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1 Introduction

Many auctions are sequential. The same type of object is sold repeatedly during a short time period: cars, apartments, cases of wines, fish, agricultural products, etc. The winning bids of these sequential auctions have often marked intra-day dynamics. In his description of wine auctions, Ashenfelter (1989) noticed a so called declining price anomaly: winning prices decrease during the day. In our data, winning bids of descending or Dutch auctions for eggplants, exhibit a regular inverse U-shape (see Table 1).

Under the usual assumptions of risk neutral bidders, private and independent values, symmetry, and bidders desiring at most one unit of the auctioned commodity, it is well known that winning bids should follow a martingale process, both for first-price or second-price auctions. This model is not rich enough to accomodate those interesting intra-day dynamics.

McAfee and Vincent (1993) have shown how risk aversion of bidders may explain declining prices, at least when the index of absolute risk aversion is non decreasing. Weber (1983) proved that, with affiliated values, the sequence of winning prices displays an upward lift.

To accomodate the inverse U-shape of the intra-day dynamics in our data set, we develop a theoretical model of sequential first-price auctions where bidders are risk-averse and values are affiliated. For constant risk-aversion utility functions and a particular specification of affiliation, we are able to obtain closed-form solutions for the symmetric equilibrium of a sequence of k first-price auctions. This model is able to generate complex intra-day dynamics, in particular inverse U-shape series of winning bids.

In a descending-price auction, the auctioneer progressively decreases his price until a buyer accepts to pay the current price.[1] This type of auction is often used for selling

[1]Strategically, descending price-auctions are similar to first-price auctions. In the context of se-

perishable goods such as flowers in Holland, fish, tobacco, and other agricultural goods. Often, a sequence of such auctions are organized within a few minutes, to sell a certain number of identical units of a good or identical lots of a given commodity. In these sequential auctions, one can observe a succession of winning prices.

We have such data for a large number of sequential auctions of eggplants. A non-linear least square estimation of a simple model explaining the distributions of bidders' valuations enables us to retrieve parameters of risk aversion as well as the distributions themselves and their degree of affiliation. This provides a possible parametric test of the existence of risk aversion or affiliation in this market.

The paper is organized as follows. In section 2, we present some general theorems related to single-unit and sequential descending-price auctions. These theorems are useful to identify classes of auctions for which symmetric equilibria are easy to compute. In Section 3, we exploit the result of the previous section and compute bidding equilibria for a specific parametric problem. A parameter of the model measures the degree of value affiliation. We allow as special cases both private values and common values. A second parameter measures the (constant) degree of risk-aversion. Simulations show that the model obtained is flexible enough to accomodate the inverse U-shape of our database. Section 4 presents the data. In Section 5, non linear least square estimates of the structural model are presented and econometric tests are performed to test for the presence of risk-aversion and affiliation. All the proofs are in the appendix.

2 The Theory of Sequential First-Price Auctions.

We consider auctions in which n bidders desire at most one unit of a given good. Each bidder possesses some information concerning the objects for sale; let $X = (X_1, \dots, X_n)$ be the vector of real-valued private informational variables observed by the individual bidders. Let $S = (S_1, \dots, S_m)$ be a vector of additional real-valued common knowledge

quential auctions, there is one noticeable difference between descending and first-price (sealed-bid) auctions; neither the auctioneer nor the other bidders can observe the losing bids.

variables which influence the value of the object for the bidders. The actual value in money of a unit of good for some bidder i will be denoted by $V_i = w_i(S, X)$. We allow for risk-averse bidders, who possess an identical wealth-dependent concave utility function, $u(\cdot)$. Hence, if bidder i receives the object being sold and pays the amount b, his realized utility is given by $u(V_i - b)$.

We make the following (standard) assumptions. There is a function w on R^{m+n} such that for all i, $w_i(S, X) = w(S, X_i, \{X_j\}_{j \neq i})$. The function w is nonnegative, and is twice continuously differentiable and non decreasing in its variables; for all i, $E[V_i] < \infty$. The function u is thrice continuously differentiable with $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. Finally, let $f(s, x)$ denote the joint probability density function of the random variables, S and X, of the model. We assume that f is twice continuously differentiable and symmetric in its last n arguments, and that the variables $S_1, \ldots, S_m, X_1, \ldots, X_n$ are affiliated according to the definition in Milgrom-Weber(1982). Roughly it implies that large values for some variables make the other variables more likely to be large than small.

We wish to compute the equilibrium bidding functions for sequential first-price auctions in cases where bidders are risk-averse and values are affiliated. One of the purposes of this paper is to offer a formulation of this auction problem with a tractable closed-form solution. In order to do so, we exploit and generalize a basic property of the simplest auction problem. We show in the following subsections how an "Indifference Property" can be exploited to compute strategic equilibria for these complex auction games.

2.1 Single-Unit Auctions.

We reconsider first the simple first-price auction model with risk-neutral bidders and private values. It is assumed that each bidder's private information and willingness to pay is independently and identically distributed according to some distribution $F(\cdot)$.

A well-known property of the strategic equilibrium for this auction game is that it leads to the same expected revenue as the Vickrey or second-price auction. This property however does not generalize when affiliation or risk-aversion are introduced. On the other hand, another related property generalizes. In equilibrium, the expected utilities of buyers are equal in both first-price and second-price auctions.

Indeed, according to the Revenue Equivalence Theorem, the expected payment obtained from a type-v bidder is the same in both auctions. Furthermore, the probability of winning the object and therefore the expected values of auctions are identical. Hence, the expected utilities of buyers in the two outcomes are equal. This equality provides a way of computing bidder v's bid in the first-price auction. Ignoring for simplicity reserve prices[2], the expected utility of a buyer with valuation v in a first-price auction is

$$[v - b^F(v)]F(v)^{n-1} \tag{1}$$

where $b^F(\cdot)$ is the optimal strategy in the first-price auction. The expected utility in a Vickrey auction where the optimal bid is the true valuation is

$$\int_0^v (v-x)dF(x)^{n-1} = \int_0^v F(x)^{n-1}dx \tag{2}$$

Equations (1) and (2) yield the familar formula:

$$b^F(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}} \tag{3}$$

Let us now reason differently and consider the strategy $b(\cdot)$ in the first-price auction which equates the expected utility in both auctions. We will show that this strategy is necessarily the optimal strategy, namely $b^F(\cdot)$. By definition,

$$[v - b(v)]F(v)^{n-1} = \int_0^v (v-x)dF(x)^{n-1} \qquad \forall v \tag{4}$$

Note that (4) implies [3]

$$[m - b(v)]F(v)^{n-1} = \int_0^v (m-x)dF(x)^{n-1} \qquad \forall v, m \tag{5}$$

[2]To simplify the formulas and since reservation prices do not play a significant role in our data, the whole paper is written without reserve prices. Extensions with reserve prices are available from the authors.

