



**HAL**  
open science

## A model of social welfare improving transfers

Brice Magdalou

► **To cite this version:**

Brice Magdalou. A model of social welfare improving transfers. *Journal of Economic Theory*, 2021, 196. hal-03287960

**HAL Id: hal-03287960**

**<https://hal.inrae.fr/hal-03287960>**

Submitted on 16 Jul 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A model of social welfare improving transfers<sup>\*</sup>

Brice Magdalou<sup>†</sup>

June 24, 2021

## Abstract

This paper provides a generalization of the Hardy-Littlewood-Polya (HLP) Theorem in the following discrete framework: a distribution counts the number of persons having each possible individual outcome –assumed to be finitely divisible– and social welfare improving transfers have the structure of a discrete cone. The generalization is abstract in the sense that individual outcomes can be unidimensional or multidimensional, each dimension can be cardinal or ordinal and no further specification is required for the transfers. It follows that most of the results in the literature, applied to discrete distributions and comparable to the HLP Theorem, are corollaries of our theorem. In addition, our model sheds new light on some surprising results in the literature on social deprivation and, in decision-making under risk, provides new arguments on the key role of the expected utility model.

*JEL Classification Numbers:* C02, D63, D81. *Keywords:* Hardy-Littlewood-Polya Theorem, stochastic dominance, risk, social welfare, inequality, welfare-improving transfers.

*‘God made the integers, all else is the work of man’*

Leopold Kronecker (1823-1891)

---

<sup>\*</sup> This paper is part of three research projects, *RediPref* (Contract ANR-15-CE26-0004), *OrdIneq* (Contract ANR-16-CE41-0005) and *ChallIneq* (ANR-18-EQUI-0003) of the French National Agency for Research (ANR), from which financial support is acknowledged. I am grateful to the Editor, Tilman Börgers, an anonymous Associate Editor and an anonymous Referee of this journal, for their useful and challenging comments. I am also grateful to John Weymark for his valuable suggestions, which helped me to improve the paper. Finally, I would like to thank Ramses Abul Naga, Alain Chateauneuf, Nicolas Gravel, Patrick Moyes, Erwin Ooghe, Alain Trannoy, Claudio Zoli and Stéphane Zuber for interesting discussions related to this paper.

<sup>†</sup> CEE-M, Univ. Montpellier, CNRS, INRAE, Institut Agro, Montpellier, France.  
E-mail: [brice.magdalou@umontpellier.fr](mailto:brice.magdalou@umontpellier.fr).

## 1. Introduction

The main challenge for any model of social welfare assessment (and related concepts such as inequality, poverty, equality of opportunity, and social mobility) is to define the transformation of the individual outcomes that the social planner considers as unambiguously social welfare improving. The second difficulty is developing measurement tools that are consistent with the social welfare views captured by such transfers. Among the normative approaches used to compare outcome distributions, the strategy that consists of providing criteria (called preorders) conclusive only when one distribution is obtained from the other by a sequence of such transfers is probably the less questionable one: The ranking of the distributions is in that case crystal clear because it is inextricably linked to our own definition of what we call ‘social welfare improvement’.

The most prominent example is the approach based on the Pigou-Dalton principle of transfers, whereby a mean-preserving income transfer from one individual to a poorer individual is deemed inequality-reducing. In this context, Hardy *et al.* (1934)’s Theorem (HLP hereafter) can be regarded as a cornerstone for comparing income distributions.<sup>1</sup> For any equal mean income distributions  $x$  and  $x'$ , by letting  $\mathbb{E}$  be the expectation operator and  $u$  a utility function, the equivalence of the following three preorders has been established:

- (a)  $x$  is obtained from  $x'$  by means of a finite sequence of Pigou-Dalton transfers,
- (b)  $\mathbb{E}[u(x)] \geq \mathbb{E}[u(x')]$  for all  $u$  that are concave,
- (c) The Lorenz curve of  $x$  does not lie below that of  $x'$ .

The first statement describes a situation where one distribution is less unequal than another, according to the principle of transfers view on inequality. The second statement describes a unanimous ranking of the distributions for all the utilitarian social planners endowed with a concave utility function. The last statement is an empirically implementable criterion, with a conclusive ranking of the distributions only when the first one is unambiguously less unequal than the other.

This paper provides a comparable equivalence result, in a general and abstract framework. First, a distribution counts the number of individuals having each possible outcome. Outcomes

---

<sup>1</sup> The first version of this result appears in Hardy *et al.* (1929), Theorem 8, but where the statement (a) is replaced by a dominance condition based on bistochastic matrices. A complete version of the theorem is provided in Hardy *et al.* (1934), even if it is not a self-contained result: It follows from Theorem 45 (Page 45), Lemmas 1 and 2 (Page 47) and Theorem 108 (Page 89). The HLP theorem has been popularized in social welfare economics by Kolm (1969), Atkinson (1970), Dasgupta *et al.* (1973), Sen (1973) and Fields and Fei (1978) and, in decision-making under risk, by Rothschild and Stiglitz (1970), where Pigou-Dalton transfers are replaced by mean-preserving spreads.

can be unidimensional or multidimensional, and each dimension can be cardinal (like income) or ordinal (as with categorical variables like health status). We only assume that each dimension is defined on an ordered and finitely divisible scale.<sup>2</sup> Then, unlike existing extensions of the HLP theorem which are based on very specific forms of welfare improving transfers (Pigou-Dalton transfers in the standard theorem), we only impose a structural property to the transfers. Precisely, we assume that the set of transfers  $\mathcal{T}$  is endowed with the following two features: (i) a transfer can be written as the difference between two distributions and, if a transfer is assumed to be welfare improving, the reverse transfer is welfare decreasing; (ii) a ‘sequence of transfers’ is described as a linear combination of transfers, with positive integer coefficients. It follows that such a sequence has the structure of a ‘discrete cone’ (the discrete analogue of a convex cone).

