Environmental Efficiency and Productivity Analysis

Arnaud Abad

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A. Abad*

Abstract

This paper introduces a general framework to analyse green efficiency and environmental productivity. Innovative environmental efficiency measures are introduced to define green productivity indices. Equivalence conditions for the additive and multiplicative green efficiency and productivity measures are displayed. In addition, the core components of environmental productivity change are defined. New implementation process of environmental efficiency and productivity assessment on convex and non convex pollution-generating technologies is proposed.

Keywords: Data Envelopment Analysis, Environmental Efficiency Indices, Environmental Productivity Indicators, Non Convexity, Pollution-generating Technology.

JEL: C61, D24, Q50

* *Université de Lorraine, Université de Strasbourg, AgroParisTech, CNRS, INRAE, BETA, 54000, Nancy, France. Contact: arnaud.abad@inrae.fr*

1 Introduction

Deteriorations of global environmental conditions induced growing interest in environmental efficiency and productivity[1] studies (Sueyoshi et al., 2017; Zhou et al., 2008). Sustainable strategies that allow to reduce impacts of production processes on the natural environment, are major concerns for private and public sectors. Indeed, they attempt to be both environmental responsible and technically efficient. It follows that, managerial efforts to promote high quality inputs and/or innovative technology are performed. These production adaptation strategies allow to support environmental efficiency.

Traditional eco-efficiency literature relies on the assumption that no polluting and polluting outputs can only be reduced simultaneously by a proportional factor; i.e. weak (or ray) disposal axiom (Shephard, 1970). This modelling of pollution-generating processes in production theory is due to Färe, Grosskopf, Lovell and Pasurka (1989). Weak Disposal (WD) approach is widely applied to numerous topics in the literature: Manello (2017), Falavigna et al. (2015), Azad and Ancev (2014), Bilsel et al. (2014), Park and Weber (2006) or Picazo-Tadeo et al. (2005). Some recent papers assuming WD models are also proposed in the literature; see for instance Pham and Zelenyuk (2019). Innovative approaches arose due to the limits of the WD models (Lauwers and Van Huylenbroeck, 2003; Coelli et al., 2007; Lauwers, 2009; Rödseth, 2017; Murty et al., 2012). In the same vein, a general class of Pollution-generating Technologies (PgT) has been defined in Abad and Briec (2019). These authors propose to model PgT using an innovative B-disposal assumption[2].

This paper aims to define innovative eco-efficiency and -productivity measures on convex and non convex environmental production processes. Equivalence conditions for the additive and multiplicative green efficiency and productivity indices are introduced. In addition, this paper shows that the new environmental efficiency and productivity measures allow to define global eco-efficiency and -productivity analysis. Indeed, green efficiency and productivity indicators are defined through various managerial adaptation strategies.

Environmental productivity advance (or deterioration) is appraised through different sources (Chung et al., 1997; Sena, 2004; Azad and Ancev 2014; Picazo-Tadeo et al., 2014; Kapelko et al., 2015; Shen et al., 2017). Knowing the main drivers of green productivity change is of particular interest for researchers (Tyteca, 1996; Aiken and Pasurka, 2003; Mahlberg and Sahoo, 2011). This paper introduces innovative eco-productivity decomposition. The components of green productivity variation are defined through convex or non convex environmental production processes.The convexity assumption of the production technology is not required to define the sources of environmental productivity change. This result brings on theoretical (Dasgupta and Mäler, 2003; Tschirhart, 2012) and empirical implications (De Borger and Kerstens, 1996). Therefore, a global framework to analyse impacts of green investments and/or

[1]Throughout this paper we use similarly the terms environmental efficiency (productivity), green efficiency (productivity) and eco-efficiency (eco-productivity).

[2]This approach considers congested production set with a relaxed disposability property (Briec et al., 2016). Hence, the term B refers to Bad outputs.

environmental policies on the components of green productivity variation is defined.

The remainder of this paper unfolds as follows. Section 2 introduces technology assumptions and definition. In addition, it defines environmental distance functions on pollution-generating production process. Section 3 introduces multiplicative and additive eco-productivity measures. Section 4 proposes a decomposition of the environmental Malmquist and Luenberger productivity indicators. Section 5 introduces new implementation process of eco-efficiency and -productivity indices. These environmental efficiency and productivity indicators are defined on convex and non convex non-parametric PgT. Finally, section 6 discusses and concludes.

2 Environmental technology and distance functions

In this section, we define the properties of the environmental production process. In addition, innovative additive and multiplicative eco-efficiency measures are introduced. Equivalence conditions between additive and multiplicative environmental efficiency indices are displayed.

2.1 Technology assumptions and definition

First, we define the notations used in this paper. Let $x_t \in \mathbb{R}^n_+$ denotes inputs used to produce no-polluting (desirable) and polluting (undesirable) outputs, $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}^m_+$ where $[m] = [m_{np}] + [m_p]$ and $[m] = card(y_t)$. In addition, assume that $B \subset [m]$ is the subset indexing polluting outputs of the technology[3]. The production possibility set is defined as follows:

$$T_t = \left\{(x_t, y_t^{np}, y_t^p) \in \mathbb{R}^{n+m}_+ : x_t \,\text{can produce}\, (y_t^{np}, y_t^p)\right\} \tag{2.1}$$

The production technology, T_t, can be similarly characterized by the output set, $P_t : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$, or the input correspondence, $L_t : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$:

$$P_t(x_t) = \left\{(y_t^{np}, y_t^p) \in \mathbb{R}^m_+ : (x_t, y_t^{np}, y_t^p) \in T_t\right\} \tag{2.2}$$

and

$$L_t(y_t^{np}, y_t^p) = \left\{x_t \in \mathbb{R}^n_+ : (x_t, y_t^{np}, y_t^p) \in T_t\right\}. \tag{2.3}$$

Therefore we have necessarily:

$$x_t \in L_t(y_t^{np}, y_t^p) \Leftrightarrow (x_t, y_t^{np}, y_t^p) \in T_t \Leftrightarrow (y_t^{np}, y_t^p) \in P_t(x_t) \tag{2.4}$$

Throughout this paper, we assume that the production set satisfies the following usual axioms (Hackman, 2008; Jacobsen, 1970; McFadden, 1978):

$P1$: For all $x_t \in \mathbb{R}^n_+$, $0 \in P_t(x_t)$ and $(y_t^{np}, y_t^p) \notin P_t(0)$, if $(y_t^{np}, y_t^p) \geq 0$ and $(y_t^{np}, y_t^p) \neq 0$.
$P2$: $P_t(x_t)$ is bounded above for all, $x_t \in \mathbb{R}^n_+$.

[3]Note that, if $B = \emptyset$ then, there is no outputs partition. It follows that, the outputs are not separated into polluting and no-polluting ones.

$P3$: $P_t(x_t)$ is closed for all, $x_t \in \mathbb{R}^n_+$.
$P4$: If $v_t \geq x_t \Rightarrow P_t(x_t) \subseteq P_t(v_t)$.
$P5$: $P_t(x_t)$ is a convex set, $\forall x_t \in \mathbb{R}^n_+$.

In addition to the properties $P1 - P4$, we assume that the outputs satisfy the B-disposal assumption (Abad and Briec, 2019):

$P6$: For all $y^{\emptyset}, y^B \in P_t(x_t)$, $y \leq^{\emptyset} y^{\emptyset}$ and $y \leq^B y^B$ implies $y \in P_t(x_t)$.

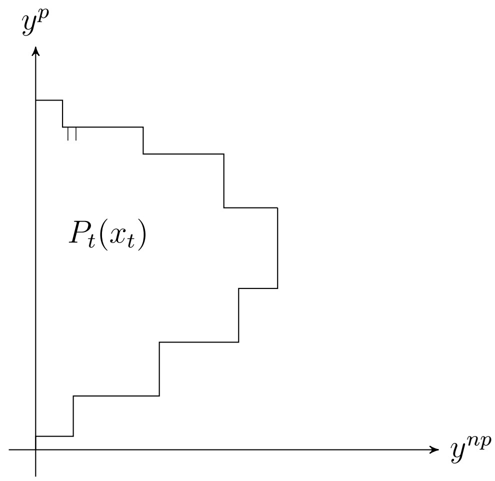

Figure 1: Non convex environmental production set ($P1 - P4$ and $P6$)

Axioms $P1 - P4$ and $P6$ define a general class of environmental output set with traditional strong disposable inputs and B-disposable outputs (polluting and no-polluting; see Figure 1). These properties are fairly weak and do not impose any convexity assumption.

2.2 Environmental efficiency measures

Let us introduce the following convex cone:

$$\mathcal{C}_t^B = \{ y_t \in \mathbb{R}^m : y_{t,j} \leq 0 \text{ if } j \in B \text{ and } y_{t,j} \geq 0 \text{ else } \}.$$

The environmental efficiency measures can be defined through the schemes below (Figure 2).

Definition 2.1 *Let $P_t(x_t)$, be an environmental output set that satisfies properties $P1 - P4$ and $P6$. For any $(x_t, y_t) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y_t^{np}, y_t^p) \in P_t(x_t)$, the environmental efficiency measures belong to the following subsets:*

i. $\mathcal{S}_1 = (y - \mathbb{R}^m_+) \cap (y + \mathcal{C}_t^B)$,

ii. $\mathcal{S}_3 = (y + \mathbb{R}^m_+) \cap (y + \mathcal{C}_t^B)$ *and*

iii. $\mathcal{S}_2 = (y + \mathcal{C}_t^B) \setminus \{\mathcal{S}_1, \mathcal{S}_3\}$.

In Figure 2, the schemes $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ underscore any production adaptation strategies in environmental efficiency analysis.

