Pratical computing in fuzzy logic

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Dub85c DUBOIS, D., PRADE, H. (1985) Fuzzy numbers : an overview. In Tech. Rep. n°219, LSI, Univ. P. Sabatier, Toulouse and to appear in The Analysis of Fuzzy Information (J.C. Bezdek, ed.), Vol. 1, Mathematics and Logic, CRC Press, Boca Raton, Florida, 1987.

Far86 FARRENY, H., PRADE, H., WYSS, E. (1986) Approximate reasoning in a rule-based system using possibility theory : a case study. Proc. 10th IFIP World Congres, (Kugler H.J., ed.), Dublin, Sept 1-5, Elsevier Sci Publ., 407-413.

Leb87 LEBAILLY, J., MARTIN-CLOUAIRE, R., PRADE, H. (1987) Use of fuzzy logic in a rule-based system in petroleum geology. In Approximate Reasoning in Intelligent Systems, Decision and Control, (E. Sanchez, L.A. Zadeh, eds.), Pergamon Press, 125-144.

Mar84 MARTIN-CLOUAIRE, R. (1984) A fast generalized modus ponens. BUSEFAL, n° 18, L.S.I., Univ. Paul Sabatier, Toulouse, 75-82.

Mar86a MARTIN-CLOUAIRE, R., PRADE H. (1986) SPII-1 a simple inference engine capable of accommodating imprecision and uncertainty. In Computer-Assisted Decision-Making (G. Mitra, ed.), pp. 117-131, North-Holland, 1986.

Mar86b MARTIN-CLOUAIRE, R. (1986) Efficient deduction in fuzzy logic. Presented at Inter. Conf on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU-86), Paris, June 30 - July 4.

Mar87a MARTIN-CLOUAIRE, R. (1987) Management of imprecision and uncertainty in a rule- based system : an application to a petroleum geology problem. To appear in "Uncertainty in Artificial Intelligence", (D. Dubois, H. Farreny, H. Prade eds.), Springer Verlag.

Mar87b MARTIN-CLOUAIRE, R. (1987) Semantics and computation of the generalized modus ponens. Proc. 2nd Inter. Fuzzy Systems Association Congress (IFSA-87), Tokyo, July 20 - 25.

Pra85 PRADE, H. (1985) A quantitative approach to approximate reasoning in rule-based expert systems. To appear in "Computer Expert Systems", (L. Bolc, M.J. Coombs, eds.) Springer Verlag. Also Tech. Rep. n° 229, LSI, Univ. P. Sabatier, Toulouse.

Zad78 ZADEH, L.A. (1978) Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets & Systems , 1, 3-28.

Zad79 ZADEH, L.A. (1979) A theory of approximate reasoning. Machine Intelligence, 9, (J.E. Hayes, D. Michie, L.I. Mikulich, eds.). Elsevier, New York,149-194.

In Figure 13, we have :
$i^* = 2$, $i'^* = 1$,
$j^* = 1$, $j'^* = 2$. Hence, $L = \{|[1, 2]|, \{1\}, \{2\}\}$.

In Figure 14, we have :
$i^* = i'^* = 1$,
$j^* = 0$, $j'^* = 2$. Hence, $L = \{|[0, 1]|, |[0, 2]|, |[1, 2]|, \{1\}\}$.
Note that $|[0, 1]|$ is not useful here.

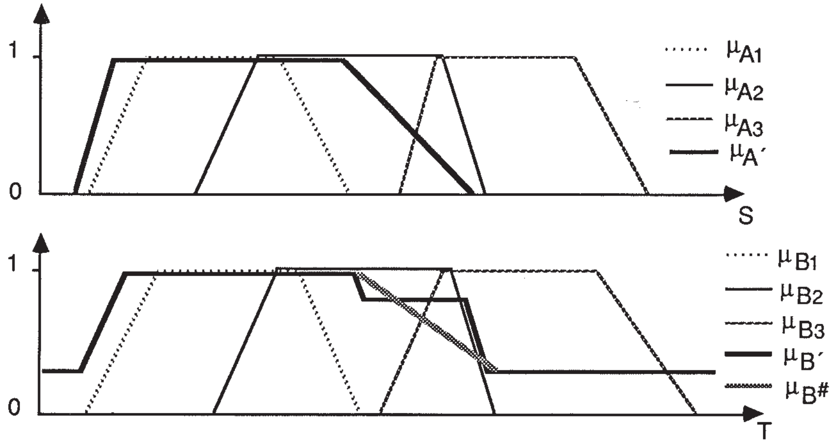


Figure 14.

5. CONCLUDING REMARKS

The main lesson of this paper is that, under some technical constraints operated on the representation of uncertain and imprecise knowledge, one can use the generalized modus ponens tool in real (i.e. computationally realistic) inference systems. Indeed, it has been shown that the deduction process with this approximate reasoning technique amounts to some simple and not numerous computations on the parameters defining the involved possibility distributions.

The approximation technique presented in this paper for the single rule case has been implemented and adapted to the inference engine SPII [Mar86a] [Mar87a] which is equipped with several other approximate reasoning capabilities. So far, our experiments with real life problems have shown that the above mentioned constraints are not limiting the representational power too much and thus are perfectly acceptable from a practical point of view. Future versions of SPII will include the treatment of collections of rules as well as the possibility of interpreting the rules in other ways [Mar87b] than the one related to the Gödel implication function and the "min" conjunction operator as used in this paper.

When processing a set of rules Ai→Bi, i=1,n, we have seen that a key point in order to infer the most precise and the less uncertain conclusion from a fact A′ is to build new rules $A_I \to B_I$ where $A_I = \cup_{i\in I}$ Ai and $B_I = \cup_{i\in I}$ Bi, or $A_I = \cap_{i\in I}$ Ai and $B_I = \cap_{i\in I}$ Bi. Rules of interest are $A_I \to B_I$ such that $A_{J^*} \supseteq A_I = C(A')$, where J^* is the minimal set such that

$A_{J^*} \supseteq A'$. This generalizes a situation which already exists when the Ai's and the Bi's are ordinary subsets.

In the general case, it may happen that A′ is included in Aj ∪ Aj+1 (for instance) but is not even weakly in Aj ∩ Aj+1, nor in one of Aj , Aj+1. An idea in order to obtain a conclusion more specific than Bj ∪ Bj+1, would be to perform a non-logical combination of the form $A_I = \lambda.Aj \oplus \lambda'.Aj{+}1$ and $B_I = \lambda.Bj \oplus \lambda'.Bj{+}1$, with $0 \le \lambda \le 1$, $0 \le \lambda' \le 1$, $\lambda + \lambda' = 1$ and where ⊕ denotes the addition of fuzzy numbers (see [Dub85a,c]). Obviously, the rule $A_I \to B_I$, constructed with the A_I and B_I that have just been defined, should only be considered as a plausible approximation (i.e. this rule cannot be proved to be valid as in the case of the logical combination-based generation) and presupposes that Y varies continuously and "gently" with X. It is just an extended linear interpolation. Adding such interpolated rules may help improve the coverage of the rule base, if needed. Clearly, $\lambda.Bj \oplus \lambda'.Bj{+}1$ is more specific than Bj ∪ Bj+1 in general. Moreover, λ and λ′ can be chosen such that $\lambda.Aj \oplus \lambda'.Aj{+}1$ contains A′ and such that $\lambda.Bj \oplus \lambda'.Bj{+}1$ is as precise as possible. Lastly, the specificity (or precision) of $\lambda.Bj \oplus \lambda'.Bj{+}1$ is intermediary between the one of Bj and Bj+1. The systematic study of interpolation techniques in a set of fuzzy rules is a topic for further research.

REFERENCES

Bui86 BUISSON, J-C., FARRENY, H., PRADE, H. (1986) Dealing with imprecision and uncertainty in the expert system DIABETO-III. Proc. 2nd Int. Conf. on Artificial Intelligence, Marseille, Dec 1-5, 705-721.

Dub80 DUBOIS, D., PRADE, H. (1980) Fuzzy Sets and Systems : Theory and Applications. Vol.144 in Mathematics in Science and Engineering Series, Academic Press, New York, 1980

Dub82 DUBOIS, D., PRADE, H. (1982) Towards the analysis and the synthesis of fuzzy mappings. In Fuzzy Set and Possibility Theory, (R. Yager, ed.), Pergamon Press, 316-326.

Dub84 DUBOIS, D., PRADE, H. (1984) Fuzzy logics and the generalized modus ponens revisited. Cybernetics & Systems, 15, n°3-4, 293-331.

Dub85a DUBOIS, D., PRADE, H. (with the collabaration of H. Farreny, R. Martin-Clouaire, C. Testemale) (1985) Théorie des Possibilités. Application à la Représentation des Connaissances en Informatique. Masson, Paris. English updated version to be published by Plenum Press, New York, 1987.

Dub85b DUBOIS, D., PRADE, H. (1985) The generalized modus ponens under sup-min composition. A theoretical study. In Approximate Reasoning in Expert Systems, (M.M. Gupta, A. Kandel, W. Bandler, J.B. Kiszka, eds.), North-Holland, 217-232.

In Figure 9 we have :

$i^* = 2$, $i'^* = 1$, $A_* = A_1 \cap A_2$

$j^* = j'^* = 1$, $A^* = A_1$.

Hence, $L = \{\{1\}, \{2\}\}$.

In Figure 10 we have :

$i^* = 1$, $i'^* = 2$,

$j^* = 0$, $j'^* = 2$.

Hence, $L = \{|[0, 2]|, |[1, 2]|\}$.

Note that here $A_{|[0,\,2]|} = S(A_1) \cup A_2 \supseteq A'$. Hence we obtain $S(B_1) \cup B_2$ by the rule $A_{|[0,\,2]|} \rightarrow B_{|[0,\,2]|}$. This is why the left hand side of $\mu_{B'}$ is not similar to its right hand side.

