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A viability analysis for the management of forest systems

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Abstract

This paper deals with sustainability in forestry management. We consider a dynamic model of a forest, structured in age and controlled by felling. We require the sequential decisions to provide a guaranteed utility level at any time, through harvesting while maintaining the total surface constant. We use the concept of viability kernel to study the compatibility between the associated dynamic control system and both environmental and economic constraints. We reveal intrinsic conditions of sustainability and, in particular, sustainable resource states and harvesting policies. Analytical results and numerical simulations illustrate the general statements.

Key words : Forestry, Discrete time, Age structured, Sustainability, Viability kernel.

JEL Classification : C61, G23.

1 Introduction

Man's perception of the forest has changed according to circumstances and throughout history. Once regarded as being a source of diseases and predators, and as an impediment to communications, the forested areas diminished in Europe thanks to the increase of the cultivated areas, or more generally the extent of the "civilized" world. Then dwellers became aware of the different advantages linked to the mere presence of trees, a source of food for men and domestic animals, their role in soil and water protection, or more recently the forest functions in the chemical equilibrium of the atmosphere. So they rapidly aimed at maintaining constant the forested areas, and once the surface stabilized, at preserving the trees in an age class structure so that year after year, the demand for goods (and among them for the different wood categories) and services may be satisfied.

This concern about sustainability appeared rather early, in forestry literature as well as in day to day management decisions. The problem was all the more difficult to apprehend than the slow growth of trees may deprive one or a few men generations from goods or services, sometimes even resulting in a relocation of entire communities. Notwithstanding these ultimate consequences, we must realize that financial markets as well as forested land markets are imperfect, and that also some service markets are inconceivable or inexistent.

Legally, forests may be classified according to their private or public ownership. For the former, if we use contemporary vocabulary, management aims at the maximization of the owner's inter-temporal utility. This utility is often directly tied to wood or to other commercial product sales. A correct assessment of the value of forests and an accurate calculation of the optimal harvest time were first put forward by Faustmann (1849), Pressler (1860) and Ohlin (1921), each taking into consideration wood sales alone.

More recently, this model was extended in several directions, for instance by Hartman (1976) to include amenities, or by Reed (1984) to take into account risks inherent to forestry.

But in these more or less detailed formulations, the objective is always to maximize a goods or service flow. One of the major difficulties arises from the necessary assessment of the discount rate. This is probably the principle reason of the development of alternative methods, some of them being fundamentally wrong (as underlined by Samuelson, 1976, and Hirshleifer 1958) such as the maximization of the internal rate of return (Boulding 1935, Goundrey, 1960). Moreover the risks and uncertainties that are inherent in forest management are impossible to totally include in such models; therefore it sharply reduces the reliability of the results and it constantly makes somewhat questionable the operational use of such tools for day to day forest management. This of course does not lessen the value of the referenced works, since each allows to answer to a specific question.

For public forests, the amenities, and also the externalities to the production of commercial goods, may become the major, or, in some cases, the exclusive concerns of the managers. The sustainability of these services is all the more important since generally there are no market which make the transfer of these services from one date to another possible.

That is why sustainability was an important issue in the forest sector well before Malthus's works (1798) on the limitation to development caused by the lack of good quality agricultural land. At the present time, the most often cited definition of sustainability is given by WCED (World Commission on Environment and Development, 1987): "paths of human progress which meet the needs and aspirations of the present generation without compromising the ability of future generations to meet their needs". But as soon as 1988, Norggard denounced the imprecision of such a definition : "With the term meaning something different to everyone, the quest for sustainable development is off to a cacophonous start."

The "dangers" inherent to such a concept are straightforward, in the forest sector at least : Because of the slow growth of the trees, this would imply maintaining certain parameters constant, which would lead to a completely different set of forest values or of forest management instructions, and in particular, to the search of the Maximum Sustainable Yield (MSY, Hartig, 1796). Samuelson (1976) highlighted the consequences of such an approach. The relation between MSY and Faustmann's calculus are now well documented (Terreaux, 1995, Terreaux, 1996 and Rapaport and al., 2000). But the former, even if not put into effect, constitutes for many managers the ideal goal to reach.

Moreover, it is clear that traditional forest management methods, along with most of the concepts linked to sustainability (Faucheux and O'Connor, 1998), give a single optimal trajectory. This solution maximizes the predetermined objective, but we do not know anything about its fragility, for examples, with regards to risks or parameters non explicitly inserted in the models.

Here we do prefer to have recourse to the viability concept. The viability approach consists in the definition of a set of state and decision constraints, that represent the "good health" of the system at any moment, and in the study of conditions which allow these constraints to be satisfied in the future. More generally, the viability approach deals with dynamic systems under state and control constraints (Aubin, 1991). The aim of the viability method is to analyze compatibility between the possibly uncertain dynamics of a system and state or control constraints, and then to determine the set of controls or decisions that would prevent this system from going into crises i.e. from violating these constraints. We refer, for example, to Bene and Doyen (2000), Bene and al.(2001), Bonneuil (1994) or to Tichit and al.(to appear) for applied models in other contexts. Toth and al.(1997) propose a similar framework called the Tolerable Windows Approach which is mainly concerned with climate change issues.

More specifically, in the sustainability context, viability implies the satisfaction of both economic and environmental constraints. In this sense, it is a multi-criteria approach sometimes known as "co-viability". Moreover, since the criteria (the viability constraints) are the same at any moment and the term horizon is infinite, an intergenerational equity feature is naturally integrated within this framework.

Here, using the concept of equivalent surface, we consider an age-structured forest system whose evolution, in discrete time, is controlled by the harvesting of "old" trees. The main environmental constraint is to maintain at a constant level the total surface of the forest. The economic constraint corresponds to a minimal annual harvest volume year after year. Then we use the mathematical concept of viability kernel to characterize the sustainability. Indeed, this kernel is the largest subset of initial resource conditions for which a feasible regime of exploitation exists and satisfies the whole constraints. More specifically, by using the viability kernel, we address the following questions :

- Given a total forest surface, what is the largest minimal harvesting value for which there exists at least one initial resource level (the initial state of the forest) and an associated sustainable policy?

