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Abstract In this paper, we consider the fixed-time flocking problem of the Cucker-Smale system. We present a variation of the standard Cucker-Smale system, and a simple fixed-time controller is designed. Based on the fixed-time stability theory, sufficient conditions for the system to achieve flocking within fixed time are obtained. Both our theoretical and numerical results demonstrate that the convergence time depends on the group size, and the group with high density can transit to ordered collective motion more rapidly.

Keywords: flocking; fixed-time; Cucker-Smale system

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1 Introduction

Nowadays, due to emerging applications in sensor networks, robot systems [1] and unmanned aerial vehicles [2], research on the collective dynamics has received growing attention from diverse scientific disciplines [3–5]. Flocking, as a typical example of collective motion, is a phenomenon in which a larger number of agents, using only limited environmental information and simple rules, organize into an ordered motion. Flocking phenomena are ubiquitous in nature and human-made systems, such as, the emergent behavior of birds, schooling of fish, and swarming of bacteria [6–8].

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Collective motion is often modeled by agent-based models which assume that each individual alters its behavior according to signals of its neighbors. The well-known Vicsek model [9], originally proposed by Vicsek in 1995, provides a systematical framework to analyze the possible mechanisms by which the flocking may form. In Vicsek's self-propelled particle model, each particle adjusts its velocity in response to the average velocity of its near neighbors. The results obtained from the Vicsek model demonstrate that disordered group movement can transit to ordered motion as the group density increases [9]. Some theoretical explanations of the Vicsek model can be found in [10]. Self-propelled particles models are nowadays commonly used to study the collective motion. Several modifications of the Vicsek model have been introduced and analyzed both with or without leaders, with or without noise [11].

In 2007, Cucker and Smale proposed a flocking model, which assumes that each particle adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of the other particles [12]. By using the graph theory and matrix theory, Cucker and Smale proved that the conditional flocking or unconditional flocking of the Cucker-Smale (C-S) system could occur, which depends on the level of communication rate [12, 13]. Various modifications of the C-S model have been addressed in recent years. For instance, in order to avoid the collisions, a repelling force between particles was added to the generic model [14]. The emergent behavior of the C-S system under hierarchical or rooted leadership was investigated [15–18]. The effect of the processing time delay was analyzed in [19–21]. The flocking dynamics of the C-S system on digraph has been investigated [22, 23]. Mu et al. investigated the hierarchical flocking of C-S systems under random interactions [24]. By adding the multiplicative or additive noise term into the original C-S model, stochastic flocking of C-S systems in noisy environments has been investigated [25–28].

Most of the previous studies focused on the asymptotical flocking, which means that the convergence time is infinite. However, in the real world, moving animal group can reach the ordered state within finite time. For example, a flock of birds can return to an orderly flight within a finite time after being influenced by occasional interference. The finite-time collective behavior of Vicsek type models has been widely studied [29–33]. However, there are a few results about the finite-time flocking problem of C-S type systems.

Recently, Han et al. [34] investigated the finite-time flocking problem of C-S systems. In their work, a simple finite-time controller is designed and sufficient conditions for the finite-time flocking are obtained by using the theory of finite-time stability for ordinary differential equations [28, 34]. However, the settling time of the finite-time flocking depends on the initial conditions which are difficult to be estimated or even can not be measured. To overcome this limitation, we will design a new controller which can drive the C-S to the desired state in a finite time and, more importantly, the estimation of the settling time is independent of the initial conditions.

The main purpose of this paper is to investigate the fixed-time flocking of a C-S system. A continuous non-Lipschitz C-S type model is proposed. Based on the fixed-time stability theory [35], sufficient conditions for the fixed-time flocking are established. It is shown that the C-S system can reach the flocking in a fixed time if the communication rate satisfies a lower bound. The upper bound of the convergence time is also obtained, which shows a power-law relationship between the convergence time and the group size. Both our analytical and numerical results imply that individuals in groups with high density can transit to ordered collective motion more rapidly. Further, the influence of the control parameters on the convergence rate is also investigated.

The remainder of the paper is organized as follows. In section 2, we give the definition of fixed-time flocking of the C-S system and the fundamental lemmas of this article. Then, we perform an analysis for proposing a theorem in section 3; after that, in section 4, we present numerical simulations to verify the theoretical results in section 3. Finally, we summary our the paper in section 5.

