Semi-semi-Markov processes : a new class of processes for formalizing and generalizing state-dependent individual-based models

Christine Jacob, Anne France Viet

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Origin : epidemiological problem

Study the propagation of a disease (BVD : Bovine Viral Diarrhoea) in a dairy herd

3 types of individual transitions :

– Branching transitions (births, deaths)

– Group changes according to physiological status and age :
4 groups : calves, heifers before breeding, heifers after breeding, dairy cows

– Health status changes which depend on the population infection

$\Longrightarrow$ individual-based stochastic model : the individual transitions are population-dependent and semi-Markovian

$\Longrightarrow$ propagation of a random process (disease) on a random graph (vertices : individuals)

Individual-based models

– used in population dynamics when individuals are marked by personal characteristics or when the next state-change is driven by complex rule decisions which depend on the current state of the population

– calculate empirical distributions at the scale of the population from simulated individual trajectories (“bottom-up approach”)

– litterature on individual-based models has considerably increased thanks to the increase of the computers capacity and the popularization of informatics

– no mathematical formalism : validation of this approach ? how is the process at the level of the population ?

Goal

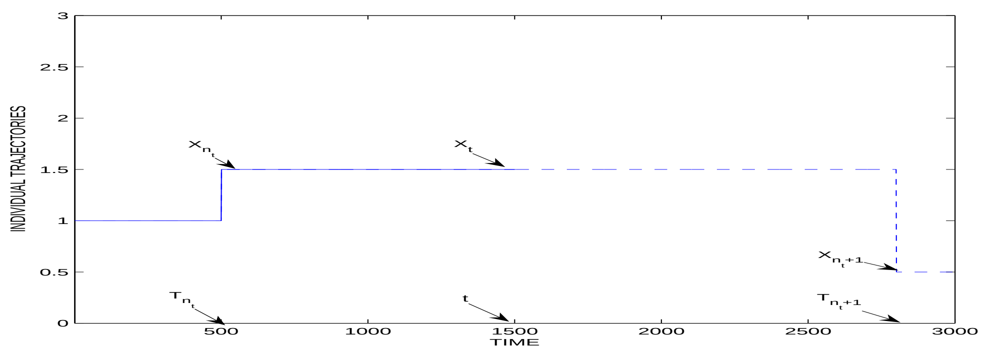
Build a rigorous mathematical formalism of these models at the population level (top-down)

$\Longrightarrow$ good readability of the different model components independently of the programming language

$\Longrightarrow$ validate and supplement the empirical distributions by analytical results

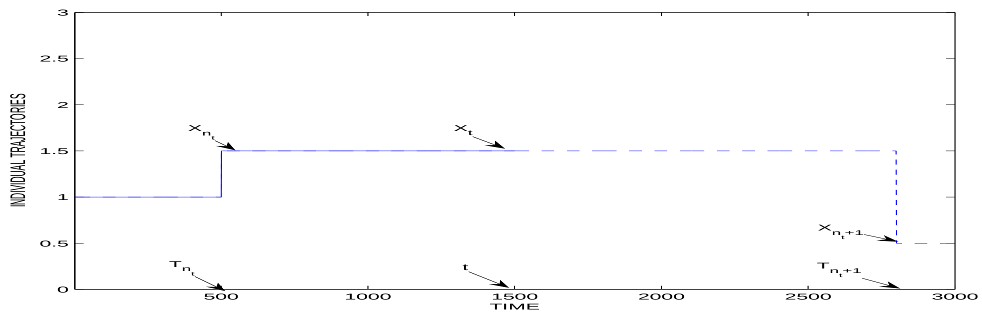
Homogeneous Semi-Markov Process (SMP) for one individual ω

Feller W. (1964), Çinlar (1975), Kulkarni, V. (1995), Becker G. *et al.* (1999), Iosifescu M. (1999)



Countable state space, jumps at random times
$\{X_t\}_{t\in\mathbb{R}^+}$ is an homogeneous semi-Markov process if

$$X_t \;=\; X_{n_t}1_{\{n_t=\sup\{n:T_n\le t\}\}},\;\; T_n:\; n\text{th jump time}, X_n:\; \text{state at } T_n,\; \{X_n,T_n\}\text{ MRP}$$



