Second-order work criterion and divergence criterion: a full equivalence for kinematically constrained systems

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NISSUNA UMANA INVESTIGAZIONE SI PUO DIMANDARE VERA SCIENZIA
S'ESSA NON PASSA PER LE MATEMATICHE DIMOSTRAZIONI
LEONARDO DA VINCI

vol. 8 no. 1 2020

MATHEMATICS AND MECHANICS of Complex Systems

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
Vol. 8, No. 1, 2020
dx.doi.org/10.2140/memocs.2020.8.1

SECOND-ORDER WORK CRITERION AND DIVERGENCE CRITERION: A FULL EQUIVALENCE FOR KINEMATICALLY CONSTRAINED SYSTEMS

JEAN LERBET, NOËL CHALLAMEL, FRANÇOIS NICOT AND FELIX DARVE

This paper presents stability results for rate-independent mechanical systems, associated with general tangent stiffness matrices including symmetric and nonsymmetric ones. Conservative and nonconservative as well as associate and nonassociate elastoplastic systems are concerned by such a theoretical study. Hill's stability criterion, also called the second-order work criterion, is here revisited in terms of kinematically constrained systems. For piecewise rate-independent mechanical systems (which may cover inelastic and elastic evolution processes), such a criterion is also a divergence Lyapunov stability criterion for any kinematic autonomous constraints. This result is here extended for systems with nonsymmetric tangent matrices. By virtue of a new type of variational formulation on the possible kinematic constraints, and thanks to the concept of kinematical structural stability (KISS), both criteria, Hill's stability criterion and the divergence stability criterion under kinematic constraints, are shown to be equivalent.

1. Introduction

The aim of this paper is to contribute to close a sixty-year-old debate initiated by Hill [1958; 1959] concerning the stability of rate-independent mechanical systems. In these papers, Hill proposed a new criterion of stability today called the second-order work criterion of stability, which leads to critical values of loading which are not always in accordance with the usual ones calculated from the divergence criterion. Among the concrete examples where the second-order work criterion performs well, one of the most demonstrative is probably the liquefaction of water-saturated loose sands. In situ, liquefaction occurs most often in a saturated sand layer during seismic events. In lab experiments, the so-called "undrained" triaxial loading simulates these in situ conditions well, where the fast seismic loading

Communicated by Angelo Luongo.
MSC2010: 34D30, 37C20.
PACS2010: 46.32.+x.
Keywords: Hill's criterion, kinematic structural stability, nonconservative systems, nonassociate materials.

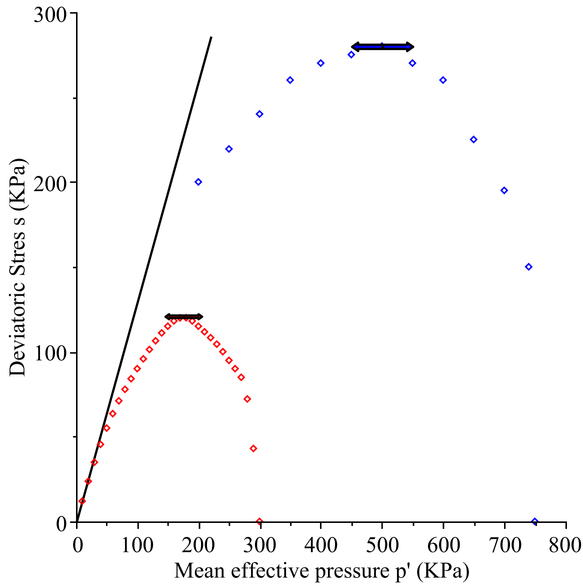

Figure 1. Undrained (isochoric) axisymmetric triaxial compressions of two loose sands plotted in blue and red: stress paths q versus the mean pressure p'. The experimental results correspond to the points, while the Mohr–Coulomb plastic limit condition is given by the straight black line. The q maxima reached along the undrained stress paths clearly occur before the Mohr–Coulomb line (the figure is reconstituted from [Daouadji et al. 2010]). In these experiments, the loading is axially strain-controlled, so q peaks can be passed. If the same loading would have been axially q-controlled, a sudden dynamic instability would have developed at the q peak.

prevents the water drainage of the layer to occur and thus enforces undrained conditions. So, in lab, if the sand is loose enough to be contractant in drained shear (for more details see [Darve 1994; 1996]), the deviatoric stress denoted by s passes through a maximum clearly before reaching the Mohr–Coulomb limit line (see Figure 1).

Thus, if the loading path is s stress-controlled (as in situ by the weight of the above constructions), at the s peak a sudden failure occurs largely before the plastic limit criterion. In this example, the second-order work criterion gives the limit load whereas the usual Mohr–Coulomb condition fails. Then, how can one explain this strong instability? First it is to be noted that the axisymmetric undrained triaxial compression is a mixed stress-strain loading, axially s stress-controlled ($ds =$ constant) with a kinematic constraint given by the isochoric condition related to the undrained conditions ($d\epsilon_1 + 2\,d\epsilon_3 = 0$). Now the second-order work for axisymmetric conditions in stress-strain principal axes can be rewritten as

$$d^2W = d\sigma_1\,d\epsilon_1 + 2\,d\sigma_3\,d\epsilon_3 = ds\,d\epsilon_1 + d\sigma_3\,d\epsilon_V$$

where $d\epsilon_V = d\epsilon_1 + 2\,d\epsilon_3$ is the relative incremental volume variation and $ds = d\sigma_1 - d\sigma_3$ characterizes the incremental deviatoric stress. Thus, for an undrained (isochoric) loading ($d\epsilon_V = 0$), the second-order work d^2W vanishes at the s peak ($ds = 0$) [Darve and Chau 1987]. So it is shown that the second-order work criterion can properly describe sand liquefaction and the explanation of the failure thanks to the second-order work criterion is linked to the kinematic constraint. Indeed, due to the kinematic constraint (constant volume), the failure is not described here by a plastic limit condition (the sand still behaves in a hardening regime) but by an instability condition given by the loss of positive-definiteness of the elastoplastic matrix. This sand liquefaction example contains all the ingredients (second-order work criterion, failure criterion, and kinematic constraint) that are involved in these investigations. However, to highlight the deep relationship between these three ingredients, we will use here a framework allowing us to perform calculations and analytical developments which are not limited to some examples but which allow us to analyze the most complex situations involving any rate-independent materials and systems.

The paper is organized as follows.

We will start in Section 2 with the simple case of a two degree of freedom system that contains all the key ingredients to well understand the foundation of both the problem and the solution as well. In this example, both stability criteria are surprisingly linked in a way which is the key of these developments. In Section 3 we focus on the different concepts of stability involved in this question. Whereas the divergence criterion is linked to the well known Lyapunov point of view on stability, the second-order work criterion can be viewed as the criterion involved in another type of stability we decided to call Hill stability. Hill [1958; 1959] did not define a concept of stability that he projected to use, but he only proposed a criterion of stability. In order to be clear from a logical point of view, we define the Hill stability by the corresponding criterion, this Hill stability definition not being in accordance with the Lyapunov one. Fortunately for the rationality of the approach, one may link this Hill stability to a type of perturbation of the equilibrium (called mixed perturbations) distinct from the perturbations (small purely kinematic perturbations) used to investigate the Lyapunov stability of the equilibrium. To summarize, we then will have on hand two types of stability, the Lyapunov stability and the Hill stability, both leading for quasistatic investigations to the divergence criterion for the first type and to the second-order work criterion for the second type. The main object of this paper is then to propose an explicit equivalence thanks to kinematic constraints.

This equivalence necessitates the investigation of Lyapunov stability of any subsystem obtained by imposing on the initial system additional kinematic constraints. Such investigations lead to the key concept of KISS, to which Section 4 is devoted and which was highlighted in the initial example in the next section. We define quite simply KISS, and we present a brief review of some results that may be related to this concept despite its relatively recent emergence in 2014. We then give its main properties involved in the solution of our problem. In Section 5, we deduce the claimed equivalence between the two criteria in a fully symmetric way which appears as a natural generalization of the well accepted equivalence of both the elastic conservative and the associated elastoplastic cases in the case of symmetric stiffness operators. Finally, in Section 6, we show how the geometric method is performing to provide the appropriate destabilizing constraint by investigating a discrete model of the Leipholz column. The nonconservative part is distributed on each link of the system through follower forces $\vec{P}_i$. This example is a model of the device realized by Bigoni and Noselli [2011]. Unlike the introductory example where the calculations could be performed by hand, the use of the geometric point of view is then essential.

2. An introductory example

Before any general consideration, we start by observing some facts on the well known example of the two degree of freedom Ziegler system Σ. Some of these facts are also well known whereas others are not or very little known.

The system Σ is made up of two bars $S_1 = OA$ and $S_2 = AB$ of the same length ℓ so that the two links of the system are torsion elastic springs at O and A with the same stiffness k (see Figure 2). The load is a follower force $\vec{P} = -P(\overrightarrow{AB}/\|\overrightarrow{AB}\|)$ with $P \geq 0$. The dimensionless expression of the load parameter P is $p = P\ell/k$, and the kinematics of Σ is described by θ_1 and θ_2 as in Figure 2.

