Stability of piecewise deterministic Markovian metapopulation processes on networks

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Abstract

The purpose of this paper is to study a Markovian metapopulation model on a directed graph with edge-supported transfers and deterministic intra-nodal population dynamics. We first state tractable stability conditions for two typical frameworks motivated by applications: constant jump rates with multiplicative transfer amplitudes, and coercive jump rates with unitary transfers. More general criteria for boundedness, petiteness and ergodicity are then given.

Keywords: metapopulation models, population dynamics, piecewise deterministic Markov processes, stability conditions, ergodicity

1 Introduction

Metapopulation models describe the behavior of a population naturally or artificially split into spatially distant patches connected through individual movements [23, 34, 25, 35]. Although most such models have been dealing with fully deterministic or fully stochastic population dynamics, network-organized systems with deterministic intra-nodal dynamics and stochastic inter-nodal transfers naturally appear as relevant to describe metapopulations with low local stochasticity (e.g. because of large population sizes) and random population transfers between patches.

The original motivation of this paper is the study of a population model on a cattle trade network where nodes are farms and commercial operators (see [15, 19] and Figure 1). Entries in such a system are deterministic and individuals move between nodes as they would in a Jackson network [20, 27, 8]. Similar models may allow for describing human transportation and animal movements causing epidemic propagation [7, 21, 32, 4].

There is, to our knowledge, little literature on general stability criteria for piecewise deterministic Markov population processes on graphs. Meaningful results on the stability of Jackson networks have been derived that are deeply rooted on considerations about the graph structure [36]. We wish to obtain similar statements in our semi-deterministic framework while allowing for state-dependent jump intensities and possibly large transfers. The total population of the system is here preserved by jumps and behaves as a non-Markovian randomly switched process [24, 26]. That is why we expect the process boundedness to arise from conditions on the deterministic inter-jump flow and the process ability to reach population-decreasing states.

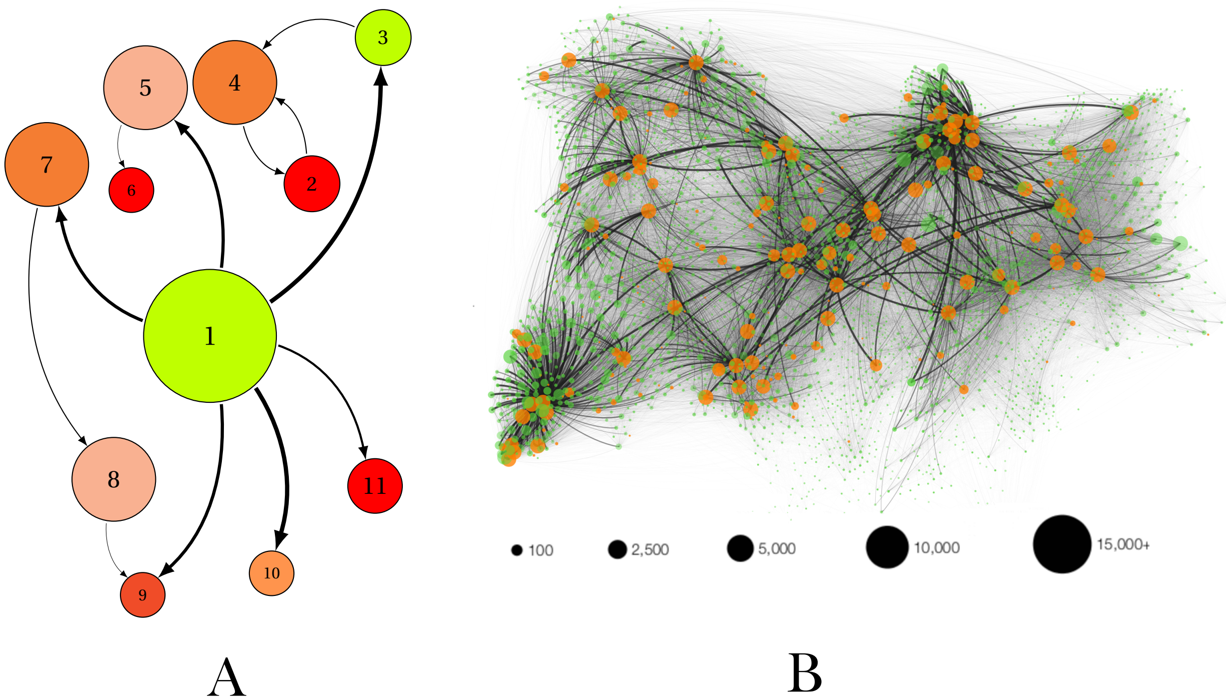

Figure 1: Panel A: stochastic "mainland-islands" Levins model [23] with unitary transfers. Edge thickness corresponds to transfer rates, while colors denote the sign of autonomous growth functions (red tones being associated to positive growth functions, green tones to positive ones). Panel B: 2008 cattle trade network in the French Auvergne-Rhône-Alpes area, densely population with cattle (courtesy of G. Beaunée). Edge thickness corresponds to the overall volume of cattle transfers within the period and colors stand for holding types (green for farms and orange for commercial operators).

Our first task is to define a general metapopulation model on a network. We will represent the $\mathbb{R}_+^n$-valued population of the system patches (that is, nodes) as a piecewise deterministic Markov process $X = (X^1, \ldots, X^n)$. The population of each patch i is associated with an autonomous growth function ϕ^i (meaning that $\mathrm{d}X_t^i = \phi^i(X_t)\mathrm{d}t$ when no jump occurs), and an instantaneous transfer from patch i to patch j at population x occurs at state-dependent rate $\theta_{i,j}(x)$, its amplitude being drawn according to a $[0, x_i]$-supported law $\mu_{i,j}(x, \cdot)$. We aim at providing sufficient conditions for the process stability as well as some asymptotic properties in the stable case.

Modelling cattle trade network dynamics and trying to allow for macroscopic jumps led to the two following motivating examples. Let $n \geqslant 1$, $\mathcal{G} = (\llbracket 1, n \rrbracket, \mathcal{A})$ a strongly connected directed graph on $\llbracket 1, n \rrbracket$ and consider the *constant growth* setting defined by:

$$\forall i \in \llbracket 1, n \rrbracket, \forall x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n, \quad \phi^i(x) = \begin{cases} c_i \text{ if } c_i \geqslant 0 \\ c_i 1_{x_i > 0} \text{ if } c_i < 0 \end{cases} \tag{1}$$

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where the c_i are real numbers that are not all equal to zero. Set $\theta_{i,j} = 0$ for $(i,j) \notin \mathscr{A}$ and either:

$$\forall (i,j) \in \mathscr{A}, \forall x \in \mathbb{R}_+^n, \quad \begin{cases} \theta_{i,j}(x) = \theta_{i,j} \\ \mu_{i,j}(x,\cdot) = \mathscr{U}([0,x_i]) \end{cases} \tag{2}$$

with $\theta_{i,j} > 0$ for all $(i,j) \in \mathscr{A}$, or:

$$\forall (i,j) \in \mathscr{A}, \forall x \in \mathbb{R}_+^n, \quad \begin{cases} \theta_{i,j}(x) = (1 \vee x_i)^\alpha \\ \mu_{i,j}(x,\cdot) = \delta_{1 \wedge x_i} \end{cases} \tag{3}$$

for some $\alpha \in (0,1]$.

Specification (2) corresponds to the case of patches transferring at constant temporal rate a random fraction of their population to each other, which fits to empirical data on cattle trade observed over large time steps. Specification (3) corresponds to unitary population transfers occurring at population-dependent rates, which is the natural framework for modelling cattle trade movements observed on a daily basis since the actual size of cattle transfers is bounded by transportation capacities. Note that the latter setting only differs from an open Jackson network by its continuous state space and the existence of the constant growth flow defined by (1).

Theorems 1 and 2 below imply that the population process is positive Harris recurrent if and only if

$$\sum_{i=1}^{n} c_i < 0 \tag{4}$$

in which case it is also F-ergodic for some exponential function $F : \mathbb{R}_+^n \to \mathbb{R}_+$. Note that equation (4) is reminiscent of the usual traffic condition for queues networks stated in equation (1.9) of [8] but does not involve any term related to inter-patch transfers. This stems from the fact that transfers along $\mathscr{G}$ happen and spread the total population among all patches, thus letting the flow on $\left(\mathbb{R}_+^*\right)^n$ bring X back towards lower population states. We will see that such a mechanism prevails in more general settings whenever jumps are large or frequent enough; in this case, a simple condition on the autonomous growth functions is required to ensure the process stability.

This paper is organized as follows. We first give a formal definition of our general piecewise deterministic metapopulation process (Section 2) of which we highlight two instances of interest referred to as *multiplicative* and *unitary* frameworks (Section 3), and state stability results for these settings. We then give Meyn-Tweedie inspired criteria for boundedness (Section 4.1), petiteness (Section 4.2) and ergodicity (Section 4.3) in the general case. Results from Section 4 apply to prove the results of Section 3. All proofs are postponed to Section 5.

From now on, let $\mathscr{I} = \{(i,j) \in [\![1,n]\!]^2 \mid i \neq j\}$. For any $i \in [\![1,n]\!]$, we denote by e_i the i-th vector of the canonical basis of $\mathbb{R}^n$. For any $d \geqslant 1$ and any $x \in \mathbb{R}_+^d$ (respectively any $\mathbb{R}^d$-valued process $X = (X_t)_{t \geqslant 0}$) we write x_i (resp. $X^i = (X_t^i)_{t \geqslant 0}$) for the i-th coordinate of x (resp. of X). For any topological space $T = (E, \mathscr{T})$, we denote by $Bor(T) = Bor(\mathscr{T})$ the Borel σ-algebra on T. If $d \geqslant 1$, λ_d will stand for the Lebesgue measure on the d-dimensional affine subsets of the $\mathbb{R}^K$ spaces (with $d \leqslant K$). Finally, we use the standard convention $\inf(\varnothing) = +\infty$.

2 A general piecewise deterministic metapopulation model

For all $i \in [\![1,n]\!]$, let $\phi^i : \mathbb{R}_+^n \to \mathbb{R}$ be a measurable and bounded function and assume that $\phi = {}^t\left(\phi^1,\dots,\phi^n\right)$ is such that the flow Φ associated to the vector field ϕ is well-defined on $\mathbb{R}_+^n$, that is, for any $x \in \mathbb{R}_+^n$ there is a unique continuous function $\Phi(x,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+^n$ such that $\Phi(x,t) = x + \int_0^t \phi(x,u)\mathrm{d}u$ for all $t \geqslant 0$.

For all $(i,j) \in \mathscr{I}$, let $\theta_{i,j} : \mathbb{R}_+^n \to \mathbb{R}$ be a positive measurable, locally bounded function. We assume that there exists an oriented graph $\mathscr{G} = ([\![1,n]\!], \mathscr{A})$, $\mathscr{A} \subset \mathscr{I}$, such that for all $(i,j) \in \mathscr{A}$, $\theta_{i,j}(x) > 0$ whenever $x_i > 0$.

For all $(i,j) \in \mathscr{I}$ and all $x \in \mathbb{R}_+^n$, finally define a probability measure $\mu_{i,j}(x,\cdot)$ on $[0,x_i]$, and for all $\xi \in [0,1]$, set

$$q_{i,j}(x,\xi) = \inf\left\{u \in [0,x^i] \mid \mu_{i,j}(x,[0,u]) \geqslant \xi\right\}$$

and assume that the $q_{i,j}$ are measurable.

Our object of interest is the family of processes solution of the following SDE:

$$\boxed{X_t = \int \phi(X_t)\mathrm{d}t + \int \sum_{(i,j)\in\mathscr{I}} \int_{\mathbb{R}_+\times[0,1]} q_{i,j}(X_{t^-},\xi)\left(e_j - e_i\right)1_{z<\theta_{i,j}(X_{t^-})} N_{i,j}(\mathrm{d}t,\mathrm{d}z,\mathrm{d}\xi)} \tag{5}$$

where $\left(N_{i,j}\right)_{(i,j)\in\mathscr{I}}$ is a family of independent homogeneous Poisson point measures on $\mathbb{R}_+ \times \mathbb{R}_+^n \times [0,1]$ with intensity $\mathrm{d}t\mathrm{d}z\mathrm{d}\xi$ defined on a probability space $(\Omega,\mathscr{A},\mathbb{P})$.

If $x \in \mathbb{R}_+^n$, the existence of a strong solution $X(x) = (X_t(x))_{t\geqslant 0}$ of (5) with initial value x follows from the explicit construction by Davis ([9] p.55). Pathwise uniqueness and uniqueness in law also hold since such solutions can naturally be expressed as deterministic functions of the atoms of the $N_{i,j}$ random measures. Moreover, we see at once that the $X(x)$ processes are non-explosive and well-defined on $\mathbb{R}_+$ because the $\theta_{i,j}$ and ϕ^i are locally bounded and jumps are mass-conserving. By Theorem 25.5 of [9] (slightly adapted to allow for zero-amplitude jumps), the strong Markov property holds for $(X(x))_{x\in\mathbb{R}_+^n}$.