Our generalized HLP theorem is an equivalence between four preorders. Statement (a) is the dominance of one distribution over another, according to the preorder induced by the discrete cone generated by  $\mathcal{T}$  (the formal definition of a sequence of transfers). Statement (b1) is the common ranking of distributions, for all social welfare functions that agree that transfers in  $\mathcal{T}$  are welfare improving. Statement (b2) is the utilitarian analogue of (b1). It describes the utilitarian ranking for all functions in  $\mathcal{U}(\mathcal{T})$ , which is the set of utility functions that agree that transfers in  $\mathcal{T}$  are welfare improving. Finally, statement (c) establishes that dominance in (b2) can be achieved by the comparison of distributions for a finite number of utility functions, precisely all the functions in the finite basis of  $\mathcal{U}(\mathcal{T})$ .

Almost all the analogous equivalence results in the literature (for discrete distributions) are special cases of the theorem provided here. This is mainly due to fact that the specific transfers which have been investigated have, with a few exceptions, the structure of a discrete cone. In that case, when the discrete cone property is satisfied, our general result has two implications. First, it ensures the existence of a full HLP theorem. Indeed, by using standard methods, such a result is sometimes hard to achieve and there is no information which suggests that this result is possible. Then, our general result can be helpful to formulate, accurately, the different statements in the equivalence theorem. As an example, it has been used in Gravel *et al.* (2021) to complete an equivalence theorem for a unidimensional ordinal variable (see Section 7).

Among the four equivalent preorders presented in the generalized theorem, statement (c) is the

---

<sup>2</sup> This discrete framework is relevant for any variable for which data are available in practice, be it cardinal or ordinal. Take income as a cardinal example. In empirical data, income is always provided on a discrete scale (euro cents, for instance). Equivalently, ordinal variables are usually defined on an ordered categorical scale. Then, focusing on the distinction between ‘continuous’ and ‘discrete’, a continuous scale can always be considered as the limit point of a ‘very narrow’ discrete scale. For instance, Gravel *et al.* (2021) provide an HLP theorem for a discrete ordinal variable by substituting the so-called ‘Hammond transfers’ for the Pigou-Dalton transfers. They demonstrate that the implementation criterion they obtain converges, when the discrete scale is ‘refined’, with the implementation criterion that can be obtained for a continuous ordinal variable, as identified by Gravel *et al.* (2019).

most relevant for applications on real data. It is known as an implementation criterion, comparable to the Lorenz criterion in the standard HLP theorem. It is not expressed in the usual form –a curve with a finite number of points to compare, for instance– due to the abstract nature of the theorem, but it can be restated for specific sets of transfers. The main challenge is to identify the finite basis of  $\mathcal{U}(\mathcal{T})$ , precisely the finite list of extreme points of this set. This task is not trivial as there does not exist, in general, an analytic solution to this problem. Nevertheless, we present in Section 6 an algorithmic approach to identify such extreme points. This approach is illustrated by the characterization of two well-known examples (for a unidimensional variable), namely first and second-order stochastic dominance.

The equivalence between our preorders (a) and (b2) is related to a comparable and well-known (but distinct) result. In the standard HLP framework, it can be stated as the equivalence between the following two statements:

- (d) For all distributions  $x$  and  $x'$ ,  $x$  being obtained from  $x'$  by means of a finite sequence of Pigou-Dalton transfers implies  $\mathbb{E}[u(x)] \geq \mathbb{E}[u(x')]$ ,
- (e)  $u$  is concave.

One can easily see that ‘(a)  $\Rightarrow$  (b)’ in the HLP theorem and ‘(e)  $\Rightarrow$  (d)’ in the result above can be proved with identical arguments. Hence, they provide the same information, which is not the case for the reverse implications. ‘(b)  $\Rightarrow$  (a)’ establishes that the unanimity of the ranking of all the utilitarian social planners, endowed with a concave utility function, is sufficient to ensure that the first distribution is obtained from the second one by means of a sequence of Pigou-Dalton transfers, whereas ‘(d)  $\Rightarrow$  (e)’ establishes that the concavity of the social planner’s utility is necessary to make his preferences consistent with the Pigou-Dalton principle of transfers. In our general framework we establish that, if we are able to identify the class of utility functions that ensures that ‘(d)  $\Leftrightarrow$  (e)’ (our Proposition 3), then ‘(a)  $\Leftrightarrow$  (b2)’ in our theorem is necessarily true. It is important to emphasize that the first equivalence is easier to achieve.<sup>3</sup> Note that this paper is the first, in the social welfare literature, which directly investigates the equivalence between statements like (a) and (b2).