2 Problem statement and preliminaries

The C-S model can be formulated by the following equations:

$$\begin{cases} \dot{x}_i = v_i, \quad 1 \le i \le N, \\ \dot{v}_i = \frac{1}{N} \sum\limits_{j=1}^{N} \psi(\|x_j - x_i\|)(v_j - v_i). \end{cases} \tag{1}$$

Here, $x_i \in R^d$ denotes the position of ith particle. Its velocity is denoted by $v_i \in R^d$. The communication rate $\psi(\cdot) : [0, \infty) \to [0, \infty)$ quantifies the influence between ith and jth particles and is a non-increasing function, the communication rate ψ is taken as $\psi(y) = K/(\sigma^2 + y)^\beta$. The results obtained in C-S model demonstrate that the phase transition between the unconditional and conditional flocking depends on the parameter β [12].

In this paper, we consider an interacting particle system of N identical self-propelled particles. Throughout this paper, we denote by x_i the position of ith particle evaluated at t, and by v_i the velocity of ith particle evaluated at t. Let $\psi_{ij} \doteq \psi(\|x_j - x_i\|)$ denote the communication rate between particles i and j. In this paper, we assume that the communication rate ψ_{ij} is locally Lipschitz continuous and uniformly bounded away from zero. This assumption means that the communications among particles always exist. The C-S type model is governed by the following differential equations:

$$\begin{cases} \dot{x}_i = v_i, \quad 1 \le i \le N \\ \dot{v}_i = u_i^{F,p} + u_i^{F,q}, \end{cases} \tag{2}$$

where $u_i^{F,\alpha} = \sum\limits_{j=1}^{N} \psi_{ij} \text{sig}(v_j - v_i)^\alpha$, $\text{sig}(v_j - v_i)^\alpha = (\text{sign}(v_{j1} - v_{i1})|v_{j1} - v_{i1}|^\alpha, \ldots, \text{sign}(v_{jd} - v_{id})|v_{jd} - v_{id}|^\alpha)^T$, $0 < p < 1 < q$, and $\text{sign} : \mathbb{R} \to \{-1, 0, 1\}$ is the sign function. The motivation and some physical explanations for the design of the controller (2) are presented in the following Remarks 1 and 2.

Remark 1. *Cucker and Smale proved that the flocking of the generic C-S model (1) can occur in an infinite time duration [12], while in [28, 34], the authors investigated the finite-time flocking* *of the C-S system by applying the theory of finite-time stability. As shown in the Introduction, the settling time of the finite-time flocking depends on the initial conditions of the system, which may be unavailable in advance. In this paper, we extend the results in [28, 34] to fixed-time flocking based on the technology of fixed-time control.*

Remark 2. *It is well known that the linear control presented in the original C-S model (1) can only ensure the asymptotic flocking, i.e., the convergence time is infinite. In order to reach the flocking in a finite-time, the nonlinear control should be used. According to the finite-time control technology, one of the typical nonlinear control is* $u_i^{F,p}$ *which means that the interactions among individuals are attractive (repulsive) when* $v_j > v_i$ (*or* $v_j < v_i$). *Using the nonlinear coupling* $u_i^{F,p}$, *the C-S system can reach the flocking in a finite time, which has been verified by Han et al. in [34]. However, the convergence time obtained by [34] depends on the initial states of individuals which is usually unknown or difficult to estimate. It is easy to see that, for* $0 < p < 1 < q$, $\|v_j - v_i\|^p < \|v_j - v_i\|^q$ *when* $\|v_j - v_i\| > 1$, *while the opposite occurs for* $\|v_j - v_i\| < 1$. *Then, the individuals will receive strong interactions by the control* $u_i^{F,q}$ *(or* $u_i^{F,p}$*) when* $\|v_j - v_i\| > 1$ *(or* $\|v_j - v_i\| < 1$*). Therefore, in this paper, we introduce both* $u_i^{F,p}$ *and* $u_i^{F,q}$ *into the protocol* (2)*, which can ensure the C-S systems reach the flocking in a finite time and the settling time is independent of the initial states of individuals. We will show that if the communication rate has a lower bound, the fixed-time flocking of system (2) can be achieved.*

The following concepts and lemmas play an important role in the proof of the main results.