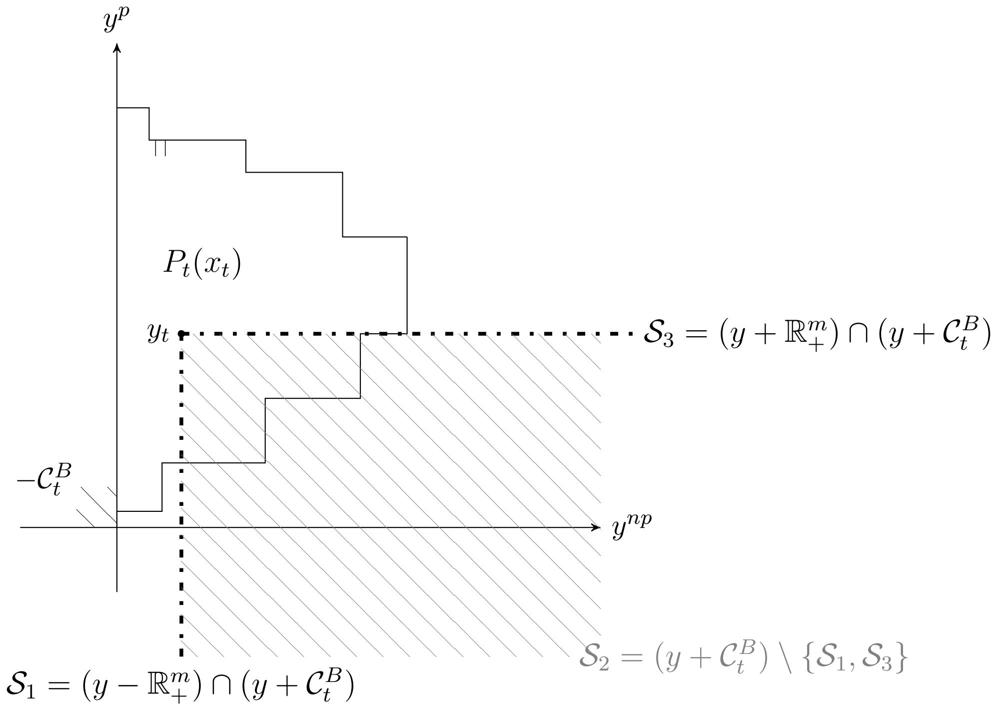

Figure 2: Environmental efficiency analysis ($P1 - P4$ and $P6$)

2.2.1 Multiplicative distance function

Shephard (1953) introduces distance functions that are the inverse of the Debreu-Farrell measures of technical efficiency (Debreu, 1951; Farrell, 1957). These distance functions can be defined in the input or the output oriented cases. The hyperbolic distance function (Färe et al., 1985) allows to extend Shephard distance functions to the graph of the production technology. Distance (or gauge) functions fully characterise the production process. Therefore, they have become standard tools to define multiplicative measures of technical efficiency.

The following definition introduces environmental multiplicative distance function (Abad, 2018).

Definition 2.2 *Let $P_t(x_t)$ be an environmental production set that satisfies properties $P1 - P4$ and $P6$. For any $(x_t, y_t) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y^{np}_t, y^p_t) \in \mathbb{R}^m_+$, the environmental multiplicative efficiency measure, $\psi^\beta_t : \mathbb{R}^{n+m}_+ \longrightarrow \mathbb{R}^+ \cup \infty$, is defined as follows:*

$$\psi^\beta_t(x_t, y_t) = \begin{cases} \sup\limits_\lambda \left\{\lambda \geq 1 : \ \left(x_t, \lambda^{\beta^p} y^p_t, \lambda^{\beta^{np}} y^{np}_t\right) \in P_t(x_t)\right\} \\ \qquad\qquad\qquad \text{if } \left(x_t, \lambda^{\beta^p} y^p_t, \lambda^{\beta^{np}} y^{np}_t\right) \in P_t(x_t), \lambda \geq 1 \\ +\infty \qquad\qquad \text{else} \end{cases} \tag{2.5}$$

with $\beta^p = \{-1, 0\}$ and $\beta^{np} = \{0, 1\}$.

The following proposition displays equivalence conditions for the environmental multiplicative distance function (2.5), desirable (D^{np}_t) and undesirable (D^p_t) outputs

Shephard distance functions (Färe et al., 2004), and hyperbolic output (H_t^o) efficiency measure (Färe et al., 1989).

Proposition 2.3 *For any* $(x_t, y_t) \in \mathbb{R}_+^{n+m}$*, such that* $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$*, we have:*

i. $\psi_t^\beta(x_t, y_t) \equiv \dfrac{1}{D_t^{np}(x_t, y_t)}$*, with* $\beta^p = 0$ *and* $\beta^{np} = 1$*.*

ii. $\psi_t^\beta(x_t, y_t) \equiv \dfrac{1}{D_t^{p}(x_t, y_t)}$*, with* $\beta^p = -1$ *and* $\beta^{np} = 0$*.*

iii. $\psi_t^\beta(x_t, y_t) \equiv \dfrac{1}{H_t^{o}(x_t, y_t)}$*, with* $\beta^p = -1$ *and* $\beta^{np} = 1$*.*

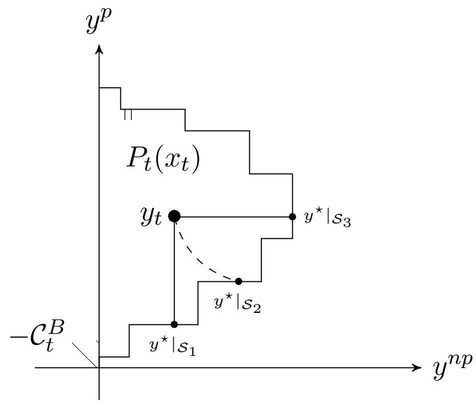

Figure 3: Environmental multiplicative distance function ($P1 - P4$ and $P6$)

The environmental multiplicative distance function in proposition 2.3 is illustrated in Figure 3. Distance between points y_t and $y^\star|_{\mathcal{S}_3}$ depicts the desirable output Shepard distance function (Färe et al., 2004). The gap between points y_t and $y^\star|_{\mathcal{S}_1}$ shows the undesirable output Shepard distance function (Färe et al., 2004). Finally, distance between points y_t and $y^\star|_{\mathcal{S}_2}$ displays the hyperbolic output efficiency measure (Färe et al., 1989).

2.2.2 Additive distance function

The directional distance function allows for simultaneous input and output variation in the direction of a pre-assigned vector $g_t = (h_t, k_t) \in \mathbb{R}_+^{n+m}$ compatible with the technology (Chambers et al., 1996, 1998). The special case $g_t = (x_t, y_t)$ is known as the Farrell proportional directional distance function (Briec, 1997) and is a generalization of the Debreu-Farrell efficiency measure[4].

Let us define the environmental additive distance function (Abad, 2018).

[4]Axiomatic properties of the proportional directional distance function are defined in Briec (1997) and Chambers, Chung and Färe (1996, 1998).

Definition 2.4 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$, the environmental additive efficiency measure, $\xi^{0,\sigma} : \mathbb{R}_+^{n+m} \times 0 \times [0,1]^{m^{np}} \times [-1,0]^{m^p} \longrightarrow \mathbb{R} \cup \infty$, is defined as follows :*

$$\xi_t^{0,\sigma}(x_t, y_t) = \begin{cases} \sup_{\delta} \left\{ \delta \geq 0 : \; \left(x_t, (1+\delta\sigma^{np})y_t^{np}, (1+\delta\sigma^p)y_t^p\right) \in P_t(x_t) \right\} \\ \qquad \text{if } \left(x_t, (1+\delta\sigma^{np})y_t^{np}, (1+\delta\sigma^p)y_t^p\right) \in P_t(x_t), \delta \geq 0 \\ +\infty \qquad \text{else} \end{cases} \tag{2.6}$$

where $\sigma = (\sigma^{np}, \sigma^p) \in [0,1]^{m^{np}} \times [-1,0]^{m^p}$.

The next results introduce equivalence conditions for the environmental additive efficiency measure (2.6), the environmental directional distance function (Chung et al., 1997) and, the desirable and undesirable sub-vector directional distance functions (Picazo-Tadeo et al., 2014).

Proposition 2.5 *For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$, we have:*

i. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t^{np}(x_t, y_t; y_t^{np}, 0)$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (1, 0)$.

ii. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t^{p}(x_t, y_t; 0, y_t^p)$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (0, -1)$.

iii. $\xi_t^{0,\sigma}(x_t, y_t) \equiv \overrightarrow{D}_t(x_t, y_t; 0, y_t^{np}, -y_t^p)$*, with* $\sigma = (\sigma^{np}, \sigma^p) = (1, -1)$.

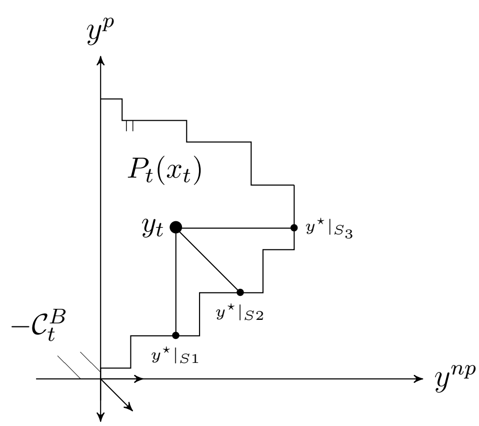

Figure 4: Environmental additive distance function ($P1-P4$ and $P6$)

The environmental additive distance function in proposition 2.5 is illustrated in Figure 4. Distance between points y_t and $y^\star|_{S3}$ depicts the sub-vector desirable output directional distance function (Picazo-Tadeo et al., 2014). The gap between points y_t and $y^\star|_{S1}$ shows the sub-vector undesirable output directional distance function (Picazo-Tadeo et al., 2014). The distance between points y_t and $y^\star|_{S2}$ represents the environmental directional distance function (Chung et al., 1997). Notify that Figure 4 just illustrates the cases (**i.-iii.**) quoted above. However, the additive environmental distance function (2.6) is defined for any $(\sigma^{np}, \sigma^p) \in [0,1]^{m^{np}} \times [-1,0]^{m^p}$.

2.3 Environmental efficiency measures: equivalence conditions

The next result defines equivalence condition for the additive and multiplicative environmental efficiency measures.