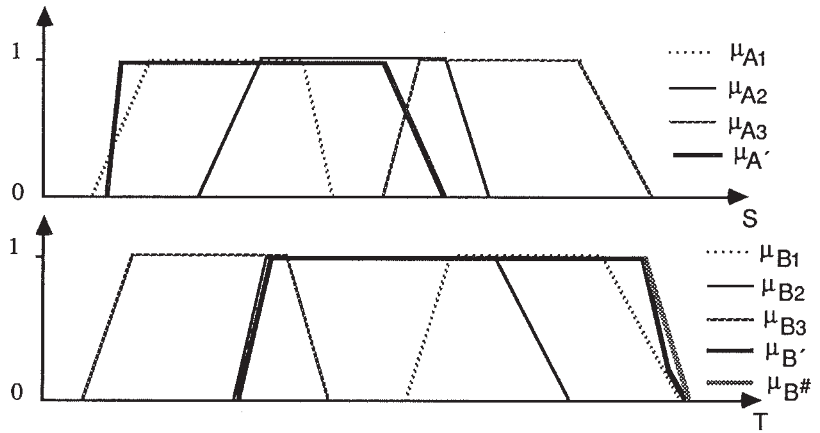


Figure 10.

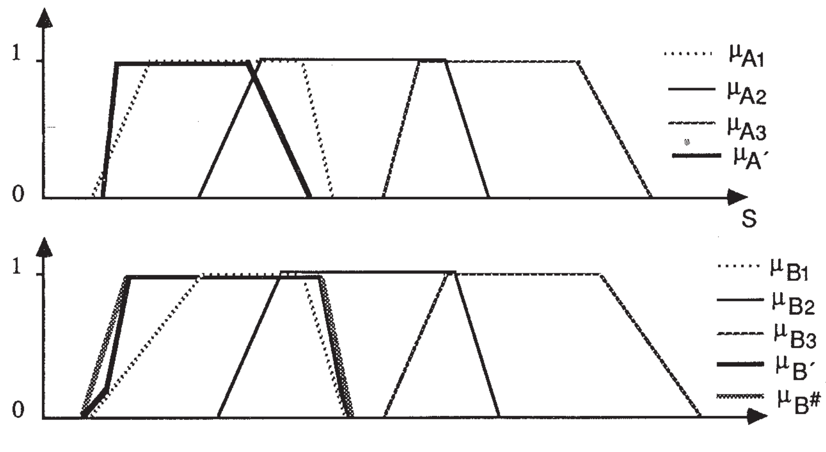


Figure 11.

In Figure 11 we have

$i^* = i'^* = 1$

$j^* = 0$, $j'^* = 1$

Hence $L = \{|[0,1]|\ ,\ \{1\}\}$.

Note that in this case $|[0,1]|$ gives nothing interesting compared with $\{1\}$ because $C(A')$ falls out of $C(A_1)$ on the left hand side only. In Figure 14, a situation where $C(A')$ is contained in $S(A_1)$ but falls out on both sides of $C(A_1)$ will be considered. In such a situation, the extra-rule $A_0 \rightarrow B_0$ plays an active role.

In Figure 12, we have :

$n = 4$

$i^* = i'^* = 3$,

$j^* = 2$, $j'^* = 4$. Hence, $L = \{|[2, 4]|, |[3, 4]|, |[2, 3]|, \{3\}\}$.

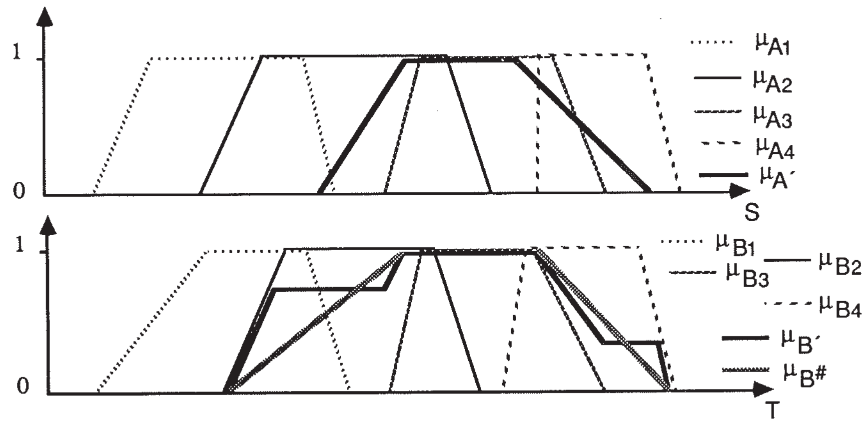


Figure 12.

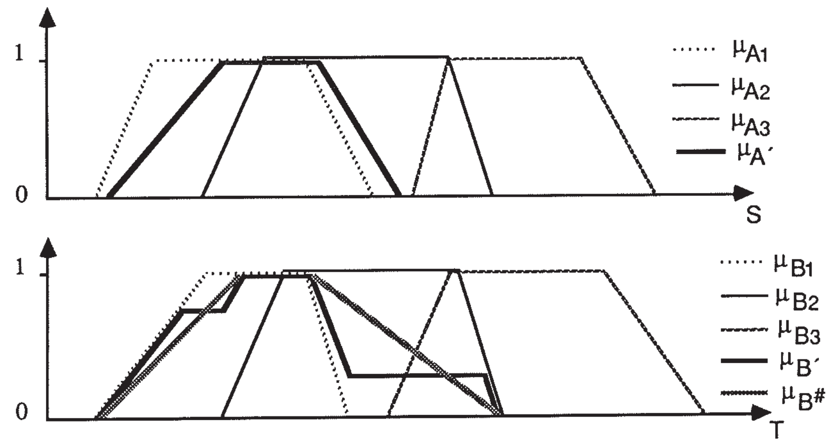


Figure 13.

$A' \circ (A_{|[j^*, i']|} \rightarrow B_{|[j^*, i']|})$ since $[A_|, +\infty) \supseteq [A^*, +\infty)$ and $(-\infty, A_|] = (-\infty, A_| \cap A^*]$. But $|[j^*, i']|$ is in $\mathfrak{I}^*$. The same reasoning applies when $i' > j'^*$. Q.E.D.

4.4.3. An algorithm for dealing with a set of fuzzy rules

The inference engine algorithm is given next. For clarity, we shall use a simple Pascal-like langage in which the keywords specific to the langage are in italics. Let us assume that $A' = (a'1\ a'2\ a'3\ a'4)$.

```
Procedure Selection   (Step 1)
   let i* = max{ i such that a_i1 ≤ a'2}
   let i'* = min{ i such that a_i4 ≥ a'3}
   if i* ≤ i'* let I* = |[i*, i'*]|  (i.e. in this case A* = A_I*)
   if i* = i'*+1 let I* = ø  (i.e. in this case A* = Ai* ∩ Ai'*)
   let j* = max{ j : a_j1 ≤ a'1 ; a_j2 ≤ a'2 }
   let j'* = min{ j : a_j3 ≥ a'3 ; a_j4 ≥ a'4}
   if j* ≤ j'* let J* = |[j*, j'*]|  (i.e. in this case A* = A_J*)
   if j* = j'*+1 let J* = ø  (i.e. in this case A* = Aj* ∩ Aj'*)

Procedure Firing   (Step 2)
   if J* = ø then
      begin
      let B'_0 = Bj* ∩ Bj'*
      let L = {0}
      end
   else (i.e. J* ≠ ø)
      begin
      let L = ø
      if I* ≠ ø then
         for all k, k' such that j* ≤ k ≤ i* and i'* ≤ k' ≤ j'*
            begin
            let K = |[k, k']|
            add the element K to the set L
            let B'_K = A' o (A_K → B_K)
            end
      else (i.e. I* = ø)
         for all k, k' such that j* ≤ k ≤ k' ≤ j'*
            begin
            let K = |[k, k']|
            add the element K to the set L
            let B'_K = A' o (A_K → B_K)
            end
      end

Procedure Combination   (Step 3)
   let B' = ∩_{K∈L} B'_K
```


Remarks :

. In Step 2, the computation technique described in Section 3 is used and the approximation $B^{\#}$ may be computed instead of the exact B'.

. In Step 3, the result of the combination may have a complicated shape. One may wish to keep only a trapezoidal approximation denoted by $B^{\#}$ in the figures exhibited in the next subsection .

. If $\cup_{i\in N}$ Ai is a trapezoidal fuzzy interval, which yet does not cover S, four situations may be encountered. Each of them requires slight modifications in the use of the above algorithm.

  . If i^* or i'^* cannot be found then the algorithm must return B' = T.

  . If j^* or j'^* cannot be found just because $a_12 > a'2$ or $a_n3 < a'3$ then, depending on the case, add the redundant rule $A_0 \rightarrow B_0$ or $A_{n+1} \rightarrow B_{n+1}$ to the rule base $\mathfrak{R}$, where $B_0 = S(B_1)$, $B_{n+1} = S(B_n)$ and $A_0 = S(A_1) - \cup_{i\in N} C(A_i)$, $A_{n+1} = S(A_n) - \cup_{i\in N} C(A_i)$, and start all over again with the new rule base. Examples are provided through the figures 10 and 11.

  . If j^* or j'^* cannot be found just because $a_11 > a'1$ or $a_n4 < a'4$ then, depending on the case, let $j^* = \min\{i / A_i \rightarrow B_i \in \mathfrak{R}\}$ or $j'^* = \max\{i / A_i \rightarrow B_i \in \mathfrak{R}\}$ and use the algorithm as in the normal case. In such a situation, a level of indetermination will pervade B'.

  . If j^* or j'^* cannot be found because $a_11 > a'1$ and $a_22 > a'2$ or $a_n4 < a'4$ and $a_33 < a'3$ then combine the tricks given in the two previous points. Figure 14 corresponds to this case.

4.5. Some examples

In all figures but Figure 12, the number of rules n is assumed to be equal to 3. For the sake of clarity we consider only cases of strong consistency. The approximation $B^{\#}$ is drawn only on the parts where it differs from B'.