- Given a sustainable harvesting value level, what are the initial resource conditions for which a sustainable policy exists?

- What are the possible sustainable policies associated with these sustainable states?

At this stage, we must confess that our paper does not deal with exogenous uncertainty, risk or stochasticity. We are convinced uncertainty due to silviculture and to economic factors should be taken into account for effective implementation of sustainability. But, for simplicity sake, we limit the present study to the development of a deterministic approach.

The content of the paper is organized as follows. In section 2, we build the model describing both the evolution of a forest and the sustainability constraints. In section 3, we study the viability kernel of the problem and the existing associated viable management policies. Section 4 provides two examples of forest management combining analytical results and numerical simulations. Concluding remarks are given in section 5. Detailed proofs of the mathematical statements can be found in the appendix (section 6).

2 The model

2.1 The endogenous dynamics of the resource

Beforehand we define the concept of equivalent surface (hereafter e-surface) of age j as the area of the forest under the same silvicultural system which bears trees of the same class j of age. In the case of even-aged management, this e-surface is simply the sum of the surfaces bearing j year old trees. In the case of uneven aged management it is the sum of the areas occupied by j year old trees.

We consider a forest which structure in age is represented at discrete time $t \in \mathbb{N}$ by a vector x of $\mathbb{R}^n_+$ (see for instance Johansson and Löfgren, 1985, for such a matrix representation) :

$$x(t) = \begin{pmatrix} x_n(t) \\ x_{n-1}(t) \\ \vdots \\ x_1(t) \end{pmatrix},$$

where $x_j(t)$ $(j = 1 \ldots n-1)$ represents the e-surface bearing trees whose age, expressed in the unit of time used to define t, is $j-1$ at time t. The last layer $x_n(t)$ represents the e-surface older than $n-1$ at time t.

We assume that the natural evolution (i.e. under no exploitation) of the vector $x(t)$ is described by a linear system :

$$x(t+1) = A\,x(t), \quad x(0) = x^0 \tag{1}$$

where the terms of the matrix A are non-negative (which ensures that $x(t)$ stays in $\mathbb{R}^n_+$ at any time). Particular instances of matrices A that we shall consider are of Leslie type :

$$A = \begin{bmatrix} 1-\alpha_n & 1-\alpha_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 1-\alpha_{n-2} & \ddots & 0 \\ & & & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & 1-\alpha_1 \\ \beta_n & \beta_{n-1} & \cdots & \cdots & \beta_1 \end{bmatrix}. \tag{2}$$

where α_i and β_i are respectively mortality and recruitement parameters, belonging to $[0, 1]$.

2.2 The controlled dynamics

Now we describe the exploitation of such a forest resource x. We assume the following main hypotheses :

1. The minimum age at which it is possible to cut trees is n. This assumption is made only in order to simplify the representation. Taking into account wood market values, it would be possible to make more precise but more complicated models.

2. Each time a plot is cut at time t, it is "immediately" (i.e. within the same unit of time) replanted, then bearing trees of age 0.

Thus let us introduce the scalar variable decision $h(t)$ that represents the e-surface harvested at time t. Previous assumptions induce the following controlled evolution :

$$(\mathcal{D}) \; : \quad x(t+1) = A\,x(t) + B\,h(t), \quad x(0) = x^0$$

where B is equal to the column vector $\begin{pmatrix} -1 & 0 & \cdots & 0 & 1 \end{pmatrix}'$.

Furthermore, since one cannot plan to harvest more than will exist at the end of the unit of time, the decision or control variable $h(t)$ is subject to the constraint :

$$(\mathcal{R}) \; : \quad 0 \leq h(t) \leq CA\,x(t), \quad \forall t \geq 0$$

where the raw vector C is equal to $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$, which ensures the non negativity of the resource x.

Thus, we have assumed implicitly that the harvesting decisions $h(t)$ are effective at the end of each unit of time t.

Remark : This assumption is reasonable in forestry management, where we may suppose that trees growth occur mainly during spring time, followed by a possible natural mortality until the end of summer time, and the forest exploitation usually occurs afterwards, during autumn and winter time[1].

2.3 Sustainability

We first consider an ecological requirement by maintaining constant the total surface of the forest :

$$(\mathcal{T}) \; : \quad \sum_{i=1}^{n} x_i(t) = S, \quad \forall t \geq 0.$$

To encompass the economic or social feature of the exploitation, we associate the harvesting $h(t)$ with an income or more generally a utility, and we require this harvesting, or equivalently the revenue or the utility to exceed some minimal threshold $\underline{h} > 0$ at any time :

$$(\mathcal{S}) \; : \quad h(t) \geq \underline{h}, \quad \forall t \geq 0.$$

Under this context, we propose the following definition of sustainable policy :

[1] For other kind of natural resources, such as water tables, it can be assumed that the extraction decisions are rather effective at the beginning of each unit of time. In this case, the modeling of $(\mathcal{D})$ and $(\mathcal{R})$ are : $x(t+1) = Ax(t) + ABh(t)$ *with* $h(t) \in [0, Cx(t)]$.

Definition 1 : Sustainable policy. Given an initial resource $x(0) = x^0$ and a minimal harvesting threshold $\underline{h}$, a harvesting policy $h(0), h(1), \cdots$ is $\underline{h}$-sustainable from x^0 if the policy $h(.)$, combined with $x(.)$ the solution of dynamics $(\mathcal{D})$ starting from x^0, fulfills the constraints $(\mathcal{R})$, $(\mathcal{T})$ and $(\mathcal{S})$.

We also derive the following definition of sustainable resource initial condition.