Proposition 2.6 *Let* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$*, for any* $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$*, we have:*

$$\xi_t^{0,\sigma}\big(\ln(x_t), \ln(y_t)\big) \equiv \ln\big(\psi_t^{\beta}(x_t, y_t)\big) \tag{2.7}$$

such that $\beta^{np} = \sigma^{np}\ln(y_t^{np})$ *and* $\beta^{p} = \sigma^{p}\ln(y_t^{p})$*.*

Proof of Proposition 2.6: For any $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$, the environmental additive distance function (2.6) is defined as follows,

$$\xi_t^{0,\sigma}(x_t, y_t) = \sup_{\delta}\left\{\delta \geq 0 : \Big(x_t, (1+\delta\sigma^{np})y_t^{np}, (1+\delta\sigma^{p})y_t^{p}\Big) \in P_t(x_t)\right\}.$$

Therefore,

$$\xi_t^{0,\sigma}\big(\ln(x_t), \ln(y_t)\big) = \sup_{\delta}\left\{\delta \geq 0 : \Big(\ln(x_t), (1+\delta\sigma^{np})\ln(y_t^{np}), (1+\delta\sigma^{p})\ln(y_t^{p})\Big) \in P_t\big(\ln(x_t)\big)\right\}.$$

Following the definition of the environmental multiplicative efficiency measure (2.5),

$$\begin{aligned}\ln\big(\psi_t^{\beta}(x_t, y_t)\big) &= \sup_{\lambda}\left\{\lambda > 0 : \big(\ln(x_t), \ln(\lambda^{\beta^{p}} y_t^{p}), \ln(\lambda^{\beta^{np}} y_t^{np})\big) \in P_t\big(\ln(x_t)\big)\right\} \\ &= \sup_{\lambda}\left\{\lambda > 0 : (\ln(x_t), \ln(y_t^{p}) + \beta^{p}\ln(\lambda), \ln(y_t^{np}) + \beta^{np}\ln(\lambda)) \in P_t\big(\ln(x_t)\big)\right\}.\end{aligned}$$

Hence, $\xi_t^{0,\sigma}\big(\ln(x_t), \ln(y_t)\big) \equiv \ln\big(\psi_t^{\beta}(x_t, y_t)\big)$ with $\beta^{np} = \sigma^{np}\ln(y_t^{np})$ and $\beta^{p} = \sigma^{p}\ln(y_t^{p})$.□

The aforementioned statement introduces connection for the additive and multiplicative green efficiency analysis. Hence, the following corollary defines equivalence conditions for the environmental additive efficiency measure, the desirable and undesirable output Shephard distance functions, and the hyperbolic output efficiency measure (Färe et al., 2004; Färe et al., 1989).

Corollary 2.7 *For any* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$*, such that* $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$*, we have:*

i. $\xi_t^{0,[\ln(y_t^{np})]^{-1},0}\big(\ln(x_t), \ln(y_t)\big) \equiv -\ln\Big(D_t^{np}(x_t, y_t)\Big)$.

ii. $\xi_t^{0,0,-[\ln(y_t^{p})]^{-1}}\big(\ln(x_t), \ln(y_t)\big) \equiv -\ln\Big(D_t^{p}(x_t, y_t)\Big)$.

iii. $\xi_t^{0,[\ln(y_t^{np})]^{-1},-[\ln(y_t^{p})]^{-1}}\big(\ln(x_t), \ln(y_t)\big) \equiv -\ln\Big(H_t^{o}(x_t, y_t)\Big)$.

3 Environmental Malmquist and Luenberger productivity measures

In this section, additive and multiplicative eco-productivity indicators are defined through the new environmental efficiency distance functions (2.5) and (2.6).

3.1 Environmental Malmquist productivity index

The Environmental Malmquist (EM) productivity index inherits the basic structure of the output Malmquist index (Färe et al., 1995). Following the multiplicative environmental efficiency measure (2.6), the EM productivity measure is defined below.

Definition 3.1 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, with $y_{t,t+1} = (y^{np}_{t,t+1}, y^{p}_{t,t+1}) \in \mathbb{R}_+^m$, the global Environmental Malmquist productivity index is defined as follows:*

$$EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \left[\frac{\psi_t^{\beta}(x_t, y_t)}{\psi_t^{\beta}(x_{t+1}, y_{t+1})} \times \frac{\psi_{t+1}^{\beta}(x_t, y_t)}{\psi_{t+1}^{\beta}(x_{t+1}, y_{t+1})} \right]^{1/2}, \tag{3.1}$$

such that $\beta^p = \{-1, 0\}$ and $\beta^{np} = \{0, 1\}$.

Notice that the cross-time multiplicative efficiency measure in (3.1) is defined as,

$$\psi_l^{\beta}(x_s, y_s) = \begin{cases} \sup\limits_{\lambda} \left\{ \lambda : \ \left(x_s, \lambda^{\beta^p} y_s^p, \lambda^{\beta^{np}} y_s^{np}\right) \in P_l(x_l) \right\} \\ \qquad\qquad \text{if } \left(x_s, \lambda^{\beta^p} y_s^p, \lambda^{\beta^{np}} y_s^{np}\right) \in P_l(x_l) \\ +\infty \qquad\quad \text{else} \end{cases} \tag{3.2}$$

where $\beta^p = \{-1, 0\}$ and $\beta^{np} = \{0, 1\}$ and, $s, l = t, t+1$ with $s \neq l$. This inter-temporal distance function allows to evaluate the efficiency of the production unit of period (s) with respect to the environmental production set of period (l).

To avoid arbitrary choice of a base time period, the global EM productivity measure is defined as a geometric mean of EM indices over periods $(t, t+1)$. When the value of the global EM index is greater (respectively lesser) than unity, then it shows environmental productivity improvement (respectively deterioration) between the periods (t) and $(t+1)$. It follows that, the decision units carry out managerial efforts (innovative environmental production processes, new green investments, innovative staff members skills etc.) according to the selected environmental production adaptation scheme.

The following proposition defines equivalence conditions for the EM index, the no polluting and polluting Malmquist and, the hyperbolic Malmquist productivity measures.

Proposition 3.2 *For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^p) \in \mathbb{R}_+^m$, we have:*

i. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\beta^p = 0$ *and* $\beta^{np} = 1$.

ii. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\beta^p = -1$ *and* $\beta^{np} = 0$.

iii. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\beta^p = -1$ *and* $\beta^{np} = 1$.

Where,

$$M^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \left[\frac{D^{np}_t(x_{t+1}, y_{t+1})}{D^{np}_t(x_t, y_t)} \times \frac{D^{np}_{t+1}(x_{t+1}, y_{t+1})}{D^{np}_{t+1}(x_t, y_t)}\right]^{1/2}, \tag{3.3}$$

$$M^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \left[\frac{D^{p}_t(x_{t+1}, y_{t+1})}{D^{p}_t(x_t, y_t)} \times \frac{D^{p}_{t+1}(x_{t+1}, y_{t+1})}{D^{p}_{t+1}(x_t, y_t)}\right]^{1/2} \tag{3.4}$$

and

$$M^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \left[\frac{H^{o}_t(x_{t+1}, y_{t+1})}{H^{o}_t(x_t, y_t)} \times \frac{H^{o}_{t+1}(x_{t+1}, y_{t+1})}{H^{o}_{t+1}(x_t, y_t)}\right]^{1/2}. \tag{3.5}$$

The next result shows that the global EM productivity index can be defined through the environmental additive efficiency measure.

Proposition 3.3 *Let* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$, *for any* $\beta^p = \{-1, 0\}$ *and* $\beta^{np} = \{0, 1\}$, *the global EM productivity measure is defined below:*

$$EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv \left[\frac{\exp\left[\xi^{0,\sigma}_t\big(\ln(x_t), \ln(y_t)\big)\right]}{\exp\left[\xi^{0,\sigma}_t\big(\ln(x_{t+1}), \ln(y_{t+1})\big)\right]} \times \frac{\exp\left[\xi^{0,\sigma}_{t+1}\big(\ln(x_t), \ln(y_t)\big)\right]}{\exp\left[\xi^{0,\sigma}_{t+1}\big(\ln(x_{t+1}), \ln(y_{t+1})\big)\right]}\right]^{1/2}, \tag{3.6}$$

where $\sigma = (\sigma^{np}, \sigma^p) = \Big(\beta^{np}[\ln(y^{np}_t)]^{-1}, \beta^p[\ln(y^p_t)]^{-1}\Big)$.

The corollary below displays equivalence conditions for the global EM productivity measure and the no polluting, polluting and hyperbolic productivity indices. Notify that this version of the global EM productivity measure substitutes environmental additive distance functions for the multiplicative eco-efficiency measures in the Malmquist index. This result is similar to the widely applied Malmquist-Luenberger methodology (Chung et al., 1997).

Corollary 3.4 *For any* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$, *such that* $\beta^p = \{-1, 0\}$ *and* $\beta^{np} = \{0, 1\}$, *we have:*

i. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\sigma = (\sigma^{np}, \sigma^p) = \Big([\ln(y^{np}_t)]^{-1}, 0\Big)$.

ii. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\sigma = (\sigma^{np}, \sigma^p) = \Big(0, -[\ln(y^{p}_t)]^{-1}\Big)$.

iii. $EM^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv M^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *with* $\sigma = (\sigma^{np}, \sigma^p) = \Big([\ln(y^{np}_t)]^{-1}, -[\ln(y^{p}_t)]^{-1}\Big)$.

3.2 Environmental Luenberger productivity indicator

The Environmental Luenberger (EL) productivity measure takes the form of the output Luenberger productivity indicator (Chambers, 1996). The EL productivity measure is defined as the arithmetic mean of difference-based environmental Luenberger productivity indicators over periods (t) and $(t+1)$. Following the additive environmental efficiency index (2.6), the EL productivity measure is defined as follows.