Law of $\{X_t\}_t \Longleftrightarrow$ law of $\{X_n, T_n\}_n \Longleftrightarrow$ transition laws of $\{X_n, T_n\}_n$

Assumption : $\{X_n, T_n\}$ is a MRP

$$\begin{aligned} P(X_{n+1} = j, \Delta T_{n+1} \leq \tau | X_n, \ldots, X_0, T_n, \ldots, T_0) &= P(X_{n+1} = j, \Delta T_{n+1} \leq \tau | X_n) \\ &\stackrel{homog.}{=} P(X_1 = j, \Delta T_1 \leq \tau | X_0) \\ \Delta T_{n+1} &\stackrel{defin.}{=} T_{n+1} - T_n \text{ (waiting time between 2 jumps)} \end{aligned}$$

Kernels (transition laws)

$$\begin{aligned} Q_{i,j}(\tau) &\stackrel{defin.}{=} P(\Delta T_1 \leq \tau, X_1 = j | X_0 = i) \\ &= P(\Delta T_1 \leq \tau | X_1 = j, X_0 = i) P(X_1 = j | X_0 = i) \\ &= F_{i,j}(\tau) P(i, j) \end{aligned}$$

$F_{i,j}(\tau)$: cdf of the sojourn time in i before jumping in j ; $P(i, j)$: transition probability of $\{X_n\}$

Kernels $\Longleftrightarrow$ Transition rates

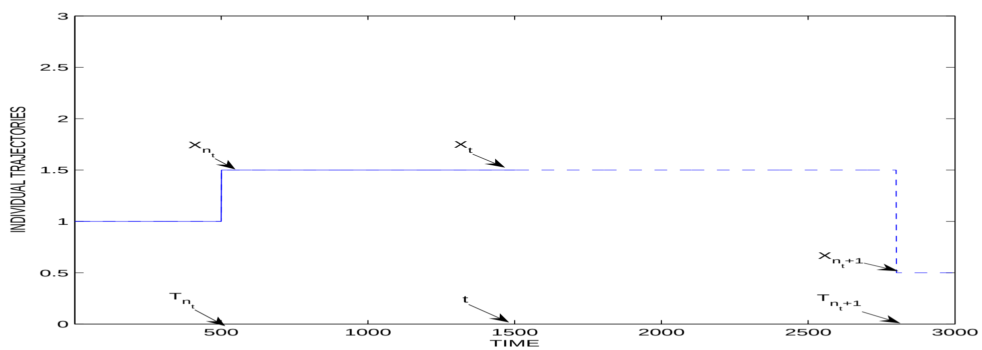
$$\lambda_{i,j}(\tau) \stackrel{def.}{=} \lim_{\Delta\tau \to 0} \frac{P(\Delta T_{n+1} \in (\tau, \tau + \Delta\tau), X_{n+1} = j | X_n = i, \Delta T_{n+1} > \tau)}{\Delta\tau}$$

$$\lambda_{i,j}(\tau) = \frac{\dot{Q}_{i,j}(\tau)}{1 - \sum_j Q_{i,j}(\tau)} \Longleftrightarrow Q_{i,j}(\tau) = \int_0^\tau \lambda_{i,j}(u) exp(-\int_0^u \sum_j \lambda_{i,j}(s) ds) du$$

Particular case : Markov process :

$\lambda_{i,j}(\tau) = \lambda_{i,j} = \lambda_i P(i,j)$, for any τ, $F_{i,j}(\tau) = 1 - \exp(-\lambda_i \, \tau)$

$\mathbf{P}(t) = \exp(\Lambda t)$, $\mathbf{P}[i,j](t) \stackrel{def.}{=} P(X(t) = j | X(0) = i)$



Event-driven simulation algorithm based on $P(i,j)F_{i,j}(\tau)$

Current jump time and jump state : $(t_n, i) \Longrightarrow$ determine the next jump time and jump state (t_{n+1}, j)

Simulate j according to $\{P(i,j')\}_{j'}$

Simulate $\tau \stackrel{def.}{=} t_{n+1} - t_n$ according to $F_{i,j}(.)$

Probability law of the process (renewal equations)

$$P(X_t = j | X_0 = i) \overset{notat.}{=} P_{i,j}(t) = [1 - \sum_{j'} Q_{i,j'}(t)] 1_{\{j=i\}} + \sum_k \int_0^t dQ_{i,k}(s) P_{k,j}(t-s)$$