We introduce $\Theta = (\theta_1, \theta_2)$, and we identify a couple (θ_1, θ_2) and the corresponding two-dimensional column vector $\binom{\theta_1}{\theta_2}$.

This system is interesting due to the double advantage of its simplicity and of the fact that it has all the characteristics of nonconservative systems. There has been a long debate about the physical meaning of follower forces [Koiter 1996; Elishakoff 2005], but Bigoni and Noselli [2011] showed the applicability of these forces by creating an experimental device illustrating the model of the two degree of freedom Ziegler system Σ used here.

The stiffness matrix at the (unique) equilibrium position $0 = (0, 0)$ reads

$$K(p) = \begin{pmatrix} 2-p & -1+p \\ -1 & 1 \end{pmatrix}$$

and we may note that $K(p)$ is nonsymmetric.

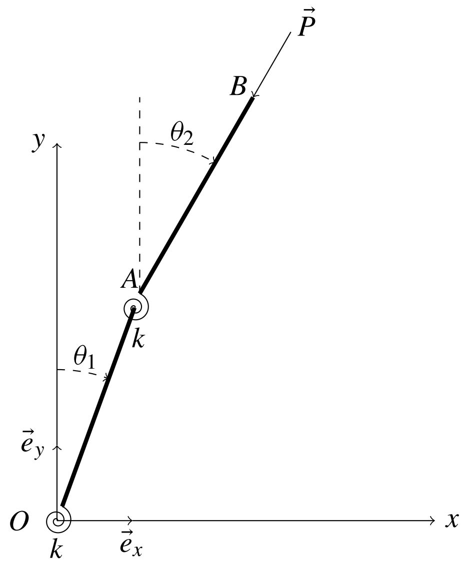

Figure 2. Two degree of freedom Ziegler system.

Referring to the usual framework of the linear Lyapunov stability approach, the divergence-type stability or static stability criterion is investigated thanks to the determinant of $K(p)$. Calculations give $\det(K(p)) = 1$ independently of p. It means that no divergence instability of the equilibrium $0 = (0, 0)$ may occur and that the critical value of divergence stability is $p^*_{\text{div}} = +\infty$. Note that it is usual to conclude that the only way to investigate the Lyapunov linear stability of $0 = (0, 0)$ is to involve inertial terms via the mass matrix. For this system, flutter-type instability occurs for the value $p^*_{\text{fl}} = 2.54$ for a uniform mass repartition.

Whereas Lyapunov stability questions the (dynamic) behavior of the system subjected to the loading of the equilibrium when it is subjected to a purely kinematic perturbation at $t = 0$ (here made up by a quadruplet $(\delta\theta_1, \delta\theta_2, \delta\dot{\theta}_1, \delta\dot{\theta}_2)$), another kind of stability may be defined thanks to the concept of mixed perturbation (see [Absi and Lerbet 2004; Challamel et al. 2009] for example). For the current example, it means that the system can be subjected to a set of perturbations involving a kinematic part related to $\delta\Theta = (\delta\theta_1, \delta\theta_2)$ and a "force" counterpart related to $\delta C = (\delta C_1, \delta C_2)$ where δC_i is any "small" torque acting on S_i for $i = 1, 2$. Both $(\delta\theta_1, \delta\theta_2)$ and $(\delta C_1, \delta C_2)$ must satisfy the fundamental principle of energy conjugation in mechanics: we cannot force together the motion and the mechanical action controlling this motion. This means that $\delta\theta_i$ and δC_i cannot be chosen arbitrarily together.

We do emphasize that, in mixed perturbations, the system is no longer subjected to only the mechanical external given actions involved in the equilibrium whose stability is investigated. For Σ, the system of external forces involved in the (unique) equilibrium position $(0, 0)$ is reduced to $\vec{P}$. For a mixed perturbation, defined by example by $\delta\theta_1$ for the kinematic part and by δC_2 for the force part, a new equilibrium position $(\delta\theta_1, \delta\theta_2) \neq (0, 0)$ is reached when Σ is subjected to $\vec{P}, \delta C_2$. This new equilibrium equation reads

$$\delta C = K(p)\,\delta\Theta \tag{1}$$

where $K(p)$ is the (tangent) stiffness matrix of Σ at the equilibrium position $(0, 0)$ (for the intrinsic geometrical meaning of $K(p)$, see [Lerbet et al. 2018]).

We then define a second type of stability of the equilibrium configuration (here $(0, 0)$), denoted Hill stability of the equilibrium (explanations of this name will appear hereafter), by requiring that the system reaches another equilibrium position close to the one whose stability is investigated when it is subjected to any mixed perturbation. The condition of Hill stability of the equilibrium position then reads (see [Absi and Lerbet 2004; Challamel et al. 2009] for example) $\delta\Theta^T\,\delta C > 0$ for any mixed perturbation. Using (1) it is equivalent to $\delta\Theta^T\,K(p)\,\delta\Theta > 0$ for any $\delta\Theta \neq 0$. Because this expression involves only the symmetric part $K_s(p)$ of $K(p)$, it is equivalent to require that $\delta\Theta^T\,K_s(p)\,\delta\Theta > 0$ for any $\delta\Theta \neq 0$. It is nothing else but the Hill second-order work criterion: the Hill stability of equilibrium is preserved as long as the symmetric part of the stiffness matrix remains positive definite. It is the reason why this type of stability is called the Hill stability. Obviously the framework of [Hill 1958; 1959] is not the same as the one considered in this example. It dealt with associate or nonassociate plasticity, and the object involved in the criterion was the tangent stiffness matrix along an incremental evolution. Nevertheless, the stiffness matrix of this elastic nonconservative system plays the role of the general tangent stiffness matrix of incremental elastoplastic evolutions and captures all the essential ingredients of the problem.

It is however worth noting that the way Hill [1958] introduced his stability criterion — which is now called the second-order work criterion — did not involve explicitly mixed perturbations of equilibrium but emerged from energetic considerations. In [Hill 1958], it is also difficult to identify a clear distinction between his definition of stability and his criterion of stability. Roughly speaking, Hill claimed the condition $\dot{f}\dot{u} \geq 0$ for all $\dot{u}$ where $f = Ku$, which explained the term of second order to characterize this criterion, the term $\dot{f}\dot{u}$ involving two terms of first order. This leads obviously to the same property of positive definiteness for the matrix K. Another approach to derive the second-order work criterion can be found for example in [Nicot et al. 2012a], where the investigation of the infinitesimal variation of the kinetic energy shows that in certain conditions, it is governed by a term $\dot{x}K\dot{x}$. However, in the two last expressions $\dot{f}\dot{u}$ and $\dot{x}^T K\dot{x}$, the dot sign over the vectors does not have the same meaning. In the first one, it means the derivative with respect to the process evolution whereas in the second one, it means the physical time.

Calculations give

$$K_s(p) = \begin{pmatrix} 2-p & -1+p/2 \\ -1+p/2 & 1 \end{pmatrix}.$$

Since the loading path is monotone, the second-order work criterion can also be investigated through the equation $\det(K_s(p)) > 0$, which is equivalent here.

We find $\det(K_s(p)) = 1 - p^2/4$ so that the critical value for the increasing loading path for the Hill stability is $p_H^* = 2$. We then have two thresholds ($p_{\mathrm{div}}^* = +\infty$ and $p_H^* = 2$) for both types of stability, and an important issue since 1958 is to find a sound bridge between them. Note that the universal relation $p_H^* \leq p_{\mathrm{div}}^*$ was well known.

We now investigate another question which will actually be the key to these issues. We propose to investigate the divergence (Lyapunov) stability of all the systems obtained from Σ by adding linear kinematic constraints. Because the system has two degrees of freedom, it can be subjected to only one kinematic constraint, which is a linear relation between the small values $x_1 = \theta_1 - \theta_{e,1}$ and $x_2 = \theta_2 - \theta_{e,2}$ of the deviation of angles θ_1 and θ_2 with respect to the equilibrium configuration $\theta_e = (\theta_{e,1}, \theta_{e,2}) = (0, 0)$. The general form reads $a_1x_1 + a_2x_2 = 0$ with $(a_1, a_2) \neq (0, 0)$. It can be stressed that the isochoric condition related to the undrained conditions as mentioned in the introduction ($d\epsilon_1 + 2d\epsilon_3 = 0$) has exactly this form.

Introducing the Lagrange multiplier λ, the (static) equation system of the constrained system reads

$$\begin{cases} (2-p)x_1 + (p-1)x_2 - \lambda a_1 = 0, \\ -x_1 + x_2 - \lambda a_2 = 0, \\ a_1x_1 + a_2x_2 = 0, \end{cases} \tag{2}$$

whose determinant reads $D = -2a_2(a_2 + a_1)p + a_2^2 + (a_2 + a_1)^2$. The divergence critical value for the constrained system is given by the (minimal positive) value $p_{\mathrm{div}}^*(a_1, a_2)$ that makes D vanish. Elementary calculations give

- for $a_2 = 0$ or $a_1 = -a_2$, $D > 0$ for all p and the corresponding critical value of divergence stability is again $p_{\mathrm{div}}^* = +\infty$, and
- for $a_2 \neq 0$ and $a_1 \neq -a_2$ then the critical value of divergence stability reads

$$p_{\mathrm{div}}^*(\alpha) = \frac{\alpha^2 + 2\alpha + 2}{\alpha + 1}, \tag{3}$$

  corresponding to a one-parameter problem with parameter $\alpha = a_1/a_2$.