Definition 3.5 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any periods $(t,t+1)$ and for any $(x_{t,t+1},y_{t,t+1}) \in \mathbb{R}^{n+m}_+$, with $y_{t,t+1} = (y^{np}_{t,t+1}, y^{p}_{t,t+1}) \in \mathbb{R}^m_+$, the global Environmental Luenberger productivity indicator is defined as:*

$$\begin{aligned} &EL^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \\ &\tfrac{1}{2}\Big[\Big(\xi^{0,\sigma}_t(x_t,y_t) - \xi^{0,\sigma}_t(x_{t+1},y_{t+1})\Big) \\ &\quad +\Big(\xi^{0,\sigma}_{t+1}(x_t,y_t) - \xi^{0,\sigma}_{t+1}(x_{t+1},y_{t+1})\Big)\Big], \end{aligned} \tag{3.7}$$

where $\sigma = (\sigma^{np},\sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^p}$.

The inter-temporal additive distance function in (3.7) is defined as,

$$\xi^{0,\sigma}_l(x_s,y_s) = \begin{cases} \sup_{\delta}\Big\{\delta : \Big(x_s,(1+\delta\sigma^{np})y^{np}_s,(1+\delta\sigma^{p})y^{p}_s\Big) \in P_l(x_l)\Big\} \\ \qquad \text{if } \Big(x_s,(1+\delta\sigma^{np})y^{np}_s,(1+\delta\sigma^{p})y^{p}_s\Big) \in P_l(x_l), \\ +\infty \qquad \text{else} \end{cases} \tag{3.8}$$

such that $s,l = t,t+1$ and $s \neq l$. The cross-time efficiency measure (3.8) estimates the environmental performance of the observation of period (s) with respect to the production process of period (l).

The global EL productivity measure is defined as an arithmetic mean of environmental Luenberger productivity indicators over periods (t) and $(t+1)$. This structure allows to avoid arbitrary selection of base period. The global EL indicator displays environmental productivity advance (respectively decrease) if it takes positive (respectively negative) value. In such a case, decision units implement positive managerial adaptation through the followed environmental strategy.

The next result introduces equivalence conditions for the global EL productivity measure (3.7) and the no polluting, polluting and environmental Luenberger indicators.

Proposition 3.6 *Let $(x_t,y_t) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y^{np}_t, y^p_t) \in \mathbb{R}^m_+$, the following equivalence conditions hold:*

i. $EL^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv L^{np}_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *such that* $\sigma = (\sigma^{np},\sigma^{p}) = (1,0)$.

ii. $EL^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv L^{p}_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *such that* $\sigma = (\sigma^{np},\sigma^{p}) = (0,-1)$.

iii. $EL^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv eL_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *such that* $\sigma = (\sigma^{np},\sigma^{p}) = (1,-1)$.

Such that,

$$L^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{1}{2}\ \Big[\Big(\overrightarrow{D}^{np}_{t}(x_t, y_t; y^{np}_t, 0) - \overrightarrow{D}^{np}_{t}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, 0)\Big) + \Big(\overrightarrow{D}^{np}_{t+1}(x_t, y_t; y^{np}_t, 0) - \overrightarrow{D}^{np}_{t+1}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, 0)\Big)\Big], \tag{3.9}$$

$$L^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{1}{2}\ \Big[\Big(\overrightarrow{D}^{p}_{t}(x_t, y_t; 0, -y^{p}_t) - \overrightarrow{D}^{p}_{t}(x_{t+1}, y_{t+1}; 0, -y^{p}_{t+1})\Big) + \Big(\overrightarrow{D}^{p}_{t+1}(x_t, y_t; 0, -y^{p}_t) - \overrightarrow{D}^{p}_{t+1}(x_{t+1}, y_{t+1}; 0, -y^{p}_{t+1})\Big)\Big] \tag{3.10}$$

and

$$eL_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{1}{2}\ \Big[\Big(\overrightarrow{D}_{t}(x_t, y_t; y^{np}_t, -y^{p}_t) - \overrightarrow{D}_{t}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, -y^{p}_{t+1})\Big) + \Big(\overrightarrow{D}_{t+1}(x_t, y_t; y^{np}_t, -y^{p}_t) - \overrightarrow{D}_{t+1}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, -y^{p}_{t+1})\Big)\Big]. \tag{3.11}$$

Note that the global EL indicator inherits the structure of the widely applied environmental additive productivity measures show in the aforementioned cases **i.-iii.** (Picazo-Tadeo et al., 2014; Azad and Ancev, 2014). However, the global EL indicator is defined for any $(\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$.

The following proposition introduces equivalence condition for the global EL indicator and the global EM productivity measure.

Proposition 3.7 *Let* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$*, for any* $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$*, equivalence condition for the global EL indicator and the global EM productivity measure is defined as:*

$$EL^{\sigma}_{t,t+1}(\ln x_t, \ln x_{t+1}, \ln y_t, \ln y_{t+1}) \equiv \ln\Big(EM^{\beta}_{t,t+1}(x_t, y_t, x_{t+1}, y_{t+1})\Big), \tag{3.12}$$

such that $\beta^{np} = \sigma^{np}\ln(y^{np}_t)$ *and* $\beta^{p} = \sigma^{p}\ln(y^{p}_t)$*.*

The above statement introduces connection for the additive and multiplicative eco-productivity analysis. Hence, the next result defines equivalence conditions for the global EL productivity measure and the no polluting, polluting and hyperbolic Malmquist indices.

Corollary 3.8 *For any* $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$*, where* $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$*, we have:*

i. $EL^{\sigma}_{t,t+1}\big(\ln(x_{t,t+1}), \ln(y_{t,t+1})\big) \equiv \ln\Big(M^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1})\Big)$*, such that* $\beta^{p} = 0$ *and* $\beta^{np} = 1$*.*

ii. $EL^{\sigma}_{t,t+1}\big(\ln(x_{t,t+1}), \ln(y_{t,t+1})\big) \equiv \ln\Big(M^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1})\Big)$*, such that* $\beta^{p} = -1$ *and* $\beta^{np} = 0$*.*

iii. $EL^{\sigma}_{t,t+1}\big(\ln(x_{t,t+1}), \ln(y_{t,t+1})\big) \equiv \ln\Big(M^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1})\Big)$*, such that* $\beta^{p} = -1$ *and* $\beta^{np} = 1$*.*

4 Decomposition of environmental productivity change

In this section, we define the main drivers of green productivity change. The sources of environmental productivity variation are defined through the new additive and multiplicative eco-productivity measures.

4.1 Environmental Malmquist productivity index

Multiplicative decomposition of the global EM productivity measure (3.1) is defined as follows.

Definition 4.1 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, where $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the global EM productivity index over periods $(t, t+1)$ is decomposed as follows:*

$$EM_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) = EMTC_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) \times EMEV_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}). \quad (4.1)$$

Such that $\beta^p = \{-1, 0\}$ and $\beta^{np} = \{0, 1\}$ and, where:

i. *$EMTC_{t,t+1}^{\beta}$ shows environmental technical change between the periods (t) and $(t+1)$.*

ii. *$EMEV_{t,t+1}^{\beta}$ displays environmental efficiency variation over the periods (t) and $(t+1)$.*

The environmental technological variation over periods $(t, t+1)$ is defined as follows,

$$EMTC_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) = \Big[\frac{\psi_{t+1}^{\beta}(x_{t+1}, y_{t+1})}{\psi_t^{\beta}(x_t, y_t)} \times \frac{\psi_{t+1}^{\beta}(x_t, y_t)}{\psi_t^{\beta}(x_{t+1}, y_{t+1})}\Big]^{1/2}. \quad (4.2)$$

If $EMTC_{t,t+1}^{\beta} > 1$ then, green technological advance arises between periods (t) and $(t+1)$. Following the eco-managerial scenarios $\mathcal{S}_1$ and $\mathcal{S}_3$, the environmental technical change is illustrated in Figure 5.

Equivalence conditions for the global EM technical variation (4.2) and the no polluting, polluting and hyperbolic Malmquist technological change are appraised in the next result.

Proposition 4.2 *For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, such that $y_t = (y_t^{np}, y_t^{p}) \in \mathbb{R}_+^m$, the following conditions hold:*

i. *$EMTC_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) \equiv MTC_{t,t+1}^{np}(x_{t,t+1}, y_{t,t+1})$, where $\beta^p = 0$ and $\beta^{np} = 1$.*

ii. *$EMTC_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) \equiv MTC_{t,t+1}^{p}(x_{t,t+1}, y_{t,t+1})$, where $\beta^p = -1$ and $\beta^{np} = 0$.*

iii. *$EMTC_{t,t+1}^{\beta}(x_{t,t+1}, y_{t,t+1}) \equiv MTC_{t,t+1}^{h}(x_{t,t+1}, y_{t,t+1})$, where $\beta^p = -1$ and $\beta^{np} = 1$.*

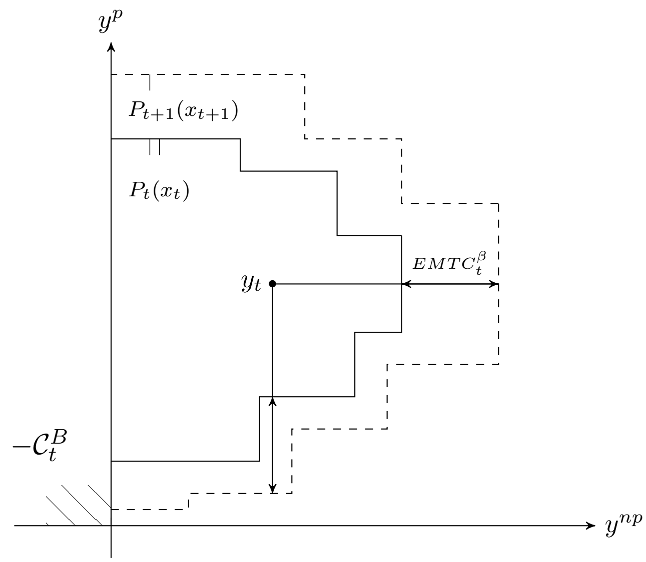

Figure 5: Environmental technological change of non convex output set $(t, t+1)$

Such that,

$$MTC^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \Big[\frac{D^{np}_t(x_t, y_t)}{D^{np}_{t+1}(x_{t+1}, y_{t+1})} \times \frac{D^{np}_t(x_{t+1}, y_{t+1})}{D^{np}_{t+1}(x_t, y_t)}\Big]^{1/2}, \tag{4.3}$$

$$MTC^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \Big[\frac{D^{p}_t(x_t, y_t)}{D^{p}_{t+1}(x_{t+1}, y_{t+1})} \times \frac{D^{p}_t(x_{t+1}, y_{t+1})}{D^{p}_{t+1}(x_t, y_t)}\Big]^{1/2} \tag{4.4}$$

and

$$MTC^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \Big[\frac{H^{o}_t(x_t, y_t)}{H^{o}_{t+1}(x_{t+1}, y_{t+1})} \times \frac{H^{o}_t(x_{t+1}, y_{t+1})}{H^{o}_{t+1}(x_t, y_t)}\Big]^{1/2}. \tag{4.5}$$