Matrix process : $\mathbf{P}[i,j](t) \overset{def.}{=} P_{i,j}(t)$

$$\mathbf{P} \ = \ (\mathbf{I} - \mathbf{Q}^{\Sigma}) + \mathbf{Q} * \mathbf{P} \Longrightarrow \mathbf{P} = \sum_{n=0}^{\infty} \mathbf{Q}^{n*} * (I - \mathbf{Q}^{\Sigma}), \ \ \mathbf{Q} * \mathbf{P}(t) = \sum_k \int_0^t Q_{i,k}(s) P_{k,j}(t-s)$$

Approximate solutions

Empirical distribution (simulations)

$\mathbf{P} \simeq \sum_{n=0}^{n_t} \mathbf{Q}^{*n} * (I - \mathbf{Q}^{\Sigma})$

Recursive solution of $\mathbf{P}$ by discretization of time

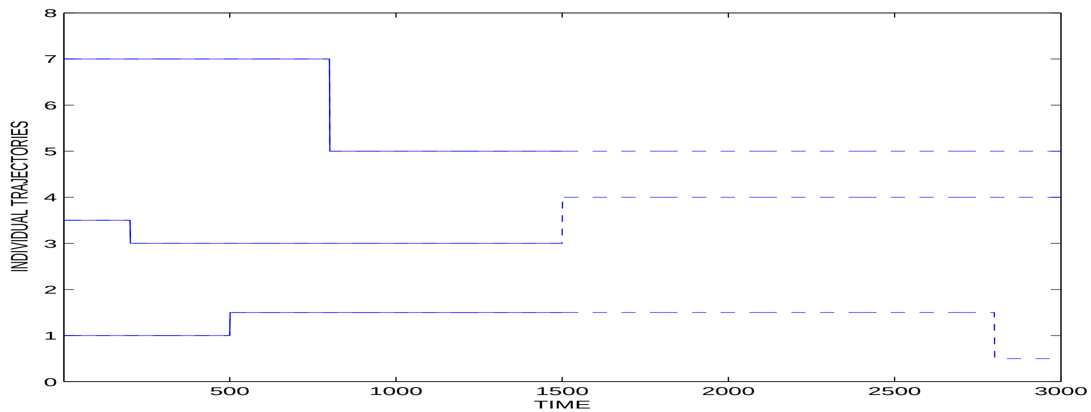
Upper and lower bounds of Li and Luo (2005)

Stationary law (all the states are recurrent) or quasi-stationary law (conditioned on staying in the nonabsorbing state)

Homogeneous Semi-Semi-Markov Processes for a closed population $\Omega = \{\omega_l\}_{l \le N}$

Set of MRP : $\{\{(X_m^{(l)}(\omega_l), T_m^{(l)}(\omega_l))\}_m\}_{l=1,\ldots,N}$, not synchronized and population-dependent

Goal : determine the distribution of the population process $\mathcal{X}_t = \{X_t^{(1)}, \ldots, X_t^{(N)}\}$



Particular case : the MRP are i.i.d.

communication networks, asymptotic distributions in the heavy-tailed case $1 - F(t) = t^{-\alpha}L(t)$ (Mikosh and Resnick, 2005, Mitov and Yanev, 2006)

