$

The maximization problem becomes:

$$(PA_{bord}) := \max_{0 \leq x \leq K} G(x, x, x_0),$$

where  $G$  is given by 12.

The first order condition for this new optimization problem is:

$$\bar{g}'_I(x)F(x) + \bar{g}_I(x)[F'(x) - r] = 0. \quad (13)$$

We denote by  $x^*$  the solution of (13).

### 3.2.3 A link with Clark's continuous control case

Having characterized the type of solutions of our impulse control model, we can establish a first link to the continuous control model. Indeed, we can see that condition (13) coincides with the equation which determines the steady state of the following singular optimal control problem:

$$\begin{aligned} \max_h \int_0^\infty e^{-rt} \bar{g}_I(x) h dt, \\ \dot{x} = F(x) - h, \end{aligned}$$

for  $x_0$  given and  $0 \leq h \leq h_{max}$  (cf. appendix 6.1, (13) and (20)). This is indeed the well known marginal productivity rule of resource exploitation [3].



**Remark 4** Concerning the harvest strategies derived from our models, and still excluding the case  $\underline{x} = 0$  and/or  $\bar{x} = K$ ,

- we have shown that if the solution of (PA) is interior there exists a cyclical solution to the problem (P). Thus, the gain from this impulse control model  $v(x^*)$ , which is associated to the profit function  $g(\underline{x}, \bar{x})$  is greater than  $G(x^*, x^*, x^*)$ , the gain of the singular control model, associated to the profit function  $\bar{g}_I(x)$ .
- we can show that if the solution of (PA) is on the boundary  $\underline{x} = \bar{x}$  then, there does not exist a solution for (P) but a sequence of cyclical impulse controls with  $\bar{x} - \underline{x} = \epsilon$  approaching the value  $G(x^*, x^*, x_0), \forall x_0$ . In other words,

$$\lim_{\epsilon \rightarrow 0} G(\underline{x}_\epsilon, \bar{x}_\epsilon, x_0) = G(x^*, x^*, x_0).$$

For the proof, see [7].

Plotting different forms of gain functions we see that the maximum may be reached for  $\bar{x} = \underline{x}$  (figure 2), or for  $\bar{x} \neq \underline{x}$  (figure 1).

Clearly, the occurrence of an interior or a boundary solution in our model depends on the profit function that applies. Once we have determined this profit function, we are able to establish a relationship between Clark's model and the impulse control model, in terms of harvest strategies (solutions), harvest conditions (first order conditions) and harvesting gains. We will now turn to the question of the type of profit function we should introduce.

## 4 The impact of cost functions on the type of solution

The cost function is the most determinant element in the explanation of many profit functions studied in the literature. Indeed, profits are described as the difference between benefits and costs, where benefits are simply expressed as yields (harvest times prices, or prices per unit of catch) and costs are more complex functions, depending on the capital investment in the sector (the number and type of vessels or the number of fishermen), the abundance of the resource stock, and the amount of harvest per unit of time. In this paper, we distinguish different profit functions by distinguishing the underlying cost functions. In particular, we study stock dependent harvest cost functions, such as the one proposed by Clark [3] for the simple continuous control model. Departing from this continuous cost function, several different types of impulse cost functions have been proposed in the literature. They do not lead to the same type of solution in our impulse control model. To see why, let us analyse some particular cost functions in the following. To ease comprehension, and in order to establish a link to the previous sections, we will sometimes refer back to profits, instead of costs.

### 4.1 A separable profit function

#### 4.1.1 Deriving the cost function from the production function

Clark [3] derives the objective function for the continuous control model. The harvest rate  $h(t)$  is determined by the stock size  $x = x(t)$  and the rate of harvesting effort  $E = E(t)$

such that  $h = Q(E, x)$ , where the function  $Q(E, x)$  is the production function of a given resource industry. Clark assumes this function to be a Cobb-Douglas type function of the form

