Metric Dimension of Radially Symmetrical Plane Graph Obtained from Heptagonal Circular Ladder

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Abstract Let $\Phi = \Phi(\mathbb{V}, \mathbb{E})$ be a basic associated (i.e., simple and connected) undirected (i.e., all edges are unidirectional, with no direction associated with them) graph (or network) with an edge set $\mathbb{E}$ and vertex set $\mathbb{V}$. A metric basis $\mathbb{F}$ of a graph Φ is the subset of nodes of least cardinality with the end goal that all different nodes are extraordinarily dictated by their separations to the nodes in $\mathbb{F}$. The metric dimension of Φ is the cardinality of the subset $\mathbb{F}$. The problem of characterizing the classes of plane graphs with a constant metric dimension is of great interest nowadays. The location number (or the metric dimension) of certain classes of plane graphs have been attained in [7] and an open problem in regards to these graphs was raised that: Depict those classes of plane graphs Υ which are acquired from the graphs Φ by including new edges in Φ with the end goal that $\mathbb{V}(\Phi) = \mathbb{V}(\Upsilon)$ and $dim(\Phi) = dim(\Upsilon)$. Then, some partial answers to this problem were given in [14]. Continuing in the same direction, in this examination, we discover another family of plane graph viz., Δ_n, which is obtained by the expansion of new edges in the graph Γ_n [14] and prove that $\mathbb{V}(\Gamma_n) = \mathbb{V}(\Delta_n)$ and $dim(\Gamma_n) = dim(\Delta_n)$. We additionally offer a response to the problems raised in [14] and demonstrate that just 3 vertices fittingly chosen are adequate to resolve all the vertices of these classes of plane graphs.

1. Introduction

Let $\Phi = \Phi(\mathbb{V}, \mathbb{E})$ be a basic associated (i.e., simple and connected) undirected (i.e., all edges are unidirectional, with no direction associated with them) graph (or network) with an edge set $\mathbb{E}$ and vertex set $\mathbb{V}$. The metric dimension of the network $\Phi = \Phi(\mathbb{V}, \mathbb{E})$ is the minimal number of nodes (vertices or hubs) in a set with the property that the rundown of good ways from any vertex to those in the set exceptionally distinguishes that vertex. By definition, for an arranged (ordered) subset $\mathbb{F} = \{\varpi_1, \varpi_2, \varpi_3, ..., \varpi_z\}$ of hubs in a graph Φ and a node ϖ of Φ, the metric code/metric representation of ϖ concerning $\mathbb{F}$ is the ordered z-tuple (or z-vector) $\zeta(\varpi|\mathbb{F}) := (d_\Phi(\varpi, \varpi_1), d_\Phi(\varpi, \varpi_2), ..., d_\Phi(\varpi, \varpi_z))$. In the event that each pair of unmistakable vertices of Φ have distinctive metric representations then the arranged set $\mathbb{F}$ is known as a resolving (locating) set of Φ. The metric dimension or the location number of Φ is the cardinality of the subset $\mathbb{F}$ i.e., location number (or metric dimension) of the graph Φ, indicated by $dim(\Phi)$ or $\beta(\Phi)$, is the smallest size of locating set on the graph Φ; formally, $\beta(\Phi) = dim(\Phi) = minimum\{|\mathbb{F}| : \mathbb{F}\ is\ locating\ set\ (or\ resolving\ set)\}$. It is realized that the issue of registering this invariant is NP-hard. If a subset $\mathbb{T}$ of the arrangement of nodes $\mathbb{V}(\Phi)$ is both resolving and independent (or autonomous), then the set $\mathbb{T}$ is known as an independent resolving set (or an autonomous resolving set) for the graph Φ.

For an organized set (or ordered set) of nodes $\mathbb{F} = \{\varpi_1, \varpi_2, \varpi_3, ..., \varpi_z\}$ of Φ, the pth component (or distance coordinate) of the code $\zeta(\varpi|\mathbb{F})$ is zero iff $\varpi = \varpi_p$. Subsequently, to watch that the set $\mathbb{F}$ is a resolving set, it is adequate to confirm that $\zeta(\varpi|\mathbb{F}) \neq \zeta(\varrho|\mathbb{F})$ for any couple of distinguishable vertices $\varrho, \varpi \in \mathbb{V}(\Phi) \setminus \mathbb{F}$.

The idea of resolving or locating set and that of metric dimension goes back to the 1950s. They were characterized by L. M. Blumenthal [2] with regards to metric space. These thoughts were acquainted with graph networks autonomously by Melter and Harary in 1976 [6], and P. J. Slater in 1975 [15]. Chart hypothesis has applications in numerous zones of figuring, social, and normal sciences and is likewise an affable play area for the investigation of the verification procedure in discrete science. Utilizations of this invariant to issues of picture preparing (or image processing) and design acknowledgment (or pattern recognition) are talked about in [12], to the route of exploring specialist (navigating agent or robots) in systems (or networks) are examined in [10], applications to science are given in [5], application to combinatorial enhancement (or optimization) is yielded in [13], and to issues of check and system revelation (or network discovery) in [1].