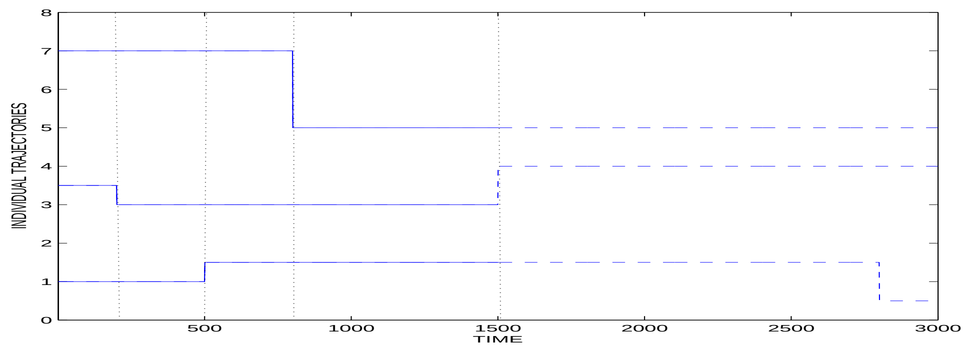
Steps

– define $\{\mathcal{X}_t(\Omega)\}_{t\in\mathbb{R}^+}$ from $\{(\mathcal{X}_n(\Omega), \mathcal{T}_n(\Omega))\}n \in \mathbb{N}$, defined itself from the MRP $\{\{(X_m^{(l)}(\omega_l), T_m^{(l)}(\omega_l))\}_m$

– deduce the kernel (transition law) of the population process from the individual kernels

– deduce transition rates, probability law, simulation algorithm,. . .

Definition of $\{\mathcal{X}_t(\Omega)\}$ from $\{(\mathcal{X}_n(\Omega), \mathcal{T}_n(\Omega))\}$, defined itself from the individual MRP



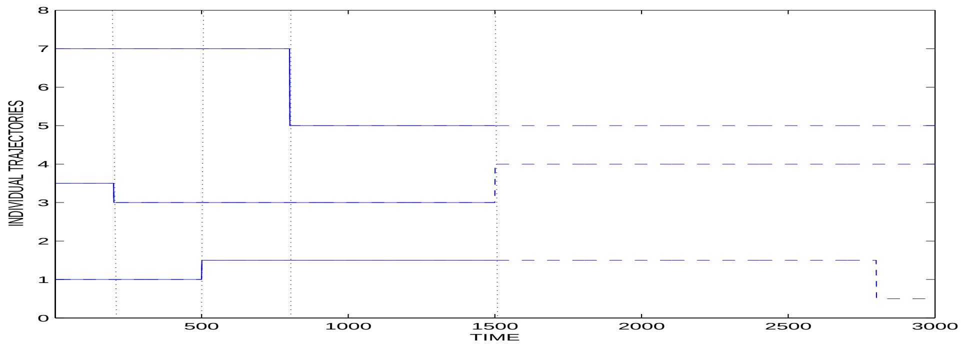
$$\mathcal{X}_t(\Omega) \stackrel{def.}{=} \mathcal{X}_{n_t}(\Omega) \stackrel{def.}{=} \{X^{(l)}_{m_{l,t}}(\omega_l)\}_l$$

$$m_{l,t}(\omega_l) \stackrel{def.}{=} \sup\{m : T^{(l)}_m(\omega_l) \leq t\}$$

$$n_t(\Omega) \stackrel{def.}{=} \sum_l m_{l,t}(\omega_l)$$

$$\mathcal{T}_{n_t} \stackrel{def.}{=} \sup_l \sup_m \{T^{(l)}_m(\omega_l) \leq t\}$$

Law of $\{\mathcal{X}_t(\Omega)\}_t \Longleftrightarrow$ law of $\{\mathcal{X}_n, \mathcal{T}_n\}_n \Longleftrightarrow \{P(\mathcal{X}_{n+1} = J, \Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{F}_n(I))\}_{n,I,J}$ (kernels)



$\mathcal{F}_n(I) = \{\mathcal{X}_n = I, \mathcal{X}_{n-1} = I_{n-1}, ..., \mathcal{X}_0 = I_0, \mathcal{T}_n = t_n, \mathcal{T}_{n-1} = t_{n-1}, ..., \mathcal{T}_0 = t_0\}$: past until (t_n, I)