$$Q(E, x) = aE^\alpha x^\beta,$$

where  $a, \alpha$  and  $\beta$  are positive constants and especially  $\alpha = 1$ . The function  $ax^\beta$  can be replaced with an arbitrary, nondecreasing function  $G(x)$ , such that  $h = Q(E, x) = G(x)E$  and  $E = \frac{h}{G(x)}$ . Next, Clark assumes that the cost of a unit of effort is also a constant:  $c = CE$ . Then, the continuous control cost function can be written as:

$$c_c(x, h) = CE = \frac{C}{G(x)}h$$

and, for  $a = 1$  and  $\beta = 1$ , we have:

$$c_c(x, h) = \frac{C}{x}h,$$

and the associated instantaneous gain is of the form:

$$g_c(x, h) = \left(p - \frac{C}{x}\right)h,$$

where  $p$  is the resource price.

#### 4.1.2 A separable profit function for our impulse control problem

If one wants to represent the harvesting process as a discrete process, one can follow the reasoning proposed by Clark (just using harvest amount and effort at a point in time). The resulting profit function will be a separable function of the form:

$$g(\underline{x}, \bar{x}) = a(\bar{x})d(\bar{x} - \underline{x}).$$

**Proposition 1** *Let  $x^*$  be solution of (13). If  $a(x^*) > 0$  and  $d''(0) + d'(0)\frac{a'(x^*)}{a(x^*)} > 0$  then there exists an interior solution to the auxiliary problem (PA).*

**Proof.** When  $g(\underline{x}, \bar{x}) = a(\bar{x})d(\bar{x} - \underline{x})$ , equation (12) is given by:

$$G(x, x, x_0) = a(x)d'(0) \frac{F(x)}{r} e^{r\tau(x_0, x)}.$$

From equation (13), we know that first order conditions hold for

$$a'(x)F(x) + a(x)(F'(x) - r) = 0. \tag{14}$$

We can develop  $G$  in a neighbourhood of the frontier  $\underline{x} = \bar{x}$  using a Taylor series

$$G(x + h, x + k, x_0) \cong G(x, x, x_0) + \frac{F(x)}{r} e^{r\tau(x_0, x)} B(x),$$

where

$$B(x) = \left\{ \frac{\epsilon}{2} a(x) \left[ d''(0) + d'(0) \frac{r - F'(x)}{F(x)} \right] + h d'(0) \left[ a'(x) - a(x) \frac{r - F'(x)}{F(x)} \right] \right\}, \quad \epsilon = h - k.$$

If there exists  $h$  and  $\epsilon > 0$  such that  $B(x^*) > 0$ , then there exists a direction pointing to the interior of the domain (where  $\underline{x} < \bar{x}$ ) with greater profit than  $G(x^*, x^*, x_0)$ . This means that in this case the solution is not on the boundary  $\underline{x} = \bar{x}$ . Given that  $x^*$  verifies (14) then sufficient conditions for obtaining  $B(x^*)$  greater than zero are:

$$a(x^*) > 0$$

and

$$d''(0) + d'(0) \frac{a'(x^*)}{a(x^*)} > 0.$$

■

**Remark 5** • *The profit function studied by Termansen [20] belongs to this class for  $g(x_i, x_i - I_i) = (p - b/x_i)I_i$  where  $a(x)$  is increasing and  $d(I)$  increasing and linear functions.*

- *We thus have identified a class of profit functions which leads to a cyclical optimal harvesting strategy in our impulse control problem. In addition, when  $a(x) = p - b/x$ ,  $d(I) = I$ , gains associated to the solution of our impulse control problem are greater than gains associated to Clark's case (in which cost function is  $c(x) = b/x$ ).*

## 4.2 An integral profit function

### 4.2.1 Deriving an impulse profit function from a continuous profit function

We can also derive an impulse control profit function  $\int_{\underline{x}}^{\bar{x}} [p - c(x)] dx$  from the standard continuous profit function  $(p - c(x))h$  (and it's continous costs) used in a singular control problem (cf. appendix 6.1). In Clark's model, the optimal approach path is a function of the maximum harvesting capacity  $h_{max}$ . Letting  $h_{max}$  become infinity, the approach path turns inwards until describing a discrete harvesting process. The corresponding profit function is derived as follows:

Compute for given quantities  $x_0$  and  $x_1$  with  $x_1 < x_0$ :

$$\lim_{h_{max} \rightarrow \infty} \int_{t_0}^{t_1} (p - c(x(t))) h_{max} dt, \quad \dot{x}(t) = F(x_t) - h_{max}, \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

Let:  $x(t) = u$ ,  $x(t_0) = x_0$ ,  $\dot{x}dt = du$ ,  $x(t_1) = x_1$ . We thus have:

$$\begin{aligned} \lim_{h_{max} \rightarrow \infty} \int_{x_0}^{x_1} (p - c(u)) \frac{h_{max}}{F(u) - h_{max}} du &= \int_{x_0}^{x_1} (p - c(u)) \lim_{h_{max} \rightarrow \infty} \frac{h_{max}}{F(u) - h_{max}} du \\ &= \int_{x_1}^{x_0} (p - c(u)) du . \end{aligned}$$

### 4.2.2 An integral profit function for our impulse control problem

Let us now suppose the above integral form for the profit function:

$$g(\underline{x}, \bar{x}) = \int_{\underline{x}}^{\bar{x}} [p - c(x)] dx, \quad c(x) = \frac{b}{x^{\alpha+1}}, \quad F(x) = g_0 x(K - x), \quad \alpha \geq 0. \quad (15)$$

In this case we obtain the following result:

**Proposition 2** For  $g$  and  $F$  given by (15),  $x^*$  solution of (13) is a global maximum.

**Proof.** We are going to prove that  $\forall \underline{x} < \bar{x}$  we have  $H(\underline{x}, \bar{x}) := \frac{\partial G}{\partial \underline{x}}(\underline{x}, \bar{x}) > 0$  hence  $\underline{x} = \bar{x} = x^*$  is a global maximum. Consider:

$$H(\underline{x}, \bar{x}) = \left[ -p + c(x) + [p(\bar{x} - \underline{x}) - \int_{\underline{x}}^{\bar{x}} c(u) du] e^{-r\tau} \frac{r}{F(\underline{x})(1 - e^{-r\tau})} \right] \frac{e^{-r\tau}}{1 - e^{-r\tau}}.$$

Where  $\tau = \tau(\underline{x}, \bar{x})$ ; Let's study the sign of  $H(\underline{x}, \bar{x})$ .

$$H(\underline{x}, \bar{x}) = \frac{H_2(\underline{x}, \bar{x}) \underline{x} (p - c(\underline{x}))(K - \underline{x}) + \alpha \underline{x} g_0 (K - \underline{x}) + H_3(\underline{x}, \bar{x})}{H_1(\underline{x}, \bar{x})}$$

where:

$$H_1(\underline{x}, \bar{x}) = \alpha \underline{x} g_0 (K - \underline{x}) \left[ \left( \frac{\bar{x}(K - \underline{x})}{\underline{x}(K - \bar{x})} \right)^{\frac{r}{g_0 K}} - 1 \right] < 0,$$

$$H_2(\underline{x}, \bar{x}) = -\alpha g_0 \left( \frac{\bar{x}(K - \underline{x})}{\underline{x}(K - \bar{x})} \right)^{\frac{r}{g_0 K}} < 0,$$

and

$$H_3(\underline{x}, \bar{x}) = r p \alpha (\bar{x} - \underline{x}) + r b c (\bar{x}^{-\alpha} - \underline{x}^{-\alpha}).$$

We have that  $H_3(\bar{x}, \bar{x}) = 0$  and  $\frac{\partial H_3}{\partial \underline{x}}(\bar{x}, \bar{x}) = r \alpha (-p + c(\underline{x}))$ . For all  $\underline{x}$  such that  $-p + c(\underline{x}) < 0$  we have  $H_3(\underline{x}, \bar{x}) > 0$ . All these inequalities imply that  $H(\underline{x}, \bar{x}) > 0$ . ■

**Remark 6** • Clark [3] studies this profit function in models where the harvesting process is supposed to be a discrete action at a point in time.

- The optimal harvesting strategy associated to the integral profit function leads to a non-cyclical optimal harvesting strategy in our impulse control problem. Gains associated to the solution tend towards the gains associated to Clark's case in which  $c(x) = \frac{b}{x^{\alpha+1}}$ .