[3]Differentiating (5) with respect to m and using (4) gives the result.

or switching notations

$$[v - b(m)]F(m)^{n-1} = \int_0^m (v - x)dF(x)^{n-1} \qquad \forall v, m \tag{6}$$

Property (6), called the generalized indifference property, guarantees that the monotone function $b(\cdot)$ constructed by solving (4) is a strategic equilibrium for the first-price auction. Indeed, let $U^F(v, b(m))$ be a type-v bidder's expected payoff when he bids $b(m)$ and others are bidding according to the bidding function $b(\cdot)$; and let $U^S(v, m)$ be the corresponding expected utility in a second-price auction for a type-v bidder bidding m. Equality (6) implies that $U^F(v, b(m)) = U^S(v, m)$ for all v and m. Since $U^S(v, v) \geq U^S(v, m)$ for all v and m, we must have $U^F(v, b(v)) \geq U^F(v, b(m))$ for all v and m. A type-v buyer is always as well-off bidding $b(v)$ than any other value $b(m) \neq b(v)$ if all other participants bid according to $b(\cdot)$. It forms the symmetric equilibrium.

The above result can be generalized. Under some conditions, so long as there exists a monotone function $b(\cdot)$ such that, for all v and m, we have $U^F(v, b(m)) = U^S(v, b^S(m))$ where $b^S(\cdot)$ is the second-price auction bidding equilibrium, this function $b(\cdot)$ will form the symmetric equilibrium strategy for the first-price auction. To formalize this we introduce some notations. We let:

$$Z(v, x, b) \equiv E[u(w(S, X_i, \{X_j\}_{j \neq i}) - b) | X_i = v,\ X_l = x\ , X_j \leq x,\ \forall j \notin \{i, l\}] \tag{7}$$

The function $Z(v, x, b)$ yields the expected utility of a bidder type $X_i = v$ when he receives the good for a payment b when the highest type among all others is given by x. Recall that the function w is increasing in its arguments; together with affiliation it implies that Z is non-decreasing in v and x (see Theorem 5 in Milgrom-Weber(1982)). Also, Z is decreasing and concave in b. Similarly, we let:

$$H(x|v) \equiv \text{Prob}[X_j \leq x,\ \forall j \neq i | X_i = v] \tag{8}$$

The function $H(x|v)$ is the distribution of the highest type among all $n-1$ other bidders conditional on i's type being equal to v. Similarly, we let $h(x|v)$ be the conditional density of x given v.

Finally, we let $b^S(\cdot)$ denote the (symmetric) equilibrium bidding strategy in the second-price auction. Following Milgrom-Weber (1982), $b^S(\cdot)$ is given by: $Z(v, v, b^s(v)) = 0$. We can now state our first main theorem: THEOREM 1: (UTILITY EQUIVALENCE THEOREM) *If $T(v, x, b) \equiv Z(v, x, b)h(x|v)$ is of the form $J(v, x) + K(v)L(x, b)$, then the unique symmetric equilibrium of the first-price auction is such that the expected utility accruing to each bidder is the same as what they would receive under a second-price auction. More precisely, it is given by the function $b^F(\cdot)$ such that for all v:*

$$\int_0^v Z(v, x, b^F(v))dH(x|v) = \int_0^v Z(v, x, b^S(x))dH(x|v). \tag{9}$$

We consider here three examples where Theorem 1 applies.

EXAMPLE 1: The special case treated above, $(v-b)h(x) = T(v, x, b)$ (risk neutrality and independent private information) satisfies the condition of Theorem 1.

EXAMPLE 2: Under private and independent values, the expression $T(v, x, b)$ equals $u(v-b)h(x)$. If $u(\cdot)$ exhibits constant absolute risk-aversion and takes the form $u(w) = 1 - e^{-rw}$, then we have: $T(v, x, b) = [h(x) - e^{-rv}(e^{rb}h(x))]$. Theorem 1 holds for utility functions with constant absolute risk aversion. [4]

EXAMPLE 3: Consider the following particular specification of affiliation for an utility function $u(w) = 1 - e^{-rw}$ with constant absolute risk-aversion. Each bidder has some private information which is parameterized by some number $v \in [0, \bar{v}]$. The v's are distributed identically and independently among agents according to a c.d.f. $F(\cdot)$, strictly increasing and differentiable.

The monetary value of a unit of good to some agent i is given by $[\alpha v_i + (1 - \alpha)\sum_{j=1}^n v_j/n]$. Note that, if $\alpha = 1$, values are private and, if $\alpha = 0$, values are common. Hence the payoff accruing to agent i who purchases one unit of the good at price b in the state of nature $\{v_1, v_2, \ldots, v_n\}$ is given by: $1 - e^{r(av_i + q(\sum_{j \neq i} v_j) - b)}$, where $a \equiv [\alpha + (1-\alpha)/n]$ and $q \equiv (1-\alpha)/n$.

[4]This result was obtained by Matthews (1987).

This expected utility given that x is the highest private information among all others, is:

$$Z(v,x,b) = 1 - e^{-r(av+qx-b)} \left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)} \right)^{(n-2)}. \tag{10}$$

The term $\left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)} \right)^{(n-2)}$ reflects the uncertainty attached to the private information of the $(n-2)$ remaining individuals. Finally, $H(x|v) = F(x)^{n-1}$ and $h(x) = (n-1)f(x)F(x)^{n-2}$. Hence, we can write:

$$T(v,x,b) = h(x) - e^{-rav} \left(e^{-r(qx-b)} \left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)} \right)^{-n-2)} h(x) \right).$$

Theorem 1 applies.

2.2 Multi-unit descending price auctions

In this subsection, we generalize the results of Theorem 1 to the cases where the auctioneer sells k identical units of a given good through a sequence of k descending price auctions. We assume that each bidder desires at most one unit of the good. It is therefore common knowledge that all previous winners will not participate in the remaining auctions. Preferences are characterized as above; they admit both risk aversion and affiliated values.

The auctions are ranked chronologically from the 1^{st} auction to the k^{th}. Since winners of previous auctions do not participate in the remaining auctions, $(n-i+1)$ participating bidders remain in the i^{th} auction, and $(k-i+1)$ units remain to be sold. At auction i, a history can be summarized by the sequence of winning bids $H_i = \{b_1^w, b_2^w, \ldots, b_{i-1}^w\}$. A symmetric bidding strategy specifies for each auction i, the history at auction i, H_i, and, for each agent's type v, a bid. For a well-defined bidding strategy, we can specify, after each history, equilibrium beliefs for the bidders. Formally, we can define the support of types still participating in the remaining auctions. Let $S_0 = [0, \bar{v}]$, we have $S_{i+1}(H_i) = \{v \in S_i(H_{i-1}) | b_i(v, H_i) \leq b_i^w\}$ where $S_{i+1}(H_i)$ defines the support of the belief after history H_i. In particular, if the bidding function is increasing in v's then along the equilibrium, the supports of beliefs are obtained by successive truncations of the initial support. Hence, beliefs can be parameterized by some number z_i such that $S_{i+1}(H_i) = [0, z_i]$ where z_i corresponds to the type of auction-i's winner.

For notational purposes, we shall represent past history as a vector of types rather than a vector of observed winning bids. Let $z_i(H_i) = \inf\{x | b_i(x|H_{i-1}) = b_i^w\}$. A symmetric equilibrium strategy can be represented by a vector $B = \{b_1(\cdot), b_2(\cdot,\cdot), \ldots, b_k(\cdot,\cdot)\}$ where $b_i(v, z^{-i})$ denotes the equilibrium bid in auction i of a bidder of type v for a history summarized by $z^{-i} = \{z_1, z_2, \ldots, z_{i-1}\}$. Note that under the assumption of monotonic bidding functions, the bidder with the i^{th} highest valuation will win, along the equilibrium path, the i^{th} auction and $\min\{z | z \in z^{-i}\} = z_{i-1}$.