For an undirected graph Φ, the line graph of the graph Φ is a graph $L(\Phi)$ with vertex set $\mathbb{V}(L(\Phi)) = \mathbb{E}(\Phi)$ and two different nodes f_i and f_j are adjacent in $L(\Phi)$ iff they have a common end vertex in Φ. Sometimes a line graph is also termed as edge graph, derived graph, or interchange graph. A polytope in elementary geometry is a geometric object with flat sides. When polytopes are having an additional property that they are convex sets and are contained in the n-dimensional space $\mathbb{R}^n$ (Euclidean space), then they are termed as convex polytopes. Depending upon the problem with which we are dealing, polytopes and convex polytopes are defined accordingly (polytopes and convex polytopes may be defined in several ways by different researchers). Convex polytopes assume a significant job both in different branches of arithmetic and in applied zones, most quite in linear programming. The location number of a few classes of convex polytopes has been considered in ([8], [9], [14]). When every edge of the given undirected graph Φ is replaced by a path of length two, the graph obtained is known as the subdivision graph of the graph Φ, denoted by $S(\Phi)$.

Let $\aleph$ establish a family of connected graphs (or charts). We call that the family $\aleph$ has steady (or constant) location number if $\beta(\Psi) = dim(\Psi)$ is free of the decision of the graph Ψ in $\aleph$ and is finite (or limited). In other words, if all the graphs in $\aleph$ have an indistinguishable location number, at that point $\aleph$ is known as a family with a steady location number [16]. In [5] Chartrand et al. demonstrated that graphs (or charts) on n nodes have location number one iff it is a path $\wp_n$. Additionally, cycle $\mathcal{C}_n$ has location number two for each positive integer n; $n \geq 3$. Henceforth, $\mathcal{C}_n$ ($n \geq 3$) and $\wp_n$ ($n \geq 2$) establish a family of graphs (or charts) with a steady location number. Additionally, Petersen graphs $P(n, 2)$ (generalized) and Harary graphs $H_{4,n}$, are also the families of graphs (or charts) with steady location numbers [9].

The join of two charts (or graphs) $\Psi_1 = \Psi_1(\mathfrak{V}_1, \mathfrak{E}_1)$ and $\Psi_2 = \Psi_2(\mathfrak{V}_2, \mathfrak{E}_2)$ is a chart $\Phi = \Phi(\mathfrak{V}, \mathfrak{E})$ with the end goal that $\mathfrak{V} = \mathfrak{V}_1 \cup \mathfrak{V}_2$ and $\mathfrak{E} = \mathfrak{E}_1 \cup \mathfrak{E}_2 \cup \{\varrho\varsigma : \varrho \in \mathfrak{V}_1 \ and \ \varsigma \in \mathfrak{V}_2\}$. Then a fan F_m is characterized as $F_m = K_1 + \wp_m$ for $m \geq 1$, a wheel W_m is characterized as $W_m = K_1 + \mathcal{C}_m$, for $m \geq 3$, and Jahangir graph J_{2m} ($m \geq 2$) is gotten from the wheel graph W_{2m} by alternately deleting m spokes of the wheel graph (which is otherwise called as a gear graph). In [4], Caceres et al. decided the location number of the fan graph F_m ($m \geq 1$) which is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{1, 2, 3, 6\}$. Tomescu and Javaid [17] acquired the location number of the Jahangir graph J_{2m} ($m \geq 4$) which is $\lfloor \frac{2m}{3} \rfloor$, and in [3] Chartrand et al. decided the location number of the wheel chart (or graph) W_m ($m \geq 3$) which is $\lfloor \frac{2m+2}{5} \rfloor$ for $m \notin \{3, 6\}$. Note that the location number of these three families (viz., Fan graph, Wheel graph, and Jahangir graph) of the plane graphs rely on the number of vertices in the graphs and in this way does not comprise the classes of plane graphs with steady location number (or constant metric dimensions).

Now, a property in regards to the metric dimension two of a connected graph $\Psi = \Psi(\mathbb{V}, \mathbb{E})$ was demonstrated by Khuller et al. in [11] and is

Theorem 1. *[11] Let $\mathbb{A} \subseteq \mathfrak{V}(\Psi)$ be the basis set of the connected graph $\Psi = \Psi(\mathbb{V}, \mathbb{E})$ of cardinality two i.e., $|\mathbb{A}| = dim(\Psi) = \beta(\Psi) = 2$, and say $\mathbb{A} = \{\varpi, \xi\}$. Then, the following listed three points are true:*

1. *Between the vertices ϖ and ξ, there exists a unique and shortest path $\wp$.*

2. *The valencies (or degrees) of the nodes ϖ and ξ can never exceed 3.*

3. *The valency of any other node on $\wp$ can never exceed 5.*

The main motivation in characterizing the classes of plane graphs with constant metric dimension (or with non-constant metric dimension) is that none of the plane graphs should be left with an unknown metric dimension (or location number). The radially symmetric plane graph Δ_n is obtained from the graph Γ_n by the insertion of new edges in the graph Γ_n between the nodes r_t and q_{t+1} $(1 \leqslant t \leqslant n)$, and is a graph on $6n$ edges and $4n$ vertices. We observed that $\mathbb{V}(\Gamma_n) = \mathbb{V}(\Delta_n)$, and in this study, we demonstrate that the location number of these two plane graphs (viz., Γ_n and Δ_n) are the same (i.e., $\beta(\Gamma_n) = \beta(\Delta_n)$) and constant. We also give the answers to the open problems that were raised in [14], by showing that there exists a minimal independent locating set with cardinality three for the plane graphs Γ_n and Σ_n. In what follows all lists t that do not fulfill the given disparities will be taken modulo n.

In the accompanying section, we acquire the location number of the radially symmetric plane graph Δ_n (see figure 1), which is obtained from the graph Γ_n by the insertion of new edges between the nodes r_t and q_{t+1} $(1 \leqslant t \leqslant n)$, and for each positive integer n: $n \geq 6$ we demonstrate that $\beta(\Delta_n) = 3$. We also prove that the minimal resolving set obtained for the graph Δ_n is independent.