It is possible (see Appendix A) to construct a Markov family $\left(X,(\mathbb{P}_x)_{x\in\mathbb{R}_+^n}\right)$ on a measurable space $(\tilde{\Omega},\tilde{\mathscr{A}})$ such that for any $x \in \mathbb{R}_+^n$, the law of X under $\mathbb{P}_x$ is the law of $X(x)$ under $\mathbb{P}$. We will mostly use this homogeneous and lighter notation.

It stems from equation (26.15) of [9] that the infinitesimal generator $\mathfrak{A}$ of $\left(X,\mathbb{P}_x,x\in\mathbb{R}_+^n\right)$ is given by

$$\mathfrak{A}f(x) = \sum_{k=1}^n \frac{\partial f}{\partial x_i}(x)\phi^i(x) + \sum_{(i,j)\in\mathscr{I}} \theta_{i,j}(x) \int \Big(f\left(x + y\cdot(e_j - e_i)\right) - f(x)\Big)\mu_{i,j}(x,\mathrm{d}y) \tag{6}$$

for every $\mathscr{C}^1$ function $f : \mathbb{R}_+^n \to \mathbb{R}$. This equation is of well-known interest in studying invariant measures for X, as we illustrate in Appendix B.

3 Stability of multiplicative and unitary models

We now turn to two particular settings for which we will state simple stability results, each of them being representative of a typical feature that our graph-based model may exhibit. The first one corresponds to constant jump rates and multiplicative transfers (as in equation (2)) while the second one, symmetrically, is defined by coercive jump rates and unitary transfers (as in equation (3)). Let us begin by stating additional assumptions that are common to both frameworks.

3.1 Additional assumptions

Assumptions on autonomous growth functions

Let $m \in [\![1, n]\!]$ and $d \in [\![1, m]\!]$. We set $V^+ = [\![1, d]\!]$, $V^0 = [\![d+1, m]\!]$ and $V^- = [\![m+1, n]\!]$. Patches in V^+, V^0 and V^- will respectively be called *sources*, *neutral patches* and *sinks* following the terminology of [33].

We make the following assumptions:

Assumption A. — [On autonomous growth functions]

(*a*) *For all* $i \in [\![1, n]\!]$ *and all* $x \in \mathbb{R}_+^n$, $\phi^i(x) = \phi^i(x_i)$ *only depends on* x_i.

(*b*) *For all* $i \in [\![1, n]\!]$:

- $\phi^i > 0$ *if* $i \in V^+$
- $\phi^i = 0$ *if* $i \in V^0$
- $\phi^i \leqslant 0$, $\phi^i(0) = 0$ *and* $\phi^i(y) < 0$ *as soon as* $y > 0$ *if* $i \in V^-$

(*c*) *For all* $i \in V^+$, ϕ^i *is continuous and piecewise* $\mathscr{C}^1$.

(*d*) *For all compact subset* $K \subset \mathbb{R}_+^n$ *there exists* $T > 0$ *such that* $\Phi^i(x, T) = 0$ *for any* $x \in K$ *if* $i \in V^-$.

This first set of assumptions will make it possible to monitor the response of a given trajectory to small variations of jump times and transfer quantiles with minor concern for the local behavior of Φ and to describe simple trajectories leading to the emptying of the system. The constant growth model defined by equation (1) obviously satisfies these conditions.

Assumptions on the topology of $\mathscr{G}$

Remind that we defined the active graph $\mathscr{G} = ([\![1, n]\!], \mathscr{A})$ such that $\theta_{i,j}(x) > 0$ for all $(i, j) \in \mathscr{A}$ and all $x \in \mathbb{R}_+^n$ with $x_i > 0$. We require a strong condition on the graph structure of $\mathscr{G}$ to hold:

Assumption B. — [On the topology of $\mathscr{G}$]

(*a*) *Any* $j \in V^-$ *can be reached from any* $i \in [\![1, n]\!]$ *by a path in* $\mathscr{G}$.

(*b*) *Any* $j \in V^0$ *can be reached from some* $i \in V^+$ *by a path in* $\mathscr{G}$.

The latter assumption is quite restrictive but will prove crucial in discussing both the boundedness of the process and the petiteness of compact subsets of $\mathbb{R}_+^n$. It allows to describe paths along which the system empties without having to discuss complex connectivity properties, while still being weaker than plain connectivity (see Figure 2).

3.2 Multiplicative models

The model is said to be *multiplicative* if the following hold:

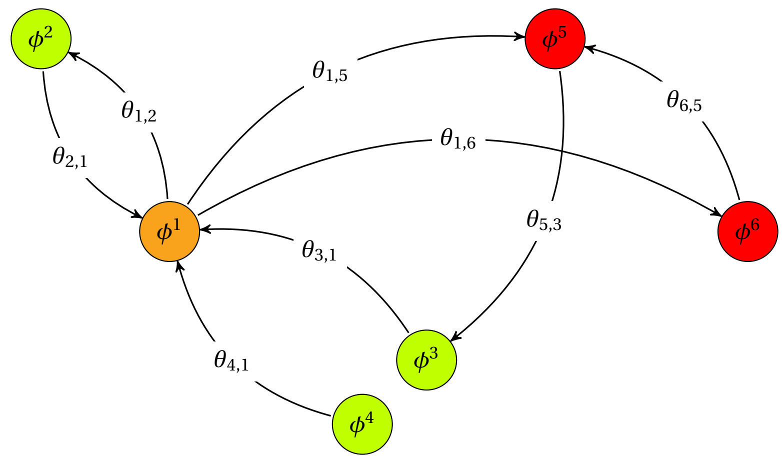

Figure 2: Graph structure complying with Assumption B. The edges in the figure are the elements of $\mathscr{A}$; sources, neutral patches and sinks are respectively green, orange and red.

Definition. — [Multiplicative setting]

1. *Assumptions A and B are fulfilled.*
2. *For all $(i,j) \in \mathscr{A}$, $\theta_{i,j}$ is constant.*
3. *For all $(i,j) \in \mathscr{I}$, there exists a probability distribution $\mu_{i,j}$ on $[0,1]$ whose restriction to some non-punctual interval $I_{i,j} \subset [0,1]$ admits a piecewise continuous and positive density with respect to the Lebesgue measure, such that*:
$$\forall x \in \mathbb{R}_+^n, \forall A \in Bor([0,1]), \quad \mu_{i,j}(x, x_i A) = \mu_{i,j}(A)$$

The "multiplicative" denomination refers to the fact that the relative population moving from patch i to patch j is drawn according to a law $\mu_{i,j}$ which is independent from x. The process therefore goes through macroscopic jumps, but the assumption on the $\theta_{i,j}$ implies that these cannot be too frequent, which results in long, uninterrupted emptying periods under suitable assumptions. Note that we required the $\theta_{i,j}$ to be constant for the sake of simplicity, but all of the results we will present hold if the $\theta_{i,j}$ take values in some compact interval of $\mathbb{R}_+^*$.

Multiplicative models encompass the introductory setting defined by equations (1) and (2) as well as a large class of *additive increase multiplicative decrease* (AIMD) models that are continuous counterparts to those studied in [14] and [18].

Our main stability result for the multiplicative setting is the following:

Theorem 1. — *If the model is multiplicative and*

$$\limsup_{\min_i x_i \to +\infty} \sum_{i=1}^{n} \phi^i(x) < 0 \tag{7}$$

then X is positive Harris recurrent and there exists $\eta > 0$ such that X is F-ergodic with

$$F : \begin{cases} \mathbb{R}_+^n \to [1, +\infty[\\ (x_1, \ldots, x_n) \mapsto e^{\eta \sqrt{\sum_{i=1}^n x_i}} \end{cases}$$

Example 1. As stated in the Introduction, a model defined by (1) and (2) if and only if $\sum_{i=1}^n c_i < 0$ and transient otherwise (remind that we assumed that $V^+ \neq \varnothing$).

Example 2. The model represented on the left in Figure 3 corresponds to an ergodic setting, while the model on the right does not.

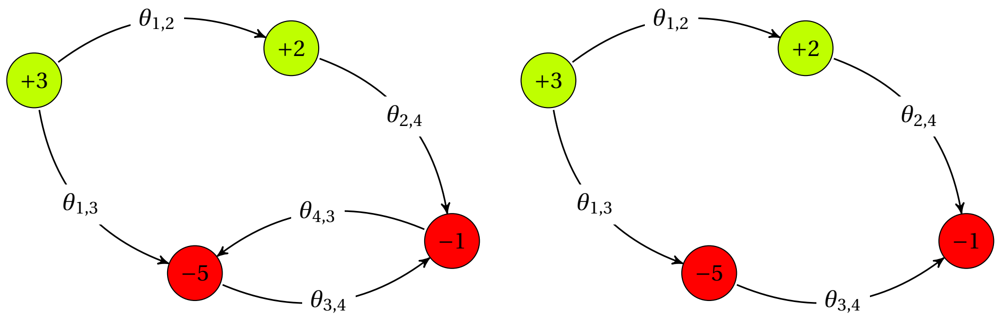

Figure 3: The importance of connectivity: if (1) and (2) are fulfilled, then the process corresponding to the graph configuration on the left is ergodic (according to Theorem 1) and that on the right is transient (consider the population on the subgraph formed by the two right patches). Figures within patches are the c_i and all $\theta_{i,j}$ represented here are strictly positive.

Let us give the main thrust of the proof of Theorem 1, the details of which will be developed in a more general framework in Section 4. Our proof strategy is based on the conceptual framework by Meyn and Tweedie [29, 30, 11]. It consists in showing that X is brought back quickly enough to some compact subset of $\mathbb{R}_+^n$ (a *boundedness* property) and that its various trajectories scan some Borel subset of $\mathbb{R}_+^n$ (a so-called *petiteness* property). The condition on the ϕ^i entails the existence of some $\mathbb{R}_+^n$ area (namely $\{x \in \mathbb{R}_+^n : \min_i x_i \geqslant R\}$ for R large enough) on which the flow commands a steady decrease of the system total population. The assumptions on the $\theta_{i,j}$ and the $\mu_{i,j}(x, \cdot)$ imply that with lower-bounded probability, the process reaches this zone quickly enough and stays in it long enough for the total population of the system to be brought back below a given threshold almost surely after some time, which yields the boundedness property. On the other hand, small variations of jump quantiles along a given path induce locally one-to-one changes of its point of arrival under suitable localization conditions, which implies the petiteness property.

In the transient case, the stability of transfer laws and the existence of macroscopic jumps make scaled multiplicative processes easy to describe. One may easily show that if the model is multiplicative and if

X is transient, then for any $x \in \mathscr{S} = \{x \in \mathbb{R}_+^n \mid \|x\|_1 = 1\}$:

$$\mathbb{E}_{Rx}\left[\sup_{t\in[0,R]}\left\|\frac{X_t}{\|X_t\|_1} - S_t\right\|_1\right] \underset{R\to+\infty}{\longrightarrow} 0 \tag{8}$$

where S is a $\mathscr{S}$-valued pure jump process which is a weak solution of the following SDE:

$$\forall (i,j) \in \mathscr{I}, \quad \mathrm{d}S_t^i = \sum_{j\neq i}\int_0^1 \xi S_t^i N_{i,j}(\mathrm{d}t,\mathrm{d}\xi) - \sum_{j\neq i}\int_0^1 \xi S_t^j N_{j,i}(\mathrm{d}t,\mathrm{d}\xi)$$

with initial value x under the $\mathbb{P}_{Rx}$ probability measures. Studying the stability of S only requires petiteness analysis, and (8) makes it possible to infer the behavior of $\frac{X}{\|X\|_1}$ (which may not be a Markov process) from that of S. As an example, it is possible to show that in the two-patch uniform constant growth model with $c_1 + c_2 \geqslant 0$, $\frac{X_t^1}{X_t^1+X_t^2}$ converges in distribution to a beta law with parameters $\left(\frac{\theta_{2,1}}{\lambda}, \frac{\theta_{1,2}}{\lambda}\right)$ as t goes to infinity.

3.3 Unitary models

The model is said to be *unitary* if the following hold:

Definition. — [Unitary setting]

1. *Assumptions A and B are fulfilled.*
2. *There exists an increasing and subadditive function* $\Theta : \mathbb{R}_+ \to \mathbb{R}_+^*$ *with* $\lim_{y\to+\infty}\Theta(y) = +\infty$ *and* $\alpha > 1$ *such that for any* $x \in \mathbb{R}_+^n$:
   - *For all* $(i,j) \in \mathscr{A}$, $\Theta(x_i) \leqslant \theta_{i,j}(x)$.
   - *For all* $(i,j) \in \mathscr{I}$, $\theta_{i,j}(x) \leqslant \alpha\Theta(x_i)$.
3. *For all* $x \in \mathbb{R}_+^n$ *and all* $(i,j) \in \mathscr{A}$, $\mu_{i,j}(x,\cdot) = \delta_{1\wedge x_i}$.