Lemma 1. *[36]. Let* $a_1, a_2, \ldots, a_n > 0$ *and* $0 < r < p$. *Then*

$$(\sum_{i=1}^{n} a_i^p)^{1/p} \le (\sum_{i=1}^{n} a_i^r)^{1/r}.$$

Lemma 2. *[36] If* $\xi_1, \xi_2, \ldots, \xi_n \ge 0$ *and* $0 < p \le 1$, *then the following inequality holds:*

$$\left(\sum_{i=1}^{n} \xi_i\right)^p \le \sum_{i=1}^{n} \xi_i^p.$$

Lemma 3. *[36] If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $q > 1$, then the following inequality holds:*

$$N^{1-q} \left(\sum_{i=1}^{n} \xi_i \right)^q \leq \sum_{i=1}^{n} \xi_i^q.$$

Lemma 4. *[35] Consider the following equation:*

$$\dot{x} = f(t, x), \quad x(0) = x_0,$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n$ is a nonlinear continuous function. Assume the origin is an equilibrium point of the above equation. If there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_+ \bigcup \{0\}$ such that

(1) $V(x) = 0 \Longleftrightarrow x = 0$;

(2) for some $\alpha > 0$, $\beta > 0$, $0 < p < 1$, $q > 1$, any solution $x(t)$ satisfies the inequality

$$\dot{V}(x(t)) \leq -\alpha V^p(x(t)) - \beta V^q(x(t)),$$

then the origin is globally fixed-time stable and $V(t) \equiv 0$ if

$$t \geq \frac{1}{\alpha(1-p)} + \frac{1}{\beta(q-1)}.$$

3 Main results

In this section, we present sufficient conditions for the existence of fixed-time flocking of system (2). Firstly, we give the definition of the fixed-time flocking of the C-S system.

Definition 1. *We say system (2) reaches a flocking in fixed time if for any initial condition $x_i(0)$, $v_i(0)$ and $1 \leq i, j \leq N$, the solutions $\{x_i, v_i\}(i = 1, \ldots, N)$ to (2) satisfy the following two conditions:* (i) *The difference of all velocities tends to zero in fixed time, i.e., $\|v_i - v_j\| = 0,\ \forall\ t \geq T_{\max}$, where $T_{\max} = \inf\{T : \|v_i - v_j\| = 0,\ \forall\ t \geq T\}$ is called the convergence time.* (ii) *The diameter of a group is bounded i.e., $\sup\limits_{0 \leq t \leq \infty} \|x_i - x_j\|^2 < \infty$.*

Theorem 1. *Consider the C-S type system (2). Assume that the communication rate function* ψ *is locally Lipschitz continuous and has a lower bound, i.e., there exists* $\psi^* > 0$ *such that* $\psi^* = \inf\limits_{1\le i,j\le N} \psi_{ij}$. *Then the system (2) can reach the flocking in fixed time. And the convergence time* T_1 *is estimated by*

$$T_1 \le T_{\max} = \frac{2}{\psi^*(2N)^{\frac{p+1}{2}}(1-p)} + \frac{2}{\psi^* d^{\frac{1-q}{2}} 2^{\frac{q+1}{2}} N^q (q-1)}.$$

Proof. From system (2), we have

$$\begin{aligned}\dot{v}_i &= \sum_{j=1}^{N} \psi(\|x_j - x_i\|)[\text{sig}(v_j - v_i)^p + \text{sig}(v_j - v_i)^q] \\ &= \sum_{j=1}^{N} \psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^p + \sum_{j=1}^{N} \psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^q.\end{aligned}$$

To investigate the flocking condition for system (2), we introduce following average variables

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,$$

$$\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i.$$

From the definition of the function $\text{sig}(\cdot)^p$, we have

$$\psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^p = -\psi(\|x_i - x_j\|)\text{sig}(v_i - v_j)^p.$$

Thus

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N} \psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^p \\ &= \sum_{i,j=1}^{N} [\psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^p \\ &+\psi(\|x_i - x_j\|)\text{sig}(v_i - v_j)^p] = 0.\end{aligned}$$

In the same way

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \psi(\|x_j - x_i\|)\text{sig}(v_j - v_i)^q = 0.$$

Therefore, we obtain from system (2) that

$$\mathrm{d}\bar{x} = \bar{v}\mathrm{d}t,$$

$$d\bar{v} = 0,$$

which yields that $\bar{x}(t) = \bar{x}(0) + t\bar{v}(0)$, $\bar{v}(t) \equiv \bar{v}(0)$.