2. The Plane Graph Δ_n

The radially symmetric plane graph Δ_n is obtained from the graph Γ_n by the insertion of new edges in the graph Γ_n between the nodes r_t and q_{t+1} $(1 \leqslant t \leqslant n$, see figure 1). At that point, the radially symmetrical plane graph Δ_n comprises of n 5 and 4-sided faces, a $2n$-sided face, and an n-sided face and it consists of $6n$ edges and $4n$ number of nodes. By $\mathbb{E}(\Delta_n)$ and $\mathbb{V}(\Delta_n)$, we signify the arrangement of edges and vertices of a radially symmetrical plane graph Δ_n separately. Consequently, we have

$$\mathbb{V}(\Delta_n) \quad = \quad \{p_t, q_t, r_t, s_t : 1 \leqslant t \leqslant n\}$$

and

$$\mathbb{E}(\Delta_n) \quad = \quad \{p_tq_t, q_tr_t, r_ts_t, p_tp_{t+1}, s_tr_{t+1}, r_tq_{t+1} : 1 \leqslant t \leqslant n\}$$

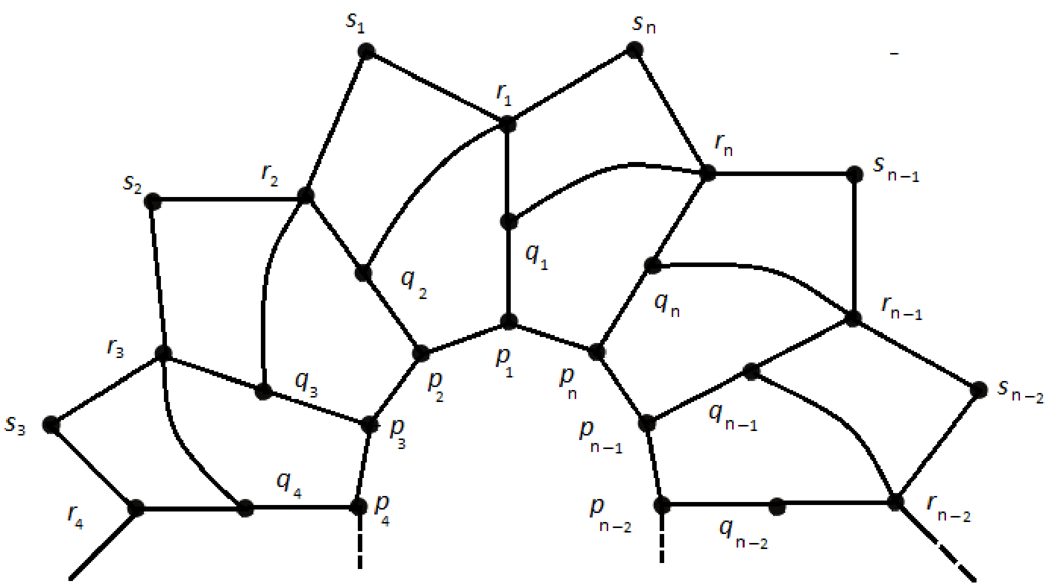

Figure 1: The radially symmetric graph Δ_n

For our motive, we call the cycle brought forth by the arrangement of vertices $\{p_t : 1 \leqslant t \leqslant n\}$ as the p-cycle, the arrangement of vertices $\{q_t : 1 \leqslant t \leqslant n\}$ as the set of core vertices, and the cycle brought forth by the arrangement of vertices $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ as the rs-cycle. In the accompanying theorem, we discover that the location number of a radially symmetrical graph, Δ_n is 3 i.e., just 3 vertices properly chosen are adequate to determine all the vertices of the radially symmetrical graph Δ_n. Note that the choice of appropriate basis node is the crux of the problem.

Theorem 2. *Let n be a positive integer such that $n \geqslant 6$ and Δ_n be the planar graph on $4n$ vertices as defined above. Then, we have $dim(\Delta_n) = 3$ i.e., it has location number 3.*

Proof. To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer n i.e., when the positive whole number n is even and when it is odd.
Case(I) When the integer n is even.
For this situation, the integer n can be written as $n = 2\vartheta$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Delta_n)$. Now, in order to unveil that $\mathfrak{R}$ is a locating or resolving set for the radially symmetrical plane graph Δ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Delta_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta - t + 1, t), & 3 \leq t \leq \vartheta \\ (2\vartheta - t + 2, t - \vartheta - 1, 2\vartheta - t), & \vartheta + 2 \leq t \leq 2\vartheta - 1. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta + 1, 2), & t = 1; \\ (t-1, \vartheta - t + 2, t + 1), & 2 \leq t \leq \vartheta; \\ (\vartheta, 1, \vartheta), & t = \vartheta + 1; \\ (2\vartheta - t + 3, t - \vartheta, 2\vartheta - t + 1), & \vartheta + 2 \leq t \leq 2\vartheta. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (2, \vartheta + 1, 3), & t = 1; \\ (t, \vartheta - t + 2, t + 2), & 2 \leq t \leq \vartheta - 1; \\ (\vartheta, 2, \vartheta + 1), & t = \vartheta; \\ (\vartheta + 1, 2, \vartheta), & t = \vartheta + 1; \\ (2\vartheta - t + 3, t - \vartheta + 1, 2\vartheta - t + 1), & \vartheta + 2 \leq t \leq 2\vartheta - 1; \\ (2\vartheta - t + 3, t - \vartheta + 1, 2), & t = 2\vartheta. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta + 1, 4), & t = 1; \\ (t + 1, \vartheta - t + 2, t + 3), & 2 \leq t \leq \vartheta - 1; \\ (\vartheta + 1, 3, \vartheta + 1), & t = \vartheta; \\ (2\vartheta - t + 3, t - \vartheta + 2, 2\vartheta - t + 1), & \vartheta + 1 \leq t \leq 2\vartheta - 2; \\ (2\vartheta - t + 3, t - \vartheta + 2, 3), & 2\vartheta - 1 \leq t \leq 2\vartheta. \end{cases}$$