The unitary setting easily is typically fit for applications to large animals trade or human transportation. A simple example is given by our introductory model defined by (1) and (3).

Our main stability result on unitary models is the following:

Theorem 2. — *If the model is unitary and if either*

$$\limsup_{\min_{i\in V^-} x_i\to+\infty} \sum_{i=1}^n \phi^i(x) < 0 \tag{9}$$

or

$$\mathscr{G} \text{ is connected and } \limsup_{\min_i x_i\to+\infty} \sum_{i=1}^n \phi^i(x) < 0 \tag{10}$$

then there exists $\eta > 0$ *such that* X *is positive Harris recurrent and* F*-ergodic with*

$$F : \begin{cases} \mathbb{R}_+^n \to [1,+\infty[\\ (x_1,\dots,x_n) \mapsto e^{\eta\sqrt{\sum_{i=1}^n x_i}} \end{cases}$$

Example 3. The following setting corresponds to an unitary metapopulations model with logistic autonomous growth in sources and jump rates taking into account the carrying capacity of target patches:

$$\begin{cases} \forall i \in V^+, \quad \phi^i(x) = \alpha_i x_i(\beta_i - x_i)_+ + c_i \\ \forall j \in V^-, \quad \phi^j(x) = -\frac{c_j x_j}{\alpha_j + x_j} \\ \forall i \in V^+, \forall j \in V^-, \quad \theta_{i,j}(x) = \gamma_{i,j} x_i \frac{\varepsilon_{i,j}+x_j}{\varepsilon'_{i,j}+x_j} \text{ and } \theta_{j,i} = 0 \\ \forall (i,j) \notin \mathscr{A}, \quad \theta_{i,j} = 0 \\ \forall (i,j) \in \mathscr{A}, \quad \mu_{i,j}(x,\cdot) = \delta_{1\wedge x_i} \end{cases} \tag{11}$$

where the α_i, β_i, $\gamma_{i,j}$, $\varepsilon_{i,j}$ and $\varepsilon'_{i,j}$ are positive constants and the c_i are such that $c_i > 0$ if $i \in V^+$, $c_i = 0$ if $i \in V^0$ and $c_i < 0$ if $i \in V^-$. Theorem 2 implies ergodicity in this setting as soon as $\sum_{i=1}^n c_i < 0$ if $\mathscr{G}$ is connected.

Example 4. As stated in our introduction, a model defined by (1) and (3) is ergodic if and only if $\sum_{i=1}^n c_i < 0$ and transient otherwise (remind that we assumed that $V^+ \neq \varnothing$).

The proof of Theorem 2 is based on arguments that only slightly differ from those put forward in the multiplicative case. If $\mathscr{G}$ is connected, boundedness is shown as in the proof of Theorem 1. If this is not the case but (9) holds, we show that X. In both cases, we then use small variations of jump times (and not jump quantiles anymore) to check for the petiteness of compacts for the resolvent chain of X.

4 Stability criteria for the general metapopulation model

We now present general Meyn-Tweedie inspired criteria for boundedness, petiteness and ergodicity for the model defined in Section 2. We will see that their application requires proof strategies that are dependent from the model specification and the system's active graph structure. However, it is easy to adapt them in a discretionary way to a large range of frameworks, which is why we endeavour to state their main thrust whenever this is possible. In particular, these criteria will apply to both multiplicative and unitary settings, yielding Theorems 1 and 2.

4.1 Criterion for boundedness

We first want to state a sufficient condition for X to be *bounded in probability on average*. Remind from [29] that this latter condition means that for all $x \in \mathbb{R}_+^n$ and all $\varepsilon > 0$ there exists a compact subset $C \subset \mathbb{R}_+^n$ such that

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{P}_x(X_s \in C)\mathrm{d}s \geqslant 1-\varepsilon.$$

Fix a Borel subset S of $\mathbb{R}_+^n$. Our first assumption implies that the flow on S drives the process at a steady rate towards the origin:

Assumption 1. — [Existence of a steady population decrease zone]

There exists $c > 0$ *such that*:

$$\forall x \in S, \quad \sum_{i=1}^n \phi^i(x) \leqslant -c$$

S corresponds to a set of population configurations in which the local deterministic dynamics cause the system to empty. For instance, in the constant growth setting defined by (1) and under condition (4), S may be defined as any Borel set of population configurations in which all sinks contain positive population.

It is worth mentioning a simple case with straightforward consequences. If Assumption 1 holds and with $S = {}^cK$ for some compact subset K of $\mathbb{R}_+^n$, it is a simple matter to show that X is bounded in probability on average using, for instance, Fatou's Lemma. In this case, $\sum \phi^i$ may play the role of a Lyapunov function for X, and it is possible to derive results on recurrence and ergodicity provided all compacts are petite and there exists an irreducible skeleton chain for X (see Theorem 4.2 of [30] and Theorem 5.2 of [11]). However, it is not always the case that $S = {}^cK$ for some compact K. In particular, it is not under Assumption A since S cannot contain any x such that $x^i = 0$ for all $i \in V^-$.

Under Assumption 1, it is natural to think that X will be bounded in probability on average if it quickly goes back to S and stays within S for a long time whenever it reaches it. This is why we put forward the following conditions:

Assumption 2. — [Bounds for the hitting time of S' and the exit time from S]

There exists a Borel subset S' of S as well as $\delta, \varepsilon, T, T' \in \mathbb{R}_+^*$ and $R \geqslant 0$ such that:

1. *For any $x \notin S$ with $\|x\|_1 \geqslant R$*:
$$\mathbb{P}_x(\exists s \in [0,T], X_s \in S') \geqslant \delta$$
2. *For any $x \in S'$ with $\|x\|_1 \geqslant R$*:
$$\mathbb{P}_x(\forall s \in [0,T'], X_s \in S) \geqslant \varepsilon$$
3. *We have*
$$\varepsilon T' c > (1-\varepsilon)\frac{T}{\delta} \sup_{\mathbb{R}_+^n} \sum_{i=1}^n \phi^i$$

Note that Assumption 2.2 may be void since we do not assume that $S \cap \{\|x\|_1 \geqslant R\}$ and $S' \cap \{\|x\|_1 \geqslant R\}$ are not empty.

One can see that under Assumptions 1, 2.1 and 2.2 and starting from a point in S with high enough total population, then with probability of at least ε, X does not leave S before time T', and the mean total population increase during an excursion from S is at most $\frac{T}{\delta}\sup_{\mathbb{R}_+^n}\sum_i \phi^i$. Assumption 2.3 therefore appears to be sufficient for the visits of S by X to bring the population process back to some compact set of $\mathbb{R}_+^n$ regardless of its original position.

The expected result follows. Its proof is deferred to Section 5 and consists in a comparison with a random walk on $\mathbb{R}$. It may easily adapt to some models that do not necessarily fulfill Assumptions 1 and 2 by considering suitable $\mathbb{R}$-valued Markov chains.

Theorem 3. — *Under Assumptions 1 and 2, X is bounded in probability on average.*

Alternate criteria for boundedness may be derived using Theorem 2.1 of [31] under irreducibility assumptions. In particular, Theorem 3.1 of [8] holds in our setting whenever compact subsets of $\mathbb{R}_+^n$ are petite. Yet, as we will see in Section 4.3, Assumptions 1 and 2 yield an upper bound for expected exponential functionals of return times, which implies strong ergodicity results.

4.2 Criterion for petiteness

Let us now consider a compact set $C \subset \mathbb{R}_+^n$. We remind (see [29]) that C being *petite* for some sampled chain $(A_n)_{n\geqslant 0}$ of X means that there exists a non-trivial Borel measure ν on $\mathbb{R}_+^n$ such that $\mathbb{P}_x(A_1 \in \cdot) \geqslant \nu(\cdot)$ for all $x \in C$. C is just said to be *petite* if it is for some $(A_n)_{n\geqslant 0}$.

Proving C is petite relies on framework-specific strategies. The criterion for petiteness we will derive in this section applies to a broad range of settings, but it is worth keeping in mind that it may prove too technical in some cases. In particular, it can be easier to identify sampled chains that dominate $\mathbb{R}_+^n$-valued Dirac measures — provided that such chains exists they exist.

In most non-pathological specifications, the distribution of inter-jump times exhibits an absolutely continuous component with respect to the Lebesgue measure. This observation drives us to look for subsets of $\mathbb{R}_+^n$ on which the Lebesgue measure is dominated by the semigroup for the resolvent chain $(R_n)_{n\in\mathbb{Z}_+}$ defined by:

$$\forall n \in \mathbb{Z}_+, \quad R_n = X_{S_n}$$

where $(S_n)_{n\in\mathbb{Z}_+}$ is the sequence of jump times of a Poisson process with density 1 independent from X.

With that aim in mind, let us first define tracking functions that return the position of our process from its inter-jump times t_k, the edges κ_k along which the transfers occur and the values of its jump distributions quantiles ξ_k. For all $x \in \mathbb{R}_+^n$, all $(i,j) \in \mathscr{I}$ and all $\theta \in [0,1]$, set

$$g_{i,j}(x,\xi) = x + q_{i,j}(x,\xi)\,(e_j - e_i).$$

For all $x \in \mathbb{R}_+^n$, let us define recursively the sequence of functions $\left(h_x^k\right)_{k\geqslant 1}$ by

$$h_x^1 : \begin{cases} \bigcup_{l\geqslant 1}\left(\mathbb{R}_+^j \times [0,1]^{l-1} \times \mathscr{I}^{l-1}\right) \to \mathbb{R}^n \\ (t,\xi,\kappa) \mapsto \Phi(x,t_1) \end{cases}$$

and for all $k \in \mathbb{N}$,

$$h_x^{k+1} : \begin{cases} \bigcup_{l\geqslant k+1}\left(\mathbb{R}_+^l \times [0,1]^{l-1} \times \mathscr{I}^{l-1}\right) \to \mathbb{R}^n \\ (t,\xi,\kappa) \mapsto \Phi\left(g_{\kappa_k}\left(h_x^k(t,\xi,\kappa),\xi_k\right), t_{k+1}\right) \end{cases}$$

where $\kappa = (\kappa_1,\ldots,\kappa_{j-1})$ denotes the generic element of $\mathscr{I}^{j-1}$. Vector $h_x^k(t,\xi,\kappa)$ is the state occupied by the process with initial condition x after having followed the flow for time t_1, undergone a transfer along edge κ_1 with amplitude given by the quantile of order ξ_1 of the appropriate $\mu_{i,j}(x,\cdot)$ law, followed the flow again for time t_2, and so on until $k-1$ transfers occurred and the process followed the flow for time t_k after its last jump.

Our strategy is to look for a subset V of an affine subspace of $\mathbb{R}_+^n$ such that we can provide a lower bound for the Lebesgue measure of the pre-image by some h_x^k of any Borel subset of V. We therefore introduce the following assumption:

Assumption 3. — [Likely paths scanning a Borel subset]

There are $M > 0$, $\overline{N} \in \mathbb{N}$, $\overline{T} \geqslant 1$, $p \in [\![1, n]\!]$, a p-dimensional affine subspace V of $\mathbb{R}^n$, a Borel subset $\mathscr{P}$ of V with non-zero Lebesgue measure, and for any $x \in C$ there are $N(x) \in [\![1, \overline{N}]\!]$ with $q(x) = 2N(x) - 1 - p \geqslant 0$, a vector of edges $\kappa_x \in \mathscr{I}^{N(x)-1}$, a $\mathbb{R}^{2N(x)-1}$-coordinates permutation σ_x and open subsets $U_1^x \subset [0, \overline{T}]^p$ and $U_2^x \subset [0, \overline{T}]^{q(x)}$ such that setting

$$\psi_{z_2}^x : \begin{cases} U_1^x \to V \\ z_1 \mapsto h_x^{N(x)}(\sigma_x(z_1, z_2), \kappa_x) \end{cases}$$

the following hold:

1. *For all $z_2 \in U_2^x$, $\psi_{z_2}^x$ is a $\mathscr{C}^1$-diffeomorphism with Jacobian determinant bounded by M on U_1^x.*
2. *For all $z_2 \in U_2^x$,*
$$\mathscr{P} \subset \psi_{z_2}^x(U_1^x)$$
3. $$\theta_0 = \inf_{x \in C} \left[\lambda_{q(x)}(U_2^x) \inf_{\substack{z_1 \in U_1^x \\ z_2 \in U_2^x}} \prod_{i=1}^{N(x)-1} \theta_{\kappa_i(x)} \left(h_x^k(\sigma_x(z_1, z_2), \kappa_x) \right) \right] > 0$$

Note that Proposition 4 below still holds if we replace p-dimensional affine subspaces by p-dimensional manifolds in the condition above. Condition 3.3 states that paths that lead X to Borel subsets of V do not correspond to unlikely sequences of jump times or jump quantiles.