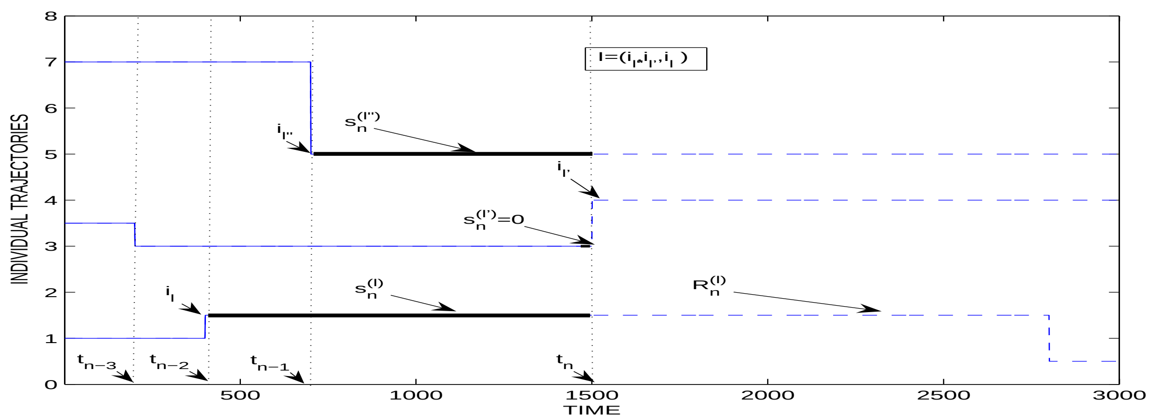
Kernel

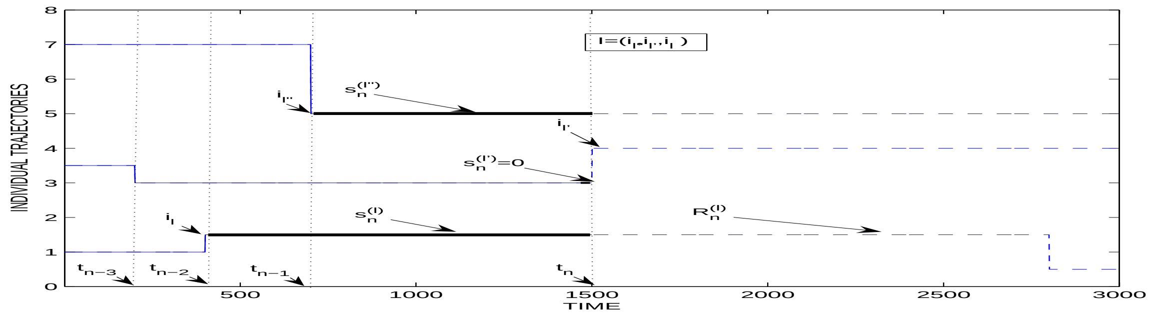
$$\begin{aligned} P(\mathcal{X}_{n+1} = J, \Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{F}_n(I)) &= P(\Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{X}_{n+1} = J, \mathcal{F}_n(I)) P(\mathcal{X}_{n+1} = J | \mathcal{F}_n(I)) \\ &\overset{notation}{=} F_{\mathcal{F}_n(I),J}(\tau) P(\mathcal{F}_n(I), J) \\ &\overset{notation}{=} Q_{\mathcal{F}_n(I),J}(\tau) \end{aligned}$$

$\Longrightarrow$ calculate $F_{\mathcal{F}_n(I),J}(.)$, $P(\mathcal{F}_n(I), J)$ from the individual transitions defined from (t_n, I)

Assumptions given the past until (t_n, I)

1. $A1$: the $\{(remaining\, waiting\, time\, R_n^{(l)},\ next\, jump\, state\, X_{m_n+1}^{(l)})\}_l$ are mutually independent
2. $A2$: for each l, the law of $R_n^{(l)}$ (remaining time in i_l before jumping in j_l) depends only on I and on $s_n^{(l)}$ (time already spent in i_l) and on I
3. $A3$: the probability for l to jump from i_l to j_l depends only on i_l, j_l, and I : $P^{(l)}(i_l|I, j_l)$
4. $A4$: the next population jump is defined by the individual jump which occurs the first
$Q_{\mathcal{F}_n(I),J_l}(\tau) = P(\min_l\{R_n^{(l')}\} = R_n^{(l)}, R_n^{(l)} \leq \tau, X_{m_n+1}^{(l)} = j_l|\mathcal{F}_n(I))$