Definition 2 : Sustainable resource. Given a minimal harvesting threshold $\underline{h}$, an initial resource x^0 is $\underline{h}$-sustainable if there exists a $\underline{h}$-sustainable harvesting policy from x^0.

3 A sustainability analysis

The previous definition and main questions mentioned in the introduction can be reformulated using the mathematical viability toolbox. In particular, let us denote by $\mathrm{Viab}_{\underline{h}}$ the viability kernel associated with dynamics $(\mathcal{D})$ and constraints $(\mathcal{R})$, $(\mathcal{S})$ namely the largest set of sustainable initial conditions :

$$\mathrm{Viab}_{\underline{h}} = \left\{ x^0 \in \mathbb{R}^n_+ \mid \text{there exists a } \underline{h}\text{-sustainable policy from } x^0 \right\}$$

In this way we reformulate the questions introduced in section 1 :

1. Find the largest value $\underline{h}^*$ for which $\mathrm{Viab}_{\underline{h}^*} \neq \emptyset$.
2. Compute the viability kernel $\mathrm{Viab}_{\underline{h}}$ for $\underline{h} \leq \underline{h}^*$.
3. Characterize the harvesting policies $\underline{h}$-sustainable from $x^0 \in \mathrm{Viab}_{\underline{h}}$.

3.1 Notations and assumptions

To provide some results related to sustainability and in particular to characterize the viability kernel, we need to introduce the following felling threshold (when it is well-defined, see below the remark after Assumption A2) :

$$\underline{h}^*(S, A) = \frac{S}{\mathbb{1}' \mathcal{O}^{-1}(\mathbb{I} - T\mathcal{O}AB)} \tag{3}$$

where the matrices T and $\mathcal{O}$, and the vector $1\!\!1$ are respectively defined by :

$$T = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}, \ \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ and } 1\!\!1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We need to consider the following assumptions on the matrices A and B :

Assumption A1. The matrix $A = [a_{ij}]$ is such that :

$$\sum_{k=1}^{n} a_{kj} = 1, \quad \forall j = 1 \cdots n.$$

Remark: Assumption A1 guarantees that the constraint $(\mathcal{T})$ is fulfilled for any control law that fulfills $(\mathcal{R})$ (see Lemma 1 in the Appendix).

Assumption A2. The matrix A is such that :

i. The matrix $\mathcal{O}$ is full rank,

ii. The vector $1\!\!1'\mathcal{O}^{-1}$ is non-negative,

iii. The vector $\mathcal{O}AB$ is non-positive .

Remark : Assumption A2 implies the positivity of the scalar $1\!\!1'\mathcal{O}^{-1}(1\!\!1 - T\mathcal{O}AB)$. Thus, $\underline{h}^*(S, A)$ is well defined and positive.

3.2 Necessary conditions of sustainability

We obtain the following result related to the vacuity of the viability kernel and hence to question 1. The proof can be found in the Appendix.

Proposition 1. Assume that the resource evolution matrix A satisfies assumptions A1 and A2. Consider the total amount of resource $S > 0$ and define the harvesting level $\underline{h}^*(S, A)$ as in equation (3). Then, for any minimal harvesting level $\underline{h}$ greater than $\underline{h}^*(S, A)$, there does not exist an initial level resource x^0 and a sustainable policy from x^0. In other words,

$$\underline{h} > \underline{h}^*(S, A) \Longrightarrow \mathrm{Viab}_{\underline{h}} = \emptyset.$$

3.3 Sufficient conditions of sustainability

We now deal with question 2 and attempt at computing the viability kernel $\mathrm{Viab}_{\underline{h}}$ namely sustainable initial resource values. We consider an additional hypothesis on the matrix A :

Assumption A3. The matrix resource A is such that

i. $CA^{n+1} = CA^n$

ii. $CA^nB = 0$

We obtain (the proof can be found in the Appendix) :

Proposition 2. We posit assumptions A1, A2 and A3 on the resource matrix A, and consider $\underline{h} \in [0, \underline{h}^*(S, A)]$. Then the viability kernel is defined by

$$\mathrm{Viab}_{\underline{h}}(S, A) = \left\{x \in \mathbb{R}^n_+ \mid \mathbb{1}'x = S \text{ and } \mathcal{O}Ax \geq (\mathbb{1} - \mathcal{T}\mathcal{O}AB)\underline{h}\right\}.$$

Remark : The linear inequalities contained in the condition $\mathcal{O}Ax \geq (\mathbb{1} - \mathcal{T}\mathcal{O}AB)\underline{h}$ advocate for the anticipation of potential crisis $CAx(t) < \underline{h}$. Indeed, these additional inequalities (the first one is $CAx(t) \geq \underline{h}$) imposes other constraints on $x(t)$ to maintain sustainability in the long term horizon (see in particular the examples in section 4).

3.4 Sustainable policies

The viability kernel revealed the states of the forest compatible with the constraints. The present step is to compute the sustainable management options (decision or control) associated with it.

3.4.1 Viable feedbacks

We consider viable policies $h(x)$ that can maintain viability for a forestry *e-surface* vector x. At any point x belonging to the viability kernel $\mathrm{Viab}_{\underline{h}}$, we know that this regulation set is not empty, under the conditions given by proposition 2.

Proposition 3. We posit assumptions A1, A2 and A3 on the resource matrix A. We consider $\underline{h} \in [0, \underline{h}^*(S, A)]$ and any $x \in \mathrm{Viab}_{\underline{h}}(S, A)$. The sustainable policies are given by the solutions $h(x)$ of the linear inequalities :

$$\begin{cases} h & \geq \underline{h} \\ h & \leq CAx \\ \mathcal{O}ABh & \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h} - \mathcal{O}A^2x \end{cases} \tag{4}$$

The proof of this proposition can be found in Appendix.