Let $\hat{x}_i = x_i - \bar{x}$, $\hat{v}_i = v_i - \bar{v}$ be the fluctuations around the center of mass system. It is easy to see that $\hat{x}_c = \frac{1}{N}\sum_{i=1}^{N}\hat{x}_i(t) = 0$, $\hat{v}_c = \frac{1}{N}\sum_{i=1}^{N}\hat{v}_i(t) = 0$. Then system (2) can be written as

$$\begin{cases} \mathrm{d}\hat{x}_i = \hat{v}_i\mathrm{d}t, \\ \mathrm{d}\hat{v}_i = \sum_{j=1}^{N}\psi(\|\hat{x}_j - \hat{x}_i\|)[\mathrm{sig}(\hat{v}_j - \hat{v}_i)^p + \mathrm{sig}(\hat{v}_j - \hat{v}_i)^q]\mathrm{d}t, \end{cases} \tag{3}$$

which means that the new variables $(\hat{x}_i, \hat{v}_i)$ satisfy the equations (2) and $\hat{x}_c \equiv 0$, $\hat{v}_c \equiv 0$. To simplify the notions we drop the hat notion in the microscopic variables and use (x_i, v_i) instead of $(\hat{x}_i, \hat{v}_i)$.

It is easy to see that

$$\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} v_i(t) = 0. \tag{4}$$

Let $x = (x_1, x_2, \ldots, x_N) \in R^{d\times N}, v = (v_1, v_2, \ldots, v_N) \in R^{d\times N}$. Take the candidate Lyapunov function:

$$V(t) = \sum_{i=1}^{N}\|v_i\|^2, \tag{5}$$

and let

$$X(t) = \sum_{i=1}^{N}\|x_i\|^2. \tag{6}$$

Let $< \cdot, \cdot >$ represent the inner product between vectors. Then from Eq. (4) we have

$$\begin{aligned}
\sum_{i,j=1}^{N} \|v_i - v_j\|^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} < v_i - v_j, v_i - v_j > \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_i||^2 - 2\sum_{i=1}^{N}\sum_{j=1}^{N} < v_i, v_j > + \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_j||^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_i||^2 - 2\sum_{i=1}^{N} < v_i, \sum_{j=1}^{N} v_j > + \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_j||^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_i||^2 - 2\sum_{i=1}^{N} < v_i, 0 > + \sum_{i=1}^{N}\sum_{j=1}^{N} ||v_j||^2 \\
&= \sum_{i=1}^{N} V(t) - 0 + \sum_{i=1}^{N} V(t) \\
&= 2NV(t). \qquad (7)
\end{aligned}$$

In the same way, we have

$$\sum_{i,j=1}^{N} \|x_i - x_j\|^2 = 2NX(t). \qquad (8)$$

It is easy to see that the difference of all individuals' velocities will tend to zero in fixed time if the function $V(t)$ tends to 0 in fixed time. And the diameter of a group is bounded if $X(t)$ is bound.

Next, we firstly prove that $V(t)$ tends to zero in fixed time. Consider the time derivative $V(t)$ along the second equation of (2),

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N} < v_i, v_i' > \\
&= 2\sum_{i=1}^{N} < v_i, \sum_{j=1}^{N} \psi(\|x_j - x_i\|)[\mathrm{sig}(v_j - v_i)^p + \mathrm{sig}(v_j - v_i)^q] > \qquad (9) \\
&= 2\sum_{i=1}^{N}\sum_{j=1}^{N} \psi(\|x_j - x_i\|) < v_i, \mathrm{sig}(v_j - v_i)^p > \\
&+ 2\sum_{i=1}^{N}\sum_{j=1}^{N} \psi(\|x_j - x_i\|) < v_i, \mathrm{sig}(v_j - v_i)^q >
\end{aligned}$$

Note that

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_i, \mathrm{sig}(v_j-v_i)^p > \\
= & \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_i - v_j, \mathrm{sig}(v_j-v_i)^p > \\
& + \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_j, \mathrm{sig}(v_j-v_i)^p > \\
= & - \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_j - v_i, \mathrm{sig}(v_j-v_i)^p > \\
& - \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_j, \mathrm{sig}(v_i-v_j)^p > .
\end{aligned} \tag{10}
$$