A change of variables argument then yields the following proposition:

Proposition 4. — *If Assumption 3 is met, then C is petite.*

Most of the technicity in applying this proposition lies on proving that Assumption 3.1 holds. Indeed, this requires describing paths of the process that lead to a given area of the state space as well as monitoring their response to small variations of jump times and jump quantiles. As we will see in the course of proving Theorem 5, this can be made simpler by a straightforward but useful result allowing for localization, which is Lemma 3.1 from [28].

4.3 Criterion for ergodicity

Theorems 3.2 (ii) and 4.1 (i) of [29] along with present Theorem 3 and Proposition 4 imply that X is positive Harris recurrent as soon as Assumptions 1 and 2 are met and all compact subsets of $\mathbb{R}_+^n$ are petite. We now look for an additional condition to ensure *F-ergodicity* for some measurable $F : \mathbb{R}_+ \to [1, +\infty[$. Remind from [29] that this property writes:

$$\forall x \in \mathbb{R}_+^n, \quad \lim_{t \to +\infty} \sup_{|g| \leqslant F} \left| \mathbb{E}_x\left(g(X_t)\right) - \pi(g) \right| = 0$$

with π standing for the invariant probability of X and the supremum being taken over all measurable $g : \mathbb{R}_+^n \to \mathbb{R}$ functions such that $|g| \leqslant F$.

Our result is the following:

Theorem 5. — *Assume that Assumptions 1 and 2, that all compact subsets of $\mathbb{R}_+^n$ are petite and that X admits an irreducible skeleton chain. Then there exists $\eta \in \mathbb{R}_+^*$ such that X is F-ergodic for*

$$F : \begin{cases} \mathbb{R}_+^n \to \mathbb{R}_+ \\ x \mapsto e^{\eta \sqrt{\sum_{i=1}^n x_i}} \end{cases} \tag{12}$$

Theorems 1 and 2 are corollaries of the latter result, as is detailed in Section 5.

Conclusion

Besides metapopulations, possible fields of application for the class of piecewise deterministic models presented here range from open Jackson networks [20, 27, 8] with deterministic inputs and outputs to communication networks based on TCP-type processes [2, 6, 18], storage [16] on a network, neuronal stimulation [13, 12] and more generally a large class of stochastic hybrid systems on graphs [5] with low stochasticity in autonomous dynamics.

Our work may be expanded in many ways. First, the multiplicative and unitary frameworks were designed as simple models that allow either for large jumps or large jump rates, but many applications may require designing and studying hybrid models. Criteria from Sections 4.1 and 4.2 will hopefully prove flexible and be useful in such settings.

Besides, some metapopulation settings do not necessarily fit with the additional hypothesis we made on the structure of the active graph $\mathcal{G}$ of the system — that is, on the graph formed by edges along which non-zero transfers occur at non-zero rate. Although the criteria for boundedness and petiteness we stated in Sections 4.1 and 4.2 do not refer to this active graph, they cannot be applied without particular sequences of transfers being made explicit, which assumes that one can describe entire paths followed by the population load. Moreover, they require some degree of connectivity so the system, loosely speaking, can "empty" and "mix". Expanding the results above to more general graphs is still a work in progress.

Finally, we chose to model deterministic intra-patch population dynamics. This assumption is only legitimate if the local demographics exhibit little stochasticity, either intrinsically or because quantities involved are so large that their aggregate evolution can be proxied by a non-random dynamic system. If this condition is not met, intra-patch birth and death or Hawkes-inspired processes ([17], [10]) should be considered. While this would require developing new tools for petiteness analysis, we are confident of our boundedness results holding without any major alteration.

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5 Proofs

This section contains the proofs of the results stated previously. We begin by setting up some notation. From now on, we will write

$$M = \sup_{\mathbb{R}_+^n} \sum_{i=1}^{n} \left|\phi^i\right| < +\infty.$$

For any Borel subset $E \subset \mathbb{R}_+^n$, we denote by E^c the complement of E in $\mathbb{R}_+^n$, and by $\left(\tau_E^k\right)_{k\geqslant 1}$ the sequence of successive visit times of E by X (with $\tau_E^1 = 0$ $\mathbb{P}_x$-a.s. if $x \in E$), with the shorthand notation $\tau_E = \tau_E^1$.

Finally, $(T_k)_{k\geqslant 0}$ and $(U_k)_{k\geqslant 1}$ will denote the sequences of the process jump times and jump quantiles (see Appendix A for detail).

5.1 Boundedness criterion 3

Proof of Theorem 3.

Step 1: study of a random walk

Assume that Assumptions 1 and 2 hold and let $y \in \mathbb{R}$. Consider a $\mathbb{R}$-valued random walk $Y = (Y_k)_{k\geqslant 1}$ with i.i.d. increments defined on some probability space in such a way that:

$$Y_1 = y \quad \text{and} \quad \forall k \geqslant 1, \quad Y_{k+1} - Y_k \overset{d}{=} -BcT' + (1-B)\Gamma \cdot TM \tag{13}$$

where B and Γ are independent random variables with respective distributions Bernoulli $\mathscr{B}(1,\varepsilon)$ and shifted geometric $\mathscr{G}(\delta)$. We first observe that

$$\mathbb{E}\left(-BcT' + (1-B)\Gamma \cdot TM\right) = -\varepsilon cT' + (1-\varepsilon)\frac{T}{\delta}M < 0$$

which entails that the hitting time σ_R of $(-\infty, R]$ by Y is almost surely finite by the law of large numbers. Moreover, there exists $r > 0$ independent from the choice of $y \in \mathbb{R}$ such that:

$$\mathbb{E}\left(e^{\gamma(r)\sigma_R}\right) \leqslant e^{r(-cT'-R+y)}.$$

Indeed, straightforward computations show that if $r > 0$ is such that $r < \frac{|\ln(1-\delta)|}{TM}$, and if we set

$$\gamma(r) = -\ln\left(\frac{\delta(1-\varepsilon)}{e^{-rTM} - (1-\delta)} + \varepsilon e^{-rcT'}\right)$$

then $\left(\exp\left(rY_k + \gamma(r)(k-1)\right)\right)_{k\geqslant 1}$ is a positive martingale with respect to its natural filtration. Using Fatou's lemma then yields the expected inequality since $\gamma(r) > 0$.

Step 2: exponential moment of the hitting time of a compact subset

Fix $C = \{x \in \mathbb{R}_+^n \mid \|x\|_1 \leqslant R\}$. We now prove that there exists $\beta > \mathbb{R}_+^*$ such that $x \mapsto \mathbb{E}_x\left(\exp\left(\beta\sqrt{\tau_C}\right)\right)$ is locally bounded on $\mathbb{R}_+^n$.

Let us first assume that $x \in S^c \cap C^c$. Then $\mathbb{P}_x$-a.s.:

$$\begin{aligned}\tau_C &\leqslant \sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_{S^c}^{k+1} - \tau_{S^c}^k\right) = \sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k + \tau_{S^c}^{k+1} - \tau_S^k\right) \\ &\leqslant \sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k + \frac{\|X_{\tau_S^k}\|_1 - R}{|c|}\right)\end{aligned}$$

since the decrease of the flow on $S \cap C^c$ is at least $|c|$. For $\beta > 0$, we thus get:

$$\left(\mathbb{E}_x\left[\exp\left(\beta\sqrt{\tau_C}\right)\right]\right)^2 \leqslant \mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k\right)}\right)\right]\mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\frac{\|X_{\tau_S^k}\|_1 - R}{c}}\right)\right] \quad (14)$$

$\mathbb{P}_x$-a.s. using the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for $(a,b) \in \mathbb{R}_+^2$ and the Cauchy-Schwarz inequality. Yet Assumption 2 entails that the increment of the total system population each time the process leaves S before τ_C is dominated by $-BcT' + (1-B)\Gamma \cdot TM$. Applying the strong Markov property to the sequences of stopping times $\left(\tau_S^k \wedge \tau_C\right)_{k \geq 1}$ and $\left(\left(\tau_{S^c}^k + pT\right) \wedge \tau_S^{k+1} \wedge \tau_C\right)_{k \geq 1, p \geq 0}$ ensures that there exists $(Y_k)_{k \geqslant 1}$ on $(\tilde{\Omega}, \tilde{A}, \mathbb{P}_x)$ satisfying (13) with $y = \|x\|_1$ such that $\mathbb{P}_x$-a.s.:

$$\forall k \geqslant 1, \quad 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k\right) \leqslant 1_{k < \sigma_R}\left(\frac{Y_{k+1} - Y_k}{M} + cT'\right) \quad (15)$$

and

$$\forall k \geqslant 1, \quad \left\|X_{\tau_{S^c}^k \wedge \tau_C}\right\|_1 \leqslant Y_{k \wedge \sigma_R}. \quad (16)$$

For any $\beta > 0$, (15) yields:

$$\mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k\right)}\right)\right] \leqslant \mathbb{E}\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{\sigma_R - 1}\left(\frac{Y_{k+1} - Y_k}{M} + cT'\right)}\right)\right]$$

hence

$$\mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k\right)}\right)\right] \leqslant \mathbb{E}\left[\exp\left(2\beta\sqrt{|c|T'\sigma_R}\right)\right]. \quad (17)$$

Now (16) entails:

$$\mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\frac{\|X_{\tau_S^k}\|_1 - R}{c}}\right)\right] \leqslant \mathbb{E}\left[\exp\left(2\beta\sqrt{\sum_{k=1}^{\sigma_R - 1}\frac{Y_k}{c}}\right)\right].$$

The increments of the Y chain are greater than $-ct$ and the value of Y at time σ_R is at least R, so we can write

$$\sum_{k=1}^{\sigma_R - 1} Y_k \leqslant \sum_{k=1}^{\sigma_R - 1}\left(R + \sigma_R cT'\right) \leqslant \sigma_R R + \sigma_R^2 cT'$$

from which we deduce that

$$\mathbb{E}_x\left[\exp\left(2\beta\sqrt{\sum_{k=0}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\frac{\|X_{\tau_S^k}\|_1 - R}{c}}\right)\right] \leqslant \mathbb{E}\left[\exp\left(2\beta\sqrt{\frac{R}{c}\sigma_R + T'\sigma_R^2}\right)\right]. \quad (18)$$

Inequality (14) combined with (17) and (18) finally yields:

$$\left(\mathbb{E}_x\left(e^{\beta\sqrt{\tau_C}}\right)\right)^2 \leqslant \mathbb{E}\left[\exp\left(2\beta\sqrt{cT'\sigma_R}\right)\right]\mathbb{E}\left[\exp\left(2\beta\sqrt{\frac{R}{c}\sigma_R + T'\sigma_R^2}\right)\right]. \tag{19}$$

Obviously $\sigma_R \leqslant \sigma_R^2$, so there exist $\alpha_1, \alpha_2 \in \mathbb{R}_+^*$ independent from our choice of x in $S^c \cap C^c$ such that

$$\left(\mathbb{E}_x\left(e^{\beta\sqrt{\tau_C}}\right)\right)^2 \leqslant \mathbb{E}\left[\exp\left(2\beta\alpha_1\sigma_R\right)\right]\mathbb{E}\left[\exp\left(2\beta\alpha_2\sigma_R\right)\right]$$

and therefore

$$\mathbb{E}_x\left(e^{\beta\sqrt{\tau_C}}\right) \leqslant \mathbb{E}\left[\exp\left(2\beta(\alpha_1 \vee \alpha_2)\sigma_R\right)\right].$$

We derive a similar inequality from

$$\tau_C \leqslant \frac{\|x\|_1 - R}{c} + \sum_{k=1}^{+\infty} 1_{\tau_{S^c}^k < \tau_C}\left(\tau_S^k - \tau_{S^c}^k\right)$$

if $x \in S \cap C^c$, and the case $x \in C$ is trivial. Step 1 now ensures that if $\beta > 0$ is small enough, then the function $x \mapsto \mathbb{E}_x\left(e^{\beta\sqrt{\tau_C}}\right)$ is locally bounded on $\mathbb{R}_+^n$. We will use this result when proving the ergodicity of X in section 4.3; for our present purpose, we will only need to know that $x \mapsto \mathbb{E}_x\left((\tau_C)^2\right)$ is locally bounded on $\mathbb{R}_+^n$.