Remark : From the calculation of viable feedbacks $h(x)$ by (4), it appears that the only mandatory unique regulation occurs exactly when

$$\mathbb{I}'x = S \text{ and } \mathcal{O}Ax = (\mathbb{I} - T\mathcal{O}AB)\underline{h}$$

In that case, the choice reduces to $\underline{h}$. For any other situations within the viability kernel, several viable regulations and sustainable policies are possible. This means that policy-makers are offered different viable alternatives which extends their flexibility with respect to the multiple, evolving (or sometimes conflicting) objectives they attempt to achieve for the forestry management. This result may represent an improvement with respect to others approaches where only one solution is usually proposed.

Two kinds of viable regulations have been used in the simulations described in the next section : the *maximal* and *inertial* viable harvestings :

3.4.2 Maximal viable harvesting

The feedback h_M consists in choosing at any $x \in \mathrm{Viab}_{\underline{h}}$ the largest value h allowed by the conditions (4) :

$$h_M(x) = \max_{h \in [0, CAx]} \left\{ h \,|\, \mathcal{O}ABh \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h} - \mathcal{O}A^2x \right\}.$$

3.4.3 Inertial viable harvesting

Given the resource state $x \in \mathrm{Viab}_{\underline{h}}$ and a current felling level h_c, the inertial viable regulation h_I consists in choosing a viable control h minimizing decision's change $|h - h_c|$:

$$h_I(x, h_c) = \arg \min_{h \in [\underline{h}, h_M(x)]} |h - h_c|.$$

Particular cases of inertial regulations are the constant ones :

3.4.4 Constant viable harvesting

Under a stronger assumption than assumption A3 :

Assumption A3b. The matrix resource A is such that

i. $CA^n = CA^{n-1}$.

ii. $CA^{n-1}B = 0$,

we have the following result, whose proof can be found in the appendix :

Proposition 4. Under Assumptions A1 and A2 and A3b, for any $\underline{h} \in [0, \underline{h}^*(S, A)]$ and from any initial state belonging to $\mathrm{Viab}_{\underline{h}}$, the constant control $\underline{h}$ is sustainable, and the trajectory converges asymptotically towards the steady state :

$$x^s(\underline{h}) = \mathcal{O}^{-1} \left[\frac{S + \mathbb{1}'\mathcal{O}^{-1}TOB\underline{h}}{\mathbb{1}'\mathcal{O}^{-1}\mathbb{1}} \mathbb{1} - TOB\underline{h} \right]$$

4 Examples

We illustrate now our sustainability analysis on two particular cases of trees population whose evolution is described by a matrix A of Leslie type.

4.1 The eternal population

We consider first the particular case where the matrix A represents an "eternal" population (i.e. without a natural mortality) :

$$A = \begin{bmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

Remark : This idealistic case can be seen as a modeling of a forest whose life expectation of trees is very large compared to the minimum age of cut.

- **Largest minimal sustainable harvesting threshold :** It is straightforward to compute :

$$\mathcal{O} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 1 & & 1 \end{bmatrix}, \quad \mathcal{O}^{-1} = \begin{bmatrix} 1 & & & 0 \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{bmatrix}.$$

$$\mathbb{1}'\mathcal{O}^{-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}' \text{ and } \mathcal{O}AB = \begin{pmatrix} -1 \\ \vdots \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Thus, Assumption A2 of Proposition 1 is fulfilled. Then, the computation gives the largest sustainable value :

$$\underline{h}^*(S, A) = \frac{S}{n-1}. \tag{5}$$

- **Viability kernel and sustainable states :** Furthermore, we have :

$$\begin{cases} CA^{n-1} = \mathbb{1}' \Rightarrow CA^n B = (CA^{n-1})AB = \mathbb{1}'AB = \mathbb{1}'B = 0 \\ CA^n = \mathbb{1}' \Rightarrow CA^{n+1} = CA^n \end{cases}$$

So Assumption A3 is fulfilled. Then, Proposition 2 allows to characterize exactly the viability kernels :

$$
\begin{aligned}
\mathrm{Viab}_{\underline{h}}(S,A) &= \left\{ x \in \mathcal{X}_S^n \mid \mathcal{O}Ax \geq \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n-1 \\ n-1 \end{pmatrix} \underline{h} \right\} \\
&= \left\{ x \in \mathcal{X}_S^n \;\middle|\; \begin{array}{ll} x_n + x_{n-1} & \geq \underline{h} \\ \vdots & \\ \sum_{i=1}^n x_i & \geq (n-1)\underline{h} \end{array} \right\}.
\end{aligned}
$$

Remark : We can see at once that particular sustainable states of the forest are :

- The constant forest structure :

$$x = \left(0, \frac{S}{n-1}, \dots, \frac{S}{n-1}\right),$$

namely we have a uniformed repartition according to the age structure.

- Non-constant forest structures of the form :

$$x = \left(0, \frac{S}{n-1}, \dots, \frac{S}{n-1} + \epsilon, \frac{S}{n-1}, \dots, \frac{S}{n-1} - \epsilon, \frac{S}{n-1}, \dots, \frac{S}{n-1}\right)$$

where $\epsilon > 0$, and any combination of such states satisfying the positivity of the e-surface vector.

- **Sustainable harvesting feedbacks :** Using Proposition 3, sustainable feedbacks $h(x)$ are defined for any $x \in \mathrm{Viab}_{\underline{h}}$ by the set-membership :

$$h(x) \in \left[\underline{h}, \min_{i=1\cdots n-1} \left\{ \left(\sum_{l=i}^{n} x_l \right) - (n-1-i)\underline{h} \right\}\right]. \tag{6}$$

(the condition $h(x) \leq CAx$ has been integrated in this formula).