In the simplest case, with risk-neutral bidders and private values, the strategic equilibrium is such that the winning prices in the sequence of auctions follows a martingale, $E[b_{i+1}^w | b_i^w] = b_i^w$ [5]. More precisely:

$$b_i(v, z^{-i}) = \int_0^v b_{i+1}(x, (z^{-i}, v)) \frac{dF(x)^{n-i}}{F(v)^{n-i}} \tag{11}$$

If a type-v bidder wins auction-i, the type of the winner of auction-$(i+1)$, x, has distribution $\frac{F(x)^{n-i}}{F(v)^{n-i}}$. x's bid in auction $i+1$ will be $b_{i+1}(x, (z^{-i+1}, v)$ where $z^{-(i+1)} = (z^{-i}, v)$ summarizes this history af ter auction-i. This property however does not generalize with risk-averse bidders or affiliation. However, another interpretation of (11) does generalize. Suppose that the bidder of type-v chooses not to participate in auction-i, lets the bidder o f type-x win instead and bids according to his equilibrium strategy, $b_{i+1}(v, (z^{-i}, x))$. We can show that $b_{i+1}(v, (z^{-i}, x)) = b_{i+1}(x, (z^{-i}, x)) = b_{i+1}(x, (z^{-i}, v))$. The first equality follows because, in auction-$i+1$, the bidder of typ e-v knows that he can win for sure if he bids $b_{i+1}(x, (z^{-i}, x))$. The second equality follows from the fact that, under private information, bidding strategies are independent of past winning bids in so far as one's type does not exceed that of pre vious winners.

[5] see Weber (1983)

If we rewrite (11), we now obtain:

$$[v - b_i(v, z^{-i})] = \int_0^v [v - b_{i+1}(v, (z^{-i}, x))] \frac{dF(x)^{n-i}}{F(v)^{n-i}} \tag{12}$$

Equation (12) expresses the indifference of bidder i between winning in auction i or waiting for auction-$(i+1)$. The left hand side gives the gain of a type-v bidder when he bids according to the equilibrium strategy and wins. Conditional on the fact that a bidder would have won auction-i, the right hand side is the expected gain of the same bidder who waits for the next auction and follows his equilibrium strategy. Putting this together with the result of Section 2.1, it provides a recursive way to construct equilibrium strategies. At the last auction k, we can construct the equilibrium strategy (in the first-price auction) from the equilibrium strategy in a Vickrey auction. At auction $k-1$, we can construct the equilibrium strategy from the equilibrium strategy of auction k, etc.

The reasoning above relies on arguments similar to the ones of the previous subsection and can be generalized to more complex games. Formally, we define for any auction i and a history summarized by the vector $z^{-i} = \{z_1, \ldots, z_{i-1}\}$, the following function:

$$\begin{aligned} Z^i(v, x, b, z^{-i}) \equiv \; & E[u(w(S, X_i, \{X_j\}_{j \neq i}) - b)| \\ & X_i = v, X_1 = z_1, \ldots, X_{i-1} = z_{i-1}, X_{i+1} = x, X_l \leq x, \forall l > i+1] \end{aligned} \tag{13}$$

The function $Z^i(v, x, b, z^{-i})$ yields the expected utility of a bidder type v when he receives the good for a payment b after a history summarized by z^{-i} and when the highest type among all $n-i$ remaining bidders is given by x. As before, Z^i is non-decreasing in all its arguments but b; it is decreasing and concave in b. Also we let:

$$\begin{aligned} H^i(x|v, z^{-i}) \equiv \; & \text{Prob}[X_j \leq x, \forall j > i | X_i = v, \\ & X_{i-1} = z_{i-1}, \ldots, X_1 = z_1, \text{and } X_j \leq \min\{z_1, \ldots, z_{i-1}\}, \forall j > i] \end{aligned} \tag{14}$$

The function $H(x|v, z^{-i})$ is the distribution of x, the highest type among $(n-i)$ bidders, conditional on v and on the $(i-1)$ highest types being $z^{-i} \equiv \{z_1, \ldots, z_{i-1}\}$. We let $h^i(x|v, z^{-i})$ be the corresponding conditional density of x given v and z^{-i}. Finally, we define:

$$T^i(v, x, b, z^{-i}) \equiv Z^i(v, x, b, z^{-i}) h^i(x|v, z^{-i}). \tag{15}$$

We can now state our second main theorem:

THEOREM 2: *If $T^i(v, x, b, z^{-i})$ takes the form $J^i(v, x, z^{-i}) + K^i(v, z^{-i}) L^i(x, b, z^{-i})$, then a symmetric equilibrium exists. It is such that bidders are indifferent between following their equilibrium strategy in auction $i < k$, and not participating in auction i and waiting for the next auction.*

More precisely, at every auction i after history summarized by z^{-i}, the equilibrium bidding function satisfies:

(i) $b_i(v, z^{-i}) = b_i(z_{i-1}, z^{-i})$ *if* $v > z_{i-1}$;
(ii) if $v \leq z_{i-1}$, $b_i(v, z^{-i})$ *solves:*

$$\int_0^v Z^i(v, x, b_i(v, z^{-i}), z^{-i})\, dH^i(x|v, z^{-i}) = \int_0^v Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i})\, dH^i(x|v, z^{-i}); \tag{16}$$

(iii) finally when $i = k$, the above holds with $b_{k+1}(\cdot)$ such that

$$Z^k(x, x, b_{k+1}(x, (z^{-k}, x)), z^{-k}) = 0.$$

Theorem 2 states that bidders receive in equilibrium what they get if they wait for the next auction. Using this "Indifference Condition" and Theorem 1 which provides a basis for the induction, we can compute the expected utility accruing to each type in equilibrium. Consider a sequence of k sequential first-price auctions. In the auction-k, bidders' equilibrium expected utility is equal to their equilibrium expected payoff from a Vickrey auction [Theorem 1 or Theorem 2(iii)]. Using the result of Theorem 2, we know that bidders are indifferent in equilibrium between following their equilibrium strategy and waiting for this last auction. Hence, we can relate the equilibrium payoff in all the sequential auctions to the expected payoff accruing to participants of a Vickrey auction. It follows that the computation and the interpretation of the equilibrium bidding strategy in sequential descending price auctions are greatly simplified.

In the next section, we use Theorem 2 to compute equilibrium strategies in sequential first-price auctions for the specification of Example 3 in Subsection 2.1.

3 Application with a Closed-Form Solution.

For a constant absolute risk-aversion utility function $u(w) = 1 - e^{-rw}$ and the stochastic structure of the section-2.1 Example 3, we obtain the expected utility of winning the i^{th} auction:

$$Z^i(v, x, b, z^{-i}) = 1 - e^{-r(av + q\sum_{j=1}^{i-1} z_j + qx - b)} \left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)} \right)^{(n-i-1)} . \tag{17}$$

Also, $H^i(x|v, z^{-i}) = F(x)^{n-i}/F(z_{i-1})^{(n-i)}$. Hence, we can write:

$$T^i(v, x, b, z^{-i}) = \frac{F(x)^{n-i}}{F(z_{i-1})^{(n-i)}} - e^{-rav - rq\sum_{j=1}^{i-1} z_j} \cdot \left(e^{-r(qx-b)} \left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)} \right)^{(n-i-1)} \frac{F(x)^{n-i}}{F(z_{i-1})^{(n-i)}} \right).$$

Theorem 2 applies.