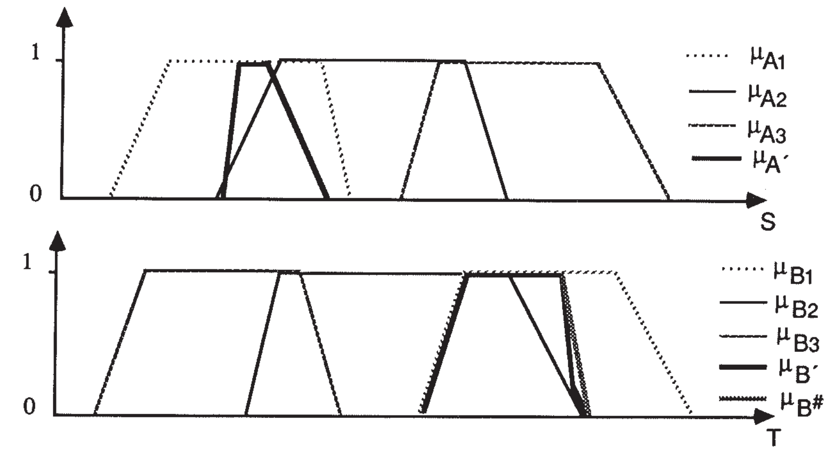


Figure 9.

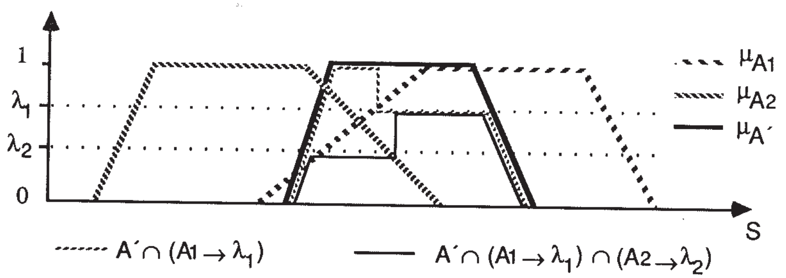


Figure 8.

4.4. Practical computing with a set of fuzzy rules

Given a set $\mathfrak{R}$ of n consistent rules, the following approach is adopted as an inference algorithm.

Step 1 *Selection* : Find $\{ I / N \supseteq I,\ A' \circ (A_I \to B_I) \neq T\} = \mathfrak{I}$ with $A_I = \cup_{i \in I} Ai$ and $B_I = \cup_{i \in I} Bi$.

Step 2 *Firing* : Compute $B'_I = A' \circ (A_I \to B_I)\ \forall I \in \mathfrak{I}$.

Step 3 *Combination* : Compute $\cap_{I \in \mathfrak{I}} B'_I$.

The combinatorial problem lies in Step 1. The following remarks help in reducing the complexity :

. If C(A´) is not included in or equal to $S(A_I)$ then $I \notin \mathfrak{I}$.

. If $J \supset I$ then $A' \circ (A_J \to B_J) \supseteq A' \circ (A_I \to B_I)$ as long as $A_I \supseteq A'$.

Hence, letting $\aleph = \{ I / N \supseteq I,\ S(A_I) \supseteq C(A')\}$, one can define $\mathfrak{I}^*$ as

$\mathfrak{I}^* = \aleph - \{ J \in \aleph / \exists L \in \aleph,\ J \supset L,\ A_L \supseteq A'\}$ and put it in place of $\mathfrak{I}$ in the above algorithm.

$\mathfrak{I}^*$ contains the "useful" rules to be triggered.

4.4.1. Assumptions for the rule base

To further reduce the selection set $\mathfrak{I}^*$ and to be in a position to use the results of Section 3 we shall make some assumptions regarding the modeling of rules and the way they cover the set S. Namely, each element of {(Ai, Bi) / i∈N} is supposed to be a pair of trapezoidal fuzzy intervals, modelled by the 4-tuples (a_i1 a_i2 a_i3 a_i4) and (b_i1 b_i2 b_i3 b_i4) respectively. $\mathfrak{R}$ is assumed to be separated ; let s_i be such that $\mu_{Ai}(s_i) = 1$, and $\mu_{Aj}(s_i) = 0\ \forall j \neq i$. The rules are supposed to be numbered so that $s_1 \le s_2 \le \ldots \le s_n$.

Note that 'separatedness' forbids the existence of Ai and Aj, j≠i, such that $Aj \supseteq Ai$. Moreover, it ensures that the rule base is not redundant in the sense of 4.2 since $\exists t \in T,\ \mu_{Ai}(s_i) \to \mu_{Bi}(t) = \mu_{Bi}(t) < \mu_{Aj}(s_i) \to \mu_{Bj}(t) = 1$, except if Bi = T! It also implies that if $s_i < s_j$ then inf S(Aj) ≥ inf C(Ai) and sup C(Aj) ≥ sup S(Ai). Consequently, $\forall \alpha$ $\widetilde{\max}(Ai(\alpha), Aj(\alpha)) = Aj(\alpha)$ where $\widetilde{\max}$ is the extended max in the sense of interval

analysis [Dub80] and Ai(α) is the α-cut of Ai.

Lastly, in order to be able to apply the results of Section 3 we want the A_I's to be trapezoidal fuzzy intervals. Therefore, we must add a covering assumption. The covering of S, that is made by the set {Ai / i∈N}, must be such that $\cup_{i \in N} C(Ai)$ is compact (i.e. there is no gap in $\cup_{i \in N} C(Ai)$). If S is a subset of the real numbers, as assumed here, then $\cup_{i \in N} C(Ai)$ must be an interval.

In order to prevent the fact that some points in S are covered by too many rules, one may require that $\forall$ i, j, k∈I, i ≠ j ≠ k ⇒ Ai ∩ Aj ∩ Ak = ø. Note that this property holds if $\mathfrak{R}$ is separated as indicated by Lemma 1.

4.4.2. Coping with complexity

Under the assumptions of consistency, separatedness and proper coverage, we are in a position to reduce the complexity of the selection step. We use the following preliminary results.

If I = {i, i+1,..., i´}, that we shall denote by ||[i , i´]|, and J = {j, j+1,..., j´} then A_I, A_J and $A_I \cap A_J$ are trapezoidal fuzzy intervals when non-empty. A_I is of the form (a_i1 $a_{i'}2$ $a_{i'}3$ a_i4). Actually, we have $A_I \cap A_J = A_{I \cap J}$ if $I \cap J \neq ø$ and $A_I \cap A_J = Ai' \cap Ai'{+1}$ if j = i´+1 ; $A_I \cap A_J = ø$ if sup I < inf J -1 or sup J < inf I -1. By convention, Ai ∪ Ai+1 is denoted A_I with I = ||[i , i+1]|. Moreover, only the rules $A_I \to B_I$ where I = ||[i , i´]| need be considered when A´ is also of a trapezoidal shape. This latter remark breaks down the complexity since the number of elements of { I / N ⊇ I , I = ||[i , i´]| for i ≤ i´∈ N} is equal to n.(n+1)/2.

Consider the minimal elements in { I / N ⊇ I, $S(A_I) \supseteq C(A')$} which is assumed non-empty (the case of emptiness corresponds to B´ = T). The minimal elements are made of a single set I* = ||[i* , i´*]| or of two singletons {i*}, {i*+1}, such that Ai* ∩ Ai*+1 ⊇ C(A´), due to separatedness. Let us define A* either as A_{I^*} or as Ai* ∩ Ai*+1 depending on the case. Now consider the set {J / $A_J \supseteq A'$}, which is assumed non-empty unless otherwise specified, and its minimal elements. They are of the same form, say J* =||[j*, j´*]| or {j*}, {j*+1}. Let us define A* either as as A_{J^*} or Aj* ∩ Aj*+1, depending on the case.

Proposition 5 : $\cap_{I,\ N \supseteq I} A' \circ (A_I \to B_I) = \cap_{I \in \mathfrak{I}^*} A' \circ (A_I \to B_I)$ where

$\mathfrak{I}^* = \{ I / N \supseteq I,\ A^* \supseteq A_I \supseteq A_*,\ I = ||[i , i']|\ \ i \le i' \}$.

Proof : As noticed earlier, if $A_I \supseteq A^*$ then $A' \circ (A_I \to B_I) \supseteq A' \circ (A^* \to B^*)$. If $A_* \supset A_I$ then $\exists s \in C(A')$, $\mu_{A_I}(s) = 0$, thus the rule $A_I \to B_I$ gives nothing. If $I \cap J^* = ø$ then $A' \circ (A_I \to B_I) =$ T. Assume J* = ||[j* , j´*]|, j* < j´*. We consider the case when $A_I \neq A_I \cap A^* \supseteq A_*$. If I = ||[i , i´]|, J* = ||[j* , j´*]|, i < j*, then $A_I \cap A^* = A_{||[j^*, i']|}$ and one can check that $A' \circ (A_I \to B_I) =$

However, one can apply this algorithm to the saturated rule base $\mathfrak{R}$, since, due to (20), we have the following inclusion relationships :

$$\cap_{i\in N} A' \circ (Ai\rightarrow Bi) \supseteq \cap_{I,\ N\supseteq I} A' \circ (A_I\rightarrow B_I) \supseteq A' \circ (\cap_{i\in N} Ai\rightarrow Bi) = B'. \qquad (25)$$

As we shall see, applying the inference engine algorithm to $\mathfrak{R}$ (i.e. as is done in the second term of (25)) will prove very powerful, provided that we can master its increased computation complexity. First, let us give some theoretical results.

<u>Proposition 2</u> : If $A' = Ai$ then $B' = Bi$, provided that $C(Ai)$ is not contained in $\cup_{j\neq i} S(Aj)$.

Proof: From (25), $Bi \supseteq B'$ is obvious. Let us show now that $B' \supseteq Bi$.