We notice that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Delta_n) \leq 3$. Now, so as to finish the evidence for this case, we show that $\beta(\Delta_n) \geq 3$ by working out that there does not exist a resolving set $\mathfrak{R}$ with the end goal that $|\mathfrak{R}| = 2$. Despite what might be expected, we guess that $\beta(\Delta_n) = 2$. Now, by A_1, A_2, A_3, and A_4, we signify the arrangement of vertices as $A_1 = \{p_t : 1 \leqslant t \leqslant n\}$, $A_2 = \{q_t : 1 \leqslant t \leqslant n\}$, $A_3 = \{r_t : 1 \leqslant t \leqslant n\}$, and $A_4 = \{s_t : 1 \leqslant t \leqslant n\}$. At that point by Theorem 1, we find that the valency of basis nodes can never exceed 3. But except the vertices of the set A_3, all other nodes of the radially symmetrical plane graph Δ_n have a valency less than or equals to 3. At that point, we have the accompanying prospects to be talked about.

- When one, as well as the other node, are in the set A_l; $l = 1, 2, 4$.

Resolving sets	Contradictions
$\{p_1, p_g\}$, p_g $(2 \leq g \leq n)$	For $2 \leq g \leq u$, we have $\zeta(q_1\|\{p_1, p_g\}) = \zeta(p_n\|\{p_1, p_g\})$, and when $g = u+1$, we have $\zeta(p_2\|\{p_1, p_{u+1}\}) = \zeta(p_n\|\{p_1, p_{u+1}\})$, a contradiction.
$\{q_1, q_g\}$, q_g $(2 \leq g \leq n)$	For $g = 2$, we have $\zeta(s_1\|\{q_1, q_2\}) = \zeta(s_n\|\{q_1, q_2\})$, when $g = 3$, we have $\zeta(p_2\|\{q_1, q_3\}) = \zeta(q_2\|\{q_1, q_3\})$, when $g = 4$, we have $\zeta(s_2\|\{q_1, q_4\}) = \zeta(r_3\|\{q_1, q_4\})$, when $5 \leq g \leq u$, we have $\zeta(s_1\|\{q_1, q_g\}) = \zeta(p_n\|\{q_1, q_g\})$, and when $g = u+1$, we have $\zeta(p_2\|\{q_1, q_{u+1}\}) = \zeta(p_n\|\{q_1, q_{u+1}\})$, a contradiction.

Resolving sets	Contradictions
$\{s_1, s_g\}$, s_g $(2 \leq g \leq n)$	For $2 \leq g \leq 3$, we have $\zeta(s_n\|\{s_1, s_g\}) = \zeta(q_1\|\{s_1, s_g\})$, when $4 \leq g \leq 5$, we have $\zeta(s_3\|\{s_1, s_g\}) = \zeta(q_4\|\{s_1, s_g\})$, and when $6 \leq g \leq u+1$, we have $\zeta(s_2\|\{s_1, s_g\}) = \zeta(q_2\|\{s_1, s_g\})$, a contradiction.

- When one node is in the set A_1 and other lies in the set A_l; $l = 2$ and 4.

Resolving sets	Contradictions
$\{p_1, q_g\}$, q_g $(1 \leq g \leq n)$	For $g = 1, u+1$, we have $\zeta(p_n\|\{p_1, q_g\}) = \zeta(p_2\|\{p_1, q_g\})$, when $g = 2$, we have $\zeta(s_2\|\{p_1, q_2\}) = \zeta(s_1\|\{p_1, q_2\})$, and when $3 \leq g \leq u+1$, we have $\zeta(s_2\|\{p_1, q_g\}) = \zeta(q_2\|\{p_1, q_g\})$, a contradiction.
$\{p_1, s_g\}$, s_g $(1 \leq g \leq n)$	For $g = 1$, we have $\zeta(s_n\|\{p_1, s_1\}) = \zeta(q_3\|\{p_1, s_1\})$, when $g = 2$, we have $\zeta(q_3\|\{p_1, s_2\}) = \zeta(s_1\|\{p_1, s_2\})$, and when $3 \leq g \leq u-1$, we have $\zeta(p_n\|\{p_1, s_g\}) = \zeta(q_1\|\{p_1, s_g\})$, when $g = u$, we have $\zeta(p_n\|\{p_1, s_u\}) = \zeta(p_2\|\{p_1, s_u\})$, and when $g = u+1$, we have $\zeta(p_3\|\{p_1, s_{u+1}\}) = \zeta(q_n\|\{p_1, s_{u+1}\})$, a contradiction.

- When one node is in the set A_2 and other lies in the set A_4.

Resolving sets	Contradictions
$\{q_1, s_g\}$, s_g $(1 \leq g \leq n)$	For $g = 1$, we have $\zeta(p_n\|\{q_1, s_1\}) = \zeta(q_2\|\{q_1, s_1\})$, when $g = 2$, we have $\zeta(q_2\|\{q_1, s_2\}) = \zeta(s_1\|\{q_1, s_2\})$, when $3 \leq g \leq 4$, we have $\zeta(s_2\|\{q_1, s_g\}) = \zeta(q_3\|\{q_1, s_g\})$, when $5 \leq g \leq u-1$, we have $\zeta(s_1\|\{q_1, s_g\}) = \zeta(p_n\|\{q_1, s_g\})$, when $g = u$, we have $\zeta(p_n\|\{q_1, s_u\}) = \zeta(p_2\|\{q_1, s_u\})$, and when $g = u+1$, we have $\zeta(p_3\|\{q_1, s_{u+1}\}) = \zeta(r_{n-1}\|\{q_1, s_{u+1}\})$, a contradiction.