- **Simulations :** We have considered for simulations a population structured in five layers with a total e-surface equal to $S = 20$ units. Then, according to (5), the largest sustainable value is :

$$\underline{h}^*(S, A) = \frac{20}{4} = 5.$$

We have chosen for the simulations $\underline{h} = 4$ as a desired sustainable value, and have simulated the trajectories provided by two kinds of selection of feedbacks, that fulfill condition (6) :

i) The maximal viable harvesting, which consists in choosing for $h(x)$ the largest value allowed by (6) :

$$h_M(x) = \min_{i=1\cdots n-1}\left\{\left(\sum_{l=i}^{n} x_l\right) - (n-1-i)\underline{h}\right\}$$

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
x_4	6.0	5.0	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0
x_3	5.0	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0
x_2	3.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0
x_1	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0
h_M	8.0	4.0	4.0	4.0	8.0	4.0	4.0	4.0	8.0	4.0	4.0

We notice that these viable trajectories becomes $(n-1)$-cyclic in finite time.

ii) The inertial viable harvesting, which consists in choosing a feedback minimizing the change, at time t :

$$h_I(x, h(t-1)) = \min_{h \in [\underline{h}, h_M(x)]} |h - h(t-1)|, \quad t \geq 1$$

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	3.5	4.0	2.5	2.0	2.0	2.0	2.0	2.0	2.0	2.0
x_4	6.0	5.0	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_3	5.0	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_2	3.0	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
x_1	4.0	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5
h_I	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5	4.5

We notice here that those viable trajectories becomes stationary in finite time, contrary to the maximal viable harvesting case.

4.2 The almost eternal population

We consider now perturbations of this previous model, for taking into account a natural mortality effect. At each unit of time, we assume that a e-surface of trees αS disappear, where $\alpha \in (0,1)$. We also assume that the dead trees (which are supposed to have no value on the market) are replaced by young ones in order to preserve the total extent of e-surface (which respects the assumption A1). We must be aware that here α is a deterministic parameter and not a probability of death, as for instance in Reed (1984). The resource matrix is then :

$$A_\alpha = \begin{bmatrix} 1-\alpha & 1-\alpha & 0 & & 0 \\ & & 1-\alpha & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 1-\alpha \\ \alpha & \cdots & \cdots & \cdots & \alpha \end{bmatrix}, \quad \alpha \in (0,1).$$

- Largest minimal sustainable harvesting threshold : An easy computation gives :

$$\mathcal{O}_\alpha = \begin{bmatrix} 1 & & & 0 \\ 1-\alpha & 1-\alpha & & \\ \vdots & & & \\ (1-\alpha)^{n-1} & \cdots & \cdots & (1-\alpha)^{n-1} \end{bmatrix}, \quad \mathbb{I}'\mathcal{O}_\alpha^{-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/(1-\alpha)^{n-1} \end{pmatrix} \text{ and}$$

$$\mathcal{O}_\alpha A_\alpha B = \begin{pmatrix} -(1-\alpha) \\ \vdots \\ -(1-\alpha)^{n-2} \\ 0 \\ 0 \end{pmatrix}.$$

So, conditions of Proposition 1 are fulfilled, and the maximal sustainable value is :

$$\underline{h}^*(S, A_\alpha) = S\frac{(1-\alpha)^{n-1}}{\sum_{i=0}^{n-2}(1-\alpha)^i} = S\frac{\alpha(1-\alpha)^{n-1}}{1-(1-\alpha)^{n-1}}. \tag{7}$$

and for small α, we have[2] :

$$\underline{h}^*(S, A_\alpha) \simeq \frac{S}{n-1}\left(1 - \frac{n}{2}\alpha\right).$$

- Viability kernel : Furthermore, we have :

$$\begin{cases} CA_\alpha^{n-1} = (1-\alpha)^{n-1}1\!\!1' \Rightarrow CA_\alpha^n B = 0 \\ CA_\alpha^n = (1-\alpha)^{n-1}1\!\!1' \Rightarrow CA_\alpha^{n+1} = CA_\alpha^n \end{cases}$$

and Assumption A3 is fulfilled (notice that Assumption A3b is also fulfilled), hence the characterization :

$$\begin{aligned} \mathrm{Viab}_{\underline{h}}(S, A_\alpha) &= \left\{ x \in \mathcal{X}_S^n \mid \mathcal{O}_\alpha A_\alpha x \geq \begin{pmatrix} 1 \\ \vdots \\ \sum_{k=0}^{i-1}(1-\alpha)^k \\ \vdots \end{pmatrix} \underline{h} \right\} \\ &= \left\{ x \in \mathcal{X}_S^n \;\middle|\; \begin{array}{ll} x_n + x_{n-1} & \geq \frac{\underline{h}}{1-\alpha} \\ \vdots & \\ \sum_{l=n-i}^{n} x_l & \geq \frac{1-(1-\alpha)^i}{\alpha(1-\alpha)^i}\underline{h} \\ \vdots & \end{array} \right\}. \end{aligned}$$

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[2] Moreover, an easy computation gives : $d\underline{h}^*(S, A_\alpha)/d\alpha = \frac{S(1-\alpha)^{n-2}[1-n\alpha-(1-\alpha)^n]}{[1-(1-\alpha)^{n-1}]^2} < 0, \ \forall \alpha \in (0,1)$. So, $\alpha \rightarrow \underline{h}^*(S, A_\alpha)$ is a decreasing function w.r.t. α.

- **Sustainable harvesting feedbacks :** Using proposition 3, sustainable feedbacks $h(x)$ are defined for any $x \in \mathrm{Viab}_{\underline{h}}$ by the set-membership :

$$h(x) \in \left[\underline{h}, \min_{i=0\cdots n-2}\left\{\left((1-\alpha)\sum_{l=n-i-1}^{n} x_l\right) - \frac{1-(1-\alpha)^i}{\alpha(1-\alpha)^i}\underline{h}\right\}\right]. \tag{8}$$

(the condition $h(x) \leq CAx$ has been integrated in this formula).