Step 3: boundedness in probability on average

If $p \geqslant 3$ is an integer, then for all $t > 0$ and $x \in \mathbb{R}_+^n$:

$$\frac{1}{t}\int_0^t 1_{X_s \notin pC} \mathrm{d}s \leqslant \frac{1}{t}\left[\tau_C + \sum_{j\geqslant 1, k\geqslant 1} 1_{\tau_C^k < \tau_{(2C)^c}^j < \tau_C^{k+1}} 1_{\tau_{(2C)^c}^j < t} \int_{\tau_{(2C)^c}^j}^{\tau_C^{k+1}} 1_{X_s \notin pC} \mathrm{d}s\right]$$

since $(pC)^c \subset (2C)^c$, so that

$$\mathbb{E}_x\left[\frac{1}{t}\int_0^t 1_{X_s \in pC} \mathrm{d}s\right] \leqslant \frac{1}{t}\mathbb{E}_x\left[\tau_C + \sum_{j,k\in\mathbb{Z}_+^*} 1_{\tau_C^k < \tau_{(2C)^c}^j < \tau_C^{k+1}} 1_{\tau_{(2C)^c}^j < t}\left(\tau_C^{k+1} - \tau_{(2C)^c}^j\right) 1_{\tau_C^{k+1} - \tau_{(2C)^c}^j \geqslant \frac{(p-2)R}{M}}\right]$$

because the process needs at least $\frac{(p-2)R}{M}$ units of time to reach pC from a state with total population $2R$. From this we deduce

$$\mathbb{E}_x\left[\frac{1}{t}\int_0^t 1_{X_s \in pC} \mathrm{d}s\right] \leqslant \frac{1}{t}\mathbb{E}_x\left[\tau_C + \sum_{j,k\in\mathbb{Z}_+^*} 1_{\tau_C^k < \tau_{(2C)^c}^j < \tau_C^{k+1}} 1_{\tau_{(2C)^c}^j < t} \mathbb{E}_{X_{\tau_{(2C)^c}^j}}\left(\tau_C 1_{\tau_C \geqslant \frac{(p-2)R}{M}}\right)\right]$$

using the strong Markov property, and, setting $\zeta = \sup_{y\in\mathbb{R}_+^n, \|y\|_1 \leqslant 2C} \mathbb{E}_y\left((\tau_C)^2\right)$ (which is finite according to Step 2) and writing that $\tau_C 1_{\tau_C \geqslant \frac{(p-2)R}{M}} \leqslant \frac{M}{(p-2)R}(\tau_C)^2$,

$$\mathbb{E}_x\left[\frac{1}{t}\int_0^t 1_{X_s \in pC} \mathrm{d}s\right] \leqslant \frac{1}{t}\left(\zeta + \mathbb{E}_x\left[\sum_{j,k\in\mathbb{Z}_+^*} 1_{\tau_C^k < \tau_{(2C)^c}^j < \tau_C^{k+1}} 1_{\tau_{(2C)^c}^j < t} \zeta\sqrt{\frac{M}{(p-2)R}}\right]\right)$$

by the Cauchy-Schwarz inequality. Thus, for any integer $p \geqslant 3$, any $t > 0$ and any $x \in \mathbb{R}_+^n$:

$$\mathbb{E}_x\left[\frac{1}{t}\int_0^t 1_{X_s \in pC}\mathrm{d}s\right] \leqslant \frac{1}{t}\left(\zeta + \frac{t}{MR}\zeta\sqrt{\frac{M}{(p-2)R}}\right)$$

as the process cannot go through more than $\frac{t}{MR}$ times the full way between C and $(2C)^c$ within time t. Finally,

$$\liminf_{t\to+\infty} \mathbb{E}_x\left[\frac{1}{t}\int_0^t 1_{X_s \in pC}\mathrm{d}s\right] \leqslant \frac{\zeta}{\sqrt{M}R^{3/2}}\sqrt{\frac{1}{p-2}}.$$

Choosing p arbitrarily large shows that X is bounded in probability on average.

5.2 Petiteness criterion 4

Proof of Proposition 4. Let us assume that Assumption 3 is met and recall that we denote the resolvent of X by $(R_n)_{n\in\mathbb{N}}$. For all $x \in C$ and all $B \in Bor(\mathscr{P})$, we can write:

$$\begin{aligned}\mathbb{P}_x(R_1 \in B) &= \int_0^{+\infty} \mathbb{P}_x(X_u \in B)e^{-u}\mathrm{d}u \\ &\geqslant e^{-N(x)\overline{T}}\int_0^{\overline{T}} \mathbb{P}_x\left[h_x^{N(x)}\left(S_u^{N(x)}, \left(U_1,\dots,U_{N(x)-1}\right), \kappa_x\right) \in B\right]\mathrm{d}u\end{aligned}$$

where $S_u^{N(x)} = \left(T_1, T_2 - T_1, \dots, T_{N(x)-1} - T_{N(x)-2}, u\right)$.

Computing the joint density of the inter-jump times and the U_k and considering a common upper bound $\overline{\theta}$ for the $\theta_{i,j}$ on $\{x' \in \mathbb{R}_+^n \mid \forall y \in C, \|x'-y\|_1 \leqslant N(x)\overline{T}M\}$ yields, for all $x \in C$ and all $B \in Bor(\mathscr{P})$:

$$\mathbb{P}_x(R_1 \in B) \geqslant e^{-\left(1+\overline{\theta}n(n-1)\right)N(x)\overline{T}}\int_{[0,\overline{T}]^{N(x)}}\int_{[0,1]^{N(x)-1}}\left[\prod_{i=1}^{N(x)-1}\theta_{\kappa_i(x)}\left(h_x^i(t,\xi,\kappa_x)\right)\right]1_{h_x^{N(x)}(t,\xi,\kappa_x)\in B}\mathrm{d}\xi\mathrm{d}t.$$

Therefore, for any $x \in C$ and $B \in Bor(\mathscr{P})$:

$$\mathbb{P}_x(R_1 \in B) \geqslant e^{-\left(1+\overline{\theta}n(n-1)\right)N(x)\overline{T}}\int_{U_2^x}\int_{U_1^x}\left[\prod_{i=1}^{N(x)-1}\theta_{\kappa_i(x)}\left(h_x^i(\sigma_x(z_1,z_2),\kappa_x)\right)\right]1_{h_x^{N(x)}(\sigma_x(z_1,z_2),\kappa_x))\in B}\mathrm{d}z_1\mathrm{d}z_2$$

by the change of variables formula. Applying this formula again using Assumption 3.1 and recalling Assumption 3.3 yields

$$\mathbb{P}_x(R_1 \in B) \geqslant e^{-\left(1+\overline{\theta}n(n-1)\right)N(x)\overline{T}}\frac{\theta_0}{\lambda_{q(x)}(U_1^x)}\int_{U_2^x}\frac{1}{M}\int_{\psi_{z_2}^x(U_2^x)}1_{y\in B}\mathrm{d}y\mathrm{d}z_2.$$

Using Assumption 3.2, we may thus write:

$$\mathbb{P}_x(R_1 \in B) \geqslant e^{-\left(1+\overline{\theta}n(n-1)\right)N(x)\overline{T}}\frac{\theta_0}{M}\cdot\lambda_p(B)$$

for any $B \in Bor(\mathscr{P})$, which entails that C is petite for $(R_n)_{n\in\mathbb{N}}$, then for X.

5.3 Ergodicity criterion 5

Proof of Theorem 5. Theorems 7.1 and 7.2 of [29] ensure that is it is sufficient for F-ergodicity to hold to prove that there are $\delta > 0$ and a compact set $C \subset \mathbb{R}_+^n$ such that

$$\sup_{x \in C} \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} F(X_t)\, dt \right] < +\infty \tag{20}$$

where $\tau_C(\delta) := \inf\{t \geq \delta \mid X_t \in C\}$, and that:

$$\forall x \in \mathbb{R}_+^n, \quad \mathbb{E}_x \left[\int_0^{\tau_C(0)} F(X_t)\, dt \right] < +\infty. \tag{21}$$

The assumptions we stated in Section 4.1 happen to entail (20) and (21). Most of the work we need to provide in order to prove this result has already been done in the process of proving Theorem 3, since we then showed that the hitting time τ_C of some compact subset $C \subset \mathbb{R}_+^n$ was such that $x \mapsto \mathbb{E}_x\left[\exp\left(\beta\sqrt{\tau_C}\right)\right] < +\infty$ was finite-valued and locally bounded on $\mathbb{R}_+^n$ for some $\beta > 0$.

We now show that there exists $\eta \in \mathbb{R}_+^*$ such that setting $C = \{x \in \mathbb{R}_+^n \mid \|x\|_1 \leqslant R\}$ and defining F by (12),

$$x \mapsto \mathbb{E}_x \left[\int_0^{\tau_C(1)} F(X_t)\, dt \right]$$

is locally bounded on $\mathbb{R}_+^n$. This will end the proof of Theorem 5.

Using the notations of the proof of Theorem 3, we set $\eta_0 = \frac{\beta}{\sqrt{M}}$ and

$$F_0 : \begin{cases} \mathbb{R}_+ \to \mathbb{R}_+ \\ y \mapsto 1_{y > \frac{1}{\eta_0^2}} \dfrac{e^{\eta_0\sqrt{y}}}{\sqrt{y}} \end{cases}$$

For any $x \in \mathbb{R}_+^n$, we observe that

$$\mathbb{E}_x \left[\int_0^{\tau_C(1)} F_0\left(\sum_{i=1}^n X_t^i\right) \mathrm{d}t \right] \leqslant \mathbb{E}_x \left[\int_0^1 F_0\left(\sum_{i=1}^n X_t^i\right) \mathrm{d}t + \mathbb{E}_{X_1}\left[\int_0^{\tau_C} F_0\left(\sum_{i=1}^n X_t^i\right) \mathrm{d}t \right]\right] \tag{22}$$

by the Markov property and that for any X-stopping time τ and any $z \in \mathbb{R}_+^n$:

$$\begin{aligned} \mathbb{E}_z \left[\int_0^\tau F_0\left(\sum_{i=1}^n X_t^i\right) \mathrm{d}t \right] &\leqslant \mathbb{E}_z \left[\int_0^\tau F_0\left(\sum_{i=1}^n z_i + tM\right) \mathrm{d}t \right] = \mathbb{E}_z\left[\frac{2}{\eta_0}\left(e^{\eta_0\sqrt{\sum_{i=1}^n z_i + \tau M}} - e\right)_+ \right] \\ &\leqslant \mathbb{E}_z\left[\frac{2}{\eta_0} e^{\eta_0\sqrt{\sum_{i=1}^n z_i + \tau M}} \right] \leqslant \frac{2}{\eta_0} e^{\eta_0\sqrt{\sum_{i=1}^n z_i}}\, \mathbb{E}_z\left[e^{\eta_0\sqrt{\tau M}} \right] \end{aligned}$$

since F_0 is nondecreasing and $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for all $(a,b) \in \mathbb{R}_+^2$.

We know from Step 3 of the proof of Theorem 3 that $z \mapsto \mathbb{E}_z\left[e^{\eta_0\sqrt{\tau_C M}}\right] = \mathbb{E}_z\left[e^{\beta\sqrt{\tau_C}}\right]$ is locally bounded on $\mathbb{R}_+^n$: so are therefore $z \mapsto \mathbb{E}_z\left[\int_0^{\tau_C} F_0\left(\sum_{i=1}^n X_t^i\right)\mathrm{d}t\right]$ and, in turn, $z \mapsto \mathbb{E}_z\left[\int_0^{\tau_C(1)} F_0\left(\sum_{i=1}^n X_t^i\right)\mathrm{d}t\right]$ by (22) since $\|X_1\|_1 \leqslant \|x\|_1 + M$ holds $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}_+^n$.

We easily find $\alpha_0, \beta_0, \gamma \in \mathbb{R}_+^*$ such that $e^{\gamma\sqrt{y}} \leqslant \alpha_0 + \beta_0 F_0(y)$ for all $y \in \mathbb{R}_+$, which entails:

$$\forall x \in \mathbb{R}_+^n, \quad \mathbb{E}_x\left[\int_0^{\tau_C(1)} e^{\gamma\sqrt{\sum_{i=1}^n X_t^i}}\mathrm{d}t\right] \leqslant \alpha_0\ \mathbb{E}_x(\tau_C(1)) + \beta_0\ \mathbb{E}_x\left[\int_0^{\tau_C(1)} F_0\left(\sum_{i=1}^n X_t^i\right)\mathrm{d}t\right].$$

This yields the expected result since $x \mapsto \mathbb{E}_x[\tau_C(1)]$ is locally bounded on $\mathbb{R}_+^n$.

5.4 Stability criterion 1 for the multiplicative setting

We now consider that Assumptions A to B hold and set some further notation.