In this manner, the above conversation explains that there does not exist a resolving set comprising of two vertices for $\mathbb{V}(\Delta_n)$ inferring that $\beta(\Delta_n) = 3$ in this case.

Case(II) When the integer n is odd.
For this situation, the integer n can be written as $n = 2\vartheta + 1$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Delta_n)$. Now, in order to unveil that $\mathfrak{R}$ is a locating or resolving set for the radially symmetrical plane graph Δ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Delta_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta - t + 1, t), & 3 \leq t \leq \vartheta \\ (\vartheta, 1, \vartheta - 1), & t = \vartheta + 2; \\ (2\vartheta - t + 3, t - \vartheta - 1, 2\vartheta - t + 1), & \vartheta + 3 \leq t \leq 2\vartheta. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta+1, 2), & t = 1; \\ (t-1, \vartheta-t+2, t+1), & 3 \leq t \leq \vartheta \\ (\vartheta, 1, \vartheta+1), & t = \vartheta+1; \\ (\vartheta+1, 2, \vartheta), & t = \vartheta+2; \\ (2\vartheta-t+4, t-\vartheta, 2\vartheta-t+2), & \vartheta+3 \leq t \leq 2\vartheta+1. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (2, \vartheta+1, 3), & t = 1; \\ (t, \vartheta-t+2, t+2), & 2 \leq t \leq \vartheta; \\ (\vartheta+1, 2, \vartheta+1), & t = \vartheta+1; \\ (2\vartheta-t+4, t-\vartheta+1, 2\vartheta-t+2), & \vartheta+2 \leq t \leq 2\vartheta; \\ (2\vartheta-t+4, t-\vartheta+1, 2), & t = 2\vartheta+1. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta+1, 4), & t = 1; \\ (t+1, \vartheta-t+2, t+3), & 2 \leq t \leq \vartheta-1; \\ (\vartheta+1, 3, \vartheta+2), & t = \vartheta; \\ (\vartheta+2, 3, \vartheta+1), & t = \vartheta+1; \\ (2\vartheta-t+4, t-\vartheta+2, 2\vartheta-t+2), & \vartheta+2 \leq t \leq 2\vartheta-1; \\ (2\vartheta-t+4, t-\vartheta+2, 3), & t = 2\vartheta; \\ (2\vartheta-t+4, \vartheta+2, 3), & t = 2\vartheta+1. \end{cases}$$

Again we see that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Delta_n) \leq 3$. Now, on expecting that $\beta(\Delta_n) = 2$, we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently, $\beta(\Delta_n) = 3$ for this situation too, which concludes the theorem. □

This result can also be written as:

Theorem 3. *Let n be the positive integer such that $n \geqslant 6$ and Δ_n be the planar graph on $4n$ vertices as defined above. Then, its independent location number is* 3.

Proof. For proof, refer theorem 2. □

In the next two sections, we prove that the minimal resolving sets obtained for the graphs Γ_n and Σ_n in [14] are independent and are of the same cardinality three i.e., $|\mathfrak{R}| = 3 = \beta(\Gamma_n) = \beta(\Sigma_n)$.

3. The Plane Graph Γ_n

The radially symmetric plane graph Γ_n was defined in [14], and by $\mathbb{E}(\Gamma_n)$ and $\mathbb{V}(\Gamma_n)$, we signify the arrangement of edges and vertices of the graph Γ_n separately. Consequently, we have

$$\mathbb{V}(\Gamma_n) = \{p_t, r_t, q_t, s_t : 1 \leqslant t \leqslant n\}$$

and

$$\mathbb{E}(\Gamma_n) = \{p_t q_t, q_t r_t, r_t s_t, p_t p_{t+1}, s_t r_{t+1} : 1 \leqslant t \leqslant n\}$$