**Literature.** This paper builds on a large set of results in the statistics literature dealing with majorization and order statistics (see Marshall *et al.*, 2011). The notion of ‘preorder induced by a convex cone’ has been introduced by Marshall *et al.* (1967) for a unidimensional real-valued distribution. They establish, in this context, necessary and sufficient conditions to be placed on a real-valued function to be order-preserving (a result in line with the equivalence between (d)

---

<sup>3</sup> Actually, only ‘(e)  $\Rightarrow$  (d)’ has to be proved. With this proof in hand, the reverse implication can be easily obtained by using a contrapositive proof.

and (e) above). Marshall (1991) investigates, in the same framework, the equivalence between statements of type (a) and (b) by using duality arguments for a convex cone and its ‘polar cone’ (Proposition 3.4, Page 237). By considering decision under risk, Muller and Scarsini (2012) extend this result to multivariate probability distributions with real-valued dimensions by defining a sequence of ‘inframodular mass transfers’ as a preorder induced by a convex cone, and Muller (2013) generalizes this result to any transfers having the same structure.<sup>4</sup> In an application to the ‘supermodular stochastic ordering’, Meyer and Strulovici (2013) obtain a comparable result for multivariate probability distributions, but defined on a finite support.

Equivalence between statements (a) and (b2) in our theorem can be seen as the discrete analogue of Muller (2013)’s result. Indeed, our paper differs from the previous literature –apart from the fact that it is the first to introduce cone orderings in the social welfare context– in that we consider discrete distributions (we count the number of individuals having each possible outcome). In Muller (2013), the comparable statements (a) and (b2) are two preorders induced by a convex cone and the equivalence is directly obtained by the duality arguments of the bipolar theorem. The duality we obtain here is more subtle, because it is between a discrete cone and a convex cone (respectively, in statements (a) and (b2)).<sup>5</sup> We establish that, in this context, a necessary and sufficient condition to have ‘(a)  $\Leftrightarrow$  (b2)’ is that the set of transfers  $\mathcal{T}$  includes its ‘Hilbert basis’ (Proposition 8). To the best of our knowledge, the notion of Hilbert basis has never been used in economic applications. Finally, statement (b1) establishes that utilitarian dominance can be extended to the largest class of social welfare functions consistent with transfers in  $\mathcal{T}$  and we provide in statement (c) an implementation criterion.

The paper is organized as follows. We present in Section 2 the framework and the main definitions, particularly the abstract notion of a ‘set of social welfare improving transfers’. Section 3 investigates social welfare functions consistent with these transfers, with a focus on the utilitarian class. In Section 4, we introduce duality relations which are used to prove our main theorem. This theorem is presented in Section 5. A strategy to identify an implementation criterion –comparable to Lorenz dominance– is proposed in Section 6. The following sections illustrate the other parts of the theorem. Section 7 considers ‘Hammond transfers’ in the case of a unidimensional ordinal variable. Section 8 deals with a bidimensional variable consisting of a transferable and cardinally measurable dimension (like income) and a nontransferable ordinal dimension (like health). Section 9 illustrates transfers outside our framework. Section 10 shows that our results can be of some interest for another economic field, theory of decision under risk. Finally, Section 11 concludes.

---

<sup>4</sup> For related works, see also Muller (1997) and Maccheroni *et al.* (2005)

<sup>5</sup> See the discussion after Proposition 5, and the fact that  $\mathcal{T} \subset \mathcal{T}^{\circ\circ}$ .

## 2. Outcomes, distributions and social welfare improving transfers

Some mathematical notions that are subsequently used hereafter are introduced in Appendix B. We also recall in this appendix some classical results, which underlie the proofs of our own results.

We use  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  to denote the sets of integers (including 0), rationals and real numbers, respectively. The symbols  $\mathbb{Z}_+$ ,  $\mathbb{Q}_+$  and  $\mathbb{R}_+$  are the non-negative restrictions of these sets, and  $\mathbb{Z}_{++}$ ,  $\mathbb{Q}_{++}$  and  $\mathbb{R}_{++}$  are the positive restrictions. The cardinality of a set  $\mathcal{A}$  is indicated by  $|\mathcal{A}|$ .

The set of *outcomes* is assumed to be a partially ordered, finite and fixed set  $\mathcal{S} \subset \mathbb{Z}_+^d$ . Precisely, we assume that each individual in the society is characterized by an outcome in  $d$  dimensions (income, health, ...), such that the outcomes in one dimension are finitely divisible, defined over an integer interval and ordered according to the usual ordering on  $\mathbb{Z}$ . Then  $\mathcal{S}$  is simply the Cartesian product of these  $d$  intervals. We also assume, without loss of generality, that the lowest outcome in each dimension is 0. Thus, the origin of the set of outcomes is  $(0, \dots, 0) \in \mathcal{S}$ . A *distribution* is described by a list  $n = (n_s)_{s \in \mathcal{S}}$ , where  $n_s \in \mathbb{Z}_+$  indicates the number of individuals having ( $d$ -dimensional) outcome  $s \in \mathcal{S}$ . In this paper, we mainly deal with the comparison of distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  of equal (and finite) population size.<sup>6</sup>

Figure 1 shows a simple example where the outcomes are bidimensional, hence points  $s = (i, j) \in \mathbb{Z}_+^2$ , and a distribution is simply a list of  $n_{(i,j)}$ .