### 4.3 An empirical outlook

Given the very different solutions and policy implications for the two cost functions studied, we may ask which function is the most probable in the field. There are not many econometric estimations of cost-functions in renewable resource economics. Several papers confirm nonetheless that costs depend on both, harvesting and resource stocks, and are of the general form:  $c(x, h)^2$ . Let's concentrate in the first place on fishery economics and focus on papers with a continuously evolving resource stock and one age class. Most studies in this field suppose that harvest is measured by the continuous harvest rate,  $h(t)$  (see for example Grafton et al. (2000) [9]), Arnason et al. (2004) [1]) or on the amount of effort (see for example Homans and Wilen (1997) [11]). One study that we considered supposes harvest to be given by the harvesting amount,  $h$ . Indeed, Singh et al. [18] have studied the case of the North Alaskan pacific halibut fishery, regressing a cost function from real-world data. They suppose costs to be stock dependent and strictly convex in harvesting, as the two functions we have studied above. Regression results confirm their assumption: the harvest cost function is convex, at least in the relevant range of vessels.<sup>3</sup> Interesting for our purpose is that the authors represent harvesting as a discrete action and use a separable cost function, in combination with the logistic growth function.

## 5 Conclusion

We have confirmed that different cost functions may trigger completely different types of optimal harvesting behaviour. We have taken the particular example of Clark's continuous control model with one state variable and we have constructed a corresponding, more general, impulse control model. We have shown that the optimal harvesting behaviour in our impulse control model can either be cyclical or converge to a Most Rapid Approach Path solution, like the one in Clark's continuous control model, depending on the cost function that is used. Finally, we have questioned the kind of impulse cost function (and profit function) that should be underlying. There are only few regression results of dynamic bioeconomic models in the literature. Those empirical studies we discussed confirm that a stock dependent harvest cost function in combination with a logistic growth function is commonly adapted to empirical data. Studies as those by Singh et al. [18] suggest that real world cost functions are separable and are non-increasing in resource stocks. They are strictly convex in harvesting for the relevant range of effort. In addition, authors as Singh et al., suppose the harvesting process to be a discrete event. However, at our knowledge, there is no detailed study that may indicate whether harvest decisions should be modelled as continuous processes or decisions at a point in time. Empirical data may be gathered in order to answer this question and to find out which kind of solutions will apply in different resource sectors.

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<sup>2</sup>cost may also depend on capital invested  $c(x, h, k)$  but this function can be translated in the former for  $k=\text{constant}$

<sup>3</sup>For a larger range, the econometric study reveals non-convexities. The corresponding parameter is not significant though.

## 6 Appendix

### 6.1 The underlying model: a singular optimal control model

In a standard Clark-Munro renewable resources model (cf. Clark [3], Clark and Munro [4]) the only control variable is the harvest rate,  $h(t)$ . The optimal harvesting problem can be stated as:

$$G_c^*(x_0) := \max_{h(\cdot)} \int_0^\infty e^{-rt} [p - c(x(t))] h(t) dt \quad (16)$$

$$\dot{x}(t) = F(x(t)) - h(t) \quad (17)$$

$$x(0) = x_0 \quad (18)$$

$$0 \leq h(t) \leq h_{max} \quad (19)$$

In the profit function (16),  $x(t)$  is the level of the resource stock at time  $t$ ,  $p$  represents the resource price,  $c(x)$  the unit harvest costs and  $r$  the discount rate. The underlying continuous control cost function  $c_c(x, h)$  is linear in  $h$ , of the form  $c_c(x, h) = c(x)h$ . Initial resource stocks,  $x_0$ , are known and the harvest capacity is bounded by  $h_{max}$ . The natural growth is stock dependent and given by:  $F(x)$ .

The profit maximising stock level leads to a steady state which is such that the marginal-productivity rule including the stock effect holds (Clark and Munro [4],[3]):

$$F'(x^*) - \frac{c'(x^*)F(x^*)}{p - c(x^*)} = r \quad (20)$$

In (20)  $x^*$  stands for the steady state level of the corresponding problem. Using the growth and cost functions, the steady state can be computed. In particular, the natural growth  $F(x(t))$  is given by (21) and is supposed to be logistic.

$$F(x) = g_0 x \left(1 - \frac{x}{K}\right) \quad (21)$$

where  $g_0$  is the intrinsic growth rate and  $K$  the carrying capacity.