A straightforward calculation shows that the minimal positive value of $p^*_{\mathrm{div}}(\alpha)$ is 2 for $\alpha = 0$, namely for the constraint $x_2 = 0$.

Surprisingly, $p = 2$ has then two distinct meanings for the two degree of freedom Ziegler system. On the one hand, it is the critical value for Hill stability of the structure Σ by applying the Hill second-order work criterion; on the other hand it is the minimal critical value regarding the Lyapunov divergence stability but for any constrained subsystem of Σ. Indeed, for $p < 2$, no constrained system can be divergence unstable. This astonishing result is in fact general and is the key result for the equivalence between both stability criteria.

It has to be stressed that this stability analysis does not involve the inertia of the system, namely, for a linear analysis, the mass matrix M. This stability analysis is full in the framework of quasistatic evolutions. This framework is the one of Hill's papers. A dynamic linear stability analysis for this two degree of freedom Ziegler system, like for any mechanical system, needs to investigate flutter-type instability. It occurs, for a homogeneous mass distribution of the two degree of freedom Ziegler system, for the critical value $p^*_{\mathrm{fl}} \approx 2.54$. When this system is subjected to a kinematic constraint, it becomes a one degree of freedom system and the flutter-type instability can no longer occur. In this paper, except for some remarks, the flutter-type instability is neither investigated nor mentioned since we are concerned by the links between the second-order work criterion and the divergence criterion.

3. Two distinct approaches of stability, and respective strengths and weaknesses

As already mentioned, both criteria refer to two distinct points of view of the stability of an equilibrium state. The apparent conflict between the two corresponding criteria should not be a real issue. However, for 60 years, the question has always been tackled in a competitive way. But, as will be shown hereafter, according to an original variational formulation, both kinds of stability will be fully reconciled.

As a usual result of linear algebra gives that $\det A_s \le \det A$ for any real matrix ($A_s = \frac{1}{2}(A + A^T)$ is the symmetric part of A), the Hill stability of any system Σ prevents the Lyapunov divergence instability of Σ. The Hill stability criterion, namely the second-order work criterion, then goes towards safety regarding the stability of the (equilibrium configuration of the) system. In fact originally, both approaches question the equilibrium from two different points of view which are complementary. Referring to the above result and reasoning with the concern of safety, the second-order work criterion should have been universally adopted for the quasistatic evolution of systems.

However, it was not. Several reasons can explain this irrational situation. A first reason is that the Lyapunov point of view of stability is an old, well established, and general framework, largely developed with a lot of deep results involving nonlinear dynamic analysis. Its theorems are applied daily with success in any field of human activity.

On the other hand, even if the Hill criterion of stability is fundamentally (since 1958) a nonlinear criterion used at each step of an incremental loading path, it has provided hitherto no natural extension to dynamics. Some recent papers such as [Nicot et al. 2011; 2012a; 2012b] suggest however that the quadratic part $\dot{X}^T K_s(p)\dot{X}$ whose sign is governed by the Hill second-order work criterion may be involved in a transition from purely incremental quasistatic evolution to a dynamic one. A complete and larger point of view, governed by the safety of structures, could then use

- the Hill criterion for quasistatic evolutions and transitions towards dynamics and
- the Lyapunov dynamic criterion for dynamic evolution.

We however must emphasize that there is no continuous transition between the two criteria. For example, for the two degree of freedom Ziegler system with a uniform mass distribution, the first threshold is $p_H^* = 2$ for the Hill stability whereas $p_{\text{fl}}^* = 2.54$ for the flutter-type stability. It can be proved for this system that no flutter-type instability may occur for $p < 2$ for any continuous mass distribution. However, there exist concentrated mass distributions such that the corresponding flutter-type instability critical value is < 2. The general conclusion is that the Hill approach is especially well founded for quasistatic evolution and Lyapunov stability for dynamic evolution, the transition between the two regimes needing to be more deeply investigated.

Note however that for conservative (and associate elastoplastic) systems, all these considerations are meaningless since the only mode of instability is the divergence-type instability; as the stiffness matrix is symmetric, the divergence stability criterion and second-order work criterion are the same and read

$$\det(K(p)) = \det(K_s(p)) > 0.$$

Finally, another mental habit also inherited from the study of conservative systems is the fact that the stability of a system ensures the stability of any system obtained from the initial system by adding kinematic constraints.

The above example shows that it is not generally right for any mechanical system and it is even a signature of the nonconservativity. This paradoxical fact may be balanced with the other paradox of the destabilizing effect of additional damping, which has been, for its part, very deeply investigated (see for example [Bolotin 1963; Kirillov and Verhulst 2010]). This paradox is called the Hermann or Ziegler paradox according to whether the damping is internal or external. Conversely, to the best of our knowledge very few references deal with the destabilizing effect of additional kinematic constraints. As will be systematically investigated in the next section, the Lyapunov criteria of stability (both divergence-type or flutter-type criteria as well) fail with respect to this property whereas the Hill stability meets this requirement. We will say that the Hill stability criterion is kinematically structurally stable.

To conclude this section, the two approaches have then their own strengths and weaknesses and none should be systematically rejected, especially since the Hill stability criterion emerges as the best criterion offering the divergence stability the required property of kinematic structural stability. It is the purpose of the next section. We first present a brief historical review about this concept. Secondly, we provide the formal definition of the so-called kinematic structural stability (KISS) and finally, we outline its main properties.

4. The KISS

4.1. ***A brief history.*** Since 2010 [Challamel et al. 2010], we have investigated the behavior of nonconservative elastic and nonassociate elastoplastic systems firstly under only one additional kinematic constraint and from 2012 [Lerbet et al. 2012] for any number of kinematic constraints. However, the key concept of kinematic structural stability (KISS) emerged only in 2014. As mentioned in the introduction of the paper, the framework for the current presentation is the linear discrete mechanics even though similar reasoning may be given for a material REV with the corresponding tangent stiffness matrix. From a historic point of view and to the best of our knowledge, only a few papers deal with some issues in relation with KISS. Thompson [1982] noted the paradoxical possibility of destabilizing a nonconservative column with an additional constraint whereas Ingerle [2018] computed approximate loadings that may destabilize nonconservative columns by investigating some special constrained systems. More accurately, for the continuous Beck column, the dimensionless divergence stability value for any kinematic constraint has been calculated in [Lerbet et al. 2017] and is equal to π^2. This value had been empirically obtained by Ingerle [2013] from a discrete approach using numerical arguments.

It is worth mentioning that the stability limit under kinematic constraints is the generalization of the one under some specific constraints especially applied to the boundaries of the system. For instance, Ingerle [1969] found a dimensionless divergence buckling load $p = 20.19$ in the presence of a specific constraint applied to the end of the column (the application point of the follower load), whereas the free Beck column admits a flutter instability value of 20.05, as calculated by Beck [1952] (see also [El Naschie 1976; 1977] for this result). However, consider that any kinematic constraint reduces this value to π^2 as mentioned above as was observed by Ingerle [2013] and as has been definitively proved by Lerbet et al. [2017].

4.2. ***General framework.*** In the chosen framework, the kinematics of the holonomic system Σ is described by a Lagrange coordinate system $q = (q_1, \dots, q_n)$, where $\tilde{q}$ is the current equilibrium configuration whose stability is investigated, and $p = (p_1, \dots, p_m)$ is a family of loading parameters. A loading path Λ_p is a one-dimensional curve in the loading parameter space $\mathcal{P} = \{p_k \geq 0 \mid k = 1, \dots, m\}$. This curve is given by $\sigma \in [0, \infty[\mapsto p(\sigma) = (p_1(\sigma), \dots, p_m(\sigma)) \in \mathcal{P}$. We suppose that $p(0) = 0 \in \mathcal{P}$ and that, for the unloaded system (namely for $p(0) = 0$), $K_s(0)$ is positive definite. The stiffness matrix at $\tilde{q}$ is then a function $K(p)$ of $p = (p_1, \dots, p_m)$. $X = q - \tilde{q}$ is the vector giving the infinitesimal or incremental (according to the point of view) response of the structure. For a complete nonlinear description with the use of differential geometry, see [Lerbet et al. 2018].

When any set $\mathcal{C} = \{C_1, \dots, C_r\}$ of r kinematic constraints is acting on Σ, we denote by $\Sigma_{\mathcal{C}}$ the corresponding kinematic constrained system and we say that $\Sigma_{\mathcal{C}}$ is r-constrained if $C_1, \dots, C_r$ viewed as vectors of $\mathbb{R}^n$ are linearly independent. That also means that the constraint corresponding to C_i reads $X^T C_i = 0$. We also suppose that the equilibrium $\tilde{q}$ is not perturbed by the additional kinematic constraints. The case where the equilibrium position is changed with the kinematic constraints is a full different mechanical problem (in this case, see for example [Tarnai 2004]) which is not investigated here. For the Hill stability, we denote by $D_H \subset \mathcal{P}$ the stability domain and $\Gamma_H = \partial D_H \subset \mathcal{P}$ is the corresponding critical domain for the system Σ. For the divergence stability, we denote by $D_{\text{div}} \subset \mathcal{P}$ the divergence stability domain and $\Gamma_{\text{div}} = \partial D_{\text{div}} \subset \mathcal{P}$ is the corresponding critical domain for the system Σ. For any constrained system $\Sigma_{\mathcal{C}}$, the corresponding domains are denoted $D_{H,\mathcal{C}}$ and $D_{\text{div},\mathcal{C}}$.