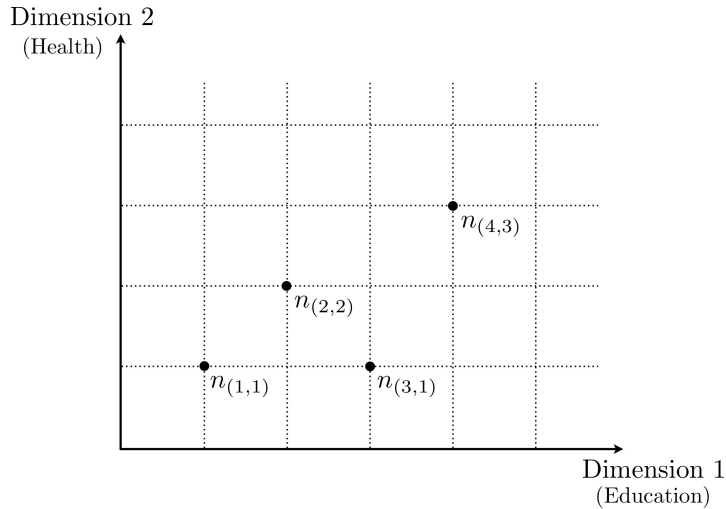


Figure 1: Example of a bidimensional variable

<sup>6</sup> There is no loss of generality to consider populations of equal size, by appealing to the Dalton principle of population which assumes that an identical replication of the population is distributionally equivalent to the initial one. See the end of Section 5 for a discussion.

We emphasize that this framework is sufficiently general to encompass most of the usual frameworks investigated in the literature on inequality, poverty and social welfare measurement. For instance, each dimension can be, independently, defined on a cardinal or ordinal scale. Income is an example of a possible cardinal outcome dimension: to satisfy the finite divisibility assumption, one has to consider the (euro) cent as the smallest income unit (which is, in practice, always assumed).

We now provide a general definition of a *set of transfers* (which can be, at this stage, welfare-improving or not). This definition is provided at an abstract level, in the sense that it specifies only the fundamental properties that such a set could satisfy. As illustrated in Sections 7 and 8, most of the sets of transfers usually considered in the literature (increments, Pigou-Dalton progressive transfers, ...) are well-described as particular cases of our general definition. The main idea is as follows. Consider two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  of equal size and assume that  $n$  is obtained from  $n'$  by means of a sequence of transfers, where a transfer is just a movement of one (or a few) individual(s) from one outcome to another. Such a sequence of transfers can be written by means of a vector  $m = n - n'$  that lists the differences between the number of individuals, before and after the transfers, in each outcome  $s \in \mathcal{S}$ .

**Definition 1** (Set  $\mathcal{T}$  of transfers).  *$\mathcal{T}$  is a finite and non-empty set of  $m = (m_s)_{s \in \mathcal{S}} \in \mathbb{Z}^{|\mathcal{S}|}$ , such that  $\sum_{s \in \mathcal{S}} m_s = 0$ , and that  $m \in \mathcal{T}$  implies  $(-m) \notin \mathcal{T}$ .*

This definition is illustrated by the standard notion of *increment*. In the bidimensional case ( $\mathcal{S} \subset \mathbb{Z}_+^2$ ), distribution  $n$  is obtained from distribution  $n'$  by means of an increment if and only if there exist  $(i, j), (k, l) \in \mathcal{S}$  such that:<sup>7</sup>

$$n_s = n'_s, \text{ for all } s \neq (i, j), (k, l), \quad (1)$$

$$n_{(i,j)} = n'_{(i,j)} - 1, n_{(k,l)} = n'_{(k,l)} + 1, \text{ and } (i, j) < (k, l). \quad (2)$$

This corresponds with a population move towards the northeast of the graph in Figure 2. We can define the set of increments, denoted as  $\mathcal{T}_I$ , by the set all vectors  $m \in \mathbb{Z}^{|\mathcal{S}|}$  such that there exist  $(i, j), (k, l) \in \mathcal{S}$  with:

$$m_s = 0, \text{ for all } s \neq (i, j), (k, l), \quad (3)$$

$$m_{(i,j)} = -1, m_{(k,l)} = 1, \text{ and } (i, j) < (k, l). \quad (4)$$

In other words, if for two arbitrary distributions  $n$  and  $n'$  of equal population size we have  $n - n' = m \in \mathcal{T}_I$ , then we can say that  $n$  is obtained from  $n'$  by means of an increment. Moreover, if an increment is assumed to be welfare-improving, the reverse transfer, which corresponds to  $(-m)$

---

<sup>7</sup> Vector inequalities are denoted by  $\leq, <$  and  $\ll$ .



according to the previous definition, is welfare-decreasing (hence not in the set  $\mathcal{T}_I$ ) and called a *decrement*. It follows that  $\mathcal{T}_I \subset \mathcal{T}$ .