First let $\gamma = (\gamma_1, \ldots, \gamma_r) \in \mathscr{A}^r$ be a cycle of $\mathscr{G}$ that visits all sinks. Then, for all $l \in [\![1, n]\!]$, let $d_-(l)$ be the $\mathscr{G}$-graph distance from patch l to V^-. Given Assumption B, there exists a vector $\kappa = (\kappa_1, \ldots, \kappa_m) \in \mathscr{A}^m$, such that, writing $\kappa_k = (i_k, j_k)$ for all $k \in [\![1, m]\!] = V^+ \cup V^0$:

- For all $k \in V^+ \cup V^0$, $d_-(i_k) > d_-(j_k)$ (edges κ are "directed towards the system exit", that is, towards sinks).
- For all $l \in V^+ \cup V^0$, there is a $k(l) \in V^+$ such that $i_{k(l)} = l$ (each source or neutral patch is the origin and one and only one edge in κ).

This choice of κ corresponds to an ordered collection of edges along which the population in every patch of the system "reaches the exit by the shortest path". A possible choice of κ when $\mathscr{G}$ is the graph in figure 2 is $\kappa = ((2,1),(3,1),(4,1),(1,5))$; if $\mathscr{G}$ is the first graph of Figure 3, then we may choose $\kappa = ((1,3),(2,4))$.

As seen in Proposition 4 and Theorem 5, it is sufficient to prove the following propositions:

Proposition 6. — *Assume that the hypotheses of Theorem 1 hold. Then:*

(i) *There exists $S \subset \mathbb{R}_+^n$ such that X fits Assumptions 1 and 2.*

(ii) *X fits Assumption 3.*

(iii) *X admits an irreducible skeleton chain.*

Proof of Proposition 6.(i). Let $q > 0$ and $a > 0$ be such that $\mu_{i,j}([a, 1-a]) > q$ for all $(i,j) \in \mathscr{A}$. Using (7), we may choose $R > 0$ such that Assumption 1 holds for $S = \{x \in \mathbb{R}_+^n \mid \min_i x_i \geqslant R\}$. Let us now set $Z_0 = 2na^{-(m+r)}(M+R)$, $M_0 = \inf_{i \in V^+} \inf_{[0,Z_0]} \phi^i$, $\underline{\theta} = \min_{(i,j)\in\mathscr{A}} \theta_{i,j}$, $\overline{\theta} = \max_{(i,j)\in\mathscr{A}} \theta_{i,j}$ and

$$S' = \left\{x \in \mathbb{R}_+^n \mid \min_{i \in [\![1,n]\!]} x_i \geqslant 2R\right\}.$$

We will show that S' meets Assumption 2.1 by considering suitable paths of X defined by transfers along the edges of κ and γ.

It is an easy matter to see that with probability at least $\exp\left(-|\mathscr{A}|\left(\frac{Z_0}{M_0}+2\right)\overline{\theta}\right)\underline{\theta}^{m+r} q^{m+r} \frac{1}{m^m r^r}$, the following holds for the path of X stemmed from any $x \in \mathbb{R}_+^n$:

- No transfer occurs before time $\frac{Z_0}{M_0}$.
- m successive transfers occur along edges $\kappa_1, \kappa_2, \ldots$ and κ_m between times $\frac{Z_0}{M_0}$ and $\frac{Z_0}{M_0}+1$. Each of these transfers, if originated from patch i with patch z and directed towards patch j, has an amplitude between z and $(1-a)z$.

- r successive transfers happen along edges of γ between times $\frac{Z_0}{M_0}+1$ and $\frac{Z_0}{M_0}+2$, the first of which is undertook from the edge of V^- with the largest population. Each of these transfers, if originated from patch i with population z and directed towards patch j, has an amplitude between az et $(1-a)\,z$.
- No other transfer than those just described occurs before time $\frac{Z_0}{M_0}+2$.

By construction, on such event one has $X_1^i \geqslant \frac{Z_0}{n} a^{m+r} - 2M = 2R$ for all $i \in [\![1,n]\!]$ so $X_1 \in S'$, which entails that Assumption 2.1 holds.

Besides, Assumption 2.2 is met for any choice of $\varepsilon \in [0,1)$ and $T'>0$ if R is large enough. Indeed, if we define

$$A_{i,j}^{\varepsilon'}(T') = \bigcup_{\substack{k\geqslant 1\\ T_k<T'}} \left(q_{i,j}(X_{T_k^-},U_k) \leqslant (1-\varepsilon')X^i_{T_k^-}\right)$$

for any $T'>0$, $(i,j)\in\mathscr{I}$ and $\varepsilon'>0$, $A_{i,j}^{\varepsilon'}(T')$ being the event that all $i\to j$ transfers before time T' are of relative amplitude less than $1-\varepsilon'$, one can write that for any $T'>0$,

$$\mathbb{P}_x\left(A_{i,j}^{\varepsilon'}(T')\right) \underset{\varepsilon'\to 0}{\longrightarrow} 1$$

uniformly in x since the $\mu_{i,j}$ assign mass 0 to $\{1\}$. Now the ϕ^i and $\theta_{i,j}$ being bounded implies that for all $T'>0$:

$$\liminf_{\substack{\|x\|_1\to+\infty\\ x\in S'}} \mathbb{P}_x(\forall s\in[0,T'], X_s\in S) \geqslant \liminf_{\substack{\|x\|_1\to+\infty\\ x\in S'}} \mathbb{P}_x\left(\bigcap_{(i,j)\in\mathscr{A}} A_{T'}^{\varepsilon'}(i,j)\right)$$

for all $\varepsilon'\in(0,1)$, hence the result as ε' tends to 0. Assumption 2.3 is then true if T' is large enough (which is possible as soon as R is), which completes our proof in the multiplicative case.

Proof of Proposition 6.(ii).

Using Lemma 3.1 from [28], we will consider the behavior of our process starting from a small ball centered on a state that corresponds to positive population levels for sources and neutral patches, and use the change of variables formula to check for Assumption 2.1.

It is clear that we may assume that C is defined as $\{x\in\mathbb{R}_+^n \mid \|x\|_1 \leqslant R\}$ for some $R>0$.

Step 1: uniform reachability of the $\mathscr{B}(x^0,\delta)$ for some x^0

For any $x\in\mathbb{R}_+^m\times\{0\}^{n-m}$ and $\delta>0$, we denote the closed ball of $\mathbb{R}_+^m\times\{0\}^{n-m}$ for the infinite norm by $\mathscr{B}(x,\delta)$. We now show that there exists $x^0\in\left(\mathbb{R}_+^*\right)^m\times\{0\}^{n-m}$ such that all $\mathscr{B}(x^0,\delta)$ be uniformly reachable from C.

Let us first assume that $V^0=\varnothing$.

By assumption on Φ, there exists $T>0$ such that $\Phi^i(x,T)=0$ for all $x\in\mathbb{R}_+^n$ with $\|x\|_1\leqslant R+M$ and all $i\in V^-$. Let $x^0=\Phi(0,T)$ and first note that $x^0=\Phi(z,T)$ for all $z\in\mathbb{R}_+^n$ such that $\|z\|_1\leqslant R+M$ and $z_i=0$ for all $i\in V^0\cup V^-$. Now consider $\delta>0$. $(\Phi^1,\dots,\Phi^m)$ is uniformly continuous on $C\times[T,T+1]$ since the non-negative ϕ^i are $\mathscr{C}^1$, so there are $\delta'\in]0,M]$ and $t_0\in(0,1]$ such that $\|\Phi(x,u)-x^0\|_\infty\leqslant\delta$ for all $x\in\mathbb{R}_+^n$ with $x_1+\ldots+x_m\leqslant\delta'$ and all $u\in[T,T+t_0]$. Set $\varepsilon\in(0,1)$ and $\eta\in(0,1)$ such that:

$$\forall (i,j)\in\mathscr{I},\quad \mu_{i,j}([0,\varepsilon))<\eta \tag{23}$$

and denote by N the smallest positive integer such that $(1-\varepsilon)^N R < \frac{\delta'}{2}$.
It is easy to see that setting $\kappa^0 = (\kappa, \ldots, \kappa) \in \mathscr{I}^{mN}$, we have:

$$\forall x \in C, \forall t \in \left[0, \frac{\delta'}{2mNM}\right]^{mN} \times [T, T+t_0], \forall \xi \in \left(\eta, 1\right]^{mN}, \quad h_x^{mN+1}(t, \xi, \kappa^0) \in \mathscr{B}\left(x^0, \delta\right).$$

Considering the joint density of the T_k and the U_k as in the proof of Proposition (i) finally yields, for all $x \in C$:

$$\mathbb{P}_x\left(\forall t \in \left[T, T+\frac{\delta'}{2M}+t_0\right], X_t \in \mathscr{B}\left(x^0, \delta\right)\right) \geqslant \exp\left(-|\mathscr{A}|\left(\frac{\delta'}{2M}+t_0\right)\overline{\theta}\right)\left(\frac{\underline{\theta}\delta'}{2mNM}\right)^{mN}(1-\eta)^{mN} \tag{24}$$

which, in turn, implies the expected property.

The proof relies on the same reasoning if $V^0 \neq \varnothing$, using paths in which some sequence of transfers would result in all neutral patches having a positive population, then the flow would bring the population of sinks to zero.

Step 2: petiteness of a $\mathscr{B}(x^0, \delta)$

According to Lemma 3.1 from [28] and Step 1 hereabove, Proposition 6.(ii) will be proved if we show that there exists some $\delta > 0$ such that $\mathscr{B}(x^0, \delta)$ is petite for the resolvent of X, which can be done by way of checking Assumption 3.

Let us thus set some $\delta \in (0, \min_{i \in V^+ \cup V^0} x_i^0)$ that we may have to reduce later on, and let $\underline{x} = \min_{i \in V^+ \cup V^0} x_i^0 - \delta$.

For simplicity reasons, we will assume that all ϕ^i associated with sources are $\mathscr{C}^1$ functions (rather than merely piecewise $\mathscr{C}^1$) on $\mathbb{R}_+$ and that $\mu_{i,j}$ measures admit continuous density functions $f_{i,j}$ on $[0,1]$ (instead of just admitting an absolutely continuous component on a subinterval of $[0,1]$). The general case only requires reducing the domain over which it is possible to consider our paths of interest, which induces an unnecessary notational inflation.

Even if it means considering a larger T than in Step 1, we may assume that $\Phi(x, T) \in \mathbb{R}_+^m \times \{0\}^{n-m}$ for all $x \in \mathscr{B}(x^0, \delta)$. As in Step 1, we define ε and η such that (23) holds.

Let us begin by proving the following inequality:

$$\forall i \in V^+, \forall y > 0, \forall u > 0, \quad \frac{\min_{[0, y+Mu]} \phi^i}{M} \leqslant \frac{\partial \Phi^i}{\partial y}(y, u) \leqslant \frac{M}{\min_{[0, y+Mu]} \phi^i}. \tag{25}$$

If $i \in V^+$, $0 < y < y+h$ and $u > 0$, the mean value inequality entails that $\Phi^i\left(y, \frac{h}{M}\right) \leqslant y+h$, from which we deduce

$$\Phi^i(y+h, u) - \Phi^i\left(y, u\right) \geqslant \Phi^i\left(y, u+\frac{h}{M}\right) - \Phi^i\left(y, u\right) \geqslant \frac{\min_{[\Phi^i(y,u), \Phi^i(y,u)+h]} \phi^i}{M} h$$

using the mean value theorem, hence the LHS of (25) by letting h go to 0. Proving the right-side inequality of (25) relies on the very same argument and is left to the reader. Moreover, it is clear that

$$\forall i \in V^0, \forall y > 0, \forall u > 0, \quad \frac{\partial \Phi^i}{\partial y}(y, u) = 1. \tag{26}$$

We now show that for any $x \in \mathscr{B}(x^0, \delta)$ and $t \in (0,1)^m \times (T, T+1)$, then

$$\psi_t^x : \begin{cases} (0,a)^m \to \mathbb{R}^m \times \{0\}^{n-m} \\ \xi \mapsto h_x^{m+1}(t,\xi,\kappa) \end{cases}$$

is a $\mathscr{C}^1$-diffeomorphism of $(0,\eta)^m$ onto its image. For fixed x and t, indeed, a simple calculation shows that

$$\frac{\partial \psi_x^t}{\partial \xi_m}(t,\xi,\kappa) = \partial_1 \Phi\left(g_{\kappa_m}(h_x^m, \xi_m), t_{m+1}\right) \cdot \partial_2 g_{\kappa_m}(h_x^m, \xi_m) \tag{27}$$

and that for any $i \in [\![1, m-1]\!]$:

$$\frac{\partial \psi_x^t}{\partial \xi_i}(t,\xi,\kappa) = \left[\prod_{k=m}^{i+1} \partial_1 \Phi\left(g_{\kappa_k}(h_x^k, \xi_k), t_{k+1}\right) \cdot \partial_1 g_{\kappa_k}\left(h_x^k, \xi_k\right)\right] \cdot \partial_1 \Phi\left(g_{\kappa_i}(h_x^i, \xi_i), t_{i+1}\right) \cdot \partial_2 g_{\kappa_i}(h_x^i, \xi_i). \tag{28}$$

In (27) and (28), we abbreviated $h_x^k(t,\xi,\kappa)$ by writing h_x^k and the matrix product $A_m \dots A_{i+1}$ by writing $\prod_{k=m}^{i+1} A_k$. We also defined

$$\partial_1 g_{i,j}(y,\zeta) = I_m + F_{\mu_{i,j}}^{-1}(\zeta) \cdot (1_{j \leqslant m} E_{j,i} - E_{i,i})$$

where $\left(E_{i,j}\right)_{i,j \in [\![1,m]\!]}$ is the canonical basis of $\mathscr{M}_m(\mathbb{R})$, as well as

$$\partial_1 \Phi(y,u) = \begin{pmatrix} \frac{\partial \Phi^1(y,u)}{\partial y_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\partial \Phi^m(y,u)}{\partial y_m} \end{pmatrix}$$

and

$$\partial_2 g_{i,j}(y,\xi) = \frac{y_i}{f_{i,j}\left(F_{\mu_{i,j}}^{-1}(\xi)\right)} \left(1_{j \leqslant m} \tilde{e}_j - \tilde{e}_i\right)$$

where $(\tilde{e}_1, \dots, \tilde{e}_m)$ is the canonical basis of $\mathbb{R}^m$.