It is clear see that

$$
\sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\|x_j-x_i\|\right) < v_i, \mathrm{sig}\left(v_j-v_i\right)^p > = \sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\|x_j-x_i\|\right) < v_j, \mathrm{sig}\left(v_i-v_j\right)^p > .
$$

Then, one has

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_i, \mathrm{sig}(v_j-v_i)^p > \\
= & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_j - v_i, \mathrm{sig}(v_j-v_i)^p > \\
= & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{p+1}.
\end{aligned} \tag{11}
$$

In the same way

$$
\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|) < v_i, \mathrm{sig}(v_j-v_i)^q > = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{q+1}.
$$

Then, we obtain from Eq. (9) that

$$
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} = & -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{p+1} \\
& -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{q+1}.
\end{aligned} \tag{12}
$$

Using Lemma 1, we can then easily show that

$$
\begin{aligned}
\left(\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{p+1}\right)^{\frac{1}{p+1}} & \geq \left(\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{2}\right)^{\frac{1}{2}} \\
& = \|v_j-v_i\|.
\end{aligned}
$$

Thus, we have

$$\sum_{k=1}^{d} |v_{jk} - v_{ik}|^{p+1} \quad \geq \quad \|v_j - v_i\|^{p+1}. \tag{13}$$

Furthermore, we obtain from Eqs. (12) and (13) that

$$-\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j - x_i\|)\sum_{k=1}^{d}|v_{jk} - v_{ik}|^{p+1} \leq -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j - x_i\|)\|v_j - v_i\|^{p+1}. \tag{14}$$

If

$$\inf_{s\geq 0}\psi(s) \geq \psi^*, \tag{15}$$

we can estimate that

$$\begin{aligned}
- & \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j - x_i\|)\sum_{k=1}^{d}|v_{jk} - v_{ik}|^{p+1} \\
\leq & -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi^*\|v_j - v_i\|^{p+1} \\
= & -\psi^*\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\|v_j - v_i\|^2\right)^{\frac{p+1}{2}}.
\end{aligned}$$

Based on the Lemma 2, we have

$$\begin{aligned}
- & \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j - x_i\|)\sum_{k=1}^{d}|v_{jk} - v_{ik}|^{p+1} \\
\leq & -\psi^*\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\|v_j - v_i\|^2\right)^{\frac{p+1}{2}} \\
= & -\psi^*\left[2NV(t)\right]^{\frac{p+1}{2}}.
\end{aligned} \tag{16}$$

Using Lemma 3, we can then easily show that

$$\begin{aligned}
\sum_{k=1}^{d}|v_{jk} - v_{ik}|^{q+1} &= \sum_{k=1}^{d}(|v_{jk} - v_{ik}|^2)^{\frac{q+1}{2}} \\
&\geq d^{\frac{1-q}{2}}(\sum_{k=1}^{d}|v_{jk} - v_{ik}|^2)^{\frac{q+1}{2}} \\
&= d^{\frac{1-q}{2}}\|v_j - v_i\|^{q+1}.
\end{aligned}$$

We can estimate that

$$
\begin{aligned}
& - \sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)\sum_{k=1}^{d}|v_{jk}-v_{ik}|^{q+1} \\
\leq & -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(\|x_j-x_i\|)d^{\frac{1-q}{2}}\|v_j-v_i\|^{q+1} \\
\leq & -\sum_{i=1}^{N}\sum_{j=1}^{N}\psi^* d^{\frac{1-q}{2}}\|v_j-v_i\|^{q+1} \\
= & -\psi^* d^{\frac{1-q}{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\|v_j-v_i\|^2\right)^{\frac{q+1}{2}} \\
\leq & -\psi^* d^{\frac{1-q}{2}}N^{\frac{1-q}{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\|v_j-v_i\|^2\right)^{\frac{q+1}{2}} \\
= & -(dN)^{\frac{1-q}{2}}\psi^*\left[2NV(t)\right]^{\frac{q+1}{2}}.
\end{aligned}
\tag{17}
$$

Then, we obtain that

$$
\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -\psi^*\left[2NV(t)\right]^{\frac{p+1}{2}} - (dN)^{\frac{1-q}{2}}\psi^*\left[2NV(t)\right]^{\frac{q+1}{2}}.
$$

Finally, from Lemma 4, we have

$$
V(t) \equiv 0, \qquad t \geq T_1,
$$

and the fixed time is estimated by

$$
T_1 \leq T_{\max} = \frac{2}{\psi^*(2N)^{\frac{p+1}{2}}(1-p)} + \frac{2}{\psi^* d^{\frac{1-q}{2}}2^{\frac{q+1}{2}}N^q(q-1)}. \tag{18}
$$

Accordingly

$$
v_i(t) \equiv 0, \qquad t \geq T_1, \qquad i = 1, 2, \ldots, N, \tag{19}
$$

which implies that the condition (i) of the definition of fixed-time flocking holds.