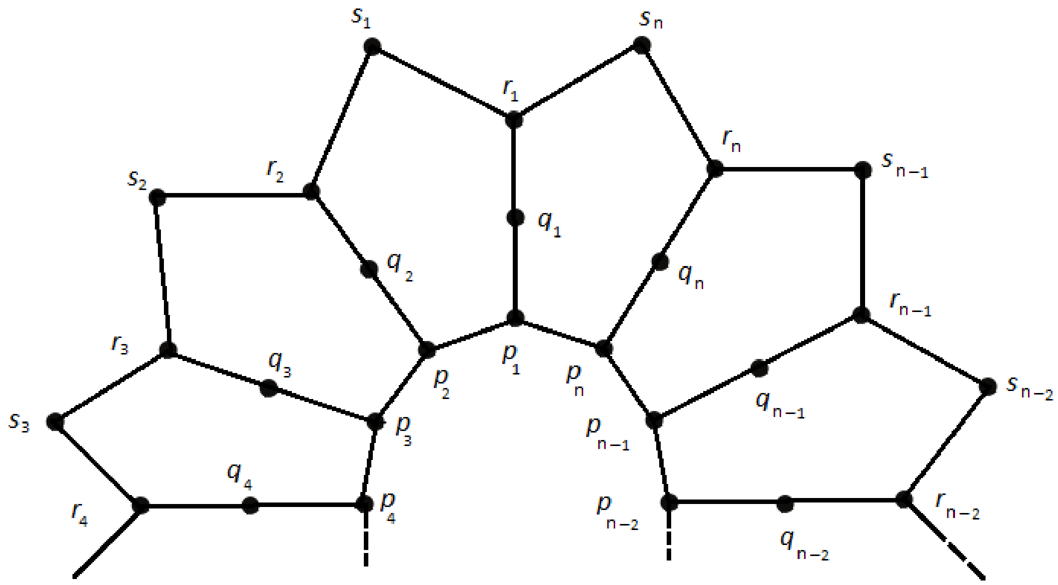

Figure 2: The radially symmetric graph Γ_n

For our motive, we call the cycle brought forth by the arrangement of vertices $\{p_t : 1 \leqslant t \leqslant n\}$ as the p-cycle, the arrangement of vertices $\{q_t : 1 \leqslant t \leqslant n\}$ as the set of core vertices, and the cycle brought forth by the arrangement of vertices $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ as the rs-cycle. In the accompanying theorem, we discover that the independent location number of a radially symmetrical graph, Γ_n is 3 i.e., just 3 vertices properly chosen are adequate to determine all the vertices of the radially symmetrical graph Γ_n. Note that the choice of appropriate basis node is the crux of the problem.

Theorem 4. *Let n be the positive integer such that $n \geqslant 6$ and Γ_n be the planar graph on $4n$ vertices as defined above. Then, its independent location number is 3.*

Proof. To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer n i.e., when the positive whole number n is even and when it is odd.
Case(I) When the integer n is even.
For this situation, the integer n can be written as $n = 2\vartheta$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Gamma_n)$. Now, in order to unveil that $\mathfrak{R}$ is a locating or resolving set for the radially symmetrical graph Γ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Gamma_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta - t + 1, t), & 3 \leq t \leq \vartheta \\ (2\vartheta - t + 2, t - \vartheta - 1, 2\vartheta - t), & \vartheta + 2 \leq t \leq 2\vartheta - 1. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta + 1, 2), & t = 1; \\ (t-1, \vartheta - t + 2, t + 1), & 2 \leq t \leq \vartheta; \\ (\vartheta, 1, \vartheta), & t = \vartheta + 1; \\ (2\vartheta - t + 3, t - \vartheta, 2\vartheta - t + 1), & \vartheta + 2 \leq t \leq 2\vartheta. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (3, \vartheta+2, 3), & t = 1; \\ (t, \vartheta-t+3, t+2), & 2 \leq t \leq \vartheta; \\ (\vartheta+1, 2, \vartheta+1), & t = \vartheta+1; \\ (2\vartheta-t+4, t-\vartheta+1, 2\vartheta-t+2), & \vartheta+2 \leq t \leq 2\vartheta. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta+2, 4), & t = 1; \\ (t+1, \vartheta-t+3, t+3), & 2 \leq t \leq \vartheta-1; \\ (\vartheta+1, 3, \vartheta+2), & t = \vartheta; \\ (\vartheta+2, 3, \vartheta+1), & t = \vartheta+1; \\ (2\vartheta-t+4, t-\vartheta+2, 2\vartheta-t+2), & \vartheta+1 \leq t \leq 2\vartheta-1; \\ (2\vartheta-t+4, t-\vartheta+2, 3), & t = 2\vartheta. \end{cases}$$

We notice that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Gamma_n) \leq 3$. Now, so as to finish the evidence for this case, we show that $\beta(\Gamma_n) \geq 3$ by working out that there does not exist a locating or resolving set $\mathfrak{R}$ with the end goal that $|\mathfrak{R}| = 2$. Despite what might be expected, we guess that $\beta(\Gamma_n) = 2$. At that point, there are indistinguishable prospects from talked about in the Case(I) of Theorem 2 in [14] and inconsistency can be concluded likewise.

In this manner, the above conversation explains that there does not exist a resolving set comprising of two vertices for $\mathbb{V}(\Gamma_n)$ inferring that $\beta(\Gamma_n) = 3$ in this case.

Case(II) When the integer n is odd.
In this case, the integer n can be written as $n = 2\vartheta + 1$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Gamma_n)$. Now, in order to unveil that $\mathfrak{R}$ is a locating (or resolving) set for the radially symmetrical plane graph Γ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Gamma_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta-t+1, t), & 3 \leq t \leq \vartheta \\ (\vartheta, 1, \vartheta-1), & t = \vartheta+2; \\ (2\vartheta-t+3, t-\vartheta-1, 2\vartheta-t+1), & \vartheta+3 \leq t \leq 2\vartheta. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta+1, 2), & t = 1; \\ (t-1, \vartheta-t+2, t+1), & 3 \leq t \leq \vartheta \\ (\vartheta+1, 2, \vartheta), & t = \vartheta+2; \\ (2\vartheta-t+4, t-\vartheta, 2\vartheta-t+2), & \vartheta+3 \leq t \leq 2\vartheta+1. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (3, \vartheta+2, 3), & t = 1; \\ (t, \vartheta-t+3, t+2), & 3 \leq t \leq \vartheta \\ (\vartheta+2, 3, \vartheta+1), & t = \vartheta+2; \\ (2\vartheta-t+5, t-\vartheta+1, 2\vartheta-t+3), & \vartheta+3 \leq t \leq 2\vartheta+1. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta+2, 4), & t = 1; \\ (t+1, \vartheta-t+3, t+3), & 2 \le t \le \vartheta; \\ (\vartheta+2, 3, \vartheta+2), & t = \vartheta+1; \\ (2\vartheta-t+5, t-\vartheta+2, 2\vartheta-t+3), & \vartheta+2 \le t \le 2\vartheta; \\ (2\vartheta-t+5, t-\vartheta+2, 3), & t = 2\vartheta+1. \end{cases}$$

Again we see that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Gamma_n) \le 3$. Now, on expecting that $\beta(\Gamma_n) = 2$, we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently, $\beta(\Gamma_n) = 3$ for this situation too, which concludes the theorem.