Environmental efficiency variation between periods (t) and $(t+1)$ is defined below:

$$EMEV^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{\psi^{\beta}_t(x_t, y_t)}{\psi^{\beta}_{t+1}(x_{t+1}, y_{t+1})}. \tag{4.6}$$

$EMEV^{\beta}_{t,t+1} > 1$ allows to define environmental efficiency improvement over periods $(t, t+1)$. The green efficiency change through the production adaptation strategies $\mathcal{S}_1$ and $\mathcal{S}_3$ is described in Figure 6.

The following proposition introduces equivalence conditions for the global EM efficiency variation (4.6) and the no polluting, polluting and hyperbolic Malmquist efficiency change.

Proposition 4.3 *Let* $(x_t, y_t) \in \mathbb{R}^{n+m}_+$*, where* $y_t = (y^{np}_t, y^p_t) \in \mathbb{R}^m_+$*, the conditions below can be defined:*

i. $EMEV^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv MEV^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$*, such that* $\beta^p = 0$ *and* $\beta^{np} = 1$.

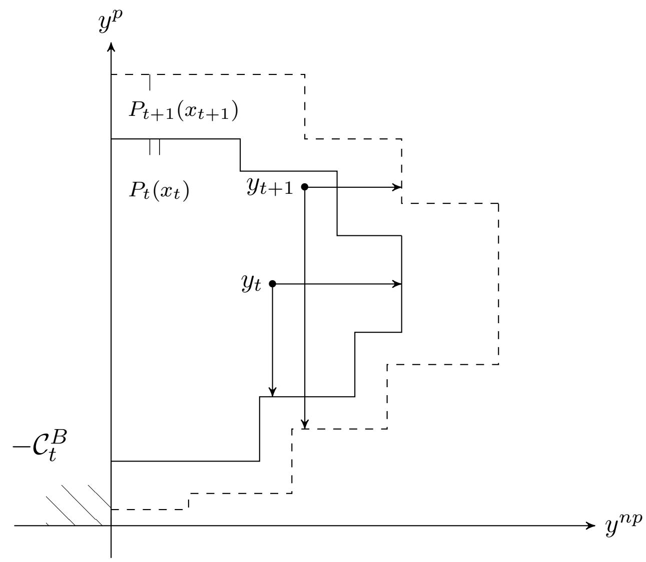

Figure 6: Environmental efficiency change of non convex output set $(t, t+1)$

ii. $EMEV^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv MEV^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *such that* $\beta^p = -1$ *and* $\beta^{np} = 0$.

iii. $EMEV^{\beta}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv MEV^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, *such that* $\beta^p = -1$ *and* $\beta^{np} = 1$.

Where,

$$MEV^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{D^{np}_{t+1}(x_{t+1}, y_{t+1})}{D^{np}_{t}(x_t, y_t)}, \tag{4.7}$$

$$MEV^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{D^{p}_{t+1}(x_{t+1}, y_{t+1})}{D^{p}_{t}(x_t, y_t)} \tag{4.8}$$

and

$$MEV^{h}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \frac{H^{o}_{t+1}(x_{t+1}, y_{t+1})}{H^{o}_{t}(x_t, y_t)}. \tag{4.9}$$

The combination of the environmental technical change and green efficiency variation displays informations about the conditions of environmental productivity change. The next table outlines these conditions.

	$EMEV^{\beta}_{t,t+1} > 1$	$EMEV^{\beta}_{t,t+1} < 1$
$EMTC^{\beta}_{t,t+1} > 1$	$EM^{\beta}_{t,t+1} > 1$	**i.** $EMTC^{\beta}_{t,t+1} > \left[EMEV^{\beta}_{t,t+1}\right]^{-1}$ then $EM^{\beta}_{t,t+1} > 1$, **ii.** $EMTC^{\beta}_{t,t+1} < \left[EMEV^{\beta}_{t,t+1}\right]^{-1}$ then $EM^{\beta}_{t,t+1} < 1$,
$EMTC^{\beta}_{t,t+1} < 1$	**i.** $\left[EMTC^{\beta}_{t,t+1}\right]^{-1} < EMEV^{\beta}_{t,t+1}$ then $EM^{\beta}_{t,t+1} > 1$, **ii.** $\left[EMTC^{\beta}_{t,t+1}\right]^{-1} > EMEV^{\beta}_{t,t+1}$ then $EM^{\beta}_{t,t+1} < 1$,	$EM^{\beta}{}_{t,t+1} < 1$,

Table 1: $EM^{\beta}_{t,t+1}$ characterization

4.2 Environmental Luenberger productivity indicator

Additive decomposition of the global EL productivity measure (3.7) is defined in the next statement.

Definition 4.4 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any periods $(t, t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}^{n+m}_+$, where $y_{t,t+1} = (y^{np}_{t,t+1}, y^{p}_{t,t+1}) \in \mathbb{R}^m_+$, the global EL productivity indicator over periods $(t, t+1)$ is decomposed as follows:*

$$EL^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = ELTC^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) + ELEV^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}). \quad (4.10)$$

Where $\sigma = (\sigma^{np}, \sigma^{p}) \in [0,1]^{m^{np}} \times [-1,0]^{m^{p}}$ and:

i. *$ELTC^{\sigma}_{t,t+1}$ shows environmental technical change between the periods (t) and $(t+1)$.*

ii. *$ELEV^{\sigma}_{t,t+1}$ denotes environmental efficiency variation over the periods (t) and $(t+1)$.*

The green technical change between periods (t) and $(t+1)$ is defined below,

$$\begin{aligned} & ELTC^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \\ & = \tfrac{1}{2} \left[\Big(\xi^{0,\sigma}_{t+1}(x_{t+1}, y_{t+1}) - \xi^{0,\sigma}_{t}(x_t, y_t) \Big) \right. \\ & \quad \left. + \Big(\xi^{0,\sigma}_{t+1}(x_t, y_t) - \xi^{0,\sigma}_{t}(x_{t+1}, y_{t+1}) \Big) \right]. \end{aligned} \quad (4.11)$$

When $ELTC^{\sigma}_{t,t+1} > 0$ then, environmental technological progress occurs over periods $(t, t+1)$. Equivalence conditions for the global EL technical change (4.11) and the no polluting, polluting and environmental Luenberger technological variation is defined in the result below.

Proposition 4.5 *For any $(x_t, y_t) \in \mathbb{R}^{n+m}_+$, with $y_t = (y^{np}_t, y^{p}_t) \in \mathbb{R}^m_+$, the following equivalence conditions hold:*

i. *$ELTC^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv LTC^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, such that $\sigma = (\sigma^{np}, \sigma^{p}) = (1, 0)$.*

ii. *$ELTC^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv LTC^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, such that $\sigma = (\sigma^{np}, \sigma^{p}) = (0, -1)$.*

iii. *$ELTC^{\sigma}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) \equiv eLTC_{t,t+1}(x_{t,t+1}, y_{t,t+1})$, such that $\sigma = (\sigma^{np}, \sigma^{p}) = (1, -1)$.*

Where,

$$\begin{aligned} LTC^{np}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \tfrac{1}{2} & \left[\Big(\overrightarrow{D}^{np}_{t+1}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, 0) - \overrightarrow{D}^{np}_{t}(x_t, y_t; y^{np}_t, 0) \Big) \right. \\ & \left. + \Big(\overrightarrow{D}^{np}_{t+1}(x_t, y_t; y^{np}_t, 0) - \overrightarrow{D}^{np}_{t}(x_{t+1}, y_{t+1}; y^{np}_{t+1}, 0) \Big) \right], \end{aligned} \quad (4.12)$$

$$\begin{aligned} LTC^{p}_{t,t+1}(x_{t,t+1}, y_{t,t+1}) = \tfrac{1}{2} & \left[\Big(\overrightarrow{D}^{p}_{t+1}(x_{t+1}, y_{t+1}; 0, -y^{p}_{t+1}) - \overrightarrow{D}^{p}_{t}(x_t, y_t; 0, -y^{p}_t) \Big) \right. \\ & \left. + \Big(\overrightarrow{D}^{p}_{t+1}(x_t, y_t; 0, -y^{p}_t) - \overrightarrow{D}^{p}_{t}(x_{t+1}, y_{t+1}; 0, -y^{p}_{t+1}) \Big) \right], \end{aligned} \quad (4.13)$$

and

$$eLTC_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \frac{1}{2}\ \Big[\Big(\overrightarrow{D}^{np}_{t+1}(x_{t+1},y_{t+1};y^{np}_{t+1},-y^{p}_{t+1}) - \overrightarrow{D}^{np}_{t}(x_t,y_t;y^{np}_t,-y^p_t)\Big) + \Big(\overrightarrow{D}^{np}_{t+1}(x_t,y_t;y^{np}_t,-y^p_t) - \overrightarrow{D}^{np}_{t}(x_{t+1},y_{t+1};y^{np}_{t+1},-y^p_{t+1})\Big)\Big], \quad (4.14)$$

The environmental efficiency change over periods $(t, t+1)$ is defined as follows,

$$ELEV^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \xi^{0,\sigma}_t(x_t,y_t) - \xi^{0,\sigma}_{t+1}(x_{t+1},y_{t+1}) \quad (4.15)$$

If $ELEV^{\sigma}_{t,t+1}$ is greater (respectively lesser) than zero it shows green efficiency advance (respectively deterioration) between periods (t) and $(t+1)$.