- **Simulations :** We have run simulations for populations with the same characteristics as those of the previous example ($n = 5$ and $S = 20$), but with $\alpha = 0.07$. The largest sustainable value, computed from (7), is :

$$\underline{h}^*(S, A_\alpha) \simeq 4.2.$$

We have chosen the same level of sustainability : $\underline{h} = 4$ and simulated trajectories from the same initial conditions :

i) Using maximal viable harvesting, provided by (8) :

t	0	1	2	3	4	5	6	7	8	9	10	$\cdots$	$+\infty$
x_5	2.0	2.7	2.9	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	$\rightarrow$	0.0
x_4	6.0	4.6	2.6	3.2	5.0	4.3	4.3	4.3	4.8	4.4	4.4	$\rightarrow$	4.5
x_3	5.0	2.8	3.5	5.3	4.7	4.7	4.7	5.2	4.7	4.7	4.7	$\rightarrow$	4.8
x_2	3.0	3.7	5.7	5.0	5.0	5.0	5.5	5.0	5.0	5.0	5.5	$\rightarrow$	5.2
x_1	4.0	6.1	5.4	5.4	5.4	6.0	5.4	5.4	5.4	5.9	5.5	$\rightarrow$	5.6
h_M	4.7	4.0	4.0	4.0	4.6	4.0	4.0	4.0	4.5	4.0	4.0	$\rightarrow$	4.2

We notice that viable trajectories do not converge in finite time towards a cyclic trajectory, but converge asymptotically towards a stationary state, which is necessarily equal to $x^s(\underline{h}^*(S, A_\alpha))$, according to Proposition 4.

ii) Using inertial viable harvesting that fulfill (8) :

t	0	1	2	3	4	5	6	7	8	9	10
x_5	2.0	3.0	2.9	1.0	0.0	0.41	0.56	0.56	0.56	0.56	0.56
x_4	6.0	4.6	2.6	3.2	4.7	4.5	4.3	4.3	4.3	4.3	4.3
x_3	5.0	2.8	3.5	5.1	4.8	4.7	4.7	4.7	4.7	4.7	4.7
x_2	3.0	3.7	5.5	5.2	5.0	5.0	5.0	5.0	5.0	5.0	5.0
x_1	4.0	6.0	5.6	5.4	5.4	5.4	5.4	5.4	5.4	5.4	5.4
h_I	4.5	4.2	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0

We notice that these viable trajectories become stationary in finite time, as for the eternal populations case, but towards different steady states.

5 Concluding remarks

In this study, we attempt to shed new light on the sustainable management of forestry systems. Our objective departs from the main body of work in forest economics, which usually aims at optimizing an objective and/or stabilizing some production of different economic goods. Our aim is to provide "easy to use" economic policies, depending on the state of the forest, so that it may fulfill the future needs of society. Our sole objective is the satisfaction of both environmental and economics constraints that represent the "good health" of the system, i.e. its viability. Intergenerational equity is taken into account since the criteria induced by the viability constraints are the same at any moment and the horizon is infinite.

The analytical and numerical results obtained constitute the first step towards the construction of managerial rules, allowing the foresters to adapt to the hazards and more generally to the (positive or negative) risks related to silviculture. Indeed, the sustainable policies suggested leave the manager some room for manoeuvre, as long as the minimal production level is fulfilled. This approach must be compared with optimal trajectories where we would have had no information on their stability with regard to the many parameters. Of course, if we gradually increase the value of this viable minimum threshold, we eventually recover some classic management criteria (for instance Hartig's criterion), which consists in maximizing the value of a constant perpetual harvest. Moreover the manager's room for manoeuvre, when there is some, may be used in order to maximize the discounted value of the production, within the bounds of his constraints. If in the last case the value of the minimum is sufficiently low, we may implement the Faustmann's and Pressler's forest value maximization criterion. So the method we present is more general in the sense that it may encompass these two solution families. In addition, if the constraint values are neither too low nor too high, it may give a solution halfway between these two sets. This solution is not dogmatic in the sense that it gives the manager some flexibility, which he can use in order to adapt to the different risks which may affect the forest, its products or the interested society.

Moreover, forests are not original entities, with specificities we would not meet in other production systems. One consequence is that even if many properties are not studied in the context of other production systems, since they do not seem of prime importance, some results of forest economics are relevant for other systems. This means in particular that our results may apply to the management of other natural resources (see for instance Clark, 1990) on an infinite horizon (Carlson and al., 1991), or to the resolution of a whole set of investment problems, insofar as there exists a delay between a command variable and a part of its effects on the state variables.

Of course there remains a lot of work to be done. First, we should tackle exogenous uncertainty since we are convinced that silvicultural and economic uncertainties are basic features to take into account for effective implementation of sustainability. However it is clear that the search for sustainability is indirectly a way of diversification against natural or economic hazards. Moreover we think it would be possible to elaborate stochastic or uncertain models showing explicitly this effect. Indeed, extensions of the viability kernel notion to the uncertain context do make sense.

On the other hand, a lot of work has to be done in order to translate this viability method into practical rules. Moreover all the viable trajectories do not hold the same interest since some of them will for instance lead to an over-capitalisation in old trees, which is not necessary good for biodiversity, or from an economic point of view. But we are confident that our approach can make a valuable contribution to other prevailing management tools.

6 Appendix

For simplicity of notation, we define the set :

$$X_S^n = \left\{ x \in \Re_+^n \ / \ \mathbb{1}'x = S \right\}$$

Lemma 1. Under assuption A1, for any positive number S, the set X_S^n is invariant under the dynamics $(\mathcal{D})$, for any control law h() that fulfill $(\mathcal{R})$.

Proof : Write $h(t) = u(t)CAx(t)$, where $u(t)$ is a new control variable. The dynamics of the evolution $(\mathcal{D})$ can then be written as follows :

$$(\mathcal{D}) \quad : \quad x(t+1) = (1 + BCu(t))Ax(t) = (1 - u(t))Ax(t).$$

The state dependent constraint on $h(t)$ is then equivalent to the constraint on $u(t)$:

$$(\mathcal{R}) \quad : \quad u(t) \in [0, 1].$$

Then, under Assumption A1, the matrix $(1 - u(t))A$ has non-negative elements, thus the invariance of $\mathbb{R}_+^n$ by the dynamics $(\mathcal{D})$.