Remarks. (1) The (second-order work) criterion of Hill stability is that $K_s(p)$ is positive definite. For elastoplastic materials, this domain is not intrinsic, namely it is path-dependent and can be defined only for each loading path Λ_p. Then, D_H is investigated through a priori an infinite number of one-dimensional (along each loading path Λ_p) analyses. At the other extreme, like in the introductory example, for elastic nonconservative systems with only one loading parameter, D_H is a simple interval of $\mathbb{R}_+$ and $\Gamma_H = \partial D_H$ is reduced to a point. For the two degree of freedom Ziegler system, $D_H = [0, 2[$ and $\Gamma_H = \{2\}$.

(2) Because of the continuity of involved applications and since $\det(K_s(0)) > 0$, the boundary $\Gamma_H = \partial D_H \subset \mathcal{P}$ of critical values for Hill stability can also be found by solving the equation $\det(K_s(p)) = 0$ which defines a nonconnected hypersurface in $\mathcal{P}$, one of whose components is $\Gamma_H = \partial D_H$.

(3) When the D_H is not path-dependent, for example for any elastic nonconservative system like for the discrete Leipholz column investigated in Section 6, some general properties for D_H can be underlined. D_H is an open set of $\mathcal{P}$. To give more properties about D_H, we need to know the dependency $p \mapsto K_s(p)$. For example, if the dependency is linear like for the discrete Leipholz column, the parametrization $p \mapsto K_s(p)$ defines an affine set S of $\mathcal{M}_n(\mathbb{R})$. Then, since the set $\mathcal{S}_n^{++}(\mathbb{R})$ of symmetric definite matrices is a convex semicone of $\mathcal{M}_n(\mathbb{R})$, $S \cap \mathcal{S}_n^{++}(\mathbb{R})$ is a convex set of $\mathcal{M}_n(\mathbb{R})$. Because the inverse of $p \mapsto K_s(p)$ from S to $\mathcal{P}$ is again an affine map, D_H is then a convex set of the loading space $\mathcal{P}$.

(4) If $p \in D_H$, the isotropic cone $C_0(p) = \{X \in \mathcal{M}_{n1}(\mathbb{R}) = \mathbb{R}^n \mid X^T K_s(p) X = 0\}$ is reduced to the zero vector of $\mathbb{R}^n$. For $p^* \in \partial D_H$, the isotropic cone $C_0(p^*) = \ker K_s(p^*)$ is always a vector space and generally a one-dimensional space. Considering $p^* = p(\sigma^*)$ belongs to a loading path Λ_p, then for $\sigma > \sigma^*$ in the vicinity of σ^*, $C_0(p) = \{X \in \mathcal{M}_{n1}(\mathbb{R}) = \mathbb{R}^n \mid X^T K_s(p) X = 0\}$ is a cone no longer reduced to a vector space. We then put $C_-(p) = \{X \in \mathcal{M}_{n1}(\mathbb{R}) = \mathbb{R}^n \mid X^T K_s(p) X \leq 0\}$ and $C_+(p) = \{X \in \mathcal{M}_{n1}(\mathbb{R}) = \mathbb{R}^n \mid X^T K_s(p) X > 0\}$.

(5) From the general statement $\det A_s \leq \det A$ it follows that $D_H \subset D_{\text{div}}$ and, in the present investigations, $D_H \subsetneqq D_{\text{div}}$.

(6) A characteristic property of elastic conservative or associate elastoplastic systems is that $D_H = D_{\text{div}}$.

(7) A characteristic property of elastic conservative or associate elastoplastic systems is that $D_{\text{div}} = D_{\text{div},\mathscr{C}}$ for any constraint system $\mathscr{C}$.

(8) In general for a system there is family of constraints $\mathscr{C}$ such that $D_{\text{div}} \subsetneqq D_{\text{div},\mathscr{C}}$.

(9) A characteristic property of the second-order Hill criterion is that $D_H \subset D_{H,\mathscr{C}}$ for any constraint system $\mathscr{C}$.

The well known properties (7) and (9) may be deduced from the Rayleigh quotient.

4.3. ***The kinematic structural stability*** **(*KISS*).** KISS refers to the behavior of the stability of the equilibrium positions when the system is subjected to additional kinematic constraints. According to path-independent or path-dependent stability domains, the definitions can be defined globally on $\mathcal{P}$ or defined for each loading path Λ_p. In order to apply this definition to the case of the discrete Leipholz column investigated in Section 6, we present both definitions:

Definition (intrinsic aspects). Suppose the system is hypoelastic such that the stability issue is intrinsic.

- The KISS is said to be universal (for the corresponding criterion and equilibrium) if and only if $D \subset D_{\mathscr{C}}$ for all $\mathscr{C}$.

- The KISS is said to be conditional (for the corresponding criterion and equilibrium) if and only if there is $D_{\mathrm{co}} \subsetneq D$ such that $D_{\mathrm{co}} \subset D_{\mathscr{C}}$ for all $\mathscr{C}$. This notation also means that there is at least a value $p^*_{\mathrm{co}} \in \partial D_{\mathrm{co}} \cap D$ and a constraint set $\mathscr{C}^*$ with $p^*_{\mathrm{co}} = p^*_{\mathscr{C}^*}$

Definition (path-dependent aspects). Let

$$\Lambda_p : \sigma \in [0, \infty[\mapsto p(\sigma) = (p_1(\sigma), \ldots, p_m(\sigma)) \in \mathscr{P}$$

be a loading path drawn in $\mathscr{P}$. The critical load for the involved stability criterion (Hill, divergence-type, or flutter-type) is supposed reached on this loading path Λ_p for $p^* = p(\sigma^*)$ (which can be infinite). Then

- the KISS is said to be universal (for the corresponding criterion and equilibrium) if and only if $p^* \leq p^*_{\mathscr{C}}$ for all $\mathscr{C}$ and
- the KISS is said to be conditional (for the corresponding criterion and equilibrium) if and only if there is $p^*_{\mathrm{co}} = p(\sigma^*_{\mathrm{co}}) < p^*$ such that $p^*_{\mathrm{co}} \leq p^*_{\mathscr{C}}$ for all $\mathscr{C}$. This notation also means that σ^*_{co} is optimal (it corresponds to the minimal value of the parameter σ with this property) and there is then a constraint set $\mathscr{C}^*$ with $p^*_{\mathrm{co}} = p^*_{\mathscr{C}^*}$

The rest of the reasoning is given with the path-dependent formalism. It means that a loading path Λ_p is fixed.

As a consequence, when the KISS is conditional, there is an appropriate set of constraints $\mathscr{C}^*$ such that $p^*_{\mathscr{C}^*} < p^*$, namely making the constrained system $\Sigma^*_{\mathscr{C}}$ unstable whereas Σ is still stable: it is the paradoxical destabilizing effect of additional kinematic constraints.

In order to highlight the link between the KISS and the second-order work criterion, the way to tackle constrained mechanical systems should be commented on further. In the example investigated in Section 2, we used, as usual, Lagrange multipliers. It leads to a problem of larger size ($2+1=3$ for the example) whereas the constrained system has a lower degree of freedom ($2-1=1$ for the example). During our investigations, it appeared that a better way to systematically tackle constrained systems consists of using so-called compressions of operators. It provides objects not only appropriate to physical systems (for example a matrix of size r for an r degree of freedom mechanical system) but also applicable to various other situations such as constrained continuous media involving infinite-dimensional spaces (see [Lerbet et al. 2017] for its use leading to the complete solution of the constrained Beck column). The formal definition of the compression of a linear map reads:

Definition. Let $u \in \mathscr{L}(E)$ be a linear map of a euclidean space E and F a vector subspace of E. The compression u_F of u on F is the element of $u \in \mathscr{L}(F)$ defined by $u_F = p_F \circ u \circ i_F$ where $i_F : F \to E$ is the canonical injection map from F to E and $p_F : E \to F$ is the orthogonal projection on F. In other words, $u_F = p_F \circ u_{|F}$ where $u_{|F}$ is the usual restriction of u to F.

The applicability of this concept is justified by:

Proposition 1. *Let Σ be a mechanical system. Suppose a mechanical property of Σ is described in a linear framework by a linear map u of $\mathbb{R}^n$. The same property for a constrained system $\Sigma_{\mathscr{C}}$ is described by the compression $u_{F_{\mathscr{C}}^{\perp}}$ of u on $F_{\mathscr{C}}^{\perp}$ where $F_{\mathscr{C}}$ is the space spanned by the vectors $C_1, \dots, C_r$ of $\mathbb{R}^n$ defining the set of constraints $\mathscr{C}$.*

Thanks to this concept of compression, the KISS issue for the divergence instability criterion may be reformulated through the following purely geometric approach.