Now, it is time to show that the function $X(t)$ is bounded. According to the first equation of (2), we have

$$
\begin{aligned}
\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{n} < x_i, v_i > \\
&\leq 2\sum_{i=1}^{n}\|x_i\|\|v_i\| \\
&\leq 2X^{\frac{1}{2}}(t)V^{\frac{1}{2}}(t).
\end{aligned}
\tag{20}
$$

By using the comparison theorem of differential equations, we have $X(t) \leq \Gamma(t)$, where $\Gamma(t)$ satisfies the following equation:

$$\frac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} = 2\sqrt{\Gamma(t)}V^{\frac{1}{2}}(t), \tag{21}$$

with the initial condition $\Gamma(0) = X(0)$. Simple calculation leads to

$$\Gamma^{\frac{1}{2}}(t) = \Gamma^{\frac{1}{2}}(0) + \int_0^t V^{\frac{1}{2}}(s)\mathrm{d}s.$$

Then

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + \int_0^t V^{\frac{1}{2}}(s)\mathrm{d}s.$$

Since $V(t) \equiv 0$ after $t > T_1$, thus

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + \int_0^{T_1} V^{\frac{1}{2}}(s)\mathrm{d}s.$$

Noting the fact that $V(t) \leq V(0)$, we have

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + V^{\frac{1}{2}}(0)T_1 < \infty.$$

or equivalently

$$\sup_{0 \leq t \leq \infty} \|x_i - x_j\|^2 < \infty. \tag{22}$$

Thus the condition (ii) of the fixed-time flocking is satisfied. This completes the proof. □

Remark 3. *Clearly, $\underline{\psi} \geq 1/(1 + D_{\max})^{\beta}$, where $D_{\max}$ denotes the maximum distance between two individuals. And $D_{\max}$ has an upper bound if the system (2) can reach the flocking. Thus, the communication rate function has a lower bound, which is a necessary condition for the emergence of flocking dynamics.*

Remark 4. *From Theorem 1, we can conclude that the convergence time of system (2) depends on the group size N, the lower bound of the communication rate, and the control parameter p, q. Especially, the estimation of the convergence time in (18) uncovers a power-law relationship between the convergence time and the group size. Our analytical results show that the group with high density can transit to ordered collective motion more rapidly.*

Remark 5. *The finite-time flocking of C-S systems has been investigated in [34], however, our study has some significant differences from Ref. [34]. On the one hand, the controllers designed are different. The controller in [34] only has a nonlinear coupling term* $u_i^{F,p}$ *and the interaction from this nonlinear control is weaker than that from the linear control when the velocity errors among individuals are large. While the controller in (2) contains two nonlinear control terms* $u^{F,p}$ *and* $u^{F,q}$*. And the individuals can receive strong interactions from the control* $u_i^{F,q}$ *when the velocity errors among individuals are very large. On the other hand, the main results are different. In the literature [34], the C-S system can reach the finite-time flocking. However, the settling time of the finite-time flocking depends on the initial conditions which are difficult to be estimated or even can not be measured. While, as a further study, in our manuscript, the setting time can be estimated to a value less than* T_{max}*, which is independent of the initial conditions. This indicates that the fixed-time flocking of the C-S system can be realized regardless of the initial conditions.*

4 Simulation results

In this section, some numerical simulations on the C-S model (2) are performed to validate our main results in Theorem 1. In the simulations, for simplicity, we present results for systems in the one-dimensional setting, i.e., $d = 1$. The initial positions and velocities $x_i(0), v_i(0)$ are uniformly taken from the interval [-3, 3], and the communication rate is taken as $\psi(s) = 1/(1+s^2)^\beta$. To characterize the transition to ordered flocking behavior we define two indicators: $\delta_v(t) = (1/N)\sum_{i=1}^N [v_i(t) - v_c(t)]^2$, and $\delta_x(t) = (1/N)\sum_{i=1}^N [x_i(t) - x_c(t)]^2$. We say the numerical flocking takes place if $\delta_v(t) < 10^{-6}$ and $\delta_x(t) < \infty$.