The vector B is such that $\mathbb{1}'B = 0$ and, by assumption A1, we have $\mathbb{1}'A = \mathbb{1}'$ thus :

$$\mathbb{1}'x(t+1) = \mathbb{1}'Ax(t) + \mathbb{1}'Bh(t) = \mathbb{1}'x(t) = S.$$

and we conclude the invariance of $\mathcal{X}_S^n$ by $(\mathcal{D})$.

6.1 Proof of Proposition 1

Assume that $\text{Viab}_{\underline{h}}$ is non empty and consider an initial condition $x^0 \in \text{Viab}_{\underline{h}}$. Then there exist $h(.)$ and $x(.)$ satisfying the constraints $(\mathcal{R})$, $(\mathcal{S})$ and $(\mathcal{D})$. In particular, we should have for the n first units of time :

$$\left\{ \begin{array}{lcll} CAx(0) & = & CAx^0 & \geq \ \underline{h} \\ CAx(1) & = & CA^2x^0 + CABh(0) & \geq \ \underline{h} \\ \vdots & & & \quad \vdots \\ CAx(n-1) & = & CA^nx^0 + CA^{n-1}Bh(0) + \cdots + CABh(n-2) & \geq \ \underline{h} \end{array} \right. \tag{9}$$

Since, by assumption A2, the numbers $CAB, \cdots, CA^{n-1}B$ are non-positive, then the inequalities (9) combined with $h(t) \geq \underline{h}$ yields :

$$\begin{cases} CAx^0 & \geq \underline{h} \\ CA^2x^0 + CAB\underline{h} & \geq \underline{h} \\ \vdots & \vdots \\ CA^nx^0 + CA^{n-1}B\underline{h} + \cdots + CAB\underline{h} & \geq \underline{h} \end{cases} \tag{10}$$

which is equivalent to write with the help of the matrix representation :

$$\mathcal{O}Ax^0 + T\mathcal{O}AB\underline{h} \geq \mathbb{I}\underline{h}.$$

Moreover, by assumption A2, each component of $\mathbb{I}'\mathcal{O}^{-1}$ are non-negative. Consequently, left multiplying the two members of this last inequality by the raw vector $\mathbb{I}'^{-1}$ induces :

$$\mathbb{I}'Ax^0 \geq \mathbb{I}'\mathcal{O}^{-1}(\mathbb{I} - T\mathcal{O}AB)\underline{h}.$$

Since $\mathbb{I}'A = \mathbb{I}'$ (by Assumption A1), $\mathbb{I}'x^0 = S$ and $\mathbb{I}'\mathcal{O}^{-1}(\mathbb{I} - T\mathcal{O}AB)$ is positive (by Assumption A2). So,

$$\underline{h} \leq \frac{S}{\mathbb{I}'\mathcal{O}^{-1}(\mathbb{I} - T\mathcal{O}AB)}$$

thus the inequality $\underline{h} \leq \underline{h}^*(S, A)$ holds true.

6.2 Proof of Proposition 2

Consider the set $\mathcal{V} = \{x \in \mathcal{X}_S^n \mid \mathcal{O}Ax \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}\}$.

(a) First let us prove that $\mathrm{Viab}_{\underline{h}} \subset \mathcal{V}$.

Consider $x^0 \in \mathrm{Viab}_{\underline{h}}$. As in the proof of proposition 1, a necessary condition for x^0 to belong to $\mathrm{Viab}_{\underline{h}}$ is to fulfill the inequalities (10) or equivalently :

$$\mathcal{O}Ax^0 \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}.$$

Consequently, $x^0 \in \mathcal{V}$.

(b) Now we claim that $\mathcal{V} \subset \mathrm{Viab}_{\underline{h}}$.

From the first inequality of $\mathcal{O}Ax \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}$, we derive that $CAx \geq \underline{h}$. Thus we deduce that

$$\mathcal{V} \subset \mathcal{K} = \{x \in \mathcal{X}_S^n \mid CAx \geq \underline{h}\}.$$

Now let us prove that $\mathcal{V}$ is a viable set.

First, by Lemma 1, the set $\mathcal{X}_S^n$ is invariant by the dynamics $(\mathcal{D})$, for any control law $h(.)$ that fulfills the constraint $(\mathcal{R})$.

Now, consider any $x \in \mathcal{X}_S^n$ such that $\mathcal{O}Ax \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}$. Then

$$\mathcal{O}A(Ax + B\underline{h}) + T\mathcal{O}AB\underline{h} = \begin{pmatrix} CA^2x + CAB\underline{h} \\ \vdots \\ CA^nx + CA^{n-1}B\underline{h} + \cdots + CAB\underline{h} \\ CA^{n+1}x + CA^nB\underline{h} + \cdots + CAB\underline{h}. \end{pmatrix}.$$

The $n-1$ last inequalities of $\mathcal{O}Ax \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}$ provide the $n-1$ inequalities :

$$(\mathcal{O}A(Ax + B\underline{h}) + T\mathcal{O}AB\underline{h})_i \geq \underline{h}, \quad (1 \leq i < n-1).$$

Assumptions A3 ensures then the last inequality :

$$\begin{aligned}(\mathcal{O}A(Ax + B\underline{h}) + T\mathcal{O}ABh)_n &= CA^{n+1}x + (CA^nB + \cdots + CAB)\underline{h} \\ &= CA^nx + (CA^{n-1}B + \cdots + CB)\underline{h} \\ &\geq \underline{h}\end{aligned}$$

Then, we conclude that

$$\mathcal{O}A(Ax + B\underline{h}) + T\mathcal{O}AB\underline{h} \geq \mathbb{I}\underline{h},$$

or that $Ax + B\underline{h} \in \mathcal{V}$. So, with the constant control $h \equiv \underline{h}$, the trajectory stays inside $\mathcal{V}$. Therefore $\mathcal{V}$ is viable in $\mathcal{K}$, and consequently $\mathcal{V} \subset \mathrm{Viab}_{\underline{h}}$ since the viability kernel is the largest viable set contained in $\mathcal{K}$.