Proposition. *Let* $I \to J_l : i_l \to j_l$. *Then*

$$dF_{\mathcal{F}_n(I),J_l}(\tau) \overset{def.}{=} \frac{dQ_{\mathcal{F}_n(I),J_l}(\tau)}{P(\mathcal{F}_n(I),J_l)} = \frac{\int_0^\tau \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)}{\int_0^\infty \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)} \quad \textit{(cdf of } \Delta\mathcal{T}_{n+1}\textit{)}$$

$$P(\mathcal{F}_n(I),J_l) \overset{def.}{=} \int_0^\infty dQ_{\mathcal{F}_n(I),J_l}(\tau) = \int_0^\infty \Pi_{l'\neq l}(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau))dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau),$$

$$Q^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau) = \frac{F^{(l)}_{i_l|I,j_l}(s_n^{(l)}+\tau) - F^{(l)}_{i_l|I,j_l}(s_n^{(l)})}{1-F^{(l)}_{i_l|I,j_l}(s_n^{(l)})} P^{(l)}(i_l|I,j_l) : \ \textit{individual prior kernel}$$

Consequence : $Q_{\mathcal{F}_n(I),J_l}(.)$ depends only on the current state I, the next state J_l and $\{s_n^{(l)}\}_l$

Time-driven simulation algorithm $\Longleftrightarrow \lambda_{\mathcal{F}_n(I),J}(.)$

Event-driven simulation algorithm $\Longleftrightarrow Q_{\mathcal{F}_n(I),J}(.) = F_{\mathcal{F}_n(I),J}(.)P(\mathcal{F}_n(I), J)$

Determine the next jump (t_{n+1}, J) from (t_n, I) and $\{s_n^{(l)}\}$, $I = (i_1, ..., i_l, ..., i_N)$: for each individual l,

1. choose j_l according to $\{P^{(l)}(i_l|I, j_l)\}_{j_l}$
2. simulate a remaining waiting time $r_n^{(l)}$ in i_l before jumping into j_l

Then $r_n^{(l)} = \min_{l'}\{r_n^{(l')}\}$ defines the next jump time and the next state $J_l = (i_1, ..., j_l, ..., i_N)$

Transition rates $\Longleftrightarrow$ kernels

$$\lambda_{\mathcal{F}_n(I),J}(\tau) \stackrel{def.}{=} \lim_{\Delta\tau \to 0} \frac{P(\Delta\mathcal{T}_{n+1} \in (\tau, \tau + \Delta\tau), \mathcal{X}_{n+1} = J | \mathcal{F}_n(I), \Delta\mathcal{T}_{n+1} > \tau)}{\Delta\tau}$$

Proposition

$$\begin{aligned}
\lambda_{\mathcal{F}_n(I),J}(\tau) &= \frac{\dot{Q}_{\mathcal{F}_n(I),J}(\tau)}{1 - \sum_J Q_{\mathcal{F}_n(I),J}(\tau)}, \tau \in \mathbb{R}^+ \\
Q_{\mathcal{F}_n(I),J}(\tau) &= \int_0^\tau \lambda_{\mathcal{F}_n(I),J}(u) exp(-\int_0^u \sum_J \lambda_{\mathcal{F}_n(I),J}(s) ds) du
\end{aligned}$$

Corollary. Assume (Exp) *:* $F^{(l)}_{i_l|I,j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\,\tau)$

Then the SSMP is a MP, and for all I *not absorbing*

$$\begin{aligned}
dF_{I,J_l}(\tau) &= dF_I(\tau) = (\sum_{l'} \lambda_{i_{l'}|I}) \exp(-\sum_{l'} \lambda_{i_{l'}|I}\,\tau) d\tau \\
P(I, J_l) &= P^{(l)}(i_l|I, j_l) \frac{\lambda_{i_l|I}}{\sum_{l'} \lambda_{i_{l'}|I}}. \\
\lambda_{I,J_l}(\tau) &= \lambda_{i_l|I} P^{(l)}(i_l|I, j_l) = \lambda_{i_l|I,j_l} \\
\lambda_I(\tau) &= \sum_J \lambda_{I,J}(\tau) = \sum_l \lambda_{i_l|I} \\
\mathbf{P}(t) &= \exp(\Lambda t)
\end{aligned}$$

Consequence. Under (Exp), if I is not an absorbing state, then

$$m_I = \Big[\sum_{l'} \lambda_{i_{l'}|I}\Big]^{-1} \text{ (mean time in } I\text{)}$$