Recalling the fact that every $i \in V^+$ is the origin of one only edge κ_i, (25) and (26) then imply that for all $t \in (0,1)^m \times (T, T+1)$ and all $\xi \in (0,\eta)$, the Jacobian matrix of ϕ_x^t at (t,ξ,κ) is a continuous function of ξ and is an invertible matrix with determinant $J(t,\xi,\kappa)$ such that

$$0 < J(t,\xi,\kappa) \leqslant \left(\frac{M}{\min_{i \in V^+} \min_{[0, x_i + (m+T+1)M]} \phi^i}\right)^{\frac{m(m+1)}{2}} \frac{(\|x\|_1 + \delta + (m+T+1)M)^m}{\prod_{i=1}^m \min_{[0,1]} f_{\kappa_k}} < +\infty. \tag{29}$$

Now ψ_t^x is injective over $(0,\eta)^m$ for fixed t and x; this stems from the fact that the cumulative distribution functions of the μ_{κ_k} are strictly increasing, so knowing $\langle \psi_t^x(t,\xi,\kappa), e_{i_1} \rangle$ makes it possible to determine ξ_1, knowing ξ_1 and $\langle \psi_t^x(t,\xi,\kappa), e_{i_2} \rangle$ yields ξ_2 and so on. This proves our claim that ψ_t^x is a $\mathscr{C}^1$-diffeomorphism.

In order to conclude, we only need to check that Assumption 3.2 holds. Let us consider $x \in \mathscr{B}(x^0, \delta)$ and for all $t \in [0,1]^m$, set $I_t = \psi_t^x(t, (0,\eta)^m, \kappa)$. Considering the possible value of the population of patches 1 to m after a series of jumps along the edges of κ such that the k-th transfer has amplitude between 0 and $k\frac{\varepsilon}{m}\underline{x}$, it is fairly easy to check that for a given $t=(t_1,\ldots,t_m)\in\left(0,\left(\frac{\varepsilon}{Mm^2}\underline{x}\right)\wedge 1\right)^m\times(T,T+1)$:

$$\prod_{i=1}^m\left[\Phi^i\left(x_i-\frac{\varepsilon}{m}+(t_1+\ldots+t_i)M,t_{i+1}+\ldots+t_{m+1}\right),\Phi^i\left(x_i,t_1+\ldots+t_{m+1}\right)\right)\subset I_t$$

and then, by (25), (26) and the mean value theorem for the $\Phi^i(\cdot,t)$:

$$\prod_{i=1}^m\left[\Phi^i(x_i,t_{i+1}+\ldots+t_{m+1})-\varepsilon_0,\Phi^i(x_i,t_1+\ldots+t_{m+1})\right)\subset I_t$$

for some positive ε_0 independent from the choice of $x\in\mathscr{B}\left(x^0,\delta\right)$. For small enough values of δ, there are $T^0\in\left(0,\left(\frac{\varepsilon}{Mm^2}\underline{x}\right)\wedge 1\right]$ and an orthotope $\mathscr{P}\subset\mathbb{R}_+^m\times\{0\}^{n-m}$ with non-zero Lebesgue measure, both independent from the choice of x within $\mathscr{B}\left(x^0,\delta\right)$, such that $\mathscr{P}\subset I_t$ for all $t\in\left(0,T^0\right)^m\times(T,T+T^0)$. This entails that Assumption 3.2 holds. Proposition 4 then implies that $\mathscr{B}(x^0,\delta)$ is petite for the resolvent of X, which ends the proof.

Note that if the $t\mapsto h_x^m(t,\xi,\kappa)$ are diffeomorphisms for suitable values of ξ (which typically is the case in a multiplicative model with constant growth!), we may want to monitor the effect of small variations of t (rather than of ξ) on $h_x^m(t,\xi,\kappa)$. This makes it possible to relax the absolute continuity assumptions on the $\mu_{i,j}$, since it is then sufficient to assume that $\mu_{i,j}(\{0\})<1$ and $\mu_{i,j}(\{1\})=0$ for petiteness to hold.

Proof of Proposition 6.(iii). We keep all notations from the proof of Proposition 6.(ii) above. First set $\Delta_0=\frac{\delta'}{2M}+t_0$. If π denotes the invariant probability of X, we may assume that $\pi(C)>0$, which entails by Birkhoff's ergodic theorem (see [1] p.169) that τ_C is $\mathbb{P}_x$-a.s. finite for all $x\in\mathbb{R}_+^n$. Inequality (24) then yields:

$$\inf_{x\in\mathbb{R}_+^n}\mathbb{P}_x\left(\forall t\in[\tau_C+T,\tau_C+T+\Delta_0],X_t\in\mathscr{B}\left(x^0,\delta\right)\right)>0.$$

Thus, according to the strong Markov property, it is sufficient to show that there exists $\Delta\in(0,\Delta_0]$ such that:

$$\forall x\in\mathscr{B}(x^0,\delta),\forall B\in Bor(\mathscr{P}),\quad \lambda_m(B)>0\Rightarrow\mathbb{P}_x\left(\exists k\geqslant 1,X_{k\Delta}\in B\right)>0.$$

Set $\Delta=\Delta_0\wedge T$ and $k\in\mathbb{Z}_+$ such that $mT^0\leqslant k\Delta<mT^0+T$, and let $x\in\mathscr{B}(x^0,\delta)$. Then for any Borel subset B of $\mathscr{P}$, the change of variables formula yields:

$$\begin{aligned}
\mathbb{P}_x\left(X_{k\Delta}\in B\right) &\geqslant \mathbb{P}_x\left[(T_m<k\Delta<T_{m+1})\cap\left(\Phi(X_{T_m},k\Delta-T_m)\in B\right)\right]\\
&\geqslant e^{-\overline{\theta}n(n-1)k\Delta}\int_{(0,T^0)^m}\int_{(0,\eta)^m}\left[\prod_{i=1}^m\theta_{\kappa_i(x)}\left(h_x^i\right)\right]1_{\Phi\left(h_x^m,k\Delta-\sum_{j=1}^m t_j\right)\in B}\mathrm{d}\xi\mathrm{d}t\\
&\geqslant e^{-\overline{\theta}n(n-1)k\Delta}\underline{\theta}^m\int_{(0,T^0)^m}\int_{(0,\eta)^m}1_{\Phi\left(h_x^m,k\Delta-\sum_{j=1}^m t_j\right)\in B}\mathrm{d}\xi\mathrm{d}t\\
&\geqslant e^{-\overline{\theta}n(n-1)k\Delta}\underline{\theta}^m\int_{(0,T^0)^m}\int_{(0,\eta)^m}1_{h_x^{m+1}\left((t,k\Delta-\sum_{j=1}^m t_j),\xi,\kappa\right)\in B}\mathrm{d}\xi\mathrm{d}t\\
&\geqslant \frac{1}{\alpha}e^{-\overline{\theta}n(n-1)k\Delta}\underline{\theta}^m\int_{(0,T^0)^m}1_{k\Delta-\sum_{j=1}^m t_j\in[T,T+T^0]}\left(\int_{\mathscr{P}}1_{y\in B}\mathrm{d}y\right)\mathrm{d}t\\
&\geqslant \frac{1}{\alpha}e^{-\overline{\theta}n(n-1)k\Delta}\underline{\theta}^m\lambda_m(S)\,\lambda_m(B)
\end{aligned}$$

where h^i_x stands for $h^i_x(t,\xi,\kappa)$, α is the upper bound of $J(t,\xi,\kappa)$ defined by (29) and

$$S = \left\{ t \in (0,T^0)^m, \sum_{j=1}^{m} t_j \in \left[k\Delta - T - T^0, k\Delta - T\right] \right\}.$$

Now $\lambda_m(S) > 0$, which ends the proof.

5.5 Stability criterion 2 for the unitary setting

We keep considering γ and κ as defined in the proof of the multiplicative case. We will show the following, which is sufficient to conclude thanks to Proposition 4 and Theorem 5.

Proposition 7. — *Assume that the hypotheses of Theorem 2 hold. Then*:

(*i*) *There exists $S \subset \mathbb{R}^n_+$ such that X fits Assumptions 1 and 2.*

(*ii*) *X fits Assumption 3.*

(*iii*) *X admits an irreducible skeleton chain.*

Proof of Proposition 7.(i). Let us first assume that (9) holds. Let $R \geqslant 2$ be an integer such that Assumption 1 holds with

$$S = \left\{ x \in \mathbb{R}^n_+ \mid \min_{i \in V^-} x_i \geqslant R \right\}$$

and set

$$S' = \left\{ x \in \mathbb{R}^n_+ \mid \min_{i \in V^-} x_i \geqslant 2R \right\}.$$

In order to verify that Assumption 2.1 holds, we consider $x \in \mathbb{R}^n_+$ as well as $T \in (0,1]$ and argue as in the proof of Proposition 6.(i). On the event we consider, transfers along edges $\kappa_1,\ldots,\kappa_m$ occur before time T and result in the total population of sinks being at least $2(n-m)R + TM$, and transfers along γ occur between time T and time $2T$ and result in the population of each sink at time $2T$ being at least $2R$. For corresponding paths, $X_{2T} \in S'$, and it is easy to show using the assumptions on Θ that the probability under $\mathbb{P}_x$ of observing such paths for a fixed T goes to 1 as $\|x\|_1$ goes to $+\infty$. As a result, T may be taken arbitrarily small in Assumption 2.1 for any fixed value of δ. Assumption 2.3 will thus be automatically fulfilled for some value of T provided that Assumption 2.2 holds. Showing the latter relies on the observation that as a sink gives up its charge, the temporal intensity of subsequent transfers from this patch is upper bounded. Simple calculations then show that the probability for a sink with original population above $2R'$ to have a population lower than R' before time $\frac{R'}{M}$ is lower than some constant in $[0,1)$, which ends our proof.

The argument is similar in the irreducible case with (10) except that it is now possible to define S and S' as in the proof of Proposition 6.(i). The irreducibility assumption then entails that the process returns to S' arbitrarily fast with a given probability when the total population in the system is high enough, and we may conclude just as before.

Proof of Proposition 7.(ii). Without loss of generality, we may assume that C writes $\{x \in \mathbb{R}^n_+ \mid \|x\|_1 \leqslant R\}$ for some $R \in \mathbb{Z}^*_+$. Let $T > 0$ be such that $\Phi^i(x,T) = 0$ for all $i \in V^-$ and all $x \in \mathbb{R}^n_+$ with $\|x\|_1 \leqslant R+1$ and define

$$\kappa' = (\kappa_1,\ldots,\kappa_1,\kappa_2,\ldots,\kappa_2,\ldots,\kappa_m,\ldots,\kappa_m) \in \mathscr{I}^{m(R+1)}$$

as the vector of edges made of $R+1$ successive repetitions of each of κ's edges.