$\forall t,\ \mu_{B'}(t) = \sup_s \min(\mu_{Ai}(s),\ \mu_{Ai}(s)\rightarrow\mu_{Bi}(t),\ \min_{j\neq i} \mu_{Aj}(s)\rightarrow\mu_{Bj}(t))$

$\geq \sup_{s\in C(Ai)} \min(\mu_{Bi}(t),\ \min_{j\neq i} \mu_{Aj}(s)\rightarrow\mu_{Bj}(t)) = \mu_{Bi}(t)$ provided that

$\sup_{s\in C(Ai)} \min_{j\neq i} \mu_{Aj}(s)\rightarrow\mu_{Bj}(t)) \geq \mu_{Bi}(t),\ \forall t$. A natural sufficient condition for that is $\exists s_i \in C(Ai),\ \forall j\neq i,\ \mu_{Aj}(s_i)\rightarrow\mu_{Bj}(t) = 1$ for all t, that is $\mu_{Aj}(s_i) = 0\ \forall j\neq i$. It all comes down to require that $C(Ai)$ is not contained in $\cup_{j\neq i} S(Aj)$. Q.E.D.

This result puts forward the question of the comparative range of applicability of the rules and leads to the following definition.

$\mathfrak{R}$ is said to be <u>separated</u> if and only if $\forall i\in N$, $C(Ai)$ is not contained in $\cup_{j\neq i} S(Aj)$.

This property means that deleting any of the rules from $\mathfrak{R}$ leaves a point in S to which no rule applies. It means that every rule is useful. Note if $\mathfrak{R}$ is not separated, then if $A' = Ai$ one may obtain B' such that $Bi \supset B'$. In this case, we might think of changing the rule $Ai\rightarrow Bi$ into $Ai\rightarrow B'$ since the former is clearly redundant with respect to the latter.

In the sequel $\mathfrak{R}$ is supposedly separated.

Let s_i denote an element of $C(Ai)$ that is not in $\cup_{j\neq i} S(Aj)$.

<u>Corollary 1</u> : If $\mathfrak{R}$ is separated, $Ai \supseteq A'$ and $\exists s_i \in C(A')$ then $B' = Bi = \cap_{j\in N} A' \circ (Aj\rightarrow Bj)$. Hence, the usual inference engine approach is good in this case.

Note that Proposition 2 can be extended to conjunctions $A' = \cap_{i\in I} Ai$, under the separatedness property. To see it we need the following lemma.

<u>Lemma 1</u> : If $\mathfrak{R}$ is separated then for any pairwise distinct Ai, Aj, Ak $Ai \cap Aj \cap Ak = \emptyset$.

Proof: Since $\mathfrak{R}$ is separated, it is possible to number the Ai's so that $s_1 \leq s_2 \leq \ldots \leq s_n$. Let i, j, k be such that $i<j<k$. $Ai \cap Aj \cap Ak \neq \emptyset$ implies that $Ai \cap Ak \neq \emptyset$ and $S(Ai \cup Ak) \supseteq S(Aj)$ since $[s_i, s_k] \supseteq S(Aj)$. But this fact contradicts separatedness according to which $\mu_{Ai}(s_j) = \mu_{Ak}(s_j) = 0$. Hence $Ai \cap Aj \cap Ak = \emptyset$. Q.E.D.

Consequently, we only have to consider conjunctive facts of the form $\cap_{i\in I} Ai$ where the set I has two elements such that $I = \{i, i+1\}$.

<u>Proposition 3</u> : If $\mathfrak{R}$ is separated and $A' = Ai \cap Ai+1$ then $B' = Bi \cap Bi+1$.

Proof: From (25) it is obvious that $Bi \cap Bi+1 \supseteq B'$. Now

$\forall t,\ \mu_{B'}(t) = \sup_s \min[\min_{j=i,\ i+1} \min(\mu_{Aj}(s),\ \mu_{Aj}(s)\rightarrow\mu_{Bj}(t)),\ \min_{k\neq i,\ i+1} \mu_{Ak}(s)\rightarrow\mu_{Bk}(t)]$

$\geq \sup_{s\in C(Ai)\cap C(Ai+1)} \min(\mu_{Bi}(t),\ \mu_{Bi+1}(t),\ \min_{k\neq i,\ i+1} \mu_{Ak}(s)\rightarrow\mu_{Bk}(t))$

$= \min(\mu_{Bi}(t),\ \mu_{Bi+1}(t))$

provided that $\exists s \in C(Ai \cap Ai+1),\ s \notin S(\cup_{k\neq i,\ i+1} Ak)$. But since $\mathfrak{R}$ is separated, $[s_i, s_{i+1}] \supseteq Ai \cap Ai+1$ and $s_{i+1} \leq \inf S(\cup_{j>i+1} Aj)$, $s_i \geq \sup S(\cup_{j<i} Aj)$, so that this property holds. Q.E.D.

Note that for any A' normalized, such that $Ai \cap Ai+1 \supseteq A'$, then $B' = Bi \cap Bi+1$.

Let us consider now disjunctive facts of the form $A' = A_I = \cup_{i\in I} Ai$.

<u>Proposition 4</u> : If $\mathfrak{R}$ is separated and $A' = A_I$ then $B' = B_I = \cup_{i\in I} Bi$.

Proof: $B_I \supseteq B'$ is obvious from the right most inequality in (25). Now

$\forall t,\ \mu_{B'}(t) = \sup_s \min(\max_{i\in I} \mu_{Ai}(s),\ \min_{i\in N} \mu_{Ai}(s)\rightarrow\mu_{Bi}(t))$

$= \max_{i\in I} \sup_s \min(\mu_{Ai}(s),\ \min_{i\in N} \mu_{Ai}(s)\rightarrow\mu_{Bi}(t))$

$= \max_{i\in I} \mu_{Bi}(t)$ since $\mathfrak{R}$ is separated. Q.E.D.

<u>Corollary 2</u> : If $\mathfrak{R}$ is separated and $A' = A_I$ then $B' = \cap_{I,\ N\supseteq I} A' \circ (A_I\rightarrow B_I)$.

This corollary indicates the potential of the inference engine algorithm (STC) applied to the saturated rule base, to deal with disjunctive facts which are not treated appropriately in the usual approach.

Note, however, that the equality $A' \circ (\cap_{i\in N} Ai\rightarrow Bi) = \cap_{I,\ N\supseteq I} A' \circ (A_I\rightarrow B_I)$ does not hold generally as shown by the following counter-example.

<u>Counter-example</u> : Let us consider two rules $A1\rightarrow B1$ and $A2\rightarrow B2$ such that $C(A1)$ is not contained in $S(A2)$ and $C(A2)$ is not contained in $S(A1)$ (Separatedness). Let A' be such that $S(A1) \supseteq A'$ and $C(A')$ is not contained in $S(A2)$. Clearly, $A' \circ (A2\rightarrow B2) = T$, $S(B1) \supseteq A' \circ (A1\rightarrow B1) \supseteq C(B1)$; $A' \circ (A1\cup A2\rightarrow B1 \cup B2) \supseteq A' \circ (A1\rightarrow B1)$. Hence, $\cap_{I,\ N\supseteq I} A' \circ (A_I\rightarrow B_I) = A' \circ (A1\rightarrow B1)$.

Now $\forall t,\ \mu_{B'}(t) = \sup_s \min(\mu_{A'}(s),\ \mu_{A1}(s)\rightarrow\mu_{B1}(t),\ \mu_{A2}(s)\rightarrow\mu_{B2}(t))$. Letting t such that $\mu_{B1}(t) = \lambda_1 > \mu_{B2}(t) = \lambda_2 > 0$. Figure 8 depicts a case where $\mu_{A'o(A1\rightarrow B1)}(t) = 1$ while $\mu_{B'}(t) = \lambda_1 = \mu_{B1}(t)$ only. Hence, in this case $N = \{1, 2\}$ and $B' = A' \circ (\cap_{i\in N} Ai\rightarrow Bi)$ is strictly included in $\cap_{I,\ N\supseteq I} A' \circ (A_I\rightarrow B_I)$.

ponding possible value of Y must be in $B1 \cap B2$. If the property (15) with $\mu_R(s, t) = \min_{i=1,2} \mu_{Ai}(s) \to \mu_{Bi}(t)$ is not satisfied it implies that $B1 \cap B2$ is empty and, thus Y cannot be assigned any value. In other words, it means that the two rules are contradictory.

Let us write (15) in the case of a set of rules expressed by means of Gödel implication :

$$(15) \Leftrightarrow \forall s \in S, \exists t \in T, \forall i \in N, \ \mu_{Ai}(s) \to \mu_{Bi}(t) = 1 \text{ where } N = \{1, 2, \ldots, n\}.$$

Let $I_S = \{i \,/\, \mu_{Ai}(s) > 0\}$; we get

$$\forall s \in S, \exists t \in T, \forall i \in I_S, \ \mu_{Ai}(s) \le \mu_{Bi}(t). \qquad (16)$$

Since the inequalities must hold for any s, let us write $\mu_{Ai}(s) = \alpha_i$ and then (16) is equivalent to :

$$\forall \text{ I such that } N \supseteq I, \ \exists t \in T, \forall i \in I, \ \mu_{Ai}(s) = \alpha_i \Rightarrow \mu_{Bi}(t) \ge \alpha_i$$

which boils down to the following consistency condition

$$\forall \text{ I such that } N \supseteq I, \ \cap_{i \in I} Ai(\alpha_i) \ne \emptyset \Rightarrow \cap_{i \in I} Bi(\alpha_i) \ne \emptyset, \ \forall (\alpha_i, i \in I), \qquad (17)$$

where $Ai(\alpha_i)$ is the α_i-cut of Ai. A consequence of (17) is that

$\forall \alpha, \forall$ I such that $N \supseteq I$, if $\cap_{i \in I} Ai(\alpha) \ne \emptyset$ then $\cap_{i \in I} Bi(\alpha) \ne \emptyset$, which is equivalent to

$$\forall \text{ I such that } N \supseteq I, \ hgt(\cap_{i \in I} Ai) \le hgt(\cap_{i \in I} Bi) \qquad (18)$$

where $hgt(F) = \sup \mu_F$, a consistency condition already sugested in [Dub82].

It is easy to find examples where (18) holds but (17) does not. So (18) may be used for inconsistency checking only.