3.1 The equilibrium bidding functions.

Using Theorem 2, we can characterize the equilibrium (symmetric and monotone) bidding functions. Note that, under the assumption of monotonic bidding functions, the bidder with the i^{th} highest valuation will win, along the equilibrium path, the i^{th} auction. Hence in equilibrium, the type of auction-i's winner, z_i, is given by the i^{th} highest value among n independent draws from a distribution $F(\cdot)$. This follows a well-known distribution law. [6] Below, we let $G_{k+1}(\cdot|z_i = v)$ denote the distribution of

[6] For more on order statistics, see H.A. David (1980).

the $(k+1)^{th}$ highest value conditional on v being the i^{th} highest statistics. The next theorem characterize the equilibrium:

THEOREM 3: *For the specification of Example 3, Equation (16) can be rewritten as follows:*

$$e^{rb_i(v,z^{-i})} \; e^{-r(q\sum_{j=1}^{i-1} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-i} = \tag{18}$$
$$\int_0^v \left[e^{rb_{i+1}(x,(z^{-i},x))} e^{-r(qx+q\sum_{j=1}^{i-1} z_j)} \left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)^{n-i-1}\right] \frac{dF(x)^{n-i}}{F(v)^{n-i}}$$

and the unique symmetric perfect Bayesian equilibrium is given by:

$$b_i(v,z^{-i}) = \begin{cases} b_i(z_{i-1}, z^{-i}), & \text{if } v > z_{i-1}, \text{ otherwise we have:} \\ q\sum_{j=1}^{i-1} z_j - \frac{(n-i)}{r}\ln\left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right) + \frac{\ln}{r}\left(\int_0^v e^{ras} dG_{k+1}(s|z_i = v)\right). \end{cases} \tag{19}$$

3.2 The path of expected winning bids.

The Indifference Condition presented above is useful to predict and understand the expected path of winning bids. With private values and risk-neutral bidders, the winning bids follow a martingale process [Weber (1983)].

Consider now the case with risk-averse agents and private values; i.e. $a=1$, $q=0$ and $r>0$. Equation (18) becomes:

$$e^{rb_i(v,z^{-i})} = \int_0^v e^{rb_{i+1}(x,(z^{-i},x))} \frac{dF(x)^{n-i}}{F(v)^{n-i}} \tag{20}$$

Conditional on being the bidder with the highest valuation, a type-v bidder is indifferent in equilibrium between bidding $b_i(v,z^{-i})$ and waiting for the next auction and bidding according to his equilibrium strategy, i.e $b_{i+1}(v,(z^{-i},x))$ w hen the history is given by $z^{-(i+1)} = (z^{-i},x)$. Recall from Theorem 2(i) that $b_{i+1}(v,(z^{-i},x)) =$

$b_{i+1}(x,(z^{-i},x))$ whenever $x < v$. Note that this latter strategy is risky; the bidder is not certain of x, the type of the bidder with the next highest valuation, hence of the price he will need to pay in auction-$(i+1)$. Since bidders are risk-averse they are ready to pay more in auction i than what he will have to pay on average in auction-$(i+1)$.

Note that we have: $b_{i+1}(x,(z^{-i},x)) = b_{i+1}(x,(z^{-i},v))$. This follows from the fact that when $q = 0$, one's bidding strategy in auction-i is independent of past winning bids in so far as $\min\{z|z \in z^{-i}\} \geq v$. Equation (20) becomes:

$$e^{rb_i(v,z^{-i})} = \int_0^v e^{rb_{i+1}(x,(z^{-i},v))} \frac{dF(x)^{n-i}}{F(v)^{n-i}} \tag{21}$$

Conditionnal on $b_i(v,z^{-i})$ being the winning bid in auction-i, the right hand side corresponds to the expected exponent-r of the winning bid in auction-$i+1$. Hence, the exponent-r of the winning bids follow, along the equilibrium path, a martingale process. Since the fonction e^{rx} is convex, we also have that $b_i^w > E[b_{i+1}^w|b_i^w]$. The path of expected winning bids is decreasing. [McAfee and Vincent, (1993)]

Conversely, consider the case with risk-neutral agents and affiliated values; i.e. $r = 0$. If we differentiate both sides of (18) with respect to r and evaluate this at $r = 0$, we obtain:[7]

$$\begin{aligned} b_i(v,z^{-i}) &= q(\sum_{j=1}^{i-1} z_j) + (n-i)q\left(\int_0^v s\frac{dF(s)}{F(v)}\right) \\ &+ \int_0^v \left[b_{i+1}(x,(z^{-i},x)) - q(x+\sum_{j=1}^{i-1} z_j) - (n-i-1)q\left(\int_0^x s\frac{dF(s)}{F(x)}\right)\right]\frac{dF(x)^{n-i}}{F(v)^{n-i}} \\ &= \int_0^v b_{i+1}(x,(z^{-i},x))\frac{dF(x)^{n-i}}{F(v)^{n-i}} \end{aligned} \tag{22}$$

Using Equation (19), one can verify that: $b_{i+1}(x,(z^{-i},x)) = b_{i+1}(x,(z^{-i},v)) - q(v-$

[7]The second equality follows from the fact that:

$$\begin{aligned} (n-i)\int_0^v s\frac{dF(s)}{F(v)} &= (n-i)\int_0^v \frac{d}{dx}\left[\left(\int_0^x s\frac{dF(s)}{F(x)}\right)\frac{F(x)^{n-i}}{F(v)^{n-i}}\right]dx \\ &= \int_0^v \left\{x + (n-i-1)\left(\int_0^x s\frac{dF(s)}{F(x)}\right)\right\}\frac{dF(x)^{n-i}}{F(v)^{n-i}} \end{aligned}$$

x). Hence we have:

$$E[b^w_{i+1}|z^{-i}\text{and } b^w_i = b_i(v,z^{-i})] \quad -b_i(v,z^{-i}) \tag{23}$$
$$= \int_0^v b_{i+1}(x,(z^{-i},v)) - b_{i+1}(x,(z^{-i},x))\frac{dF(x)^{n-i}}{F(v)^{n-i}}$$
$$= q\int_0^v (v-x)\frac{dF(x)^{n-i}}{F(v)^{n-i}} > 0$$

In the case with risk-neutral bidders and affiliated values, the winning bids follow a martingale process with an upward drift. The drift is given by q times the expected difference between the i^{th} and the $i+1^{th}$ highest value. Consider again the implication of the Indifference Condition. Suppose that v were the i^{th} highest value and that the bidder of type v were to deviate and not to participate in auction-i. Everybody else would be fooled as they would believe that the true $i+1^{th}$ value is the i^{th} highest value. Under affiliated values, the deviation will lower the bids in auction-$(i+1)$. In order to satisfy the indifference condition, v's equilibrium bid in auction i must be equal to the expected pri ce he will have to pay in auction $(i+1)$ if he deviates, which is less than the expected winning bid if he were to follow his equilibrium strategy. Formally, $b^w_i < E[b^w_{i+1}|b^w_i]$.