Marginal probability law of $\{\mathcal{X}_t\}_t$: renewal equations

$$
\begin{aligned}
P(\mathcal{X}_t = J | I_0, \{s_0^{(l)}\}, t_0) &= P(\Delta\mathcal{T}_1 > t - t_0 | I_0, \{s_0^{(l)}\}) 1_{\{J = I_0\}} + \\
&\sum_{I_1 \neq I_0} \int_{t_1 \in (t_0, t)} dP(\mathcal{X}_1 = I_1, \Delta\mathcal{T}_1 = t_1 - t_0 | I_0, \{s_0^{(l)}\}) P(\mathcal{X}_t = J | I_1, \{s_1^{(l)}\}, t_1) \\
&= P(\Delta\mathcal{T}_1 > t - t_0 | I_0, \{s_0^{(l)}\}) 1_{\{J = I_0\}} + \\
&\sum_{I_1 \neq I_0} \int_{t_1 \in (t_0, t)} dP((\mathcal{X}_1, S_1) = (I_1, \{s_1^{(l)}\}), \Delta\mathcal{T}_1 = t_1 - t_0 | I_0, \{s_0^{(l)}\}) P(\mathcal{X}_t = J | I_1, \{s \\
\mathbf{P} &= (\mathbf{I} - \mathbf{Q}^{\Sigma}) + \mathbf{Q}^{\mathcal{Y}} * \mathbf{P}; \;\; \mathcal{Y} = (\mathcal{X}, S) \\
\mathbf{P} &= \sum_{n \geq 0} \mathbf{Q}^{\mathcal{Y} * n} * (\mathbf{I} - \mathbf{Q}^{\Sigma})
\end{aligned}
$$

$\Longrightarrow$ Approximate solution : $\mathbf{P}(t) = [\sum_{n \geq 0}^{n_t} \mathbf{Q}^{\mathcal{Y} * n} * (\mathbf{I} - \mathbf{Q}^{\Sigma}))(t)$

Approximate solution : discretization of time

$s_h^{(l)} = [s_{h-1}^{(l)} + t_h - t_{h-1}]1_{\{t_h \notin \{T_m^{(l)}\}_m\}} \Longrightarrow \{s_h^{(l)}\}_l \overset{not.}{=} \Delta_{t_h - t_{h-1}}$; $\{s_0^{(l)}\}_l \overset{not.}{=} \Delta_0$

$$\mathbf{P}_{\Delta_0}(t-t_0) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(t-t_0) + \int\limits_{t_1 \in (t_0,t)} d\mathbf{Q}_{\Delta_0}(t_1 - t_0)\mathbf{P}_{\Delta_{t_1 - t_0}}(t - t_1).$$

The discretization of the system using $t - t_0 = nh$, $t_1 - t_0 \in \{ih\}_{i \leq n}$, leads to the solution

$$\begin{pmatrix} \mathbf{P}_{\Delta_0}(nh) \\ \\ \mathbf{P}_{\Delta_h}((n-1)h) \\ \\ \ldots \\ \\ \mathbf{P}_{\Delta_{(n-1)h}}(h) \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{\Delta_0}(0) & \mathbf{R}_{\Delta_0}(h)\ldots\mathbf{R}_{\Delta_0}((n-1)h) \\ 0 & \mathbf{R}_{\Delta_h}(0)\ldots\mathbf{R}_{\Delta_h}((n-2)h) \\ . & .\quad \ldots\ldots\ldots \\ 0 & 0\ldots\ldots\ldots\mathbf{R}_{\Delta_{(n-1)h}}(0) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{B}_{\Delta_0,n} \\ \\ \mathbf{B}_{\Delta_h,n} \\ \\ \ldots \\ \\ \mathbf{B}_{\Delta_{(n-1)h},n} \end{pmatrix}$$

$\mathbf{R}_{\Delta}(ih) = \mathbf{I}\delta_{0,i} - a_i \dot{\mathbf{Q}}_{0,\Delta}(ih)(1 - \delta_{0,i}),\ \delta_{0,i} = 1$ when $i = 0$ (and is 0 otherwise), $i = 0, \ldots, n-1$
$\mathbf{B}_{\Delta_{jh},n} = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_{jh}}((n-j)h),\ j = 0, \ldots, n-1$
$\{a_i\}_i$ depends on the numerical integration scheme,