If (20) has a unique solution, then the optimal policy is the most rapid approach path to the steady state, i.e. the harvester choses maximum harvest capacity  $h_{max}$  if  $x$  greater than  $x^*$ , he waits if  $x$  smaller than  $x^*$  and he takes off just the additional growth once the steady state is reached,  $x = x^*$ .

### 6.2 Flow and reaching time: derivation of $\tau(x, y)$ .

Consider a flow  $\phi(t; x)$ , solution of the ODE  $\dot{x} = F(x)$ . We have the following property:  $\phi(t; \phi(s; x)) = \phi(t + s; x)$ , which implies that:

$$\frac{\partial \phi}{\partial x}(t; \phi(s; x)) \frac{\partial \phi}{\partial t}(s; x) = \frac{\partial \phi}{\partial t}(t + s; x), \quad \frac{\partial \phi}{\partial t}(t; x) = \dot{x}(t) = F(x(t)) = F(\phi(t; x)),$$

and hence:

$$\frac{\partial \phi}{\partial x}(t; x) = \frac{\partial \phi}{\partial t}(t; x) / \frac{\partial \phi}{\partial t}(0; x) = F(\phi(t; x)) / F(x).$$

Consider now  $\tau(x, y)$  the time to get from  $x$  to  $y$  for the flow  $\phi$ . We have

$$\phi(\tau(x, y); x) = y. \quad (22)$$

Differentiating (22) with respect to  $x$ :

$$\frac{\partial \tau}{\partial x}(x, y) \frac{\partial \phi}{\partial t}(\tau(x, y); x) + \frac{\partial \phi}{\partial x}(\tau(x, y); x) = 0,$$

and from this equation we obtain:

$$\frac{\partial \tau}{\partial x}(x, y) = -\frac{\partial \phi}{\partial x}(\tau(x, y); x) / \frac{\partial \phi}{\partial t}(\tau(x, y); x) = -\frac{F(y)}{F(x)} \frac{1}{F(y)} = -\frac{1}{F(x)}.$$

Differentiating (22) with respect to  $y$ :

$$\frac{\partial \tau}{\partial y}(x, y) \frac{\partial \phi}{\partial t}(\tau(x, y); x) = 1 \quad \Rightarrow \quad \frac{\partial \tau}{\partial y} = \frac{1}{F(y)}.$$

Then  $\tau(x, y)$  is the solution of the following partial differential equation:

$$\frac{\partial \tau}{\partial x} = -\frac{1}{F(x)}; \quad \frac{\partial \tau}{\partial y} = \frac{1}{F(y)}; \quad \tau(x, x) = 0.$$

Assuming that

$$\tau(x, y) = A(y) - A(x)$$

we conclude that:

$$A'(x) = \frac{1}{F(x)}, \quad A(x) = \int_0^x \frac{1}{F(u)} du$$

and

$$\tau(x, y) = \int_x^y \frac{1}{F(u)} du. \quad (23)$$

### 6.3 Study of function G

Let us analyze

$$G(\underline{x}, \bar{x}, x_0) = g(\bar{x}, \underline{x}) \frac{e^{-r\tau(x_0, \bar{x})}}{1 - e^{-r\tau(\underline{x}, \bar{x})}} = g(\bar{x}, \underline{x}) \frac{e^{-r\tau(\underline{x}, \bar{x}) - \tau(\underline{x}, x_0)}}{1 - e^{-r\tau(\underline{x}, \bar{x})}}$$

if  $x \leq x_0 \leq \bar{x}$ , in the light of the identity  $\tau(\underline{x}, \bar{x}) = \tau(\underline{x}, x_0) + \tau(x_0, \bar{x})$  and using the logarithmic derivative:

$$\ln G = \ln g - r\tau - \ln(1 - e^{-r\tau}), \quad \tau = \tau(\underline{x}, \bar{x}).$$