The next statement defines equivalence conditions for the global EL efficiency variation (4.15) and the no polluting, polluting and environmental Luenberger efficiency change.

Proposition 4.6 *Let $(x_t, y_t) \in \mathbb{R}^{n+m}_+$, such that $y_t = (y^{np}_t, y^p_t) \in \mathbb{R}^m_+$, the next equivalence conditions hold:*

i. $ELEV^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv LEV^{np}_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *where* $\sigma = (\sigma^{np},\sigma^p) = (1,0)$.

ii. $ELEV^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv LEV^{p}_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *where* $\sigma = (\sigma^{np},\sigma^p) = (0,-1)$.

iii. $ELEV^{\sigma}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \equiv eLEV_{t,t+1}(x_{t,t+1},y_{t,t+1})$, *where* $\sigma = (\sigma^{np},\sigma^p) = (1,-1)$.

With,

$$LEV^{np}_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \overrightarrow{D}^{np}_t(x_t,y_t;y^{np}_t,0) - \overrightarrow{D}^{np}_{t+1}(x_{t+1},y_{t+1};y^{np}_{t+1},0), \quad (4.16)$$

$$LEV^{p}_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \overrightarrow{D}^{p}_t(x_t,y_t;0,-y^p_t) - \overrightarrow{D}^{p}_{t+1}(x_{t+1},y_{t+1};0,-y^p_{t+1}) \quad (4.17)$$

and

$$eLEV_{t,t+1}(x_{t,t+1},y_{t,t+1}) = \overrightarrow{D}_t(x_t,y_t;y^{np}_t,-y^p_t) - \overrightarrow{D}_{t+1}(x_{t+1},y_{t+1};y^{np}_{t+1},-y^p_t). \quad (4.18)$$

The following proposition introduces equivalence condition for the decomposition of the global EL indicator and the decomposition of the global EM productivity measure.

Proposition 4.7 *Let $(x_t, y_t) \in \mathbb{R}^{n+m}_{++}$, such that $\sigma = (\sigma^{np},\sigma^p) \in [0,1]^{m^{np}} \times [-1,0]^{m^p}$, equivalence condition for the decomposition of the global EL indicator and the decomposition of the global EM productivity measure is defined as:*

$$ELTC^{\sigma}_{t,t+1}\Big(\ln(x_{t,t+1}),\ln(y_{t,t+1})\Big) + ELEV^{\sigma}_{t,t+1}\Big(\ln(x_{t,t+1}),\ln(y_{t,t+1})\Big) \equiv \ln\Big(EMTC^{\beta}_{t,t+1}(x_{t,t+1},y_{t,t+1}) \times EMEV^{\beta}_{t,t+1}(x_{t,t+1},y_{t,t+1})\Big), \quad (4.19)$$

where $\beta^{np} = \sigma^{np}\ln(y^{np}_t)$ and $\beta^p = \sigma^p\ln(y^p_t)$.

Notify that the aforementioned result introduces connection for the additive and multiplicative sources of eco-productivity change. The conditions of environmental technological variation and green efficiency change provide informations about the global environmental productivity variation. The table below summarizes the conditions of green productivity change.

	$ELEV^{\sigma}_{t,t+1} > 0$	$ELEV^{\sigma}_{t,t+1} < 0$
$ELTC^{\sigma}_{t,t+1} > 0$	$EL^{\sigma}_{t,t+1} > 0$	**i.** $\lvert ELTC^{\sigma}_{t,t+1}\rvert > \lvert ELEV^{\sigma}_{t,t+1}\rvert$ then $EL^{\sigma}_{t,t+1} > 0$, **ii.** $\lvert ELTC^{\sigma}_{t,t+1}\rvert < \lvert ELEV^{\sigma}_{t,t+1}\rvert$ then $EL^{\sigma}_{t,t+1} < 0$,
$ELTC^{\sigma}_{t,t+1} < 0$	**i.** $\lvert ELTC^{\sigma}_{t,t+1}\rvert < \lvert ELEV^{\sigma}_{t,t+1}\rvert$ then $EL^{\sigma}_{t,t+1} > 0$, **ii.** $\lvert ELTC^{\sigma}_{t,t+1}\rvert > \lvert ELEV^{\sigma}_{t,t+1}\rvert$ then $EL^{\sigma}_{t,t+1} < 0$,	$EL^{\sigma}{}_{t,t+1} < 0$,

Table 2: $EL^{\sigma}_{t,t+1}$ characterization

5 Environmental productivity measures on non-parametric technologies

In this section, we focus on convex and non-convex non-parametric environmental production processes (Abad and Briec, 2019). The new green efficiency measures are defined through the Data Envelopment Analysis (DEA) model (Banker, Charnes and Cooper, 1984) and the Free Disposal Hull (FDH) non-convex production model (Tulkens, 1993).

5.1 Non-parametric convex and non-convex environmental technologies

Let us consider the following notation : $(x_t, y_t) = (x, y)$ et $(x_{t+1}, y_{t+1}) = (\hat{x}, \hat{y})$. In addition, assume that $\mathcal{A} = \{(x_z, y_z) : z \in \mathcal{Z}\}$ is a set of Decision Making Units (DMUs), such that $\mathcal{Z}$ is an index set of natural number. For any $(x_0, y_0) \in \mathcal{A}$, non-parametric convex environmental output set of period (t) is defined as follows: $P_t^{\{\emptyset,B\},DEA}(x_0) = P_t^{\emptyset,DEA}(x_0) \cap P_t^{B,DEA}(x_0) = \Big(\big(P_t^{DEA}(x_0) - \mathbb{R}^m_+\big) \cap \big(P_t^{DEA}(x_0) - \mathcal{C}_t^B\big) \Big) \cap \mathbb{R}^m_+$. Therefore, the following program holds:

$$\begin{aligned}P_t^{\{\emptyset,B\},DEA}(x_0) = \Big\{ y : x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \theta_z x_{z,i}, \;\; i=1,...,n \\ x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \mu_z x_{z,i}, \;\; i=1,...,n \\ y_j &\geq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j}, \;\; j\in B \\ y_j &\leq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j}, \;\; j\notin B \\ y_j &\leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j}, \;\; j=1,...,m \\ \sum_{z\in\mathcal{Z}} \theta_z &= \sum_{z\in\mathcal{Z}} \mu_z = 1, \;\; \theta,\mu\geq 0 \Big\}. \end{aligned} \tag{5.1}$$

For any $z \in \mathcal{Z}$, let us introduce the following individual production possibility set:

$$\begin{aligned} I^{\emptyset}(x_z,y_z) = \Big\{ (x,y)\in\mathbb{R}_+^{n+m} : x_i &\geq x_{z,i}, \; i=1,...,n \\ y_j &\leq y_{z,j}, \; j=1,...,m \Big\} \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} I^{B}(x_z,y_z) = \Big\{ (x,y)\in\mathbb{R}_+^{n+m} : x_i &\geq x_{z,i}, \; i=1,...,n \\ y_j &\leq y_{z,j}, \; j\notin B \\ y_j &\geq y_{z,j}, \; j\in B \Big\}. \end{aligned} \tag{5.3}$$

FDH non-convex environmental output set of period (t) is defined as follows (Figures 7-9):

$$P_{nc}^{\{\emptyset,B\},DEA}(x) = \Big\{ y : (x,y) \in \big(\cup_{z\in\mathcal{Z}} I^{\emptyset}(x_z,y_z)\big) \cap \big(\cup_{z\in\mathcal{Z}} I^{B}(x_z,y_z)\big) \Big\}. \tag{5.4}$$

Notify that the aforementioned non-parametric convex and non-convex environmental technologies are defined as an intersection of sub-technologies. For given values of input and no polluting output, these environmental production processes display upper and lower bounds on polluting output.

5.2 Non-parametric convex and non convex environmental productivity measures

For any $(x_z, y_z) \in \mathcal{A}$, additive and multiplicative eco-productivity measures are defined below for convex and non convex environmental production set.

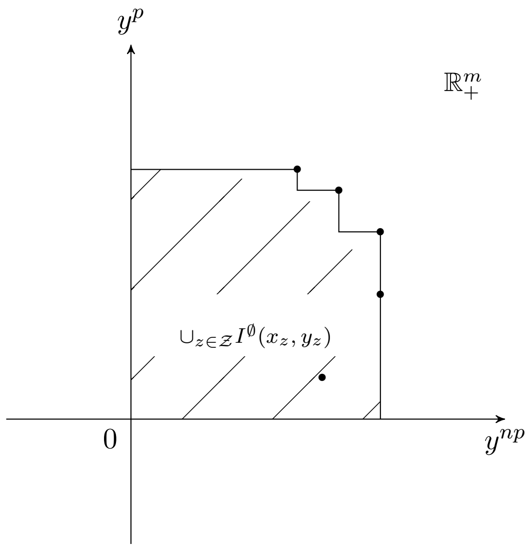

Figure 7: Non convex union of (5.2)

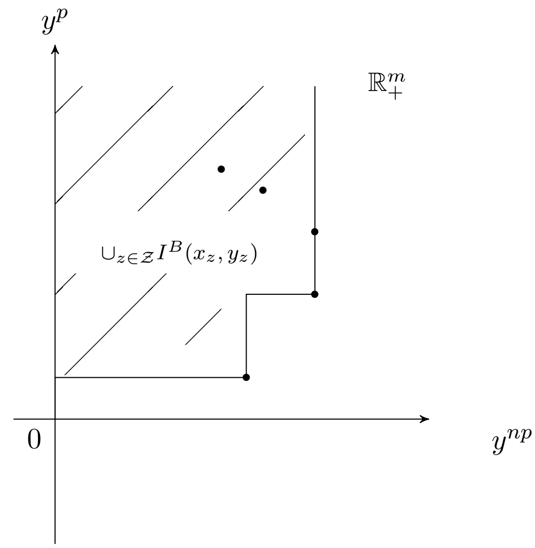

Figure 8: Non convex union of (5.3)