In the general case, we have:

$$E[b^w_{i+1}|z^{-i} \text{ and } b^w_i = b_i(v,z^{-i})] - b_i(v,z^{-i}) = q\left[v - \int_0^v x\frac{dF(x)^{n-i}}{F(v)^{n-i}}\right] \tag{24}$$
$$-\left[\frac{\ln}{r}\int_0^v\left(\int_0^x e^{ras}dG_{k+1}(s|z_{i+1}=x)\right)\frac{dF(x)^{n-i}}{F(v)^{n-i}} - \int_0^v\left(\frac{\ln}{r}\int_0^x e^{ras}dG_{k+1}(s|z_{i+1}=x)\right)\frac{dF(x)^{n-i}}{F(v)^{n-i}}\right]$$
$$+\left[\frac{(n-i)}{r}\ln\left(\int_0^v e^{-rqx}\frac{dF(x)}{F(v)}\right) + \int_0^v\left[qx - \frac{(n-i-1)}{r}\ln\left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)\right]\frac{dF(x)^{n-i}}{F(v)^{n-i}}\right]$$

The difference between the expected winning bid in auction-$(i+1)$ and the observed auction-i's winning bid can be decomposed into three elements. The first element measures the drift due to affiliated values; this element is positive if and only if $q \equiv \frac{(1-a)}{(n-1)} > 0$. The second follows from the downward trend caused by risk-aversion and is negative by Jensen's inequality if and only if $r \neq 0$. The rationale for these two opposite effects were discussed above. The third and last elem ent follows from the cross-effect of both risk-aversion and affiliated values. At any one time, bidders are uncertain of the true value (for them) of a unit of good, because they remain unaware of the information held by other unsuccessful bidders. One of the benefits of waiting an extra period is to learn the type of one extra participant reducing part of the undesired uncertainty. Hence, all other effects being considered a risk-averse bidder would accept to bid more on average in future auction. Under the *Indifference Condition*, this cross effect must therefore lead to a positive drift. Indeed, we have:

$$\begin{aligned} \tfrac{\ln}{r}\left(\int_0^v e^{-rqx}\tfrac{dF(x)}{F(v)}\right)^{(n-i)} &= \frac{\ln}{r}\int_0^v \frac{d}{dx}\left(\int_0^x e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-i} dx \qquad (25) \\ &= \frac{\ln}{r}\left(\int_0^v \left[e^{-rqx}\cdot\left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)^{n-i-1}\right]\frac{dF(x)^{n-i}}{F(v)^{n-i}}\right) \\ &> \int_0^v \left[-qx + \frac{(n-i-1)}{r}\ln\left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)\right]\frac{dF(x)^{n-i}}{F(v)^{n-i}} \end{aligned}$$

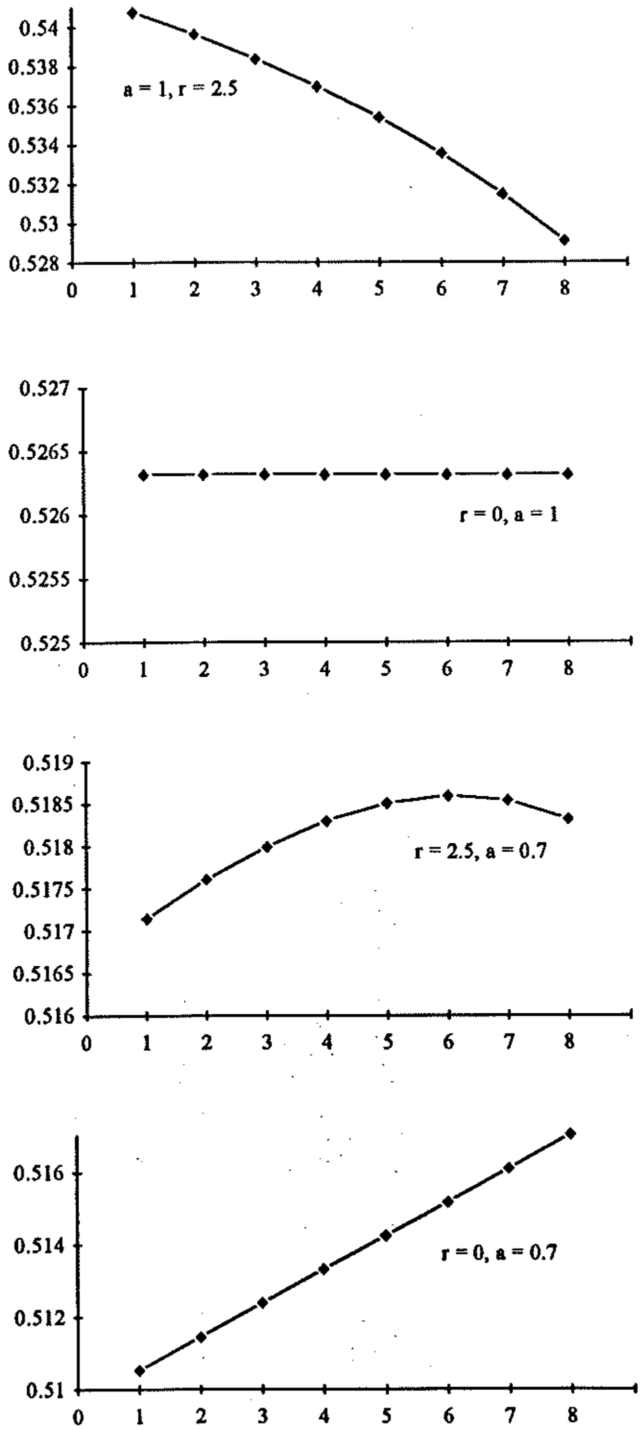
The last inequality follows from Jensen's inequality. Note that because of this cross-effect, an increase in risk-aversion may lead to a greater upward drift in the path of winning bids.

In conclusion, the path of observed winning bids may be decreasing or increasing according to the various parameters, hence the importance of estimating empirically these parameters. In order to explore further the properties of the path of expected winn ing bids, we have computed expected winning bids. For the uniform distribution, $F(v) = v$, we obtain: (i) the path of expected winning bids is increasing from auction 1 to k and is linear when $r = 0$ and $a < 1$; (ii) the path of expected winning bids is d ecreasing and concave when $r > 0$ and $a = 1$; (iii) in the general case, the path is concave, increasing or decreasing according to the parameters and exhibits in some case an inverse U-shape. See the Figure 1.

Figure 1

Paths of Winning Bids

η = number of bidders = 18
k = number of units to be sold = 8
$F(v) = v$



4 Application to Sequential Auctions of Eggplants

We provide in this section a description of the data we will be using to estimate the model of bidding behavior developed above. The auction itself is first presented, then the data set and finally some summary statistics.

4.1 Sequential auctions of eggplants in Marmande

We study a descending auction of greenhouse eggplants in Marmande, France. The sellers are farmers and the buyers are resale trade firms. The buyers can be considered as agents of retail sellers who have placed orders at specific prices before the opening of the market. Those prices are the retail sellers' willingnesses to pay in the auction. At each round of the auction, several cases - from 15 to 350 kilos each - are displayed. The products to be auctioned are described (size, name of seller, weight of case in the general catalog of the day given to each buyer before the market). The seller announces a reservation unit price, i.e. a price for one kilogram, for all the cases of the day. Because of the highly perishable nature of eggplants, it is reasonable to postulate no inter-day dynamics and consider that the markets are different from one day to the next. However, when several cases are sold the same day we assume that buyers optimize the bidding behavior over all the auctions of the day.