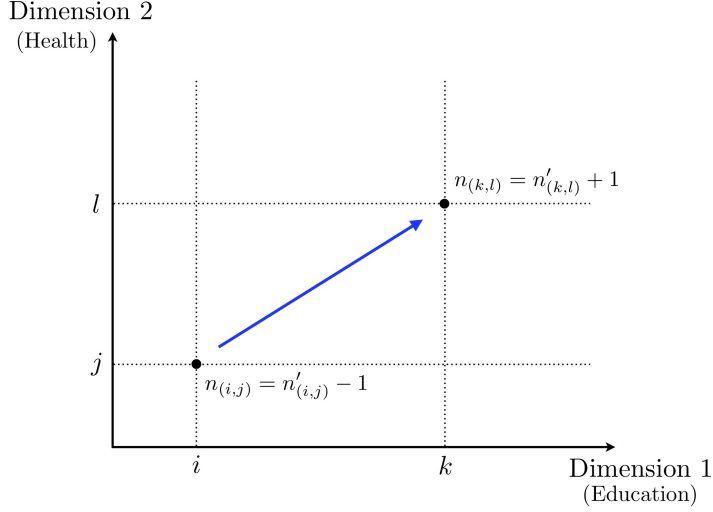


Figure 2: An increment

The first restriction in Definition 1 results from the fact that a transfer involves two equal size distributions, so that  $\sum_{s \in \mathcal{S}} m_s = \sum_{s \in \mathcal{S}} (n_s - n'_s) = 0$ . The second restriction reflects the idea that a welfare-improving transfer is, in some sense, *directional*: If  $m \in \mathcal{T}$  is considered as a welfare improvement, then it cannot be also true for  $(-m)$ . Note that another consequence of the last restriction is that transfers are considered in a strict sense, involving the movement of at least one individual ( $0 \notin \mathcal{T}$ ).

Even if Definition 1 is sufficiency flexible to encompass most of the welfare-improving transfers considered in the literature, it imposes some restrictions, including the following one.

**Remark 1** (Independence). *Let  $n, n', n'', n''' \in \mathbb{Z}_+^{|\mathcal{S}|}$  be four distributions of equal population size, such that  $n'' = n + \epsilon$  and  $n''' = n' + \epsilon$ . If  $n$  is obtained from  $n'$  by means of a transfer in  $\mathcal{T}$ , then  $n''$  is also obtained from  $n'''$  by means of a transfer in  $\mathcal{T}$ .*

This observation trivially follows from the fact that if  $n - n' \in \mathcal{T}$ , then  $n'' - n''' \in \mathcal{T}$  whenever  $n - n' = n'' - n'''$ . Remark 1 has important implications: It states that admissible transfers in  $\mathcal{T}$  are, necessarily, not impacted by the common parts of the distributions under comparison. This *independence requirement* is not satisfied by some sets of transfers considered in the literature, as illustrated in Section 9.

If the set of welfare-improving transfers we are interested in can be written as in Definition 1, one can infer from  $\mathcal{T}$  a dominance relation that can help a social planner to rank pairs of distri-

butions. Indeed, if one distribution is obtained from another by a sequence of welfare-improving transfers, then the social planner can unambiguously consider the first one as socially better. If  $\mathcal{T}$  defines the admissible transfers, then the discrete cone generated by  $\mathcal{T}$  is the natural way to define a sequence of such transfers.<sup>8</sup>

**Definition 2** (Sequence of transfers in  $\mathcal{T}$ ). *For all  $n, n' \in \mathbb{Z}^{|\mathcal{S}|}$ , we write  $n \succeq_{\mathcal{T}} n'$  if and only if  $\succeq_{\mathcal{T}}$  is induced by the discrete cone generated by  $\mathcal{T}$ . Formally,  $n \succeq_{\mathcal{T}} n'$  if and only if  $n - n' \in \mathcal{D}(\mathcal{T})$ , where  $\mathcal{D}(\mathcal{T}) = \{\sum_{t=1}^{|\mathcal{T}|} \lambda_t m_{.t} \mid \lambda_t \in \mathbb{Z}_+, m_{.t} \in \mathcal{T}\}$  with  $m_{.t} = (m_{st})_{s \in \mathcal{S}}$ .*

We emphasize that the relation  $\succeq_{\mathcal{T}}$  is defined on  $\mathbb{Z}^{|\mathcal{S}|}$  instead of  $\mathbb{Z}_+^{|\mathcal{S}|}$  even if it is used, as in this paper, only to compare distributions in  $\mathbb{Z}_+^{|\mathcal{S}|}$ . This is a consequence of the basic properties of a discrete cone: For instance we have, by definition,  $n \succeq_{\mathcal{T}} n'$  if and only if  $n - n' \succeq_{\mathcal{T}} 0$ . Nevertheless, note that the equal population size condition for distributions compared according to  $\succeq_{\mathcal{T}}$  is implicitly assumed.

**Remark 2** (Same population size). *For any pair of distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$ , a necessary condition for  $n \succeq_{\mathcal{T}} n'$  is that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ .*

**Proof.** Let  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  and  $n \succeq_{\mathcal{T}} n'$ , or equivalently  $n - n' \in \mathcal{D}(\mathcal{T})$ . By definition  $n - n' = \sum_{t=1}^{|\mathcal{T}|} \lambda_t m_{.t}$  for some  $\lambda_t \in \mathbb{Z}_+$  and  $m_{.t} = (m_{st})_{s \in \mathcal{S}} \in \mathcal{T}$ . Moreover, we have  $\sum_{s \in \mathcal{S}} m_{st} = 0$  for any  $m_{.t} \in \mathcal{T}$ . One concludes that  $\sum_{s \in \mathcal{S}} (n_s - n'_s) = \sum_{t=1}^{|\mathcal{T}|} \lambda_t \sum_{s \in \mathcal{S}} m_{st} = 0$ .  $\square$

We now present the main properties satisfied by  $\succeq_{\mathcal{T}}$ . This proposition is a mere transposition of Lemma 4 (in Appendix B) to our discrete cones. Note that Equation (5) is a weak scale-invariance property, as it only applies to scale parameters in  $\mathbb{Z}_+$  (instead of  $\mathbb{R}_+$ ).