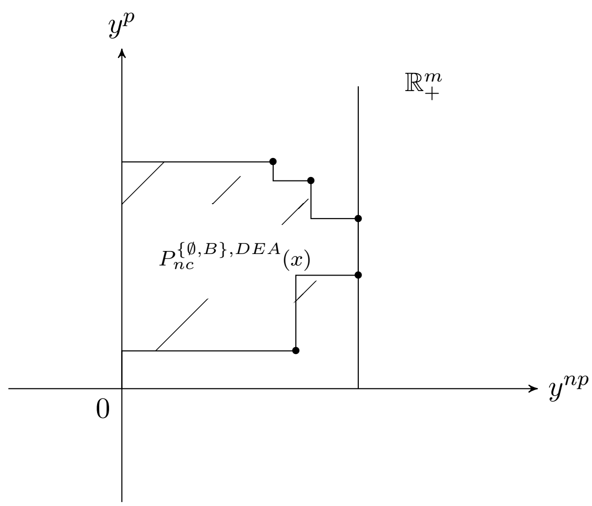

Figure 9: FDH non-convex environmental production set ($P1 - P4$ and $P6$)

5.2.1 Environmental multiplicative efficiency measure

The next statement introduces non-parametric environmental multiplicative distance function.

Proposition 5.1 *Let $P_t(x_t)$ be an environmental production set that satisfies properties $P1-P6$. For any periods $(t,t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, where $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the non-parametric environmental multiplicative efficiency measure is the solution of the following mathematical program:*

$$\begin{aligned}
\psi_t^{\beta,DEA}(x_0, y_0^{np}, y_0^{p}) = \sup \lambda & \\
s.t.\ x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \theta_z x_{z,i},\ \ i=1,...,n \\
x_{0,i} &\geq \sum_{z\in\mathcal{Z}} \mu_z x_{z,i},\ \ i=1,...,n \\
\lambda^{\beta^p} y_{0,j}^{p} &\geq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j},\ \ j \in B \\
\lambda^{\beta^{np}} y_{0,j}^{np} &\leq \sum_{z\in\mathcal{Z}} \theta_z y_{z,j},\ \ j \notin B \\
\lambda^{\beta^p} y_{0,j}^{p} &\leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j},\ \ j \in B \\
\lambda^{\beta^{np}} y_{0,j}^{np} &\leq \sum_{z\in\mathcal{Z}} \mu_z y_{z,j},\ \ j \notin B \\
\sum_{z\in\mathcal{Z}} \theta_z &= \sum_{z\in\mathcal{Z}} \mu_z = 1,\ \ \theta, \mu \geq 0
\end{aligned} \tag{5.5}$$

with $\beta^p = \{-1, 0\}$ and $\beta^{np} = \{0, 1\}$ and $(x_t, y_t) = (x, y)$.

The following proposition allows to define green multiplicative distance function on FDH non-convex environmental production set.

Proposition 5.2 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P4$ and $P6$. For any periods $(t,t+1)$ and for any $(x_{t,t+1}, y_{t,t+1}) \in \mathbb{R}_+^{n+m}$, such that $y_{t,t+1} = (y_{t,t+1}^{np}, y_{t,t+1}^{p}) \in \mathbb{R}_+^m$, the non-parametric environmental multiplicative efficiency* *measure of period (t) is defined as follows:*

$$\psi_{t_{nc}}^{\beta,DEA}(x,y) = \begin{cases} \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), & \textit{if } \beta^{np}=1 \textit{ and } \beta^{p}=0. \\ \left\{ \begin{array}{ll} \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), & \textit{if } \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right) (A) \\ & \geq \max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right) (B) \\ \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right), & \textit{if} (A) < (B) \textit{ and } \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right) \\ & < \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right) \\ \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), & \textit{if} (A) < (B) \textit{ and } \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right) \\ & \geq \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right) \end{array} \right\} & \textit{if } \beta^{np}=-1 \textit{ and } \beta^{p}=1. \\ \min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right), & \textit{if } \beta^{np}=0 \textit{ and } \beta^{p}=-1. \end{cases} \tag{5.6}$$

where $(x_t, y_t) = (x, y)$.

Proof of Proposition 5.2: Let $\beta^{np} = -1$ and $\beta^p = 1$,

$$\begin{aligned} \psi_{t_{nc}}^{\beta,DEA}(x,y) &= \min_{z\in\mathcal{Z}} \Big\{ \max_{\lambda} \big\{\lambda \geq 1 : x \geq x_z, \quad \lambda y^{np} \leq y_z^{np}, \frac{y^p}{\lambda} \leq y_z^p\big\}; \\ &\qquad \max_{\lambda} \big\{\lambda \geq 1 : x \geq x_z, \quad \lambda y^{np} \leq y_z^{np}, \frac{y^p}{\lambda} \geq y_z^p\big\} \Big\} \\ &= \min_{z\in\mathcal{Z}} \Big\{ \max_{\lambda} \Big\{\lambda \geq 1 : x \geq x_z, \quad \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), \lambda \geq \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right)\Big\}; \\ &\qquad \max_{\lambda} \Big\{\lambda \geq 1 : x \geq x_z, \quad \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right)\Big\} \Big\} \end{aligned}$$

such that $(x_t, y_t) = (x, y)$. If $\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right) \leq \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right)$ then,

$$\max_{\lambda} \Big\{\lambda \geq 1 : x \geq x_z, \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), \lambda \geq \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right)\Big\} = \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right)$$

and

$$\max_{\lambda} \Big\{\lambda \geq 1 : x \geq x_z, \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right), \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \left(\frac{y_j}{y_{z,j}}\right)\Big\} = \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right).$$

Hence,

$$\begin{aligned} \psi_{t_{nc}}^{\beta,DEA}(x,y) &= \min \Big\{ \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right); \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right) \Big\} \\ &= \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \left(\frac{y_{z,j}}{y_j}\right). \end{aligned}$$

Contrariwise, if $\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right) > \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\left(\frac{y_{z,j}}{y_j}\right)$ then,

$$\max_{\lambda}\left\{\lambda \geq 1 : x \geq x_z, \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}}\left(\frac{y_{z,j}}{y_j}\right), \lambda \geq \max_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right)\right\} = +\infty$$

and

$$\max_{\lambda}\left\{\lambda \geq 1 : x \geq x_z, \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\notin B}}\left(\frac{y_{z,j}}{y_j}\right), \lambda \leq \min_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right)\right\} = \min_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right)$$

with $\min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right) > \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\left(\frac{y_{z,j}}{y_j}\right)$. Hence,

$$\begin{aligned}\psi_{t_{nc}}^{\beta,DEA}(x,y) &= \min\left\{+\infty\,;\min_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right)\right\}\\ &= \min_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_{z,j}}{y_j}\right).\end{aligned}$$

If $\min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\left(\frac{y_j}{y_{z,j}}\right) \leq \min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\left(\frac{y_{z,j}}{y_j}\right)$, the same reasoning holds. The proof for $\beta^{np} = 1$ and $\beta^p = 0$ (respectively, $\beta^{np} = 0$ and $\beta^p = -1$) can be directly deduced from the aforementioned proof. □

The next result introduces equivalence conditions for the non-parametric multiplicative eco-efficiency (-productivity) measure and, the polluting, no polluting and hyperbolic non-parametric green multiplicative efficiency (productivity) indices.

Corollary 5.3 *Let $P_{t_{nc}}^{\{\emptyset,B\},DEA}(x_t)$ be an environmental production set that satisfies properties $P1-P4$ and $P6$. For any $(x_t, y_t) \in \mathbb{R}_+^{n+m}$, the following equivalence conditions hold:*

i. $\psi_{t_{nc}}^{\beta,DEA}(x_t,y_t) \equiv D_{t_{nc}}^{np,DEA}(x_t,y_t)$ *and* $EM_{t,t+1_{nc}}^{\beta,DEA}(x_{t,t+1},y_{t,t+1}) \equiv M_{t,t+1_{nc}}^{np,DEA}(x_{t,t+1},y_{t,t+1})$, *such that* $\beta^p = 0$ *and* $\beta^{np} = 1$.

ii. $\psi_{t_{nc}}^{\beta,DEA}(x_t,y_t) \equiv D_{t_{nc}}^{p,DEA}(x_t,y_t)$ *and* $EM_{t,t+1_{nc}}^{\beta,DEA}(x_{t,t+1},y_{t,t+1}) \equiv M_{t,t+1_{nc}}^{p,DEA}(x_{t,t+1},y_{t,t+1})$, *such that* $\beta^p = -1$ *and* $\beta^{np} = 0$.

iii. $\psi_{t_{nc}}^{\beta,DEA}(x_t,y_t) \equiv H_{t_{nc}}^{o,DEA}(x_t,y_t)$ *and* $EM_{t,t+1_{nc}}^{\beta,DEA}(x_{t,t+1},y_{t,t+1}) \equiv M_{t,t+1_{nc}}^{h,DEA}(x_{t,t+1},y_{t,t+1})$, *such that* $\beta^p = -1$ *and* $\beta^{np} = 1$.

Note that similar conditions hold for convex environmental production set (see proposition 5.1). In such a case, the convexity assumption is required to study the environmental efficiency and productivity changes.

5.2.2 Environmental additive efficiency measure

The next statement defines non-parametric environmental additive distance function.