The general principles of the auction are the same as in any descending auction. It starts from a very high unit price, for example, fifteen francs when the usual unit price of the commodity is about ten francs per kilo. From then on, the price drops very quickly until one of the bidders makes a bid. At that instant the reserve clock is stopped. The first buyer who makes a bid before the reservation price is reached wins the auction. His unit payment equals his bid.

4.2 Data

We have data from daily sales of greenhouse eggplants for the year 1991. For each lot (several each day) we observe the reserve price (which is the same during the day) and various characteristics as the date, the winning bid if any, the weight of the lot, the quality defined by the average weight of one fruit, the sellers' name, the buyer's name, the number of lots offered in one day (all qualities included), the quantity of eggplants offered in all of the day's lots, etc.

To study the dynamics of intra-day bidding we select the days when 3 or 4 lots are sold during the day.[8] We also restricted the estimation to the case of caliber 300 and 59 auctions to limit somewhat the numerical work. The reserve price contains a lot of the information included in the specific heterogeneity of the day. A rough description of the average intra-day dynamics is obtained by averaging the differences between the winning bid and the reserve price. We obtain :

Table 1

Average Intra-Day Dynamics

	Average Winning Bids				
Number of lots	1^{st} LOT	2^{e}LOT	3^{e}LOT	4^{e}LOT	5^{e}LOT
3 (23 units)	0.4826	0.7539	0.5922		
4 (36 units)	0.4978	0.7100	0.7967	0.6344	

We observe an inverted U-shape for the normalized winning bids averages.

[8]On the contrary, in Laffont-Ossard-Vuong (1995) we selected the days with a single lot to avoid the difficulties of the intra-day dynamics.

There are 15 buyers but one of them is a large company, and the 14 others are of a similar size. To solve this difficulty we follow the approach used by Laffont et al. (1995): we assume that the big buyer represents several small buyers. We consider each buyer to be a perfect agent for the final buyer. The big buyer receives orders from several buyers. Since he buys about half of the lots we assume that he represents 14 buyers and that he bids independently for each of them. In total it amounts to having 28 buyers, for which we further assume each one desires only one case. [9]

5 Estimation

5.1 The structural model

Using Equation (19), we can calculate the expected winning bid of the i^{th} auction. Recall that, along the equilibrium, the winner of the first auction is the bidder with the highest private information, the winner of the second auction is the one with the second highest type, etc. So the winning price of auction-i corresponds to the bid of the participant with the i^{th} highest type. His bid will depend on the history up to auction-i summarized by z^{-i}, where the elements in z^{-i} correspond to the $(i-1)^{th}$ highest types among n independent draws from the distribution $F(\cdot)$. If v is the i^{th} order statistics, we have:

$$E[\sum_{j=1}^{i-1} z_j \,|\, z_i = v] = (i-1)\int_v^{\infty} s\frac{dF(s)}{[1-F(v)]} \tag{26}$$

Hence, we can write the expected value of the i^{th} winning bid, b_i^w:

$$\begin{aligned} E[b_i^w] \; = \int_0^{\infty} \; & \left[(i-1)\left(\int_v^{\infty} s\frac{dF(s)}{[1-F(v)]}\right) - \frac{(n-i)}{r}\ln\left(\int_0^v s\frac{dF(s)}{F(v)}\right)\right. \\ & \left. +\frac{\ln}{r}\left(\int_0^v e^{ras} dG_{k+1}(s|z_i = v)\right)\right] dG_i(v) \end{aligned} \tag{27}$$

[9] Clearly, the analysis should be pursued in two directions which raise serious new difficulties: sequential auctions where a buyer wishes to buy several units (see Robert (1995)) and asymmetric auctions.

where $G_i(\cdot)$ is the distribution of the i^{th} order statistics and $G_{k+1}(\cdot|z_i = v)$ is the distribution of the $(k+1)^{th}$ highest value given that the i^{th} highest value is v.[10]

5.2 Parametrization and Estimation Method

The structural econometric model is completed by taking into account the observed heterogeneity across lots. For each period t, let Z_t denote the vector of relevant characteristics of all the lots sold during the day. Then the conditional distributions $F(\cdot|Z_t)$ represent the distributions of the agents' valuations (denoted v_i in the previous sections) normalized by the reserve price of the day (i.e. v_i-reserve price).

We take $F(\cdot|\cdot)$ to be Weibull distributions[11] with density for period t

$$f_t(v) = \frac{\eta_t}{\lambda_t}\left(\frac{v}{\lambda_t}\right)^{\eta_t - 1} \exp\left(-\left(\frac{v}{\lambda_t}\right)^{\eta_t}\right)$$

where (η_t, λ_t) are the usual Weibull parameters.

To complete the specification we take one explanatory variable only x_t, the reserve price of the previous date (or the quantity offered on the market the previous date). From classical formulas (see Johnson and Kotz (1970)), we have the implicit equations for η_t, λ_t:

$$\alpha + \beta x_t = \frac{\lambda_t}{\eta_t}\Gamma\left(\frac{1}{\eta_t}\right)$$

$$\sigma^2 = \lambda_t^2\left(\frac{2}{\eta_t}\Gamma\left(\frac{2}{\eta_t}\right) - \frac{1}{\eta_t^2}\Gamma\left(\frac{1}{\eta_t}\right)\right)$$

[10] We can show that: $G_i(y) = P(z_i \leq y) = P($ At least $n-i+1$ of n draws are $\leq y)$ which is equal to: $\sum_{j=n-i+1}^{n} P($Exactly j of n draws are $\leq y)$ which is equal to:

$$= \sum_{j=n-i+1}^{n} \left(\frac{n!}{j!(n-j)!}\right)(F(y))^j(1-F(y))^{n-j} = \left(\frac{n!}{(n-i)!(i+1)!}\right)\int_0^{F(y)} t^{n-i}(1-t)^{i-1}dt,$$

The last line uses what is called the *incomplete beta function*. Similarly, we can show that: $G_{k+1}(y|z_i = v) = P($ At least $(n-k)$ of $(n-i)$ draws are $\leq y|$ all these draws are $\leq v)$ which is:

$$\left(\frac{(n-i)!}{(n-k-1)!(k-i)!}\right)\int_0^{F(y)/F(v)} t^{n-k-1}(1-t)^{k-i}\,dt$$

[11] See Elyakime et alii (1997) for ce justification of this choice.

where α, β, σ are unknown parameters. Solving these equations gives the specification of η_t, λ_t we use.

The structural parameters of the model are then

$$\theta = (\alpha, \beta, \sigma, r, \alpha).$$

To estimate these parameters we use the nonlinear least squares (NLLS) for the system (28) explaining the bids where b_{it}^w denotes now the winning bid when selling the i^{th} lot at date t. The resulting estimator is consistent and asymptotic normal as the number T of periods increases.