6.3 Proof of Proposition 3

The sustainable policies $h \in [0, CAx]$ at $x \in \mathrm{Viab}_{\underline{h}}$ have to fulfill the constraint $(\mathcal{S})$ and to ensure that the next state $Ax + Bh$ belongs also to $\mathrm{Viab}_{\underline{h}}$, which is equivalent to require that

$$\mathcal{O}A(Ax + Bh) \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h}$$

or that

$$\mathcal{O}ABh \geq (\mathbb{I} - T\mathcal{O}AB)\underline{h} - \mathcal{O}A^2x$$

thus the linear inequalities (4).

6.4 Proof of Proposition 4

Fix $\underline{h} \geq 0$ and make the change of variable :

$$\begin{aligned} z(0) &= \mathcal{O}x^0 - (\mathbb{I} - T\mathcal{O}B)\underline{h}, \\ z(t) &= \mathcal{O}x(t) - (\mathbb{I} - T\mathcal{O}B)\underline{h} \quad (t > 0) \end{aligned}$$

Then, the dynamics of $(\mathcal{D})$ becomes

$$\begin{aligned} z(t+1) &= \mathcal{O}[Ax(t) + Bh(t)] - (\mathbb{I} - T\mathcal{O}B)\underline{h} \\ &= \mathcal{O}[A\mathcal{O}^{-1}(z(t) + (\mathbb{I} - T\mathcal{O}B)\underline{h}) + Bh(t)] - (\mathbb{I} - T\mathcal{O}B)\underline{h} \\ &= \mathcal{O}A\mathcal{O}^{-1}z(t) + \mathcal{O}Bh(t) + (\mathcal{O}A\mathcal{O}^{-1} - Id)(\mathbb{I} - T\mathcal{O}B)\underline{h} \end{aligned} \tag{11}$$

Name $\tilde{A} = \mathcal{O}A\mathcal{O}^{-1}$, then

$$\mathcal{O}A = \tilde{A}\mathcal{O} \Longrightarrow \begin{pmatrix} CA \\ \vdots \\ CA^n \end{pmatrix} = \tilde{A} \begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

By Assumption A3b, we have $CA^n = CA^{n-1}$, so

$$\tilde{A} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 1 \end{pmatrix}.$$

We notice that

$$\tilde{A}\,\mathbb{I} = \mathbb{I}, \tag{12}$$

$$\tilde{A}^k = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \forall k \geq n-1, \tag{13}$$

and

$$(\tilde{A} - Id)T = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Due to Assumption A3b, the last component of $\mathcal{O}B$ is always null. So, we have also

$$(\tilde{A} - Id)T\mathcal{O}B = \mathcal{O}B. \tag{14}$$

Using the equalities (12) and (14), equation (11) can be rewritten as follows :

$$z(t+1) = \tilde{A}z(t) + \mathcal{O}B(h(t) - \underline{h}). \tag{15}$$

Using (12), the equation (15) admits $k\mathbb{1}$ ($k \in \mathbb{R}$) as fix points, when the control $h(t)$ is constantly equal to $\underline{h}$. This is also equivalent to claim that $x^e(k) = \mathcal{O}^{-1}(k\,\mathbb{1} - T\mathcal{O}B\underline{h})$ are fix points for $(\mathcal{D})$ under the constant control $\underline{h}$. We prove now that $x^s(\underline{h})$ is the only fix point belonging to $\mathcal{X}^n_S$:

First, we have :

$$\mathbb{1}'x^e(k) = S \implies \mathbb{1}'\mathcal{O}^{-1}(k\,\mathbb{1} - T\mathcal{O}B\underline{h}) = S \tag{16}$$

By Assumption A2, the vector $\mathbb{1}'\mathcal{O}^{-1}$ is non-negative (and non-null because $\mathcal{O}^{-1}$ is non-singular). So $\mathbb{1}'\mathcal{O}^{-1}\mathbb{1} \neq 0$ and then equation (16) possesses a unique solution $k = \tilde{k}$:

$$\tilde{k} = \frac{S + \mathbb{1}'\mathcal{O}^{-1}T\mathcal{O}B\underline{h}}{\mathbb{1}'\mathcal{O}^{-1}\mathbb{1}} \implies x^e(\tilde{k}) = x^s(\underline{h}).$$

We show now that $x^s(\underline{h}) \in \mathbb{R}^n_+$ when $\underline{h} \in [0, \underline{h}^*(S, A)]$.

Take $x^0 \in \mathcal{X}^n_S$. By assumption A1, as in the proof of lemma 1, the solution of $(\mathcal{D})$ with any control law (which does not necessarily fulfill $(\mathcal{R})$) satisfies $\mathbb{1}'x(t) = S$ at any time (but $x(t)$ does not necessarily stay non-negative). The eigenvalues of $\tilde{A}$ being 0 (with multiplicity $n-1$) and 1 (whose associated eigenvector is $\mathbb{1}$), we deduce that the solution of (15) converges necessarily towards the vector $\tilde{k}\,\mathbb{1}$ when the control $h(t)$ is constant equal to $\underline{h}$, which is equivalent to claim that the trajectory $x(t)$ converges towards $x^s(\underline{h})$.

When $\mathrm{Viab}_{\underline{h}}$ is not empty, take $x^0 \in \mathrm{Viab}_{\underline{h}}$. As in the proof of Proposition 2, with the constant control $\underline{h}$, the solution $x(t)$ stays inside $\mathrm{Viab}_{\underline{h}}$. We conclude then that $x^s(\underline{h}) \in \mathcal{X}_S^n$.

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