Proposition 5.4 *Let $P_t(x_t)$ be an environmental output set that satisfies properties $P1-P6$. For any periods $(t,t+1)$ and for any $(x_{t,t+1},y_{t,t+1})\in\mathbb{R}_+^{n+m}$, where $y_{t,t+1}=(y^{np}_{t,t+1},y^{p}_{t,t+1})\in\mathbb{R}^m_+$, the non-parametric environmental additive efficiency measure is the solution of the following mathematical program:*

$$\begin{aligned}\xi_t^{\{0,\sigma\},DEA}(x_0,y_0) = \max\delta&\\ s.t.\ \ x_{0,i}\geq \sum_{z\in\mathcal{Z}}\theta_z x_{z,i},&\ \ i=1,...,n\\ x_{0,i}\geq \sum_{z\in\mathcal{Z}}\mu_z x_{z,i},&\ \ i=1,...,n\\ (1+\delta\sigma^p)y_{0,j}\geq \sum_{z\in\mathcal{Z}}\theta_z y_{z,j},&\ \ j\in B\\ (1+\delta\sigma^{np})y_{0,j}\leq \sum_{z\in\mathcal{Z}}\theta_z y_{z,j},&\ \ j\notin B\\ (1+\delta\sigma^p)y_{0,j}\leq \sum_{z\in\mathcal{Z}}\mu_z y_{z,j},&\ \ j\in B\\ (1+\delta\sigma^{np})y_{0,j}\leq \sum_{z\in\mathcal{Z}}\mu_z y_{z,j},&\ \ j\notin B\\ \sum_{z\in\mathcal{Z}}\theta_z=\sum_{z\in\mathcal{Z}}\mu_k=1,&\ \ \theta,\mu\geq 0\end{aligned}\tag{5.7}$$

such that $\sigma=(\sigma^{np},\sigma^p)\in[0,1]^{m^{np}}\times[-1,0]^{m^p}$ and $(x_t,y_t)=(x,y)$.

Environmental additive distance function on FDH non-convex environmental output set is defined in the result below.

Proposition 5.5 *Let $P_t(x_t)$ be an environmental production set that satisfies properties $P1-P4$ and $P6$. For any periods $(t,t+1)$ and for any $(x_{t,t+1},y_{t,t+1})\in\mathbb{R}_+^{n+m}$, with $y_{t,t+1}=(y^{np}_{t,t+1},y^{p}_{t,t+1})\in\mathbb{R}^m_+$, the non-parametric environmental additive efficiency measure is defined as follows:*

$$\xi_{tnc}^{\{0,\sigma\},DEA}(x,y)=\begin{cases}\frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big), & \text{if } \frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)\geq\frac{1}{\sigma^{p}}\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big).\\ \frac{1}{\sigma^{p}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big), & \text{if } \frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)<\frac{1}{\sigma^{p}}\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)\\ & \text{and } \frac{1}{\sigma^{p}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)>\frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big).\\ \frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big), & \text{if } \frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)<\frac{1}{\sigma^{p}}\max\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)\\ & \text{and } \frac{1}{\sigma^{p}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\in B}}\Big(\frac{y_{z,j}}{y_j}-1\Big)\leq\frac{1}{\sigma^{np}}\min\limits_{\substack{z\in\mathcal{Z}\\ j\notin B}}\Big(\frac{y_{z,j}}{y_j}-1\Big).\end{cases}\tag{5.8}$$

where $(x_t,y_t)=(x,y)$ and $\sigma=(\sigma^{np},\sigma^p)\in]0,1]^{m^{np}}\times[-1,0[^{m^p}$.

Proof of Proposition 5.5: Let $\sigma = (\sigma^{np}, \sigma^p) \in]0,1]^{m^{np}} \times [-1,0[^{m^p}$,

$$\begin{aligned}
\xi_{t_{nc}}^{\{0,\sigma\},DEA}(x,y) &= \min_{z\in\mathcal{Z}} \Big\{ \max_{\delta} \left\{\delta \geq 0 : x \geq x_z, y^{np} + \sigma^{np}\delta y^{np} \leq y_z^{np}, y^p + \sigma^p \delta y^p \leq y_z^p\right\}; \\
&\qquad \max_{\delta} \left\{\delta \geq 0 : x \geq x_z, y^{np} + \sigma^{np}\delta y^{np} \leq y_z^{np}, y^p + \sigma^p \delta y^p \geq y_z^p\right\} \Big\} \\
&= \min_{z\in\mathcal{Z}} \Big\{ \max_{\delta} \Big\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big), \\
&\qquad \delta \geq \frac{1}{\sigma^p} \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\}; \\
&\qquad \max_{\delta} \Big\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big), \\
&\qquad \delta \leq \frac{1}{\sigma^p} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\}\Big\}
\end{aligned}$$

where $(x_t, y_t) = (x, y)$. If $\frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big) \geq \frac{1}{\sigma^p} \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)$ then,

$$\begin{aligned}
\max_{\delta} \Big\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big), \delta \geq \frac{1}{\sigma^p} \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\} \\
= \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)
\end{aligned}$$

and

$$\begin{aligned}
\max_{\delta} \Big\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big), \delta \leq \frac{1}{\sigma^p} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\} \\
= \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\xi_{t_{nc}}^{\{0,\sigma\},DEA}(x,y) &= \min \Big\{ \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big); \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\} \\
&= \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big).
\end{aligned}$$

Contrarily, if $\frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big) < \frac{1}{\sigma^p} \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)$ then,

$$\max_{\delta} \Big\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big), \delta \geq \frac{1}{\sigma^p} \max_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j} - 1\Big)\Big\} = +\infty$$

and

$$\max_{\delta}\left\{\delta \geq 0 : x \geq x_z, \delta \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j}-1\Big), \delta \leq \frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)\right\}$$
$$= \frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big),$$

when $\frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big) > \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)$. Hence, $\xi_{tnc}^{\{0,\sigma\},DEA}(x,y) = \frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)$.
If $\frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big) \leq \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)$, a similar reasoning holds. □

Notify that the sub-vectors no polluting and polluting non-parametric additive efficiency measure are respectively defined as $\xi_{tnc}^{\{0,\sigma^{np},0\},DEA}(x,y) = \frac{1}{\sigma^{np}} \min_{\substack{z\in\mathcal{Z}\\ j\notin B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)$ and $\xi_{tnc}^{\{0,0,\sigma^{p}\},DEA}(x,y) = \frac{1}{\sigma^{p}} \min_{\substack{z\in\mathcal{Z}\\ j\in B}} \Big(\frac{y_{z,j}}{y_j}-1\Big)$. These results can be immediately deduced from the proof of Proposition 5.5.

Equivalence conditions for the non-parametric additive eco-efficiency (respectively -productivity) measure and the multiplicative non-parametric environmental efficiency (respectively productivity) indicators are introduced below.

Corollary 5.6 *Let $P_{tnc}^{\{\emptyset,B\},DEA}(x_t)$ be a non-parametric environmental output set that satisfies properties $P1-P4$ and $P6$. For any $(x_t,y_t)\in\mathbb{R}^{n+m}_{++}$, the next results hold:*

	Efficiency Analysis	**Productivity Analysis**
(i.) $1=\sigma^{np}\ln(y_t^{np})$ and $0=\sigma^{p}\ln(y_t^{p})$	$\xi_{tnc}^{\{0,[\ln(y_t^{np})]^{-1},0\},DEA}(\ln(x_t),\ln(y_t)) \equiv -\ln\Big(D_{tnc}^{np,DEA}(x_t,y_t)\Big)$	$EL_{t,t+1}{}_{nc}^{\sigma,DEA}(\ln(x_{t,t+1}),\ln(y_{t,t+1})) \equiv \ln\Big(M_{t,t+1}{}_{nc}^{np,DEA}(x_{t,t+1},y_{t,t+1})\Big)$
(ii.) $0=\sigma^{np}\ln(y_t^{np})$ and $-1=\sigma^{p}\ln(y_t^{p})$	$\xi_{tnc}^{\{0,0,-[\ln(y_t^{p})]^{-1}\},DEA}(\ln(x_t),\ln(y_t)) \equiv -\ln\Big(D_{tnc}^{p,DEA}(x_t,y_t)\Big)$	$EL_{t,t+1}{}_{nc}^{\sigma,DEA}(\ln(x_{t,t+1}),\ln(y_{t,t+1}) \equiv \ln\Big(M_{t,t+1}{}_{nc}^{p,DEA}(x_{t,t+1},y_{t,t+1})\Big)$
(iii.) $1=\sigma^{np}\ln(y_t^{np})$ and $-1=\sigma^{p}\ln(y_t^{p})$	$\xi_{tnc}^{\{0,[\ln(y_t^{np})]^{-1},-[\ln(y_t^{p})]^{-1}\},DEA}(\ln(x_t),\ln(y_t)) \equiv -\ln\Big(H_{tnc}^{o,DEA}(x_t,y_t)\Big)$	$EL_{t,t+1}{}_{nc}^{\sigma,DEA}(\ln(x_{t,t+1}),\ln(y_{t,t+1})) \equiv \ln\Big(M_{t,t+1}{}_{nc}^{h,DEA}(x_{t,t+1},y_{t,t+1})\Big)$

Note that similar conditions are satisfied in the context of convex environmental production processes (see proposition 5.4).

6 Concluding Comments

This paper gives a general representation of environmental issues in production economics. Consecutively, environmental efficiency measure and green productivity change are analysed. Indeed, additive and multiplicative eco-efficiency and -productivity indicators are defined. In addition, some equivalence conditions for the new additive and multiplicative environmental efficiency and productivity measures are introduced.

Knowing components of green productivity variation is a major concern to define environmental recommendations. This paper defines a procedure to decompose environmental additive and multiplicative productivity measures. Two main drivers of eco-productivity change are defined: environmental technological change and green efficiency variation. The combination of these sources of environmental productivity change displays informations about the conditions of eco-productivity variation. This result could be of particular interest for the production of empirical studies. In addition, the new environmental efficiency and productivity analysis does not require to assume the convexity of the production process. This result brings out some theoretical and empirical issues as in De Borger and Kerstens (1996), Dasgupta and Mäler (2003), Tschirhart (2012) or Chavas and Briec (2018).

Environmental efficiency and productivity measures are defined through convex and non-convex non-parametric models. Hence, an extension of this paper could be the production of an empirical application. Such investigation is left for future research.

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