Let $I(t)$ denote the number of lots at date t. The statistical objective that we maximize with respect to θ is

$$Q(\theta) = \sum_{t=1}^{T} \sum_{i \in I(t)} (b_{it}^w - E_t b_{it}^w)^2$$

where $E_t(\cdot)$ denotes the expectation operator with respect to the structural distributions taken at the beginning of period t.

The unconstrained estimates are:

$$\begin{aligned} \hat{\alpha} &= 0.5206 \\ \hat{\beta} &= 0.2973 \\ \hat{\sigma} &= 1.6020 \\ \hat{r} &= 0 \\ \hat{\alpha} &= 0. \end{aligned}$$

i.e. the estimation converges despite numerous attempts towards no risk aversion and common values.

The fitted expected values (to be compared with Table 1) are:

For 3 units:	0.5147	0.7076	0.7513	
4 units:	0.4990	0.6453	0.6874	0.7093
Criterion:	10.877			

The fit is reasonable except that we do not catch the descending part of the invested U shape since we have no risk aversion.

To test the significance of affiliation we reestimate the model with α constrained to be one and we obtain:

$$\begin{aligned} \hat{\hat{\alpha}} &= 0.3673 \\ \hat{\hat{\beta}} &= 0.1227 \\ \hat{\sigma} &= 1.6073 \\ \hat{r} &= 0 \end{aligned}$$

with the following fitted values

3 units	0.6918	0.8369	0.8413	
4 units	0.4948	0.5431	0.5455	0.5456

and a criterium of 12.152 which shows that the private value assumption is rejected[12].

6 Conclusion

We have provided a flexible model which can be used to study the intra-day dynamics of first price auctions and estimate the degree of risk aversion of the bidders and the degree of affiliation of the distribution of valuations. The numerical burden of the estimation is overwhelming when using ordinary computers, so that our empirical results must be considered as preliminary. Only the use of fast computers will enable us to reach

[12]The $\chi^2(1)$ statistics takes the value 40 (with a significance level of 4).

definitive results and hopefully an estimated model which, as our simulations, exhibit an invested U shape.

References

ASHENFELTER, O. (1989); "How Auctions Work for Wine and Art", *Journal of Economic Perspectives*, **3**, 39-64.

DAVID, H.A. (1980); *"Order Statistics"*, second Edition, Wiley Series in Probability and Mathematical Statistics.

ELYAKIME, B., J.J. LAFFONT, P. LOISEL AND Q. VUONG (1997); "Auctioning and Bargaining: An Econometric Study of Timber Auctions with Secret Reservation Prices", *Journal of Business and Economic Statistics*, 15, 209-220.

JOHNSON, N. AND KOTZ, S. (1970); Distributions in Statistics, New York, Wiley.

LAFFONT, J.J., OSSARD, H. AND VUONG, Q. (1995); "Econometrics of First-Price Auctions", *Econometrica*, 63, 953-980.

MATTHEWS, S.A. (1987); "Comparing auctions for risk-averse buyers: A buyer's point of view", *Econometrica*, **55**, 633-646.

MCAFEE, P.R. AND D.VINCENT (1993); "The declining price anomaly", *Journal of Economic Theory*, **60**, 200-213.

MILGROM, P. AND R. WEBER (1982); "A general theory of auctions and bidding", *Econometrica* **59**, 1089-1122.

ROBERT, J. (1985); "Sequential Descending-Price Auction with Multi-Unit Demand", mimeo.

WEBER, R. (1983); "Multiple-object auctions", in R.Engelbrecht-Wiggans, M. Shubik, and R.Stark eds, *Auction, bidding and Contracting: Uses and Theory.* NewYork University press, NewYork.

WILSON, R. (1992); "Strategic Analysis of Auctions", chap. 8 in R.Aumann and S. Hart eds, *Handbook of Game Theory: with economic applications.* Nort-Holland.

Appendix
Proof

PROOF OF THEOREM 1: The claim is that if $T(v,x,b)$ takes the form $J(v,x)+K(v)L(x,b)$, the solution, $b^F(\cdot)$, for all v of the following functional equation is the unique symmetric equilibrium of the first-price auction:

$$\int_0^v Z(v,x,b^F(v))dH(x|v) = \int_0^v Z(v,x,b^S(x))dH(x|v) \qquad \forall v \tag{28}$$

where $b^S(\cdot)$ is the optimal strategy in the second-price auction, i.e. the solution of $Z(v,v,b^S(v))=0$ for all v. Let us first show the following lemma.

LEMMA 1: *If the function $b^F(\cdot)$ satisfies condition (29), it also satisfies:*

$$W(v,m) \equiv \int_0^m Z(v,x,b^F(m)) - Z(v,x,b^S(x))dH(x|v) = 0 \qquad \forall v,m \tag{29}$$

PROOF OF LEMMA 1: Using that fact the $T(v,x,b) = J(v,x)+K(v)L(v,b)$, we have:

$$\begin{aligned}
W(v,v) &= \int_0^v \left[T(v,x,b^F(v)) - T(v,x,b^S(x))dx\right] \\
&= \int_0^v \left[[J(v,x)+K(v)L(x,b^F(v))] - [J(v,x)+K(v)L(x,b^S(x))]\right] dx \\
&= \int_0^v K(v)[L(x,b^F(v)) - L(x,b^S(x))]\, dx \\
&= \frac{K(v)}{K(m)}\int_0^v K(m)[L(x,b^F(v)) - L(x,b^S(x))]\, dx \\
&= \frac{K(v)}{K(m)}\int_0^v \left[[J(m,x)+K(m)L(x,b^F(v))] - [J(m,x)+K(m)L(x,b^S(x))]\right] dx \\
&= \frac{K(v)}{K(m)}W(m,v) \qquad ,
\end{aligned} \tag{30}$$

Since by assumption, $T_3(v,x,b) < 0$ for all v, so we must have $K(v) \neq 0$ for all v. Hence, if $W(v,v)=0$ for all v then we must also have $W(m,v)=0$ for all v and m. *Q.E.D.*

Lemma 1 implies that, under the conditions of Theorem 1, we have:

$$\int_0^m Z(v,x,b^F(m))dH(x|v) = \int_0^m Z(v,x,b^S(x))dH(x|v) \qquad \forall v,m \tag{31}$$

The left hand side of expression (32) is the expected gain, denoted by $U^F(v,b(m))$, to a type-v bidder bidding $b^F(m)$ in the first-price auction when all other participants use the bidding strategy $b^F(\cdot)$. The right hand side is the gain, denoted by $U^S(v,m)$, for a type-v bidder in a second-price auction when he deviates and uses the equilibrium strategy of type-m. By the definition of an equilibrium, $U^S(v,v) \geq U^S(v,m)$. So (32) implies

$$U^F(v,b^F(v)) = U^S(v,v) \geq U^S(v,m) = U^F(v,b^F(m)) \tag{32}$$

for all m and v. From (33), $b^F(\cdot)$ is indeed a symmetric equilibrium of the first-price auction. We know from Wilson (1992) that there exists at most one symmetric bidding equilibrium. *Q.E.D*

Proof of Theorem 2: (i) Consider first the case where $v > z_{i-1}$. Recall that $x \leq z_{i-1}$ along the equilibrium path, the type-v bidder will win for sure if he bids $b_i(z_{i-1}, z^{-i})$. He can only lose if he bids more. Finally, if a type-$z_{i-1} < v$ bidder does not find it profitable to bid lower neither will a type-v. For the remainder of this proof, when we refer to a type v we will be referring to the minimum between v and z_{i-1}.