□

4. The Plane Graph Σ_n

The radially symmetric plane graph Σ_n was defined in [14], and by $\mathbb{E}(\Sigma_n)$ and $\mathbb{V}(\Sigma_n)$, we signify the arrangement of edges and vertices of the graph Σ_n separately. Consequently, we have

$$\mathbb{V}(\Sigma_n) = \{p_t, r_t, q_t, s_t : 1 \leqslant t \leqslant n\}$$

and

$$\mathbb{E}(\Sigma_n) = \{p_t q_t, q_t r_t, r_t s_t, p_t p_{t+1}, s_t r_{t+1}, r_t r_{t+1} : 1 \leqslant t \leqslant n\}$$

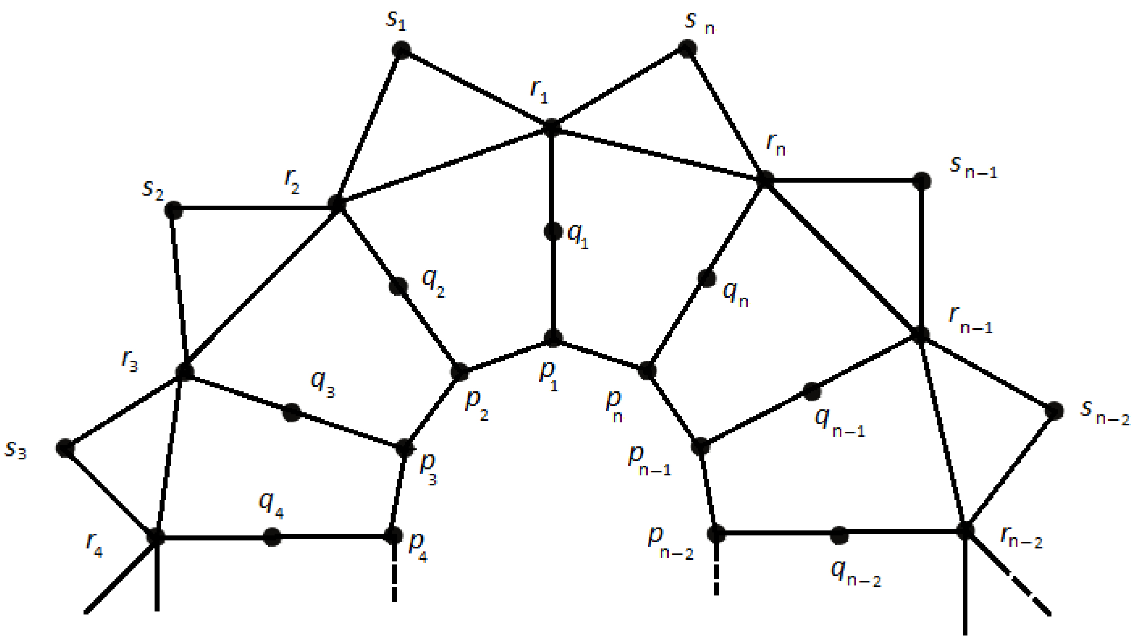

Figure 3: The radially symmetrical graph Σ_n

For our motive, we call the cycle brought forth by the arrangement of vertices $\{p_t : 1 \leqslant t \leqslant n\}$ as the p-cycle, the arrangement of vertices $\{q_t : 1 \leqslant t \leqslant n\}$ as the set of core vertices, and the cycle brought forth by the arrangement of vertices $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ as the rs-cycle. In the accompanying Theorem, we discover that the independent location number of a radially symmetrical graph, Σ_n is 3 i.e., just 3 vertices properly chosen are adequate to determine all the vertices of the radially symmetrical graph Σ_n. Note that the choice of appropriate basis node is the crux of the problem.

Theorem 5. *Let n be a positive integer such that $n \geqslant 6$ and Σ_n be the planar graph on $4n$ vertices as defined above. Then, its independent location number is 3.*

Proof. To demonstrate this, we consider the resulting two cases relying on the positive integer n i.e., when the positive whole number n is even and when it is odd.
Case(I) When the integer n is even.
For this Situation, the integer n can be written as $n = 2\vartheta$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Sigma_n)$. Now, in order to unveil that $\mathfrak{R}$ is a locating or resolving set for the radially symmetrical graph Σ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Sigma_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta - t + 1, t), & 3 \leq t \leq \vartheta \\ (2\vartheta - t + 2, t - \vartheta - 1, 2\vartheta - t), & \vartheta + 2 \leq t \leq 2\vartheta - 1. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta + 1, 2), & t = 1; \\ (t-1, \vartheta - t + 2, t + 1), & 2 \leq t \leq \vartheta; \\ (\vartheta, 1, \vartheta), & t = \vartheta + 1; \\ (2\vartheta - t + 3, t - \vartheta, 2\vartheta - t + 1), & \vartheta + 2 \leq t \leq 2\vartheta. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (3, \vartheta + 2, 3), & t = 1; \\ (t, \vartheta - t + 3, t + 2), & 2 \leq t \leq \vartheta; \\ (\vartheta + 1, 2, \vartheta + 1), & t = \vartheta + 1; \\ (2\vartheta - t + 4, t - \vartheta + 1, 2\vartheta - t + 2), & \vartheta + 2 \leq t \leq 2\vartheta. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta + 2, 4), & t = 1; \\ (t+1, \vartheta - t + 3, t + 3), & 2 \leq t \leq \vartheta - 1; \\ (\vartheta + 1, 3, \vartheta + 2), & t = \vartheta; \\ (\vartheta + 2, 3, \vartheta + 1), & t = \vartheta + 1; \\ (2\vartheta - t + 4, t - \vartheta + 2, 2\vartheta - t + 2), & \vartheta + 1 \leq t \leq 2\vartheta - 1; \\ (2\vartheta - t + 4, t - \vartheta + 2, 3), & t = 2\vartheta. \end{cases}$$