Let Σ be a mechanical system and $u(p)$ the linear map of $\mathbb{R}^n$ associated with the stiffness matrix $K(p)$. The divergence stability of Σ means the invertibility of $K(p)$, namely the one-to-one property of $u(p)$. The KISS issue means

(1) find a threshold p^*_{co} such that all the compressions u_F are still one-to-one for all $p < p^*_{\mathrm{co}}$ and for all subspaces F of $\mathbb{R}^n$ and

(2) as $p = p^*_{\mathrm{co}}$, find a subspace F^* of $\mathbb{R}^n$ such that u_{F^*} is no longer one-to-one. The corresponding set of critical constraints $\mathscr{C}^*$ will be a generator system of the orthogonal $(F^*)^{\perp}$ of F^*.

The solution of these issues is contained in:

Theorem 2. *Let $u \in \mathscr{L}(E)$ be an one-to-one linear map of a euclidean space E. All these compressions on (strict) subspaces are still one-to-one if and only if the symmetric part u_s of u is definite. As u_s loses its definiteness, one may build a compression on a hyperplane F^* of E for example such that u_{F^*} is not one-to-one. More specifically, if $x^* \in E$ is a nonzero vector on the isotropic cone of u_s, one may choose $F^* = \langle u(x^*)\rangle^{\perp}$ ($u(x^*) \neq 0$ because u is supposed to be one-to-one).*

This result extends to Hilbert spaces with compressions to closed spaces. It has been used in [Lerbet et al. 2017].

Geometric proof. Suppose first that u_s ceases to be definite for a value p^* of the parameter. Then, there is $x^* \neq 0$ such that $u_s(x^*) = 0$. Let $F^* = \langle u(x^*)\rangle^{\perp}$, which is a hyperplane because $u(x^*) \neq 0$. Then $F^* = \ker c^*$ with c^* a linear form (namely the appropriate constraint) $c^*(y) = (u(x^*) \mid y)$. We have to prove that the constrained system is divergence unstable or equivalently that the compression u_{F^*} of u on F^* is not one-to-one. But $u_s(x^*) = 0$ implies $(u(x^*) \mid x^*) = (u_s(x^*) \mid x^*) = (0 \mid x^*) = 0$, which proves that $x^* \in F^*$.

Thus, $(u_{F^*}(x^*) \mid y) = (p_{F^*} \circ u(x^*) \mid y) = (u(x^*) \mid p_{F^*}(y)) = (u(x^*) \mid y) = 0$ for all $y \in F^*$, which means that $u_{F^*}(x^*)$ is not one-to-one as an endomorphism of F^*.

Reciprocally, suppose now that there is a compression u_{G^*} which is not one-to-one with G^* a subspace of E. There is $x^* \in G^*$, $x^* \neq 0$, such that $u_{G^*}(x^*) = 0$ or equivalently such that $(p_{G^*} \circ u(x^*) \mid y) = (u(x^*) \mid y) = 0$ for all $y \in G^*$. Applying this relation for $y = x^*$ shows that u_s is no longer definite. Moreover, we also deduce that $G^* \subset F^* = \langle u(x^*)\rangle^{\perp}$, which proves that only one constraint is sufficient to investigate the question. □

To be closer to the usual language of mechanics, we now propose a direct definition of the compression for the constrained system $\Sigma_{\mathscr{C}}$ of the stiffness matrix $K(p)$ of the system Σ. Moreover, since the theorem shows that the variational formulation with only one constraint is necessary and sufficient, we suppose that $\mathscr{C}$ is reduced to one constraint C which reads $C^T X = 0$. We put $F_C = \{X \in \mathbb{R}^n \mid C^T X = 0\}$. It is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ identified with the vector space of n-dimensional column matrices. F_C describes the kinematics of the constrained system Σ_C. We then provide:

Definition. Let $\mathscr{B}_C$ be an orthonormal basis of F_C. The compression $K_{\mathscr{B}_C}(p)$ of $K(p)$ on the kinematic space F_C of the constrained system Σ_C in $\mathscr{B}_C$ is the $n-1$ square matrix defined by

$$X^T_{\mathscr{B}_C} K_{\mathscr{B}_C}(p) Y_{\mathscr{B}_C} = X^T K(p) Y \quad \text{for all } X, Y \in F_C$$

where $X_{\mathscr{B}_C}, Y_{\mathscr{B}_C}$ are the coordinate column vectors of $X, Y \in F_C$ in the basis $\mathscr{B}_C$. So, obviously the (compressed) stiffness matrix $K_{\mathscr{B}_C}(p)$ of the constrained system $\Sigma_{\mathscr{C}}$ depends on $\mathscr{B}_C$ but not its invertibility nor its determinant, which depend only on F_C itself.

The main theorem then reads:

Theorem 3. *Suppose we have an increasing loading parameter p starting from* 0 *with* $\det K_s(0) > 0$. *As long as* $\det K_s(p) > 0$, *there is no kinematic constraint C destabilizing by divergence the corresponding constrained system*: $\det K_C(p) \neq 0$ *for all constraints C. As soon as $p = p^*$ such that* $\det K_s(p^*) = 0$ (*Hill's second-order work criterion failure*), *then there is a divergence destabilizing constraint C^** (*namely* $\det K_{C^*}(p^*) = 0$) *and the kinematic constraint C^* is explicit.*

Proof. We only prove the construction of the destabilizing constraint.

Thus, suppose that $\det K_s(p^*) = 0$ (failure of the second-order work criterion for Σ) and $\det K(p^*) \neq 0$ (divergence stability of Σ). Let $X^* \neq 0$ be in $\ker K_s(p^*)$ or equivalently on the isotropic cone of $K_s(p^*)$, and let us put $C^* = K(p^*)X^*$ so that $X^{*T} C^* = X^{*T} K(p^*) X^* = X^{*T} K_s(p^*) X^* = 0$. But, since $K(p^*)$ is invertible, then $C^* \neq 0$ and since $X^{*T} C^* = C^{*T} X^* = 0$, then $X^* \in F_{C^*} = \{Z \mid Z^T C^* = 0\}$. Moreover, $X^* \neq 0$ implies that $X^*_{\mathscr{B}_{C^*}} \neq 0$ in any orthonormal basis $\mathscr{B}_{C^*}$ of F_{C^*}.

But if $Y \in F_{C^*}$, then by a similar calculation, $0 = Y^T C^* = Y^T K(p^*) X^* = Y^T_{\mathscr{B}_{C^*}} K_{\mathscr{B}_{C^*}}(p^*) X^{*T}_{\mathscr{B}_{C^*}}$, which means that $K_{\mathscr{B}_{C^*}}(p^*) X^{*T}_{\mathscr{B}_{C^*}}$ is (the column vector of the coordinates in $\mathscr{B}_{C^*}$ of) a vector of F_{C^*} orthogonal to any vector of F_{C^*}. It is then the zero vector of F_{C^*}, which means that $K_{\mathscr{B}_{C^*}}(p^*) X^{*T}_{\mathscr{B}_{C^*}} = 0$. But, since $X^* \neq 0$, then $X^{*T}_{\mathscr{B}_{C^*}} \neq 0$ and then $K_{\mathscr{B}_{C^*}}(p^*)$ is not invertible, which implies the divergence instability of the constrained system Σ_{C^*}. □

Remark. We find nowhere in the literature neither the result nor its proof.

This theorem explicitly solves the two KISS issues. First it gives the equivalence between the divergence KISS and the Hill second-order work criterion: $p^*_{\text{co}} = p^*_H$. It also shows that the case of constrained systems with only one constraint (one-constrained system according to the above definition) is sufficient to investigate the KISS issues. This fact had been proved in [Lerbet et al. 2012] with the language of Lagrange multipliers thanks to the concept of r-definite matrices. A comparison between both approaches shows that the language of compressions greatly simplifies the problem. Finally, this above theorem also gives a constructive way to find the destabilizing constraint, because thanks to this theorem, the destabilizing constraint is given by the vector $K(p^*_H)X^*$ where X^* is any nonzero vector on the isotropic cone of $K_s(p^*_H)$. Before highlighting in the next section, thanks to this result, the announced full equivalence between divergence Lyapunov stability and Hill stability, let us conclude this current section by noting that the KISS issue has been investigated for various frameworks like flutter-type instability [Lerbet et al. 2016b], divergence-type instabilities for continuum mechanics [Lerbet et al. 2017], and instabilities of nonlinear incremental discrete mechanics [Lerbet et al. 2018].

5. Equivalence of the two criteria via an original variational approach

We now tackle the claimed equivalence regarding the Lyapunov divergence stability criterion and the Hill second-order work criterion. According to the previous section, we are led to investigate this question in terms of the variational formulation on all the possible kinematic constraints $\mathscr{C}$ that may be applied on the system Σ, keeping in mind that this large variational formulation may be reduced to families built by only one constraint, namely for one-constrained systems $\Sigma_{\mathscr{C}}$ as has been defined above. We first tackle the question of the kinematic structural stability of the Hill criterion itself.

A usual result of (bi)linear algebra — also called Sylvester's conditions for symmetric positive definite matrices — means in terms of compressions that all the compressions of a symmetric positive definite map are also positive definite. It is the exact characterization of the kinematic structural stability of the Hill second-order work criterion: if a mechanical system Σ is Hill stable, any constrained system is still Hill stable.