**Proposition 1.** *The discrete cone ordering  $\succeq_{\mathcal{T}}$  is an additive preorder and satisfies the following property:*

$$\forall n, n' \in \mathbb{Z}^{|\mathcal{S}|} : \quad n \succeq_{\mathcal{T}} n' \implies \lambda n \succeq_{\mathcal{T}} \lambda n', \quad \forall \lambda \in \mathbb{Z}_+. \quad (5)$$

*As it is also antisymmetric, the discrete cone  $\mathcal{D}(\mathcal{T})$  is pointed.*

**Proof.** For any  $n \in \mathbb{Z}^{|\mathcal{S}|}$ , we have  $n - n = 0 \in \mathcal{D}(\mathcal{T})$ . Hence,  $n \succeq_{\mathcal{T}} n$ , so  $\succeq_{\mathcal{T}}$  is reflexive. Now, suppose that  $n \succeq_{\mathcal{T}} n'$  and  $n' \succeq_{\mathcal{T}} n''$ . It follows that  $n - n' \in \mathcal{D}(\mathcal{T})$  and  $n' - n'' \in \mathcal{D}(\mathcal{T})$ . Moreover,  $n - n'' = (n - n') + (n' - n'') \in \mathcal{D}(\mathcal{T})$ , or equivalently,  $n \succeq_{\mathcal{T}} n''$ . Hence,  $\succeq_{\mathcal{T}}$  is transitive. As it is reflexive and transitive, it is a preorder. Additivity is immediately obtained by noting that  $n - n' \in \mathcal{D}(\mathcal{T})$  if and only if  $(n + n'') - (n' + n'') \in \mathcal{D}(\mathcal{T})$ . The implication in Equation (5) also follows from the definition of a discrete cone, so that  $n - n' \in \mathcal{D}(\mathcal{T})$  implies  $\lambda(n - n') \in \mathcal{D}(\mathcal{T})$  as soon as  $\lambda \in \mathbb{Z}_+$ . To prove antisymmetry, suppose that  $n \succeq_{\mathcal{T}} n'$  and  $n' \succeq_{\mathcal{T}} n$ , or equivalently

<sup>8</sup> Note that an empty sequence of transfers is allowed in the definition, so that  $0 \in \mathcal{D}(\mathcal{T})$  even if  $0 \notin \mathcal{T}$ .

$n - n' \in \mathcal{D}(\mathcal{T})$  and  $n' - n \in \mathcal{D}(\mathcal{T})$ , and let us show that  $n = n'$ . By contradiction, assume that  $n \neq n'$ . In that case, there exist  $T \leq |\mathcal{T}|$ ,  $\lambda_t \in \mathbb{Z}_{++}$  and  $m_{.t} \in \mathcal{T}$  such that  $n - n' = \sum_{t=1}^T \lambda_t m_{.t}$ . Moreover,  $n' - n = -(n - n') = \sum_{t=1}^T \lambda_t (-m_{.t})$ , and because  $n' - n \in \mathcal{D}(\mathcal{T})$  we should also have  $(-m_{.t}) \in \mathcal{T}$ , which is by definition impossible because  $m_{.t} \in \mathcal{T}$ . Hence,  $n = n'$ , so  $\succeq_{\mathcal{T}}$  is antisymmetric. Because  $\mathcal{T} \subseteq \mathcal{D}(\mathcal{T})$ , the previous argument equivalently states that  $\mathcal{D}(\mathcal{T})$  is pointed.  $\square$

Proposition 1 has some implications when distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  of equal size are compared. Following Equation (5), if  $n \succeq_{\mathcal{T}} n'$ , then an identical replication of the population has no impact on the ranking of the distributions  $n$  and  $n'$ . This property corresponds to the weak version of the Dalton principle of population, which is a standard property in normative economics.<sup>9</sup> Additivity is somewhat stronger and related to the independence condition, as described in Remark 1 (see Section 9 for a discussion).

### 3. Social welfare functions

The previous section provides an abstract model of welfare-improving transfers, which encompass most of the transfers considered in the literature (such as increments and Pigou-Dalton progressive transfers). This model leads to a preorder, which can be used to rank the distributions of individuals according to their outcomes. In this section, we investigate another approach that can be used to compare distributions, building on the preferences of a social planner or a group of social planners. Such preferences are characterized by a set of ethical considerations, which are not restricted to the mere notion of welfare-improving transfers. The goal here is to introduce a preorder that reflects the unanimity of the rankings among a large class of social planners who share some ethical views.