We notice that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Sigma_n) \leq 3$. Now, so as to finish the evidence for this case, we show that $\beta(\Sigma_n) \geq 3$ by working out that there does not exist a locating or resolving set $\mathfrak{R}$ with the end goal that $|\mathfrak{R}| = 2$. Despite what might be expected, we guess that $\beta(\Sigma_n) = 2$. At that point, there are indistinguishable prospects from talked about in the Case(I) of Theorem 3 in [14] and inconsistency can be concluded likewise.

In this manner, the above conversation explains that there does not exist a resolving set comprising of two vertices for $\mathbb{V}(\Sigma_n)$ inferring that $\beta(\Sigma_n) = 3$ in this case.

Case(II) When the integer n is odd.
For this situation, the integer n can be written as $n = 2\vartheta + 1$, where $\vartheta \in \mathbb{N}$ and $\vartheta \geq 3$. Let $\mathfrak{R} = \{p_2, p_{\vartheta+1}, p_n\} \subset \mathbb{V}(\Sigma_n)$. Now, to unveil that $\mathfrak{R}$ is a locating or resolving set for the radially symmetrical graph Σ_n, we consign the metric codes for each vertex of $\mathbb{V}(\Sigma_n) \setminus \mathfrak{R}$ regarding the set $\mathfrak{R}$.

Presently, the metric codes for the vertices of p-cycle $\{p_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(p_t|\mathfrak{R}) = \begin{cases} (1, \vartheta, 1), & t = 1; \\ (t-2, \vartheta - t + 1, t), & 3 \leq t \leq \vartheta \\ (\vartheta, 1, \vartheta - 1), & t = \vartheta + 2; \\ (2\vartheta - t + 3, t - \vartheta - 1, 2\vartheta - t + 1), & \vartheta + 3 \leq t \leq 2\vartheta. \end{cases}$$

The metric codes for the set of core nodes $\{q_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(q_t|\mathfrak{R}) = \begin{cases} (2, \vartheta + 1, 2), & t = 1; \\ (t-1, \vartheta - t + 2, t + 1), & 3 \leq t \leq \vartheta \\ (\vartheta + 1, 2, \vartheta), & t = \vartheta + 2; \\ (2\vartheta - t + 4, t - \vartheta, 2\vartheta - t + 2), & \vartheta + 3 \leq t \leq 2\vartheta + 1. \end{cases}$$

At last, the metric codes for the nodes of rs-cycle $\{r_t : 1 \leqslant t \leqslant n\} \cup \{s_t : 1 \leqslant t \leqslant n\}$ are

$$\zeta(r_t|\mathfrak{R}) = \begin{cases} (3, \vartheta + 2, 3), & t = 1; \\ (t, \vartheta - t + 3, t + 2), & 3 \leq t \leq \vartheta \\ (\vartheta + 2, 3, \vartheta + 1), & t = \vartheta + 2; \\ (2\vartheta - t + 5, t - \vartheta + 1, 2\vartheta - t + 3), & \vartheta + 3 \leq t \leq 2\vartheta + 1. \end{cases}$$

and

$$\zeta(s_t|\mathfrak{R}) = \begin{cases} (3, \vartheta + 2, 4), & t = 1; \\ (t+1, \vartheta - t + 3, t + 3), & 2 \leq t \leq \vartheta; \\ (\vartheta + 2, 3, \vartheta + 2), & t = \vartheta + 1; \\ (2\vartheta - t + 5, t - \vartheta + 2, 2\vartheta - t + 3), & \vartheta + 2 \leq t \leq 2\vartheta; \\ (2\vartheta - t + 5, t - \vartheta + 2, 3), & t = 2\vartheta + 1. \end{cases}$$

Again we see that no two vertices are having indistinguishable metric codes, suggesting that $\beta(\Sigma_n) \leq 3$. Now, on expecting that $\beta(\Sigma_n) = 2$, we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently, $\beta(\Sigma_n) = 3$ for this situation too, which concludes the theorem. □

5. Conclusion

In this paper, we got another family of plane graph viz., Δ_n, which is acquired by the expansion of new edges in a radially symmetrical graph Γ_n [14], and demonstrate that $\mathbb{V}(\Gamma_n) = \mathbb{V}(\Delta_n)$ and $dim(\Gamma_n) = dim(\Delta_n)$. We likewise offer responses to the problems brought up in [14], that is there exist a self-governing (or an independent) locating set for the two classes of the radially symmetrical graphs (viz., Γ_n, and Σ_n). We have furthermore observed that the location number of these three classes of the radially symmetrical graphs viz., Δ_n, Γ_n, and Σ_n, is finite (or limited) and is independent (or free) of the number of vertices of these radially symmetrical graphs, and just 3 vertices sensibly picked are agreeable to choose all the vertices of these classes of radially symmetrical plane graphs. We besides saw that the basis set $\mathfrak{R}$ is self-governing (or independent) for these three radially even families of plane graphs.

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