First, we present results of numerical simulations for system (2) with $\beta = 1/4,\ p = 0.7,\ q = 2,\ N = 30$. Figs. 1(a)-(c) display the evolutions of velocities $v_i(t)$, flocking indicators $\delta_v(t)$ and $\delta_x(t)$ for group size, respectively. It can be observed that the difference of all velocities tends

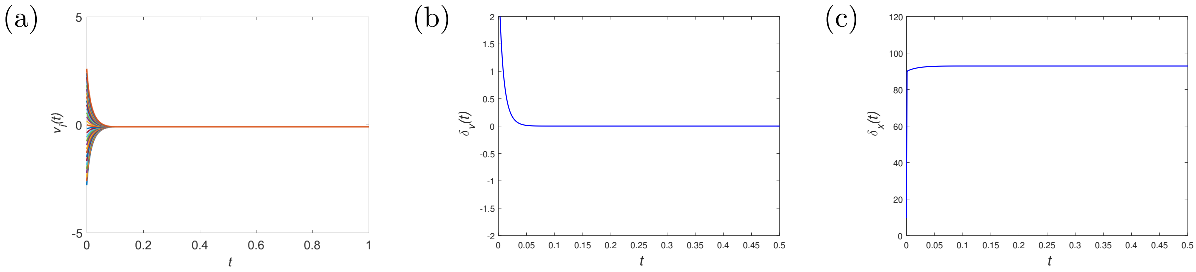

Figure 1: Flocking of the C-S type model (2) with communication rate $\psi(s) = 1/(1+s^2)^\beta$. (a) The evolutions of agents' velocities $v_i(t)$; (b) $\delta_v(t)$ vs t; (c) $\delta_x(t)$ vs t. Parameter values used are $\beta = 1/4,\ p = 0.7,\ q = 2,\ N = 30$.

to zero after $t = 0.005$ and the diameter of the group is bounded. The simulation results in Fig. 1 show that system (2) reaches the flocking in fixed time, which verifies the correctness of Theorem 1.

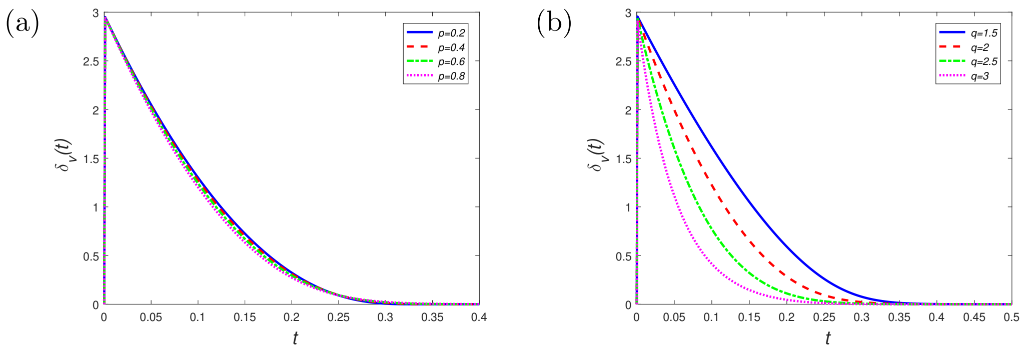

Figure 2: The influence of control parameter $p,\ q$ on the convergence speed. (a) The flocking indicator $\delta_v(t)$ for simulations of system (2) with $q = 2,\ \beta = 1$, and $p = 0.2,\ 0.4,\ 0.6,\ 0.8$. (b) The flocking indicator $\delta_v(t)$ for simulations of system (2) with $p = 0.7,\ \beta = 1$, and $q = 1.5,\ 2.0,\ 2.5,\ 3.0$.