and

$$\frac{1}{G} \frac{\partial G}{\partial \underline{x}} = \frac{1}{g} \frac{\partial g}{\partial \underline{x}} - r \frac{\partial \tau}{\partial \underline{x}}(\underline{x}, \bar{x}) \frac{e^{-r\tau}}{1 - e^{-r\tau}}, \quad \frac{1}{G} \frac{\partial G}{\partial \bar{x}} = \frac{1}{g} \frac{\partial g}{\partial \bar{x}} - r \frac{\partial \tau}{\partial \bar{x}}(\underline{x}, \bar{x}) \frac{e^{-r\tau}}{1 - e^{-r\tau}}.$$

With the identity on  $\tau$ , we have:

$$\frac{1}{G} \frac{\partial G}{\partial \underline{x}} = \frac{1}{g} \frac{\partial g}{\partial \underline{x}} + r \frac{1}{F(\underline{x})} \frac{e^{-r\tau}}{1 - e^{-r\tau}}, \quad \frac{1}{G} \frac{\partial G}{\partial \bar{x}} = \frac{1}{g} \frac{\partial g}{\partial \bar{x}} - r \frac{1}{F(\bar{x})} \frac{1}{1 - e^{-r\tau}}.$$

Suppose we have an interior solution (cf. equations (10), (11) of lemma 1):

$$\frac{1}{g} \frac{\partial g}{\partial \underline{x}} = -r \frac{1}{F(\underline{x})} \frac{e^{-r\tau}}{1 - e^{-r\tau}}, \quad \frac{1}{g} \frac{\partial g}{\partial \bar{x}} = r \frac{1}{F(\bar{x})} \frac{e^{-r\tau}}{1 - e^{-r\tau}}. \quad (24)$$

These equations imply

$$\frac{\partial g}{\partial \underline{x}} / \frac{\partial g}{\partial \bar{x}} = -\frac{F(\bar{x})}{F(\underline{x})} e^{-r\tau(\underline{x}, \bar{x})}.$$

Note that this last equation coincides with equation (8).

## 6.4 Proof of theorem (3)

First, consider  $x_0 < \bar{x}^*$ . According to theorem 1, the value function of problem P verifies:

$$v(x) = \max_{y \in [0, K], t \geq 0} e^{-rt} [g(\phi(t, x), y) + v(y)], \quad (25)$$

where  $\phi(t, x)$  is the dynamics of the system. In our problem  $\phi(t, x) = \bar{x}$  and  $t = \tau(x, \bar{x})$ .

We show that

$$v(x) = G(\underline{x}^*, \bar{x}^*, x) = g(\bar{x}^*, \underline{x}^*) \frac{e^{-r\tau(x, \bar{x}^*)}}{1 - e^{-r\tau(\underline{x}^*, \bar{x}^*)}}$$

verifies equation (25), i.e. verifies:

$$v(x) = \max_{\bar{x}, y \in [0, K]} \psi(x, \bar{x}, y), \quad \psi(x, \bar{x}, y) := e^{-r\tau(x, \bar{x})} [g(\bar{x}, y) + v(y)]. \quad (26)$$

First of all we are going to prove that the maximum of  $\psi$  is reached in  $(\underline{x}^*, \bar{x}^*)$ , in fact:

$$\frac{\partial}{\partial y} \psi(x, \bar{x}, y) = e^{-r\tau(x, \bar{x})} \left[ \frac{\partial g}{\partial y} + r \frac{g(\bar{x}^*, \underline{x}^*)}{F(y)} \frac{e^{-r\tau(y, \bar{x}^*)}}{1 - e^{-r\tau(y, \bar{x}^*)}} \right] = 0. \quad (27)$$

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} \psi(x, \bar{x}, y) &= \frac{-r e^{-r\tau(x, \bar{x})}}{F(\bar{x})} [g(\bar{x}, y) + v(y)] + e^{-r\tau(x, \bar{x})} \frac{\partial g}{\partial \bar{x}} = 0 \\ &\Leftrightarrow \frac{-r}{F(\bar{x})} \left[ g(\bar{x}, y) + g(\bar{x}^*, \underline{x}^*) \frac{e^{-r\tau(y, \bar{x}^*)}}{1 - e^{-r\tau(\underline{x}^*, \bar{x}^*)}} \right] = 0. \end{aligned} \quad (28)$$

Equations (27) and (28) correspond to the first order conditions of PA.