It is important to ensure at first sight that a set of rules is consistent. A sufficient condition for that can be

$$\forall \text{ I such that } N \supseteq I, \text{ if } \cap_{i \in I} S(Ai) \ne \emptyset \text{ then } \cap_{i \in I} C(Bi) \ne \emptyset, \qquad (19)$$

where S(F) and C(F) denote the support and the core of F, respectively defined by $S(F) = \{u, \ \mu_F(u) > 0\}$ and $C(F) = \{u, \ \mu_F(u) = 1\}$; it is obvious that (19) implies (17). (19) is called strong consistency.

Note that consistency implies $\forall$ A´ normalized, B´ is normalized too.

4.2. Redundancy

A rule r = "if X is A then Y is B", or $r = A \to B$ for short, is redundant with respect to $\{r_i = Ai \to Bi \,/\, i = 1,n\}$ if it brings nothing new that was not already in the knowledge base. Let us denote (9) as $B' = A' \circ (\cap_{i \in N} Ai \to Bi) = A' \circ R$, for short. The rule $A \to B$ is redundant if and only if

$$\forall A', \ A' \circ R = A' \circ (R \cap (A \to B))$$

which is equivalent to $R = R \cap (A \to B)$, or in more detailed terms

$$\forall s, \forall t, \exists i \ \ \mu_A(s) \to \mu_B(t) \ge \mu_{Ai}(s) \to \mu_{Bi}(t).$$

Examples of redundant rules with respect to $\Re = \{Ai \to Bi \,/\, i = 1,n\}$ are created as follows.

Proposition 1 : The rule $Ai \cup Aj \to Bi \cup Bj$ where $\mu_{Ai \cup Aj} = \max(\mu_{Ai}, \mu_{Aj})$ is redundant with respect to $\Re$.

Proof : It is sufficient to prove that $\forall$ a, b, c, d $\min(a \to b, c \to d) \le \max(a, c) \to \max(b, d)$. Indeed if $a \le b$, $c \le d$ then $\max(a, c) \le \max(b, d)$ and both sides are 1, otherwise $\min(a \to b, c \to d) \le \max(b, d)$. Q.E.D.

Consequently, for any I included in or equal to N, the rule $A_I \to B_I$, where $A_I = \cup_{i \in I} Ai$, is redundant with respect to $\Re$ and we have the following equality :

$$\forall A', \ A' \circ (\cap_{i \in N} Ai \to Bi) = A' \circ (\cap_{I, N \supseteq I} A_I \to B_I). \qquad (20)$$

A rule base $\Re$ is said to be saturated if and only if

$$\forall (Ai \to Bi), (Aj \to Bj) \in \Re, \ Ai \cup Aj \to Bi \cup Bj \in \Re.$$

If $\Re$ is not saturated then $\underline{\Re} = \Re \cup \{A_I \to B_I \,/\, N \supseteq I\}$ is called its saturation. This concept will prove very useful to derive an efficient inference technique.

In order to make a first step to redundancy checking, let us examine conditions under which a rule $Aj \to Bj$ is redundant with respect to another rule $Ai \to Bi$. A necessary and sufficient condition is provided by Formula (21).

$$\forall s, \forall t, \ \mu_{Aj}(s) \to \mu_{Bj}(t) \ge \mu_{Ai}(s) \to \mu_{Bi}(t). \qquad (21)$$

It is easy to check that $a \to b \ge c \to d \Leftrightarrow (a \le b)$ or $(a > b$ and $b \ge c \to d)$.

Hence (21) is equivalent to

$$\forall s, t, \text{ if } \mu_{Ai}(s) \le \mu_{Bi}(t) \text{ then } \mu_{Aj}(s) \le \mu_{Bj}(t), \qquad (22)$$

$$\text{if } \mu_{Ai}(s) > \mu_{Bi}(t) \text{ then } (\mu_{Aj}(s) \le \mu_{Bj}(t)) \text{ or } (\mu_{Bj}(t) \ge \mu_{Bi}(t)). \qquad (23)$$

. $\mu_{Aj}(s) \le \mu_{Bj}(t)$ $\forall$ s, t, implies that Bj = T since Aj is normalized. This is the trivial rule "if X is Aj then Y is T" which is redundant with anything.

. The other condition leads to $\mu_{Bj}(t) \ge \mu_{Bi}(t)$, $\forall$ t such that $1 > \mu_{Bi}(t)$, choosing s sò that $\mu_{Ai}(s) = 1$.

Moreover, (22) implies that $\forall t \in C(Bi)$, $\mu_{Bj}(t) \ge \max_s \mu_{Aj}(s)$, i.e. $\mu_{Bj}(t) = 1$ since Aj is normalized. So (22), (23) imply that $Bj \supseteq Bi$. Now, given Aj , Bi, and $Bj \supseteq Bi$, (22) implies

$$\forall s, \ \mu_{Aj}(s) \le \inf\{\mu_{Bj}(t) \,/\, \mu_{Bi}(t) \ge \mu_{Ai}(s)\}. \qquad (24)$$

When Bj = Bi, (24) leads to $Ai \supseteq Aj$, which is a rather expected result. So, $Aj \to Bj$ is redundant with respect to $Ai \to Bi$ if and only if $Bj \supseteq Bi$ and Aj satisfies (24).

4.3. A new approach to multiple-rule inference

As recalled in Subsection 2.2, given a set $\Re$ of n rules relating two universes S and T, and given a fact "X is A´" we have the following inequality :

$$\cap_{i \in N} A' \circ (Ai \to Bi) \supseteq A' \circ (\cap_{i \in N} Ai \to Bi)$$

which expresses that triggering the rules separately, and combining the partial results in a second step does not provide conclusions as precise as one could get by combining the rules in a first step. The usual inference engine algorithm "select, trigger and combine" (STC) is especially inadequate when facts are disjunctive. Indeed, in general, a disjunctive fact of the form $A' = A_I = \cup_{i \in I} Ai$, such that $N \supseteq I$, yields

$\cap_{i \in N} A_I \circ (Ai \to Bi) = T$ as long as the set I contains more than one element.

As a consequence, $\mu_{B^\#}$ differs from $\mu_{B'}$ only in the area of transition from the core to the values having a uniform degree of possibility induced by the level of uncertainty.

Concludingly, in any case, the inference process amounts to a simple transformation of B into $B^\#$. In the first case (i.e. total uncertainty), the transformation yields the universe T. In the fifth case (i.e. the fact perfectly satisfies the condition), the transformation is the identity. In cases 2 to 4 (i.e. partial matching), the distribution associated to $B^\#$ is of a θ-trapezoidal shape as shown in Figure 7. Its θ-support is the support of B and the transformation consists at most in two operations, each directly linked to the compution of ω_0 or ω_1. Indeed, from ω_0 one gets the level of uncertainty (or indetermination) θ pervading the conclusion and from ω_1 one can derive quantitatively the widening (or enlargement) to be performed on the core of B in order to obtain the core of $B^\#$. More explicitly, the five-tuple describing $B^\#$ is such that

$$\begin{aligned} b'1 &= b1 \\ b'2 &= b2 - (1-\omega_1)(b2-b1) \\ b'3 &= b3 + (1-\omega_1)(b4-b3) \qquad (14) \\ b'4 &= b4 \\ \theta &= \omega_0 \end{aligned}$$

where $\omega_1 = \min(\mu_A(a'2), \mu_A(a'3))$ and $\omega_0 = \max(\mu_{A'}(a1), \mu_{A'}(a4))$.

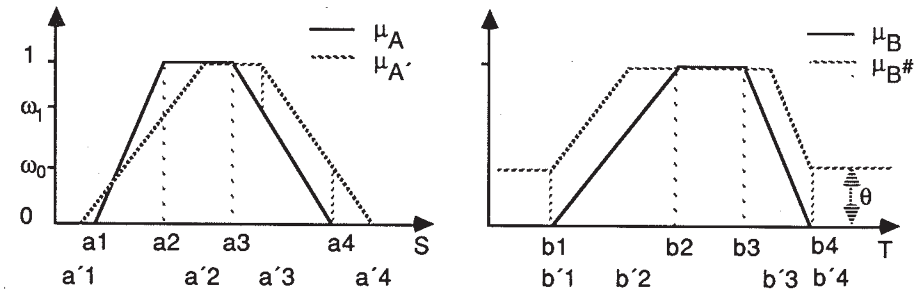


Figure 7.

3.4. Uncertain facts and compound variables

Assume now that A′ is pervaded by a level of uncertainty λ and is defined via $\mu_{A'}(s) = \max(\mu_\Lambda(s), \lambda)$ where μ_Λ is represented by the four tuple (a′1 a′2 a′3 a′4). As shown at the very beginning of this section, the mapping f_λ that modifies B into B_λ' is defined by $f_\lambda = \max(f, \lambda)$ where f is obtained out of Λ only, as has been explained in the above subsections. Therefore, an approximation $\mu_{B_\lambda\#}$ of $\mu_{B_\lambda'}$ is readily obtained as the five tuple (b′1 b′2 b′3 b′4 θ_λ) where the first four parameters are defined as in (14), and $\theta_\lambda = \max(\omega_0, \lambda)$ with $\omega_0 = \max(\mu_\Lambda(a1), \mu_\Lambda(a4))$.

An important feature of the proposed approximation is that the distributions involved in facts and those deduced through the generalized modus ponens have the same λ-trapezoidal shape, which is very useful for chaining rules.