(ii) Let define:

$$W^i(v,m,z^{-i}) = \int_0^m \left[Z^i(v,x,b_i(m,z^{-i}),z^{-i}) - Z^i(v,x,b_{i+1}(x,(z^{-i},x)),z^{-i})\right] dH^i(x|v,z^{-i}) \tag{33}$$

One can show using an argument similar to that in Lemma 1 that if $T^i(v,x,b,z^{-i}) = J^i(v,x,z^{-i}) + K^i(v,z^{-i})L^i(x,b,z^{-i})$, then if $W^i(v,v,z^{-i}) = 0$ for all v and z^{-i} we have $W^i(v,m,z^{-i}) = 0$ for all v and m. Hence we must have for all m and v:

$$\int_0^m Z^i(v,x,b_i(m,z^{-i}),z^{-i})dH^i(x|v,z^{-i}) = \int_0^m Z^i(v,x,b_{i+1}(x,(z^{-i},x)),z^{-i})dH^i(x|v,z^{-i}) \tag{34}$$

If $b_i(\cdot)$ satisfies (35), the expected utility of a type-v bidder when he bids $b_i(m, z^{-i})$ in auction-i is given by:

$$\int_0^m Z^i(v, x, b_i(m, z^{-i}), z^{-i})\, dH^i(x|v, z_{i-1}) + \int_m^{z_{i-1}} U^*_{i+1}(v, (z^{-i}, x))\, dH^i(x|v, z_{i-1})$$
$$= \int_0^m Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i})\, dH^i(x|v, z_{i-1})$$
$$+ \int_m^{z_{i-1}} U^*_{i+1}(v, (z^{-i}, x))\, dH^i(x|v, z_{i-1}) \qquad (35)$$

where $U^*_{i+1}(v, (z^{-i}, x))$ denotes the expected equilibrium utility accruing to a type-v bidder in auction-$i+1$ after history (z^{-i}, x).

Hence, if a type-v bidder deviates from his equilibrium strategy and instead bids in auction i according to m's strategy, his benefit from deviation will be:

$$\begin{cases} \int_m^v \left[U^*_{i+1}(v, (z^{-i}, x)) - Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i})\right] dH^i(x|v, z_{i-1}) \text{ if } m < v \\ \int_v^m \left[Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i}) - U^*_{i+1}(v, (z^{-i}, x))\right] dH^i(x|v, z_{i-1}) \text{ if } m > v \end{cases} \qquad (36)$$

The above is obtained by subtracting the left-side of (36) for any arbitrary m with the same expression evaluated at $m = v$. It suffices to show that this is non-positive for all m and v. When $m < x < v$, the best strategy for a type v in auction $i+1$ is to bid $b_{i+1}(x, (z^{-i}, x))$ and win for sure, hence $U^*_{i+1}(v, (z^{-i}, x)) = Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i})$. It follows that the above expression is equal to 0 for all $m < v$. In equilibrium, bidders are indifferent between following t heir equilibrium strategy and bidding any price below. Conversely, suppose that $m > v$. Since in auction-$i+1$, the type-v bidder can always bid $b_{i+1}(x, (z^{-i}, x))$ and win for sure, we have $U^*_{i+1}(v, (z^{-i}, x)) \geq Z^i(v, x, b_{i+1}(x, (z^{-i}, x)), z^{-i})$. Hence the gain from deviating is non-positive. Note that when $i+1 = k$, we have a strict inequality if $x > v$, then by induction it holds with strict inequality for all i. In conclusion, no bidder can gain by deviating from the above equi libria. *Q.E.D.*

Proof of Theorem 3: Given the definition of Z^i and H^i, equation (17) corresponds to:

$$\int_0^v \quad 1 - \left[e^{rb_i(v,z^{-i})} e^{-r(av+qx+q\sum_{j=1}^{i-1} z_j)} \left(\int_0^x e^{-rqs} \frac{dF(s)}{F(x)}\right)^{n-i-1}\right] \frac{dF(x)^{n-i}}{F(v)^{n-i}} =$$

$$\int_0^v 1 - \left[e^{rb_{i+1}(x,(z^{-i},x))}e^{-r(av+qx+q\sum_{j=1}^{i-1} z_j)} \left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)^{n-i-1}\right] \frac{dF(x)^{n-i}}{F(v)^{n-i}} \qquad (37)$$

Integrating the left-side and simplifying, we obtain (19):

$$e^{rb_i(v,z^{-i})} \quad e^{-r(q\sum_{j=1}^{i-1} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-i} = \qquad (38)$$

$$\int_0^v \left[e^{rb_{i+1}(x,(z^{-i},x))}e^{-r(qx+q\sum_{j=1}^{i-1} z_j)} \left(\int_0^x e^{-rqs}\frac{dF(s)}{F(x)}\right)^{n-i-1}\right] \frac{dF(x)^{n-i}}{F(v)^{n-i}}.$$

Using this we can proceed recursively. The definition of b_{k+1} is such that

$$Z^k(x,x,b_{k+1}(x(z^{-k},x)),z^{-k}) = 0. \qquad (39)$$

This implies that

$$\left[e^{rb_{k+1}(x,(z^{-k},x))}e^{-r(qx+q\sum_{j=1}^{k-1} z_j)} \left(\int_0^x e^{-rqs}\frac{dF(s)}{F(s)}\right)^{n-k-1}\right] = e^{rax} \qquad (40)$$

Hence, for $i = k$, we have:

$$e^{rb_k(v,z^{-k})}e^{-r(q\sum_{j=1}^{k-1} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-k} = \int_0^v e^{rax}\frac{dF(x)^{n-k}}{F(v)^{n-k}} \qquad (41)$$

Note that the expression $e^{rb_k(v,z^{-k})}e^{-r(q\sum_{j=1}^{k-1} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-k}$ is independent of z^{-k}. So we can then write for $i = k-1$:

$$e^{rb_{k-1}(v,z^{-(k-1)})}e^{-r(q\sum_{j=1}^{k-2} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(z_v)}\right)^{n-k+1} = \int_0^v \left(\int_0^x e^{ras}\frac{dF(s)^{n-k}}{F(x)^{n-k}}\right) \frac{dF(x)^{n-k+1}}{F(v)^{n-k+1}} \qquad (42)$$

Applying this recursively, we obtain:

$$e^{rb_i(v,z^{-i})}e^{-r(q\sum_{j=1}^{i-1} z_j)} \left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right)^{n-i} = \int_0^v e^{ras}dG_{k+1}(s|z_i = v). \qquad (43)$$

Where $G_{k+1}(\cdot|z_i = v)$ denotes the distribution of the $(k+1)^{th}$ highest value conditional on v being the i^{th} highest statistics. Hence, we obtain as claimed that for all $i \in 1,\ldots,k$:

$$b_i(v,z^{-i}) = q\sum_{j=1}^{i-1} z_i - \frac{(n-i)}{r}ln\left(\int_0^v e^{-rqs}\frac{dF(s)}{F(v)}\right) + \frac{1}{r}ln\int_0^v e^{ras}dG_{k+1}(s|z_i = v). \quad (44)$$

Q.E.D.