Finally we prove that  $v(x) = G(\underline{x}^*, \bar{x}^*, x)$  satisfies the dynamic programming equation, in



fact:

$$\begin{aligned}
& \max_{y \in [0, K]} \{ e^{-r\tau(x, \bar{x})} [g(\bar{x}, y) + G(\underline{x}^*, \bar{x}^*, y)] \} \\
&= e^{-r\tau(x, \bar{x}^*)} \left[ g(\bar{x}^*, \underline{x}^*) + g(\bar{x}^*, \underline{x}^*) \frac{e^{-r\tau(\underline{x}^*, \bar{x}^*)}}{1 - e^{-r\tau(\underline{x}^*, \bar{x}^*)}} \right] \\
&= e^{-r\tau(x, \bar{x}^*)} \left[ g(\bar{x}^*, \underline{x}^*) \frac{1}{1 - e^{-r\tau(\underline{x}^*, \bar{x}^*)}} \right] = G(\underline{x}^*, \bar{x}^*, x)
\end{aligned}$$

**Remark 7** for  $x > \bar{x}^*$  it is easy to verify that the value function  $v$  is:

$$v(x) = g(x, \underline{x}^*) + v(\underline{x}^*).$$

to conclude:

$$v(x) = \begin{cases} G(\underline{x}^*, \bar{x}^*, x) & \text{if } x \leq \bar{x}^* \\ g(x, \underline{x}^*) + G(\underline{x}^*, \bar{x}^*, x) & \text{if } x > \bar{x}^* \end{cases}$$

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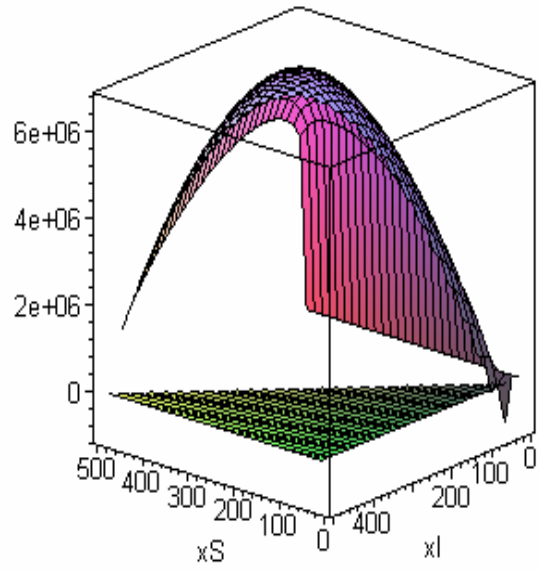


Figure 1: Gain function for different values of  $\bar{x}$  and  $\underline{x}$  : an interior solution

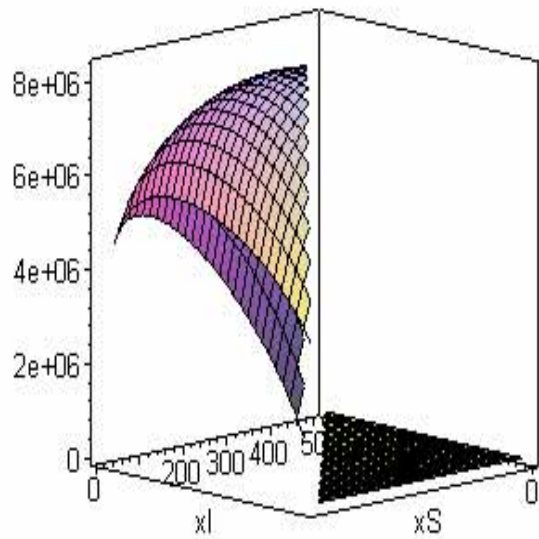


Figure 2: Gain function for different values of  $\bar{x}$  and  $\underline{x}$  : a boundary solution