For all $t \in \left[0, \frac{1}{mM(R+1)}\right)^{m(R+1)} \times (T, T+1)$ and all $\xi \in (0,1]^{m(R+1)}$, we may write:

$$\forall i \in V^+, \quad \left(h_x\left(t,\xi,\kappa'\right)\right)_i = \Phi^i\left(0, t_{(i+1)(R+1)} + t_{(i+2)(R+1)} + \ldots + t_{m(R+1)} + t_{m(R+1)+1}\right)$$

and

$$\forall i \in V^0 \cup V^-, \quad \left(h_x\left(t,\xi,\kappa'\right)\right)_i = 0.$$

For fixed $t' = (t_1, \ldots, t_R, t_{R+2}, \ldots, t_{2(R+1)-1}, t_{2(R+1)+1}, \ldots, t_{m(R+1)-1}, t_{m(R+1)+1}) \in \left[0, \frac{1}{mM(R+1)}\right)^{mR} \times (T, T+1)$ and $\xi \in (0,1]^{mR}$, we easily see that

$$\psi^x_{t'} : \begin{cases} \left(0, \frac{1}{mM(R+1)}\right)^m \to \mathbb{R}^d \times \mathbb{R}^{n-d} \\ \left(t_{R+1}, t_{2(R+1)} \ldots, t_{m(R+1)}\right) \mapsto h_x^{m(R+1)+1}(t,\xi,\kappa) \end{cases}$$

is a $\mathscr{C}^1$-diffeomorphism of $\left(0, \frac{1}{mM(R+1)}\right)^m$ onto its image. It is independent from the choice of x in C and has its Jacobian determinant upper bounded by $\alpha = \prod_{i=1}^d \max_{[0,R+1+(T+1)M]} \phi^i$. It is then possible to define, as in the proof of Proposition 6.(ii), $T^0 \in \left(0, \frac{1}{mM(R+1)}\right)$ and an orthotope $\mathscr{P} \subset \mathbb{R}_+^d \times \{0\}^{n-d}$ with non-zero Lebesgue measure, independently from the choice of $x \in C$, such that $\mathscr{P}$ be included in the image of $\left(0, T^0\right)^m$ by $\psi^x_{t'}$ for all $t' \in \left(0, T^0\right)^{mR} \times (T, T+T^0)$. Assumption 3.3 is also met since Θ is lower bounded by a positive real number, and Proposition 4 therefore implies that C is petite.

Proof of Proposition 7.(iii). We use the notations of the proof of Proposition 7.(ii) above. It is sufficient to prove the irreducibility property for initial conditions within C because of the same argument as in the proof of Proposition 6.(iii). Even if it means considering smaller T^0 and $\mathscr{P}$, one may assume that:

$$\forall t' \in \left(0, \frac{T^0}{2m(R+1)}\right)^{mR} \times \left(0, T^0\right), \quad \mathscr{P} \subset \psi^x_{t'}\left(0, \frac{T^0}{2m(R+1)}\right)^m.$$

Let us consider $\Delta \in \left(0, \frac{T^0}{2}\right)$ and $k \geqslant 1$ such that $k\Delta \in \left(T + \frac{T^0}{2}, T + T^0\right)$. Reproducing the calculations of the proof of Proposition 6.(iii) yields for any $x \in C$ and any $B \in Bor(\mathscr{P})$:

$$\begin{aligned} \mathbb{P}_x\left(X_{k\Delta} \in B\right) &\geqslant e^{-k\Delta\overline{\theta}} \Theta(0)^{m(R+1)} \int_{\left(0,T^0\right)^{m(R+1)}} 1_{h_x^{m(R+1)}((t, k\Delta - T - \sum_{j=1}^{m(R+1)} t_j), \xi, \kappa') \in B} \mathrm{d}t \\ &\geqslant \frac{1}{\alpha} e^{-k\Delta\overline{\theta}} \Theta(0)^{m(R+1)} \int_{\left(0,T^0\right)^{m(R+1)}} \int_{\left(0, \frac{T^0}{2m(R+1)}\right)^{mR}} \lambda_d(\mathscr{P} \cap B) \mathrm{d}t' \\ &\geqslant \frac{1}{\alpha} e^{-k\Delta\overline{\theta}} \Theta(0)^{m(R+1)} \left(\frac{T^0}{2m(R+1)}\right)^{mR} \lambda_d(B) \end{aligned}$$

where $\overline{\theta}$ is a common upper bound for the $\theta_{i,j}$ on $\{y \in \mathbb{R}_+^n \mid \|y\|_1 \leqslant R + k\Delta M\}$. Then $\mathbb{P}_x\left(X_{k\Delta} \in B\right) > 0$ if $\lambda_d(B) > 0$, which ends our proof.

Appendix A: A Markov family of weak solutions of (5)

For any $x \in \mathbb{R}_+^n$, let us define $(T_k(x))_{k\in\mathbb{Z}_+}$ as the sequence of jump times of the counting process

$$(Z(x)_t)_{t\geqslant 0} = \left(\sum_{(i,j)\in\mathcal{I}} \int_{[0,t]\times\mathbb{R}_+} 1_{z<\theta_{i,j}(X_{u^-}(x))} N_{i,j}(\mathrm{d}u, \mathrm{d}z, [0,1]) \right)_{t\geqslant 0}$$

with $T_0(x) = 0$, so $T_k(x)$ accounts for the k-th (potentially null) transfer between patches of $[\![1, n]\!]$. Similarly, for all $x \in \mathbb{R}_+^n$ we define the jump quantiles sequence $(U_k(x))_{k\in\mathbb{Z}_+}$ as follows:

$$\forall k \in \mathbb{Z}_+, \quad U_k = \sum_{(i,j)\in\mathcal{I}} \int_{[0,1]} N_{i,j}(T_k, \mathbb{R}_+, \mathrm{d}\xi).$$

The $U_k(x)$ are i.i.d. with uniform law over $[0,1]$ and the amplitude of the jump of $X(x)$ at time $T_k(x)$ is given by $\sum_{(i,j)\in\mathcal{I}} q_{i,j}(X_{T_k^-}(x), U_k)$. Let $\tilde{\Omega} = \mathbb{D} \times \mathbb{R}_+^{\mathbb{Z}_+} \times [0,1]^{\mathbb{Z}_+^*}$, where $\mathbb{D}$ stands for the set of *càdlàg* $\mathbb{R}_+ \to \mathbb{R}_+^n$ functions, and define $\tilde{\mathcal{A}} = Bor(Sk) \otimes \mathcal{C} \otimes \mathcal{C}'$, where Sk is the Skorokhod topology on $\mathbb{D}$ (see [3] p.168) and $\mathcal{C}$ (resp. $\mathcal{C}'$) is the cylindrical σ-algebra on $\mathbb{R}_+^{\mathbb{Z}_+}$ (resp. $[0,1]^{\mathbb{Z}_+^*}$). We finally build the Markov family $\big((X,T,U), \mathbb{P}_x, x \in \mathbb{R}_+^n\big)$ on $\big(\tilde{\Omega}, \tilde{\mathcal{A}}\big)$ by writing (X,T,U) for the canonical process on $\tilde{\Omega}$ and defining $\mathbb{P}_x$ as the distribution of $(X(x), T(x), U(x))$ under $\mathbb{P}$ for all $x \in \mathbb{R}_+^n$. It is then straightforward to see that $\big(X, (\mathbb{P}_x)_{x\in\mathbb{R}_+^n}\big)$ fits the conditions given in section 2.

Appendix B: On the invariant probability of X

It is clear by the proof of Proposition 4 that if Assumption 3 holds and if X admits an invariant probability π, then π dominates the Lebesgue measure on some open subset of an affine subspace of $\mathbb{R}^n$. Its support is yet not necessarily included in this subspace, as can be seen in Figure 4 below. Moreover, determining conditions for the absolute continuity of π restricted to given areas of $\mathbb{R}_+^n$ is a non-trivial matter that still has to be discussed. Interested readers are referred to [22] for a discussion about the absolute continuity of π in a *house of cards* model that has common specifications with our setting.

If X is positive Harris recurrent with invariant probability π and $\int \|x\|_1 \mathrm{d}\pi(x) < +\infty$, we see by (6) that for any $\mathcal{C}^1$ function $f : \mathbb{R}_+^n \to \mathbb{R}$ with bounded differential the classical relation hereafter holds:

$$\int \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\phi^i(x) + \sum_{(i,j)\in\mathcal{I}} \theta_{i,j}(x) \int \big(f(x+\xi(e_j - e_i)) - f(x)\big)\, \mu_{i,j}(x, \mathrm{d}\xi_i) \right) \mathrm{d}\pi(x) = 0. \tag{30}$$

It is worth noting that the second term of the integrand in the LHS above is zero whenever $f(x)$ only depends on $x_1 + \ldots + x_n$. Equation (30) then takes a simple form that may provide valuable information on π.

In particular, applying (30) to the projection functions $\langle \cdot, e_i \rangle$ yields:

$$\forall t \geqslant 0, \forall i \in [\![1, n]\!], \quad \mathbb{E}_\pi\left(\phi^i(X_t)\right) + \sum_{j\neq i} \mathbb{E}_\pi\left(d_{j,i}(X_t) - d_{i,j}(X_t)\right) = 0 \tag{31}$$

where $d_{i,j} : \mathbb{R}_+^n \to \mathbb{R}$ is the *debit function* defined by

$$d_{i,j}(x) = \theta_{i,j}(x) \int \xi\, \mu_{i,j}(x, \mathrm{d}\xi). \tag{32}$$

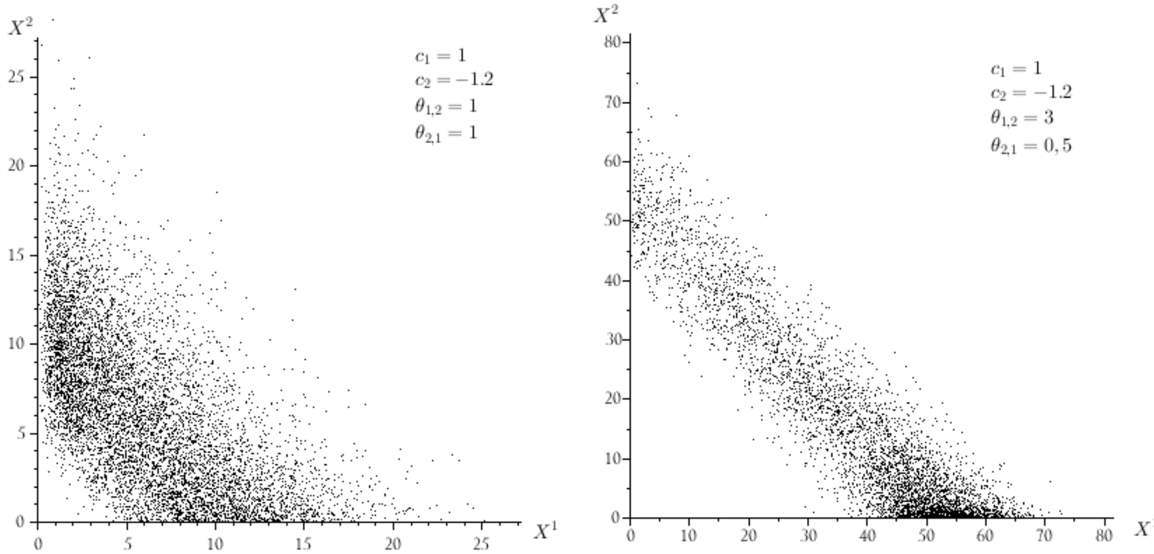

Figure 4: Plotting of 10000 simulated instances of X_{100} in the two-patch uniform model with constant growth for specified parameters, with initial value $(5,5)$. In accordance with equation (33) of example 6, about 17.67% of points in both figures lie on the x axis.

Example 5. Let us assume that the following conditions hold:

$$\begin{cases} \phi^i(x) = a_i - x_i \\ \theta_{i,j}(x) = 1 \\ \int \xi \mu_{i,j}(x, \mathrm{d}\xi) = m_i x_i \end{cases}$$

where the a_i and m_i are non-negative real numbers, and that X is positive Harris recurrent and integrable under its invariant probability. Then (31) becomes:

$$\forall t \geqslant 0, \forall i \in [\![1, n]\!], \quad (1 + m_i(n-1))\,\mathbb{E}_\pi\left(X_t^i\right) - \sum_{j \neq i} m_j \mathbb{E}_\pi\left(X_t^j\right) = a_i$$

and can be written as a dominant diagonal linear system, which makes it possible to determine the $\mathbb{E}_\pi\left(X_t^i\right)$ explicitly.

Example 6. Let us consider the one-exit constant growth framework, that is, a constant growth model with $c_i \geqslant 0$ for all $i \in [\![1, n-1]\!]$ and $c_n < 0$, and assume that X is positive Harris recurrent with invariant probability π and $\int \|x\|_1 \mathrm{d}\pi(x) < +\infty$. Proving that

$$\mathbb{P}_\pi\left(X_n^t > 0\right) = \frac{\sum_{j=1}^{n-1} c_j}{|c_n|} \tag{33}$$

and that the event $(X_t^n > 0)$ is independent from the variable $\sum_{i=1}^n X_t^i$ for all $t \geqslant 0$ under $\mathbb{P}_\pi$ is left to the reader as an (easy) exercise.

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