We first assume that the preferences of a social planner can be represented by a social welfare function  $W : \mathbb{Z}_+^{|\mathcal{S}|} \rightarrow \mathbb{R}$ . We use  $\mathcal{W}$  to denote the set of all these social welfare functions. We now focus on the social welfare functions consistent with  $\succeq_{\mathcal{T}}$  or, equivalently, on all the social planners who agree that transfers in  $\mathcal{T}$  are unambiguously welfare-improving:

$$\mathcal{W}(\mathcal{T}) = \left\{ W \in \mathcal{W} \mid W(n) \geq W(n'), \forall n, n' \in \mathbb{Z}_+^{|\mathcal{S}|} : n - n' \in \mathcal{T} \right\}. \quad (6)$$

We emphasize that  $\mathcal{W}(\mathcal{T})$  identifies the set of all social planners sharing the ethical views captured by the transfers in  $\mathcal{T}$ , but their preferences are not necessarily equivalent elsewhere. As an

---

<sup>9</sup> The standard Dalton principle of population would require that  $n \sim_{\mathcal{T}} \lambda n$ , which is not true in our case. Indeed, as  $\succeq_{\mathcal{T}}$  is antisymmetric, we have  $n \sim_{\mathcal{T}} n'$  if and only if  $n = n'$ .

illustration in the unidimensional case, if  $\mathcal{T}$  is the set Pigou-Dalton transfers, then  $\mathcal{W}(\mathcal{T})$  is the set of *Schur-concave* social welfare functions (see Dasgupta *et al.*, 1973).

One first observes that  $\mathcal{W}(\mathcal{T})$  has the structure of a convex cone. Because  $\mathcal{W}(\mathcal{T})$  is defined as the largest set of functions consistent with  $\succeq_{\mathcal{T}}$ , if  $W, W' \in \mathcal{W}(\mathcal{T})$ , then  $\lambda W + \lambda' W' \in \mathcal{W}(\mathcal{T})$  for all  $\lambda, \lambda' \in \mathbb{R}_+$ . Another important property of the class  $\mathcal{W}(\mathcal{T})$ , related to the additivity of  $\succeq_{\mathcal{T}}$  (see Remark 1 and Proposition 1), is the following. For all  $W \in \mathcal{W}(\mathcal{T})$ , all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $n \succeq_{\mathcal{T}} n'$ , and all  $\epsilon \in \mathbb{Z}^{|\mathcal{S}|}$  such that  $n + \epsilon, n' + \epsilon \in \mathbb{Z}_+^{|\mathcal{S}|}$ , we have by definition:

$$W(n) \geq W(n') \Leftrightarrow W(n + \epsilon) \geq W(n' + \epsilon). \quad (7)$$

This property has interesting implications, which will be used in our results. We first have the following, quite obvious, equivalence.

**Proposition 2.** *The following two statements are equivalent.*<sup>10</sup>

$$(d) \forall n, n' \in \mathbb{Z}_+^{|\mathcal{S}|} : n \succeq_{\mathcal{T}} n' \implies W(n) \geq W(n'),$$

$$(e) W \in \mathcal{W}(\mathcal{T}).$$

**Proof.** We first prove that (e)  $\implies$  (d). Let  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$ . The implication is immediate if  $n = n'$ , hence suppose that  $n \neq n'$ . Then assume that  $W \in \mathcal{W}(\mathcal{T})$  and  $n \succeq_{\mathcal{T}} n'$ , or equivalently  $n - n' \in \mathcal{D}(\mathcal{T})$ . Hence, there exist  $T \leq |\mathcal{T}|$ ,  $\lambda_t \in \mathbb{Z}_{++}$  and  $m_{.t} \in \mathcal{T}$  such that  $n - n' = \sum_{t=1}^T \lambda_t m_{.t}$ . Because  $W \in \mathcal{W}(\mathcal{T})$  and  $m_{.t} \in \mathcal{T}$ , we have:

$$W(n') \leq W(n' + m_{.1}) \leq W(n' + \lambda_1 m_{.1}) \leq \dots \leq W\left(n' + \sum_{t=1}^T \lambda_t m_{.t}\right) = W(n). \quad (8)$$

Note that all the intermediate distributions described in (8) belong to  $\mathbb{Z}^{|\mathcal{S}|}$ , but not necessarily to  $\mathbb{Z}_+^{|\mathcal{S}|}$ . If non-negativity is not satisfied, instead of comparing  $n$  and  $n'$  in (8), it suffices to choose the appropriate  $\epsilon \in \mathbb{Z}_+^{|\mathcal{S}|}$  and to compare  $n + \epsilon$  and  $n' + \epsilon$ , by noting that  $(n + \epsilon) - (n' + \epsilon) = \sum_{t=1}^T \lambda_t m_{.t}$ . Then,  $W(n) \geq W(n')$  is obtained by applying the equivalence described in (7).<sup>11</sup>

To prove the reverse implication, we prove that  $\neg(e) \implies \neg(d)$ . Assume that  $W \notin \mathcal{W}(\mathcal{T})$ , so that there exist  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $n - n' \in \mathcal{T}$  and  $W(n) < W(n')$ . Clearly for two such  $n$  and  $n'$ , we have  $n \succeq_{\mathcal{T}} n'$  and  $W(n) < W(n')$ , so that statement (d) is not true for all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$ .  $\square$

<sup>10</sup> Statement (d) formally defines the notion of ‘function  $W$  consistent with  $\succeq_{\mathcal{T}}$ ’. Note that a function consistent with a preorder is sometimes called *order-preserving* (see Marshall *et al.*, 1967).