All above results of this paper may be summarized (for a monotone load path) as

(1) $\det K_s(p) \le \det K(p)$, that is, the Lyapunov divergence instability of Σ leads to the Hill instability of Σ,

(2) the Hill instability of Σ by loss of definiteness of $K_s(p)$ leads to the existence of a set of constraints $\mathscr{C}$ such $\Sigma_{\mathscr{C}}$ is not Lyapunov divergence stable, and

(3) the Hill stability of Σ is equivalent to the Hill stability of $\Sigma_{\mathscr{C}}$ for any set of constraints $\mathscr{C}$.

These three results allow us to formulate the first statement for any rate-independent mechanical system Σ

$$\boxed{\begin{array}{l}\text{Hill stability (s.o.w. criterion) of } \Sigma \\ \qquad\qquad \iff \text{Lyapunov divergence stability of } \Sigma_{\mathscr{C}} \quad \text{for all } \mathscr{C}\end{array}}$$

and finally to conclude with the full and symmetric equivalence

$$\boxed{\begin{array}{l}\text{Hill stability (s.o.w. criterion) of } \Sigma_{\mathscr{C}} \quad \text{for all } \mathscr{C} \\ \qquad\qquad \iff \text{Lyapunov divergence stability of } \Sigma_{\mathscr{C}} \quad \text{for all } \mathscr{C}\end{array}}$$

which, due to the remark on the compressions on hyperplanes, is also equivalent to

$$\boxed{\begin{array}{l}\text{Hill stability (s.o.w. criterion)} \quad \text{for all one-constrained } \Sigma_{\mathscr{C}} \\ \qquad \iff \text{Lyapunov divergence stability} \quad \text{for all one-constrained } \Sigma_{\mathscr{C}}.\end{array}}$$

These last two equivalences may be interestingly compared with the usual statement valid for any conservative or associate elastoplastic system Σ:

$$\boxed{\text{Hill stability (s.o.w. criterion) of } \Sigma \iff \text{Lyapunov divergence stability of } \Sigma.}$$

6. The discrete Leipholz column

The system Σ_n consists of n bars $OA_1, A_1A_2, \dots, A_{n-1}A_n$ with $OA_1 = A_1A_2 = \cdots = A_{n-1}A_n = h$ linked with n elastic springs with the same stiffness k. Adopting the same device at the end of each bar of Σ leads to a family of follower forces $\vec{P}_1, \dots, \vec{P}_n$. Figure 3 illustrates the case $n = 3$.

The pure follower forces $\vec{P}_1, \vec{P}_2, \dots, \vec{P}_n$ are applied at the ends of OA_1, A_1A_2, $\dots$, $A_{n-1}A_n$. The equilibrium position is $\theta = (\theta_1, \theta_2, \dots, \theta_n) = (0, 0, \dots, 0)$.

Adopting a dimensionless format, $p_i = \|\vec{P}_i\| h/k$ for $i = 1, \dots, n$ are used as loading parameters. The corresponding physical system has been realized and described in [Bigoni and Noselli 2011]. For our concern, this system is interesting because of its nonreduced geometric degree of nonconservativity d (see [Lerbet et al. 2014; 2016a] for this concept). Roughly speaking, the geometric degree of nonconservativity d of a system is the minimal number of kinematic constraints necessary to make the system conservative. Indeed, unlike Ziegler systems, whose

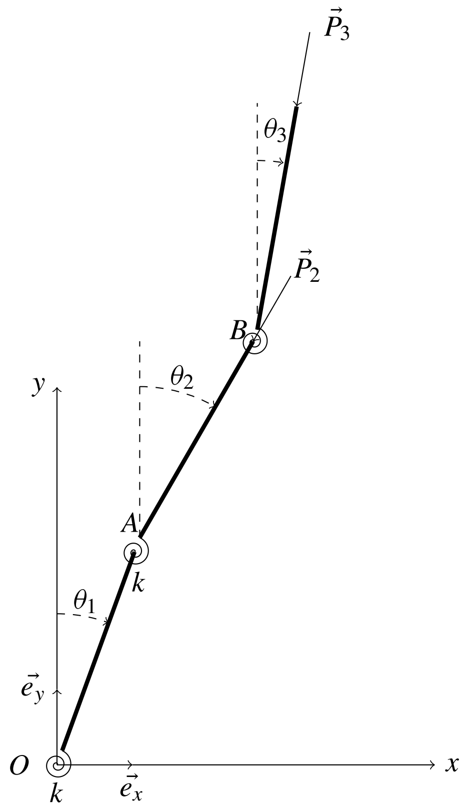

Figure 3. Three degree of freedom discrete Leipholz system.

geometric degree of nonconservativity is always equal to one whatever the degree of freedom is, the geometric degree of nonconservativity of the discrete Leipholz column is increasing as $\lfloor n/2 \rfloor$, the integer part of $n/2$.

The stiffness matrix K reads $K(p) = K(p_1, p_2, \ldots, p_n)$:

$$K(p) = \begin{pmatrix} Q_2 & -1+p_2 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -1 & Q_3 & -1+p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ 0 & -1 & Q_4 & -1+p_4 & p_5 & \cdots & p_{n-1} & p_n \\ 0 & 0 & -1 & Q_5 & -1+p_5 & \cdots & p_{n-1} & p_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1+p_{n-1} & p_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2-p_n & -1+p_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

where $Q_j = 2 - \sum_{i=j}^{n} p_i$.

We see that K does not depend on p_1, which is obvious from a mechanical point of view. We then may suppose that $p_1 = 0$.

A remarkable property of $K(p_1, p_2, \ldots, p_n)$ is that its determinant does not depend on p:

$$\det K(p_1, p_2, \ldots, p_n) = 1 \quad \text{for all } p_1, p_2, \ldots, p_n.$$

That may be proved by applying $n-1$ times from $k = n$ to $k = 2$ (in this order) the same rule: the column C_{k-1} of the matrix at the step number k is replaced by $C_{k-1} + C_k$. At the end of the process, the determinant of the matrix is unchanged but the matrix is then upper-triangular with a diagonal of 1 and then its determinant is 1.

We may deduce that the condition "$K(p)$ invertible" of the main theorem (Theorem 2) holds without any condition on the loading parameters p_i, $i = 1, \ldots, n$. It also means that $D_{\text{div}} = \mathcal{P}$.

The symmetric part of K then reads

$$K_s(p) = \begin{pmatrix} Q_2 & -1+R_2 & R_3 & R_4 & R_5 & \cdots & R_{n-1} & R_n \\ -1+R_2 & Q_3 & -1+R_3 & R_4 & R_5 & \cdots & R_{n-1} & R_n \\ R_3 & -1+R_3 & Q_4 & -1+R_4 & R_5 & \cdots & R_{n-1} & R_n \\ R_4 & R_3 & -1+R_4 & Q_5 & -1+R_5 & \cdots & R_{n-1} & R_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{n-2} & R_{n-2} & R_{n-2} & R_{n-2} & R_{n-2} & \cdots & -1+R_{n_1} & R_n \\ R_{n-1} & R_{n-1} & R_{n-1} & R_{n-1} & R_{n-1} & \cdots & 2-p_n & -1+R_n \\ R_n & R_n & R_n & R_n & R_n & \cdots & -1+R_n & 1 \end{pmatrix}$$

where $R_j = p_j/2$. Then $p \mapsto K_s(p)$ is an affine map. So, D_H is a convex set of $\mathcal{P}$, but there is no chance to find a general formula for $\det(K_s(p))$. Thus, to make the computations analytically, we must fix a value of n. We investigate successively $n = 3$ and $n = 4$.

Case $n = 3$. The stiffness matrix then reads

$$K(p_2, p_3) = \begin{pmatrix} 2-(p_2+p_3) & -1+p_2 & p_3 \\ -1 & 2-p_3 & -1+p_3 \\ 0 & -1 & 1 \end{pmatrix}$$

and its symmetric part $K_s(p_2, p_3)$

$$K_s(p_2, p_3) = \begin{pmatrix} 2-(p_2+p_3) & -1+p_2/2 & p_3/2 \\ -1+p_2/2 & 2-p_3 & -1+p_3/2 \\ p_3/2 & -1+p_3/2 & 1 \end{pmatrix}.$$

The domain of investigation lies in the quadrant $\mathcal{P} = (p_2 \geq 0,\ p_3 \geq 0)$. As usual we suppose that any loading path starts from $(0, 0)$. The convex domain D_H of Hill