Next, we show the influence of parameters p, q and β on the convergence speed. We ran the simulations of system (2) with different values of p, q and β. First, we fixed $q = 2,\ \beta = 1$, and took $p = 0.2,\ 0.4,\ 0.6,\ 0.8$, respectively. From Fig. 2(a) we can observe a decreasing relation between convergence time and control parameter p; we fixed $p = 2,\ \beta = 1$, and took $q =$

1.5, 2.0, 2.5, 3.0, respectively. Fig. 2(b) displays the decreasing relation between convergence time and control parameter q, which implies that the system (2) can quickly achieve flocking with large value of parameter $p,\ q$. In Fig. 3, we fixed $p = 0.7,\ q = 2$, and took $\beta = 0.5,\ 1,\ 1.5,\ 2$. The definition of the communication rate function ψ shows that small values of β means strong interactions between agents. The simulation result suggests that the C-S system with strong coupling can reach flocking with high speed.

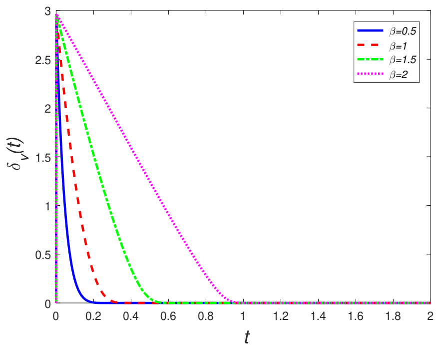

Figure 3: The influence of the parameter β on the convergence speed. The flocking indicator $\delta_v(t)$ for simulations of system (2) with $p = 0.7,\ q = 2$, and $\beta = 0.5,\ 1,\ 1.5,\ 2$.

Understanding the influence of group density on the convergence time is the major goal of this paper. In the following simulations, we study the dependence of the flocking behaviour on the number of individuals N. To gain the clear understanding of the influence of group size, we ran the simulation with $N \in \{2, 4, 6, \ldots, 40\}$ and $\beta = 1/4,\ p = 0.7,\ q = 2$. The convergence time $T_1(N)$ derived from the simulations and calculated by using the analytical estimation in (18) are shown in Fig.4. Both the numerical and analytical results show that as the number of individuals increases the convergence time is significantly decreased. Our result implies that the individuals in groups with high density can transit to ordered collective motion more rapidly. This is similar to the simulation results obtained from the Vicsek-type model. It is intuitively imaginable that a group with higher density leads to better connectivity of the network, underpinning the dynamics of the interacting moving agents. Thus, in a group with high density, each individual can receive more explicit information from the group, leading to better information exchanging. Therefore, a group with higher density leads to a faster convergence rate of flocking. This observation is in accord with previous experimental and theoretical studies. For example, Vicsek et al. show that moving groups can transit to ordered motion when the group density increases [9]. Buhl et al. found that, as locust density increases, the swarming locusts exhibit highly alignment [37].

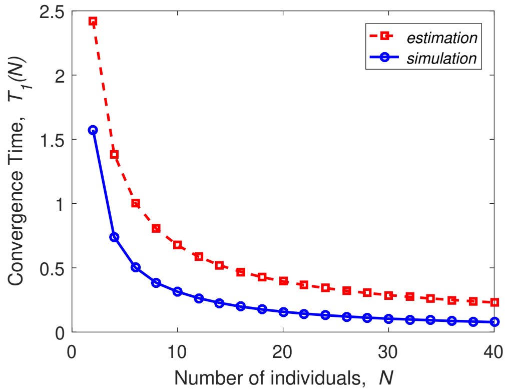

Figure 4: Variation of the convergence time with the number of individuals, N, calculated by using the analytical estimation in (18)(squares with dashed line) and derived by stochastically simulating the model (circles with full line). Parameter values are $\beta = 1/4,\ p = 0.7,\ q = 2$.

5 Conclusion

In this paper, we have studied the flocking problem of the C-S model with a continuous and non-Lipschitz protocol. By using the fixed-time stability theorem of differential equations, sufficient conditions for the fixed-time flocking are given. Our results demonstrate that the convergence time depends on the group size and control parameters. Especially, both our analytical and numerical results uncover a power-law relationship between the convergence time and the group size, which implies that individuals in groups with high density can transit rapidly to ordered collective motion. Here, we focus on the influence of the convergence speed of flocking dynamics. The fixed-time flocking defined in this paper does not require that the collision avoidance of the group. Real bird flocks can not only achieve the fixed-time flocking defined in this paper but also individuals will not collide with each other. Thus, the collision avoidance is another important problem in flocking dynamics, which will be investigated in our future work.

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