Finally, let us consider the case of X being a compound variable of the form $X = (X_1,\ldots,X_p)$ where the X_r's $r=1,\ldots,p$, are non-interactive variables. Such a situation corresponds to a conjunction "X_1 is A_1 and...and X_p is A_p" of non-interactive conditions in the antecedent part of a rule. Let the fact "X is A′" be of the form "X_1 is A'_1 and...and X_p is A'_p" or equivalently "X is $A'_1 \times \ldots \times A'_p$" and let $\Pi'_1,\ldots,\Pi'_p$ be the possibility measures associated with $A'_1,\ldots,A'_p$ respectively. Noting that $A_\varpi = A_{1\varpi} \times \ldots \times A_{p\varpi}$ it is easy to verify that, in Formula (12) we have $\Pi'(\neg A_\varpi) = \max_{r=1,\ldots,p} \Pi'_r(\neg A_{r\varpi})$. Let f_r denote the mapping that modifies B into B'_r through the artificial rule $A_r \to B$ and the fact "X_r is A'_r". Then, for any ω in [0,1], $f(\omega) = \max_{r=1,\ldots,p} f_r(\omega)$ and the mapping $f^\#$ is thus easily obtained from $f^\#_r$, $r=1,\ldots,p$. The pair (ω_0, ω_1) associated with $f^\#$ is easily calculated from the pair (ω^r_0, ω^r_1) associated with $f^\#_r$, as follows :

$\omega_0 = \max_{r=1,\ldots,p} \omega^r_0$ and $\omega_1 = \min_{r=1,\ldots,p} \omega^r_1$.

4. USING A COLLECTION OF RULES : PRELIMINARY RESULTS

This section addresses the problem of the practical computation of the B′ deduced in the pattern (8). For the sake of clarity, all along this section it is assumed that X is a non-compound variable.

When using a set of rules one has to be careful about their consistency. A natural consistency condition to be satisfied by a collection of rules is given in Subsection 4.1. The complementary question is that of redundancy. It is the topic of Subsection 4.2. Subsection 4.3 gives some theoretical results about the general case of dependency between X and Y. Subsection 4.4 provides a practical computation technique when the involved fuzzy sets are trapezoidal fuzzy numbers. Some examples are closing the section.

4.1. Consistency of rules

With the Gödel implication function and the hypotheses of normality of the involved distributions, any rule "if X is Ai then Y is Bi " verifies that for a given s in S there is at least one corresponding element t in T such that $\mu_{Ai}(s) \le \mu_{Bi}(t)$. In other words, such a rule, considered in isolation, represents a fuzzy relation R defined by

$\mu_R(s,t) = \mu_{Ai}(s) \to \mu_{Bi}(t)$ as specified by Formula (3) and having the property :

$$\forall s \in S, \exists t \in T \text{ so that } \mu_R(s,t) = 1. \qquad (15)$$

A collection of rules is inconsistent if the fuzzy relation R, defined this time by the conditional possibility distribution $\min_{1\le i\le n} \mu_{Ai}(s) \to \mu_{Bi}(t)$, no longer satisfies the property (15). Let us illustrate with an example involving two non-fuzzy rules, say "if X is A1 then Y is B1" and "if X is A2 then Y is B2" where A1, A2, B1 and B2 are crisp sets, how the property (15) is related to their consistency. Notice that for each rule (i=1 or i=2) we have $\mu_{Ai}(s) \to \mu_{Bi}(t) = 1$ when $\mu_{Ai}(s) = 1$ provided that $\mu_{Bi}(t) = 1$. This simply expresses that if we consider a value of X in Ai the corresponding possible value of Y is in Bi. Now, if a given value of X is in A1 ∩ A2 it can be proved that the corresthen $f^+(\omega) = 1$ if $\omega \ge \omega^+_1$

$= \varpi^+ + (\omega - \varpi^+)(1-\varpi^+) / (\omega^+_1 - \varpi^+)$ if $\varpi^+ < \omega \le \omega^+_1$

$= \omega$ if $\omega \le \varpi^+$.

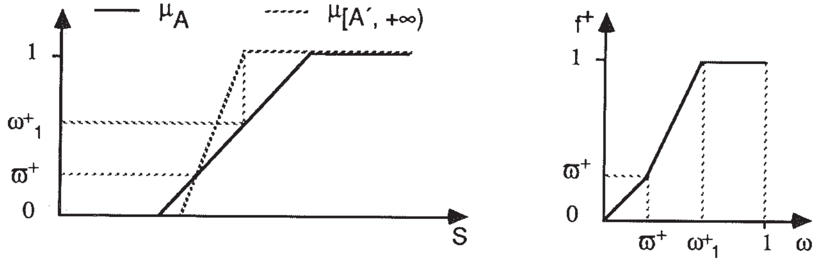


Figure 4.

4th case $a'1 < a1$; $a2 < a'2$. See Figure5.

Then $f^+(\omega) = \omega$ if $\omega \ge \varpi^+$

$= \omega^+_0 + \omega\,(\varpi^+ - \omega^+_0) / \varpi^+$ otherwise.

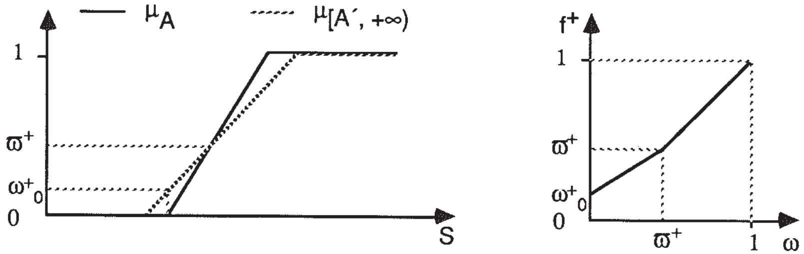


Figure 5.

5th case $a1 \le a'1$; $a2 < a'2$

Then $f^+(\omega) = \omega \;\forall \omega \in [0, 1]$.

The calculation of f^- is similar, changing (a1, a2) into (a4, a3) , (a´1, a´2) into (a´4, a´3) and reversing the inequalities. The quantities ω^-_0, ω^-_1, ϖ^- are evaluated similarly to ω^+_0, ω^+_1, ϖ^+.

Hence both f^+ and f^- are piecewise linear, and f is thus also piecewise linear, but possibly involving more break-points.

3.3. Approximating the exact computation

In order to simplify the calculation and facilitate the chaining of rules, only an approximation of f, say $f^\#$, is computed. The construction of $f^\#$ which is such that $f^\# > f$ is given next. The five cases elicited in the previous subsection are considered in turn.

First, note that, in case 1, $\omega^+_0 = 1$, $\omega^+_1 = 0$, $\varpi^+ \notin\,]0,1[$;

in case 2, $\varpi^+ \notin\,]0,1[$;

in case 3, $\omega^+_0 = 0$;

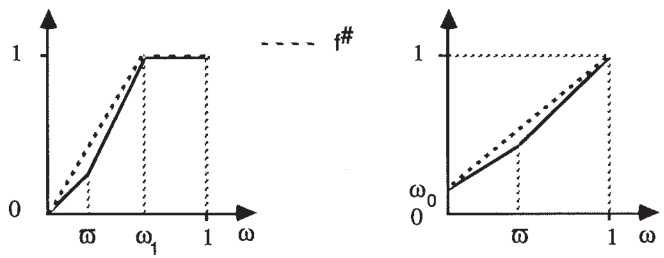
in case 4, $\omega^+_1 = 1$;

in case 5, $\omega^+_0 = 0$, $\omega^+_1 = 1$, $\varpi^+ \notin\,]0,1[$.

Thus, the quantities ω^ε_0, ω^ε_1, ϖ^ε, $\varepsilon \in \{-,+\}$ are always defined.

Let $\omega_0 = \max(\omega^+_0, \omega^-_0)$ and $\omega_1 = \min(\omega^+_1, \omega^-_1)$. The following tests are made to specify the shape of $f^\#$.

1. If $\omega_0 = 1$ or $\omega_1 = 0$ then $f^\# = f$ and $f^\#(\omega) = 1$, as in case 1. This case results in total uncertainty about the conclusion.
2. If $0 < \omega_0 < 1$ or $0 < \omega_1 < 1$ then $f^\# \ge f$ and $f^\#$ is defined as f^+ in case 2 (deleting the +).
3. If $\omega_0 = 0$ and $0 < \omega_1 < 1$ then set $\varpi = \min(\varpi^+, \varpi^-)$ and $f^\#$ could be defined as f^+ in case 3 (deleting the +). We obviate the need for computing ϖ by setting $\varpi = 0$ (i.e. the dashed line in Figure 6).
4. If $0 < \omega_0 < 1$ and $\omega_1 = 1$ then set $\varpi = \max(\varpi^+, \varpi^-)$ and $f^\#$ could be defined as f^+ in case 4 (deleting the +). We obviate the need for computing ϖ by setting $\varpi = 1$ (i.e. the dashed line in Figure 6).
5. If $\omega_0 = 0$ and $\omega_1 = 1$ then $f^\# = f$ as in case 5. This situation corresponds to A´ included in or equal to A (which implies B´ = B).



case 3 case 4

Figure 6.

Thus, $f^\#$ is fully characterized by the pair (ω_0, ω_1). Let us define $B^\#$ by $\mu_{B^\#}(t) = f^\#(\mu_B(t))$ for any t in T. $B^\#$ compares to the actual result B´ as follows :

- $B^\#$ contains B´ since $f^\# \ge f$; therefore, deducing $B^\#$ instead of B´ is logically valid.
- The core of $B^\#$ is equal to the core of B´ ; indeed, the peak of B´ is dictated through the set $\{\omega \,/\, f(\omega)=1\}$ whose lower bound is clearly $\min(\omega^+_1, \omega^-_1)$ which, by construction, is also the lower bound of $\{\omega \,/\, f^\#(\omega)=1\}$.
- $B^\#$ and B´ have the same level of uncertainty given by $\inf \mu_{B'} = \inf \mu_{B^\#} = \omega_0$ since $\inf\{ f(\omega) \,/\, \omega \in [0,1]\} = \omega_0 = \inf\{ f^\#(\omega) \,/\, \omega \in [0,1]\}$.
- The points where $\mu_{B^\#}$ reaches its level of uncertainty are the same as for B´ since they coincide with the endpoints of the support of B.

$\mu_{B'}(t) = \max [\sup_{s\in S} \min(\mu_A(s) , \mu_A(s)\rightarrow\mu_B(t)) , \sup_{s\in S} \min(\lambda , \mu_A(s)\rightarrow\mu_B(t))] =$

$\max [\sup_{s\in S} \min(\mu_A(s), \mu_A(s)\rightarrow\mu_B(t)) , \lambda].$

Consequently, B´ is easy to obtain provided one knows how to compute (10) when A´ is given by a trapezoidal distribution.