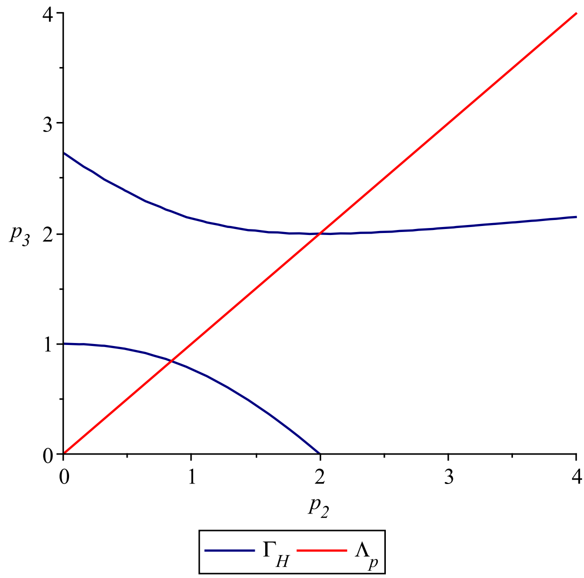

Figure 4. Hill stability domain D_H of the three degree of freedom discrete Leipholz column (D_{div} is the whole quadrant ($p_2 \geq 0$, $p_3 \geq 0$)). Inside D_H, $\det K_s(p) > 0$. On $\partial D_H = \Gamma_H$, for example for $p^* = ((1+\sqrt{33})/8, (1+\sqrt{33})/8) = \Gamma_H \cap \Lambda_p$, $\det K_s(p) = 0$. For $p_d = \left(\frac{3}{2}, \frac{3}{2}\right)$, $\det K_s(p) < 0$.

stability is delimited by the blue curve Γ_H in the vicinity of (0, 0) (see Figure 4). D_H is given by the set of inequalities

$$\left\{0 \leq p_2 \leq 2,\ 0 \leq p_3 \leq 1,\ 1 - \tfrac{3}{2} p_3^2 + \tfrac{1}{2} p_2 p_3^2 + \tfrac{1}{2} p_3^3 - \tfrac{1}{4} p_2^2 - \tfrac{1}{2} p_2 p_3 \geq 0\right\},$$

and the explicit equation of the curve $\Gamma_H = \partial D_H$ delimiting the domain D_H is

$$p_2 = f(p_3) = p_3^2 - p_3 + \sqrt{p_3^4 - 5p_3^2 + 4}.$$

As proved in the general case (see above point (3) on page 12)), D_H is a convex set of the quadrant which can be directly checked by calculating the second derivative of f, which is always negative for $p_3 \in [0, 1]$. The main result (Theorem 2) means that, inside D_H defined by the second-order work criterion, there is no kinematic constraint that destabilizes the system Σ_3. On the contrary, on any $p^* \in \partial D_H = \Gamma_H$, there is a constraint $\mathscr{C}^* = \{c^*\}$ such that the constrained system $\Sigma_{3,\mathscr{C}^*}$ is Lyapunov divergence unstable.

The case $p_3 = 0$ corresponds to the introductory example whereas the case $p_2 = 0$ has been investigated in [Lerbet et al. 2012]. A loading path is determined by a curve Λ_p in the quadrant $\mathscr{P} = (p_2 \geq 0,\ p_3 \geq 0)$ and starting from (0, 0).

Suppose for example that the loading path is given by the curve $\Lambda_p : p_2 = p_3$ or $\sigma \mapsto p(\sigma) = (\sigma, \sigma)$ plotted in red in the Figure 4. It is in accordance with Bigoni's device [Bigoni and Noselli 2011] where the friction forces are the same at each joint and because of the same velocity of the support which induces equal force friction at each joint. The line $\Lambda_p : p_2 = p_3$ intersects the curve Γ_H at the point $p^* = (p_2^*, p_3^*) = ((1+\sqrt{33})/8, (1+\sqrt{33})/8) = \Gamma_H \cap \Lambda_p$. For this critical value, the isotropic cone of $K_s(p^*)$ is no longer reduced to 0 but is reduced to one single direction. It is a vector space generated by the vector

$$X^* = \begin{pmatrix} 1986 - 350\sqrt{33} \\ 3678 - 642\sqrt{33} \\ 5370 - 934\sqrt{33} \end{pmatrix}.$$

Thus, applying the main theorem, the next step consists of calculating

$$K(X^*) = \begin{pmatrix} -2685 + 467\sqrt{33} \\ -993 + 175\sqrt{33} \\ 1692 - 292\sqrt{33} \end{pmatrix},$$

which means that the system Σ_3 subjected to the kinematic constraint

$$\begin{aligned} &c^*(x_1, x_2, x_3) \\ &\qquad = (-2685 + 467\sqrt{33})x_1 + (-993 + 175\sqrt{33})x_2 + (1692 - 292\sqrt{33})x_3 = 0 \end{aligned}$$

is divergence unstable when subjected to the load $p^* = ((1+\sqrt{33})/8, (1+\sqrt{33})/8)$.

Remarks. (1) If we would like to naively proceed as in the introductory example in order to investigate the divergence stability of the constrained systems, instead of searching for the maximum of one real function on one real variable as in (3), we should now solve a four-dimensional extremum formal (namely parametrized and nonnumerical) problem (in order to define any system of two linear constraints on three variables we need four variables) whose objective function is the determinant of a 5×5 formal matrix whose coefficients are functions of the parameter $t = p_2 = p_3$. Indeed, the involved matrix is no longer a 3×3 matrix as in (2) but is built with the five unknowns $(x_1, x_2, x_3, \lambda_1, \lambda_2)$ where λ_1 and λ_2 should be the corresponding Lagrange multipliers. Moreover, it should be calculated for all the possible loading paths $t \mapsto (p_2 = p_2(t),\ p_3 = p_3(t))$ or $h(p_2, p_3) = 0$, which becomes a quasi-impossible task. For the next case $n = 4$, the format of the involved matrix should be 7×7 whereas the loading path is described by any function $t \mapsto (p_2 = p_2(t),\ p_3 = p_3(t),\ p_4 = p_4(t))$. That shows that the second-order work criterion is an appropriate tool to tackle the constrained problem and that the main result (Theorem 2) is the appropriate geometrical approach that shortcuts these algebraic computations.

(2) Obviously in order to "see appear" the unstable behavior for the corresponding constrained system, the constrained system must be disturbed in one of the directions of the isotropic cone. At the boundary p^* it means in the X^* direction. If at $t = 0$ the perturbation is $U(0) = 0$ and $\dot{U}(0) = X^*$, then $t \mapsto U(t) = tX^*$ is a divergent solution of the (linear) dynamic equation of the constrained system and $\|U(t)\| \to +\infty$. On the contrary, in another direction of perturbation, no unstable behavior will be observed. It means that for concretely observing a divergence unstable evolution we have to first reach a threshold p^* in accordance with the loading path and given by the second-order work criterion, which also provides a direction X^*, second constrain the system by the appropriate constraint KX^*, and third perturb the system in the X^* direction. This direction is compatible with the kinematic constraint because, by construction, $(X^*)^T KX^* = 0$ since X^* is in the isotropic cone.

(3) From a geometric point of view, on the boundary ∂D_H the isotropic cone is degenerated into a vector space (a one-dimensional vector space "generally", which means that "generally" the rank of the matrix $K_s(p)$ drops from n to $n-1$ as p reaches p^* or as the determinant of $K_s(p)$ vanishes). Beyond the boundary, it is a real cone. For example, in Figure 4, consider the load $p = \left(\frac{3}{2}, \frac{3}{2}\right)$ which belongs to the domain $\det K_s(p) < 0$. According to Figure 4, for this loading we have $\det(K_s(p_d)) < 0$. For this loading, the isotropic cone is the blue surface plotted in Figure 5. The green line is in the direction

$$X_g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

for which $X_g^T K(p_d) X_g > 0$, $X_g \in C_+\left(\left(\frac{3}{2}, \frac{3}{2}\right)\right)$, whereas the red line is in the direction

$$X_r = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

for which $X_r^T K(p_d) X_r < 0$, $X_r \in C_-\left(\left(\frac{3}{2}, \frac{3}{2}\right)\right)$.

Case $n = 4$. The stiffness matrix reads

$$K(p_2, p_3, p_4) = \begin{pmatrix} 2-(p_2+p_3+p_4) & -1+p_2 & p_3 & p_4 \\ -1 & 2-(p_3+p_4) & -1+p_3 & p_4 \\ 0 & -1 & 2-p_4 & -1+p_4 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

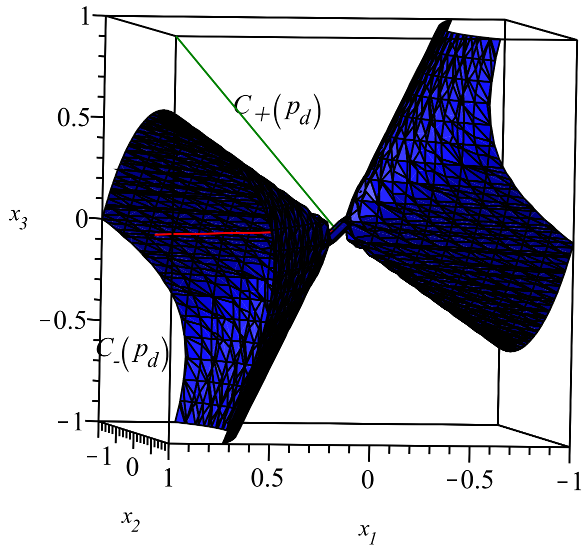

Figure 5. Isotropic cone (blue surface) for the load $p_d = \left(\frac{3}{2}, \frac{3}{2}\right)$ of the three degree of freedom discrete Leipholz column: $\det K_s(p_d) < 0$. Outside the isotropic cone (for example in the green line direction), then $X \in C_+(p_d)$: $X^T K_s(p_d) X > 0$. Inside the isotropic cone (for example in the red direction), then $X \in C_-(p_d)$: $X^T K_s(p_d) X < 0$. On the isotropic cone (blue surface), $X^T K_s(p_d) X = 0$.

and its symmetric part

$$K_s(p_2, p_3, p_4) = \begin{pmatrix} 2-(p_2+p_3) & -1+p_2/2 & p_3/2 & p_4/2 \\ -1+p_2/2 & 2-(p_3+p_4) & -1+p_3/2 & p_4/2 \\ p_3/2 & -1+p_3/2 & 2-p_4 & -1+p_4/2 \\ p_4/2 & p_4/2 & -1+p_4/2 & 1 \end{pmatrix}.$$

The domain of investigation lies now in the infinite cube $(p_2 \geq 0,\ p_3 \geq 0,\ p_4 \geq 0)$. As above we suppose that any loading path starts from $(0, 0)$. The domain D_H of Hill stability is delimited by the blue surface Γ_H in the vicinity of $(0, 0)$ (see Figure 6).