Semi-semi-Markovian branching process for individuals with a pregnancy period

$\mathcal{X}_t(\Omega)$ takes values in $\mathcal{X} = \{\{(P_l)\}_{l\in\mathcal{L}}\}_{\mathcal{L}}$, where $P_l \in \mathcal{P} = \{pregnant, not\,pregnant, R\}$
$l \in \mathcal{L}$, l : (date of birth, number u of the individual among the individuals born at this date)

$$\mathcal{X}_t(\Omega) \overset{def.}{=} \mathcal{X}_{n_t}(\Omega) \tag{1}$$

$$n_t(\Omega) \overset{def.}{=} \sum_{l\in\mathcal{L}_{n_t-1}(\Omega)} m_{p,l,t} \tag{2}$$

$$m_{p,l,t} \overset{def.}{=} \sup\{m : T_m^{(p,l)} \leq t\},\ l \in \mathcal{L}_{n_t-1}(\Omega) \tag{3}$$

$$\mathcal{X}_{n_t}(\Omega) \overset{def.}{=} \{X_{m_{p,l,t}}^{(B,l)}\}_{l\in\mathcal{L}_{n_t-1}(\Omega)} \tag{4}$$

$$X_{m_{p,l,t}}^{(B,l)} \overset{def.}{=} \{X_{m_{p,l,t}}^{(p,l)} \neq R, \{X_0^{(p,l')} \neq R\}_{l'\in\widetilde{Y}_{n_t,l}}\} \tag{5}$$

$$T_0^{(p,l')} \overset{def.}{=} T_{m_{p,l,t}}^{(p,l)},\ l' \in \widetilde{Y}_{n_t,l},\ l \in \mathcal{L}_{n_t-1}(\Omega) \tag{6}$$

$$\mathcal{T}_{n_t}(\Omega) \overset{def.}{=} \sup_{l\in\mathcal{L}_{n_t-1}(\Omega)} \{T_{m_{p,l,t}}^{(p,l)}\}. \tag{7}$$

$$\mathcal{L}_{n_t}(\Omega) \overset{def.}{=} \{l, \{labels\{\widetilde{Y}_{n_t,l}\}\}\}_{l\in\mathcal{L}_{n_t-1}(\Omega)} \tag{8}$$

Example : if $i_l^p \to j_l^B$: $pregnant \to not\,pregnant$, then

$F^{(B,l)}_{i_l^p|I,j_l^B}(.) = F^{(B,l)}_{pregnant,not\,pregnant}(.)$ (cdf of the pregnancy period),

$P^{(B,l)}(i_l^p|I, j_l^B)$ is the probability for l to give birth to $\widetilde{Y}_{n,l}$ newborns at his next "jump" among the states $\{alive\,with\,\widetilde{Y}\,newborns\}_{\widetilde{Y}}, R\}$

Spread of a disease in a branching population structured in groups

(4), (5), (6) replaced by

$$\begin{aligned}
\mathcal{X}_{n_t}(\Omega) &\overset{def.}{=} \{(X^{(B,l)}_{m_{p,l,t}}, X^{(h,l)}_{m_{h,l,t}}, X^{(g,l)}_{m_{g,l,t}})1_{\{X^{(g,l)}_{m_{g,l,t}}\neq R\}}\}_{l\in\mathcal{L}_{n_t-1}(\Omega)} \\
X^{(B,l)}_{m_{p,l,t}} &\overset{def.}{=} \{X^{(p,l)}_{m_{p,l,t}}, \{(X_0^{(p,l')}, X_0^{(h,l')}, X_0^{(g,l')})1_{\{X_0^{(g,l')}\neq R\}}\}_{l'\in\widetilde{Y}_{n_t,l}}\},\ l\in\mathcal{L}_{n_t-1}(\Omega) \\
T_0^{(c,l')} &\overset{def.}{=} T^{(p,l)}_{m_{p,l,t}},\ l'\in\widetilde{Y}_{n_t,l},\ l\in\mathcal{L}_{n_t-1}(\Omega),\ c\in\{p,h,g\}
\end{aligned}$$

Conclusion

–Individual based models : empirical distributions based on individual simulated trajectories

–Population process : kernel, simulation algorithm, probabilty law, approximated probability law, asymptotic behavior ?

THANK YOU FOR YOUR ATTENTION !