In the next three subsections, it is assumed that X is a non-compound variable and that A´ is not pervaded by a level of uncertainty ; A´ is represented by the four-tuple (a´1 a´2 a´3 a´4). Subsection 3.1 shows that the computation of the supremum expressed by (10) can be performed via a generic treatment applied independently on two subparts of S where the interesting phenomenas occur. Subsection 3.2 presents the above-mentioned generic treatment which permits an exact computation of (10). Usually, an approximation of the exact results is sufficient and convenient. Subsection 3.3 provides such an approximation. Finally, Subsection 3.4 extends the computation of the approximation of the generalized modus ponens to situations where A´ is given by a λ-trapezoidal distribution or X is a compound variable.

3.1. Decomposing the problem

Letting $\omega = \mu_B(t)$, the calculation of (10) is split into two steps :

i) calculate the mapping of [0,1] into [0,1] expressed by

$\forall \omega, f(\omega) = \sup_{s\in S} \min(\mu_{A'}(s), \mu_A(s)\rightarrow\omega)$;

ii) calculate $\mu_{B'}$ as $\mu_{B'}(t) = f(\mu_B(t))$.

Let us focus on the first step. Using the definition of →, expressed by (3), leads to

$f(\omega) = \max(\sup_{s\notin A_\varpi} \mu_{A'}(s) , \sup_{s\in A_\varpi} \min(\omega, \mu_{A'}(s)))$

$= \max(\Pi'(\neg A_\varpi), \min(\omega,\Pi'(A_\varpi)))$ (11)

where A_ϖ is the strong ω-cut of A (i.e. $A_\varpi = \{s\in S / \mu_A(s) > \omega\}$), and Π´ is the possibility measure based on the distribution $\mu_{A'}$ (i.e. for any fuzzy or non-fuzzy subset F of S we have $\Pi'(F) = \sup_{s\in S} \min(\mu_F(s), \mu_{A'}(s))$).

Note that if $\Pi'(\neg A_\varpi) \geq \omega$ then $f(\omega) = \Pi'(\neg A_\varpi) \geq \omega$,

whereas if $\Pi'(\neg A_\varpi) < \omega$ then $\Pi'(\neg A_\varpi) = 1$ and $f(\omega) = \omega$.

Hence, in the general case, (11) can be simplified into

$f(\omega) = \max(\Pi'(\neg A_\varpi), \omega)$. (12)

N.B. : Incidentally, Formula (12) shows, if needed, that B´ ⊇ B whatever A´.

Denoting A_ϖ by the open interval $]\nabla_\omega, \Delta_\omega[$, it is not difficult to figure out that

$\Pi'(\neg A_\varpi) = \max(\mu_{[A', +\infty)}(\nabla_\omega) , \mu_{(-\infty, A']}(\Delta_\omega))$

where $\mu_{[A', +\infty)}$ and $\mu_{(-\infty, A']}$ are fuzzy intervals defined by (see Figure 2)

$\mu_{[A', +\infty)}(s) = \sup_{u\leq s} \mu_{A'}(u)$

$\mu_{(-\infty, A']}(s) = \inf_{u\geq s} 1 - \mu_{A'}(u)$.

Introducing the auxilliary functions f^+ and f^- defined by

$f^+(\omega) = \max(\mu_{[A', +\infty)}(\nabla_\omega), \omega)$ (13a)

$f^-(\omega) = \max(\mu_{(-\infty, A']}(\Delta_\omega), \omega)$ (13b)

we have $f = \max(f^+, f^-)$.

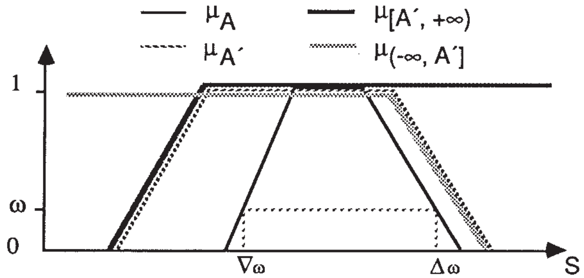


Figure 2.

Consequently, the computation of the generalized modus ponens can be decomposed into two independent calculations corresponding to local evaluations of what is happening on each side of A. These two calculations are obviously similar. In the next subsection, we give the complete results pertaining to one of them. Namely, we show how to compute the effects of the mismatching between A´ and A in the case where the latter takes place on the side of the lower possible values constrained by A ; this side is henceforth referred to as the left hand side of A (respecting the left to right orientation usually employed in graphical representations of the real line).

3.2. Evaluating the effect of what is happening on the left hand side of A

This subsection gives a detailed presentation of how to calculate f^+ as defined by (13a). The shape of f^+ depends upon the respective locations of (a1, a2) with respect to (a´1 a´2).

1st case a´2 ≤ a1. See Figure2.

Then clearly $f^+(\omega) = 1 \quad \forall \omega \in [0, 1]$.

2nd case a´1 ≤ a1 < a´2 ≤ a2. See Figure3.

Let $\omega^+_0 = \mu_{[A', +\infty)}(a1)$ and $\omega^+_1 = \mu_{[A, +\infty)}(a'2)$

then $f^+(\omega) = 1$ if $\omega \geq \omega^+_1$

$= \omega^+_0 + \omega (1- \omega^+_0) / \omega^+_1$ otherwise.

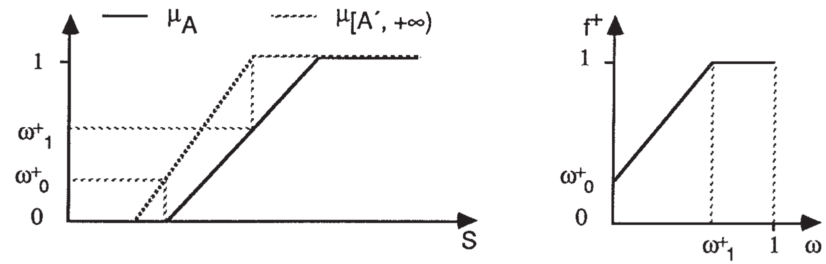


Figure 3.

3rd case a1 < a´1 ; a´2 < a2. See Figure4.

Let ϖ^+ such that $\mu_A(s) = \mu_{A'}(s) = \varpi^+ < 1$

associated implication function, denoted by $\mu_A(s) \to \mu_B(t)$, which is known as the Gödel operator given by (3).

$$\mu_A(s) \to \mu_B(t) = \begin{cases} 1 \text{ if } \mu_A(s) \le \mu_B(t) \\ \mu_B(t) \text{ otherwise} \end{cases} \quad \text{for any } s \text{ in } S \text{ and any } t \text{ in } T. \qquad (3)$$

Actually, within the interpretation linked to the t-norm 'min', there is another possible semantic acceptation of the dependency between X and Y ; this latter acceptation being associated to the contrapositive form of the rule (i.e "if Y is not B then X is not A") since, in fuzzy logic, a rule is not necessarily equivalent to its contrapositive form. Thus, according to this acceptation and with the usual definition of complementation (i.e. $\mu_{\neg A}(s) = 1 - \mu_A(s)$), the conditional possibility distribution is taken as

$$\pi_{Y/X}(s, t) = \begin{cases} 1 \text{ if } 1-\mu_B(t) \le 1-\mu_A(s) \\ 1-\mu_A(s) \text{ otherwise} \end{cases} \quad \text{for any } s \text{ in } S \text{ and any } t \text{ in } T. \qquad (4)$$

For instance, if A is a fuzzy set and B is a crisp set, Formula (4) gives

$$\pi_{Y/X}(s, t) = \max(\mu_B(t), 1-\mu_A(s)). \qquad (5)$$

Thus, for a given s_o in A, $\pi_{Y/X}(s_o, t) = \max(\mu_B(t), \lambda)$ with $\lambda = 1-\mu_A(s_o)$ and therefore $\pi_{Y/X}(s_o, .)$ is of the λ-trapezoidal form ; this means that the smaller λ the more certain that the value of Y is in B. In other words, the contrapositive form of the rule expresses a conditional piece of knowledge of the form "the more X is A the more certain the proposition Y is B". Formula (3) does not support the above interpretation.

In the sequel, we shall only consider the acceptation in agreement with (3). Basically, according to this acceptation, the generalized modus ponens is characterized by the following properties :

- B′= B when A ⊇ A′.
- A uniform level of indetermination (or uncertainty) appears in B′ as soon as the support of A′ is not contained in A, that is, if it is somewhat possible that the value of X is completly outside the scope of the rule.
- If the core of A′ is not contained in the core of A, then it causes the core of B′ to be larger than the core of B, yet still being contained in the support of B. In other words, if any completely possible value for X is outside the class of values that are surely within the scope of the rule, then the set of the completely possible values for Y becomes less specific that the one expressed in the rule.

If X is a compound variable of the form $X = (X_1,...,X_p)$ where the X_r's $r=1,...,p$, are non-interactive variables (i.e. the possible value of one variable does not depend on the possible values of the others) it means that $A = A_1 x...x A_p$ and $A' = A'_1 x...x A'_p$. In such a case, μ_A and $\mu_{A'}$ in Formula (3) and in the definition of $\pi_{Y/X}$ must be taken as follows [Zad78] :

$$\forall s = (s_1,...,s_p) \in S,\ \mu_A(s) = \min_{r=1,...,p} \mu_{A_r}(s_r) \qquad (6)$$

$$\forall s = (s_1,...,s_p) \in S,\ \mu_{A'}(s) = \min_{r=1,...,p} \mu_{A'_r}(s_r) \qquad (7).$$

Each of the A_r's and A'_r's is respectively represented by a four-tuple and a five-tuple.

Usually, the dependency between X and Y is described through a collection of rules rather than a single one. The generalized modus ponens problem corresponds then to the following pattern of reasoning.