D_H is given by the set of inequalities

$$D_H = \Big\{0 \leq p_2 \leq 2,\ 0 \leq p_1 \leq 1,\ 0 \leq p_4 \leq 2-\sqrt{2},$$
$$1+\tfrac{1}{2}p_3^3-\tfrac{3}{4}p_4^4+4p_4^3+p_2p_3p_4-\tfrac{3}{4}p_2p_3p_4^2-\tfrac{1}{4}p_2^2+\tfrac{1}{16}p_2^2p_4^2-\tfrac{3}{4}p_2p_4^3+\tfrac{1}{2}p_2p_3^2-\tfrac{3}{4}p_3^2p_4^2$$
$$-\tfrac{3}{2}p_3p_4^3+\tfrac{3}{2}p_3^2p_4-\tfrac{1}{2}p_2p_3-\tfrac{1}{2}p_2p_4+2p_2p_4^2-\tfrac{3}{2}p_3^2-3p_3*p_4-5p_4^2+5p_3p_4^2 \geq 0\Big\}.$$

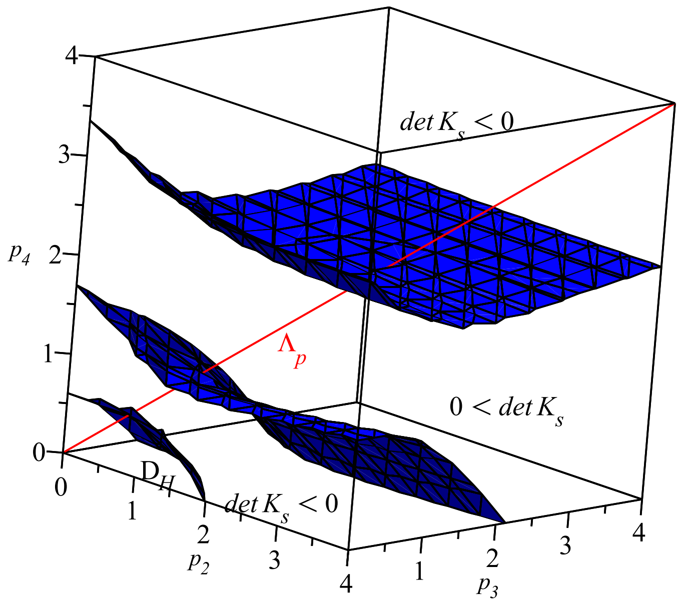

Figure 6. Hill stability domain D_H of the four degree of freedom discrete Leipholz column (D_{div} is the whole infinite cube ($p_2 \geq 0$, $p_3 \geq 0,\ p_4 \geq 0$)). Red line: Loading path $\Lambda_p : p_2 = p_3 = p_4$. Blue surface: three components of the hypersurface $\det(K_s(p)) = 0$.

Because the last inequality is a two-degree polynomial in the variable p_2, it can be solved explicitly. The explicit expression $p_2 = g(p_3, p_4)$ which is the explicit equation of the boundary $\partial D_H = \Gamma_H$ can be used to prove the convexity of D_H. The thresholds for the three intervals of variation for the variables p_2, p_3, p_4 are obtained by vanishing the two other variables and solving the remaining equalities.

As above, now choose the loading path $\Lambda_p = \{p = (p_2, p_3, p_4) \mid p_2 = p_3 = p_4\}$ plotted in red color in Figure 6. This line intersects the boundary $\partial D_H = \Gamma_H$ of the Hill stability domain at the point $p^* = (p_2^*, p_3^*, p_4^*)$ with $p_2^* = p_3^* = p_4^* \approx 0.4351852922$. Here p_2^*, p_3^*, p_4^* are solutions of the fourth-degree polynomial $\phi(t) = 1 - \frac{43}{4}t^2 + \frac{29}{2}t^3 - \frac{71}{16}t^4$ whose curve is plotted in Figure 7 and whose zeros give the values of the intersections of the red line Λ_p and the blue surface Γ_H in Figure 6. Then 0.4351852922 is a numerical approximation of the first positive root.

For this critical value, the isotropic cone of $K_s(p^*)$ is no longer reduced to $\{0\}$. It is generated by the vector

$$X^* = \begin{pmatrix} 0.742133032540456 \\ 0.625378368942899 \\ 0.0997261713362030 \\ -0.219533934555109 \end{pmatrix}.$$

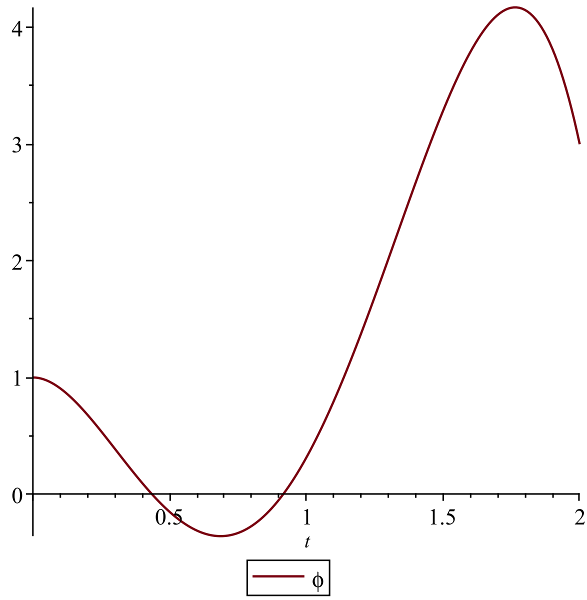

Figure 7. Function ϕ giving the critical load p^* on the loading path $\Lambda_p : p_2 = p_3 = p_4$.

Thus, applying the main theorem, the next step consists of calculating

$$K(X^*) = \begin{pmatrix} 0.110008445768715 \\ -0.187551978632908 \\ -0.345329394165552 \\ -0.319260105891312 \end{pmatrix},$$

which means that the system Σ_3 subjected to the kinematic constraint

$$c^*(\theta_1, \theta_2, \theta_3, \theta_4) = 0.110008445768715\theta_1 - 0.187551978632908\theta_2 \\ - 0.345329394165552\theta_3 - 0.345329394165552\theta_4 = 0$$

becomes, on the loading path $\Lambda_p = \{p = (p_2, p_3, p_4) \mid p_2 = p_3 = p_4\}$, divergence unstable when it is subjected to the critical load $p^* = (0.4351852922, 0.4351852922, 0.4351852922)$.

7. Conclusion

In this paper, the KISS concept is introduced and applied to shed new light on the sixty-year-old "competition" between the two criteria of stability investigated here: the second-order work criterion and the divergence criterion. We first start with an introductory example which possess all the necessary ingredients. Second we stress that the two criteria do not question exactly the same approach of stability. Third, by use of the KISS concept and thanks to the geometrical approach of constrained mechanics involving the compression of operators, a way is proposed to obtain a full equivalence.

This final equivalence between both criteria allowed us to highlight the significant original variational formulation on all the possible constrained systems keeping in mind that only one-constrained systems are finally involved in the solution. The multiparameter discrete Leipholz column example illustrates the involved concepts and shows the power of the geometric solution associated with the variational formulation.

The extension of wider frameworks (linear continuum mechanics and nonlinear discrete mechanics) have already been performed whereas the flutter-type instability does not lead to such beautiful results regarding the KISS issue. Indeed, according to the mass distribution, the full variety of situations may occur (universal or conditional KISS as well) and the KISS must then be investigated case by case. We believe that one of the last real challenges regarding both of these criteria is the investigation of the transition from the Hill stability criterion well adapted to a purely incremental quasistatic evolution to the Lyapunov dynamic stability approach.

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Received 4 Jul 2019. Revised 6 Nov 2019. Accepted 7 Jan 2020.

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS

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MEMOCS (ISSN 2325-3444 electronic, 2326-7186 printed) is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

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MATHEMATICS AND MECHANICS OF COMPLEX SYSTEMS
vol. 8 no. 1 2020

MEMOCS is a journal of the International Research Center for the Mathematics and Mechanics of Complex Systems at the Università dell'Aquila, Italy.

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