X is A′
if X is Ai then Y is Bi $\quad i=1,...,n \qquad$ (8)

Y is B′

One way of processing a collection of rules could consist, first, in making as many inferences as there are rules and, second, in combining the results provided by each of them via a fuzzy set intersection. In reality, it has been demonstrated [Dub84] [Dub85b] that in such a case, it is better to combine the rules before the inference is performed. This prior combination permits to take into account the fact that the rules complement each other. The main advantage of the technique involving a prior combination is that it may provide a more specific conclusion in many situations (and an as specific one in the others). Indeed, consider a relationship between X and Y described by a collection of rules as indicated in Pattern (8). Assume the fact "X is A′" is such that $A' = A_j \cup A_k$ with $1 \le j < k \le n$. Then, the technique involving a prior combination of the n rules yields the conclusion "Y is B′" where B′ is included in or equal to $B_j \cup B_k$. In the same situation, the other way of processing does not necessarily preserve this desirable property (actually, it may likely yield a completely indeterminate conclusion). If one sticks to the interpretation associated to the min-Gödel operators, the computation of the generalized modus ponens with a collection of rules (used jointly) is done according to the following formula [Dub84], [Dub85b] :

$$\mu_{B'}(t) = \sup_{s \in S} \min(\mu_{A'}(s), \min_{i=1,...,n} \pi^i_{Y/X}(s,t)) \quad \text{for any } t \text{ in } T \qquad (9)$$

where $\pi^i_{Y/X}$ is the conditional possibility distribution built from the i-th rule with the Gödel function. The prior combination of rules stands in the use of $\min_{i=1,...,n} \pi^i_{Y/X}(s,t)$ as the global conditional possibility distribution. In this paper, it is assumed that $\pi^i_{Y/X}$ is in agreement with the acceptation defined by Formula (3).

Works oriented toward the practical computation of $\mu_{B'}$ in Formula (9) are still in progress. Some simple--yet having practical significance--cases of this problem are dealt with in Section 4. The next section gives the complete results concerning the case where the deduction is based on a rule used isolately.

3. THE SINGLE RULE CASE

This section gives a technique for obtaining B′ in a situation fitting Pattern (1).
For any t in T, $\mu_{B'}(t)$ is given by

$$\mu_{B'}(t) = \sup_{s \in S} \min(\mu_{A'}(s), \mu_A(s) \to \mu_B(t)) \qquad (10)$$

where $\mu_A(s) \to \mu_B(t)$ is the Gödel implication function that approximates the conditional possibility distribution according to (3).

As explained in Subsection 2.1, the possibility distribution $\mu_{A'}$ may be such that, for any s in S, $\mu_{A'}(s) = \max(\mu_A(s), \lambda)$. If so, by distributivity of "min" over "max", we have

X is A´
<u>if X is A then Y is B</u> (1)
Y is B´

Basically, this means that from a rule which associates a variable X --for which we consider the class of values represented by the fuzzy set A-- with a variable Y --that takes then a value loosely specified by the elastic constraint B-- and a fact "X is A´" expressing what information is available about the value of X, one can infer that "Y is B´" where B´ is the deduced elastic constraint on Y. Any elastic constraint is represented by a fuzzy set.

Our approach exploits the fact that, from a pratical point of view, it is sufficient to use parametrized functions for representing the involved fuzzy sets in the given rule and the observed datum. By means of a simple approximation and without any costly computation one can then derive a parametrized function of the same kind for representing the deduced value of Y. The technique is developed for the particular setting where the involved possibility distributions are continuous, normalized and unimodal. Moreover, the rules are interpreted according to the 'sup-min' composition with the Gödel implication function.

The next section provides some background about the generalized modus ponens and introduces the parametrized functions used as representation tools. In Section 3 the computation of the generalized modus ponens is treated in the case where a single rule is used at one time. When the dependency between the two considered variables is expressed via a collection of rules it has been shown that the best result is obtained through a global treatment of the rules rather than an independent and isolate use of each of them. Section 4 deals with the computation of the generalized modus ponens when such a global treatment of a collection of rules is performed.

2. KNOWLEDGE REPRESENTATION AND THE GENERALIZED MODUS PONENS

2.1. Unconditional information

First, let us consider the pattern of reasoning expressed by (1) and let us suppose that X is not a compound variable (i.e. X is not of the form $X = (X_1,..., X_p)$). The variables X and Y are taking their values in S and T which are both assumed to be subsets of the real line.

The imprecise value A´ of X is represented by the possibility distribution π_X. If it is known with absolute certainty that the value of X lies somewhere in a particular bounded subpart of S then π_X is expressed via a trapezoidal distribution which, practically, requires the identification of only four parameters ; two for bounding the core (i.e. the set of completely possible values) and two for representing the support (the set of values that are not completely impossible). Now, if one is not sure that the value of X is in the considered bounded subpart of S it means (without any further information) that it is possible at a degree greater than or equal to a non-zero constant, say λ, that the value of X is anywhere in S. The distribution π_X is then expressed via a so-called λ-trapezoidal function that can be encoded in a five-tuple of the form (a´1 a´2 a´3 a´4 λ) as shown in Figure 1. The interval [a´2, a´3] is the core of A´ and]a´1, a´4[is called its λ-support.

The fuzzy set A appearing in the condition part of the rule represents a class having ill-defined boundaries. It is modeled by a trapezoidal distribution encoded in a four-tuple (a1 a2 a3 a4) equivalent to a five-tuple encoding with the last parameter being equal to 0 (the 0-support is simply the support) ; see Figure 1. The possibility distribution associated to B in Pattern (1) is represented by the four-tuple (b1 b2 b3 b4) and, as we shall see, B´ can be approximated by a five-tuple distribution denoted by (b´1 b´2 b´3 b´4 θ).

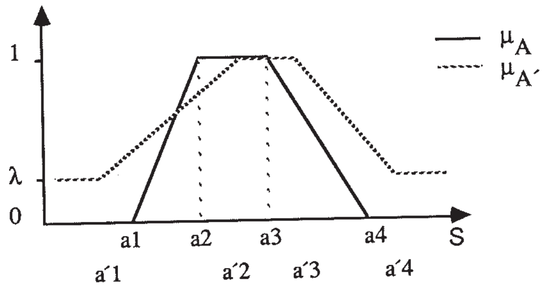


Figure 1.

Thus, the possibility distribution $\mu_{A'}$ depicted in Figure 1 represents an imprecise and uncertain information item [Dub85a] [Pra85]. For example, if the value of X is obtained through a physical measure, the imprecision may come from the technical limits of the measuring devices and the uncertainty may result from the possibility of having misused some of the involved instruments.

The imprecision, considered isolately, may be conveyed by an interval or a fuzzy interval, say Λ, that can be represented by a trapezoidal distribution. The possibility distribution $\mu_{A'}$ is then constructed as $\mu_{A'}(s) = \max(\mu_\Lambda(s) , \lambda)$ for any s in S ; it tells immediately that there is a certainty (or necessity) equal to 1-λ that X is in the support of Λ. Such a representation of unconditional information has been successfully used in several knowledge-based systems equipped with approximate reasoning capabilities ; see, for instance, [Bui86], [Far86], [Leb87] or [Mar87a].

2.2. Conditional information and the generalized modus ponens

As explained in some detailed presentations [Dub84] [Dub85b] the possibility distribution associated to B´, restricting the possible values of Y, is computed as

$$\mu_{B'}(t) = \sup_{s \in S} \text{t-norm} (\mu_{A'}(s), \pi_{Y/X}(s,t)) \quad \text{for any } t \text{ in } T \qquad (2)$$

where "t-norm" is a triangular norm operator (i.e. a conjunction operator) and $\pi_{Y/X}$ is a conditional possibility distribution of Y given X. $\pi_{Y/X}$ is usually unknown, however, one can obtain an approximation of it by considering any implication function built from the rule "if X is A then Y is B". It has been shown in [Dub84] [Dub85b] that, in order to ensure a suitable behavior of the generalized modus ponens technique (i.e. B´= B when A´ is equal to or included in A), one must chose the implication function according to which t-norm is used. Different t-norms correspond to different interpretations of the dependency expressed by the rule "if X is A then Y is B" [Mar87b]. In this paper, we consider only the interpretation linked to the t-norm 'min' and its

Fuzzy Computing
M.M. Gupta and T. Yamakawa (Editors)

PRACTICAL COMPUTING IN FUZZY LOGIC

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Abstract : This paper presents an efficient technique for performing deduction with the generalized modus ponens. This fuzzy logic reasoning tool is considered in the particular setting where the involved possibility distributions are normalized, unimodal, defined on continuums and continuous. In addition, the rules are interpreted according to the 'sup-min' composition with the Gödel implication. The efficiency of the method stems from the fact that the distributions, involved in rules and data, are represented by parametrized functions. The deduction process consits then in some simple computations performed on the parameters. Moreover, the technique involves an approximation that aims at keeping only the meaningful and essential features of the deduced information and at yielding a conclusion of the same form than what is processed as data (thus permitting to chain rules). The paper treats in detail the case where the deduction is based on a single rule. The more general situation where several rules are available for describing the dependency between two variables is considered under some restrictive conditions.

Keywords : Fuzzy logic, Generalized modus ponens, Rule-based systems

1. INTRODUCTION

Fuzzy logic or fuzzy arithmetics make an extensive use of sup-min composition or convolution. Especially when the universes on which the fuzzy sets are defined, are continuums, the practical computation of supremums on such domains may appear to be time-consuming. However it is has been shown for several years that it is possible to perform arithmetic operations on fuzzy numbers defined on the real line, using parametrized representations [Dub80], [Dub85c]. Then the computation comes down to standard arithmetic operations on the parameters. It can be proved that the obtained result is always (at worst) a good approximation of the theoretical one.

This paper presents a similar approach to the computational handling of the generalized modus ponens [Zad79]. In its simplest form the generalized modus ponens is an approximate reasoning technique that can be expressed by the following syllogism.

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* R. Martin-Clouaire is supported by a grant from the Elf Aquitaine company.