<sup>11</sup> It is worth emphasizing that this last remark refers to the *phantom individuals* which are sometimes required to properly characterize a sequence of transfers in some specific cases such as, for instance, in Gravel and Moyes (2012) or Decancq (2012).

In what follows, we focus on the utilitarian social welfare functions, the prevailing approach used in the literature to compare distributions. Because the sets of welfare-improving transfers we consider are discrete cones, this subclass of  $\mathcal{W}(\mathcal{T})$  plays, actually, a particular role. Indeed, we establish in our main result (following section) that it is possible to identify a class of utilitarian social welfare functions such that the unanimity of the ranking among this class is necessary but also sufficient to have dominance according to  $\succeq_{\mathcal{T}}$ . Intuitively, the independence property satisfied by the discrete cones and the fact that the utilitarian approach involves additive separable functions can, at least partially, explain this relationship.

We use  $u = (u_s)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$  to denote a vector that assigns utility  $u_s \in \mathbb{R}$  to any outcome  $s \in \mathcal{S}$ . In the utilitarian framework, utilities are usually defined up to an increasing affine transformation. As  $(0, \dots, 0) \in \mathcal{S}$  is the origin of  $\mathcal{S}$ , we propose to normalize  $u_{(0, \dots, 0)} = 0$  and to define the utilities up to an increasing linear transformation. The set of such  $u$  is denoted by  $\mathcal{U}$ . Given  $u \in \mathcal{U}$ , a *utilitarian social welfare function* is defined, for all  $n \in \mathbb{Z}_+^{|\mathcal{S}|}$ , by  $W(n) = \sum_{s \in \mathcal{S}} n_s u_s$ . We now adapt the notion of consistency with  $\succeq_{\mathcal{T}}$  to this class of functions. We let:

$$\mathcal{U}(\mathcal{T}) = \left\{ u \in \mathcal{U} \mid \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall n, n' \in \mathbb{Z}_+^{|\mathcal{S}|} : n - n' \in \mathcal{T} \right\}. \quad (9)$$

In the unidimensional case it is well-know that, if  $\mathcal{T}$  is the set Pigou-Dalton transfers, then  $\mathcal{U}(\mathcal{T})$  is the set of concave utility functions. Notice that  $\mathcal{U}(\mathcal{T})$  has also the structure of a convex cone. Then, one immediately observes that the consistency of a utilitarian social welfare function with  $\succeq_{\mathcal{T}}$  is obtained if and only if  $u \in \mathcal{U}(\mathcal{T})$ , as stated in the following result. The proof is similar to the previous one, thus omitted.

**Proposition 3.** *The following two statements are equivalent:*

- (d)  $\forall n, n' \in \mathbb{Z}_+^{|\mathcal{S}|} : n \succeq_{\mathcal{T}} n' \implies \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s,$
- (e)  $u \in \mathcal{U}(\mathcal{T}).$

For specific set of transfers  $\mathcal{T}$ , the subclasses  $\mathcal{W}(\mathcal{T})$  and  $\mathcal{U}(\mathcal{T})$  are usually not too difficult to identify. The strength of the approach described in this paper is that, having these subclasses in hand we obtain, with only a few (duality) arguments, equivalences between statements (a), (b1) and (b2) stated in our generalized HLP theorem (Section 5).

#### 4. Duality relations and bases

It is advised, before reading this section, to see first the mathematical concepts presented in Appendix B (they are required to understand the proofs).

We introduce this section by the notion of *minimality* of a set of transfers. Welfare-improving transfers are usually defined in such a way that they involve a small number of individuals, and that their ethical meaning is transparent. However, the transparency of a transfer does not guarantee that the transformation underlying the transfer is minimal from a mathematical point of view. This distinction is important in our context. Here we appeal to the notion of Hilbert basis.

**Proposition 4.** *Let  $\mathcal{C}(\mathcal{T})$  denote the convex cone generated by  $\mathcal{T}$ . This set is pointed and admits a unique Hilbert basis  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$ . Moreover any transfer in  $\mathcal{T}$  can be written as a linear combination of elements in  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$ , with non-negative integer coefficients.*

**Proof.** By using the arguments used in Proposition 1 for  $\mathcal{D}(\mathcal{T})$ , one observes that  $\mathcal{C}(\mathcal{T})$  is also pointed. As  $\mathcal{C}(\mathcal{T})$  is generated by a set  $\mathcal{T} \subset \mathbb{Z}^{|\mathcal{S}|}$ , it is a rational cone. Thus, from Lemma 2 in Appendix B,  $\mathcal{C}(\mathcal{T})$  is generated by a unique Hilbert basis  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$ . The second assertion is a direct consequence of the fact that  $\mathcal{T} \subset \mathcal{C}(\mathcal{T}) \cap \mathbb{Z}^{|\mathcal{S}|}$ , and the definition of an Hilbert basis.  $\square$

It follows that the Hilbert basis  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$  can be viewed as a set of irreducible elements that can be used to construct any transfer in  $\mathcal{T}$ . Actually, the proof of our main theorem requires that the set of transfers we consider includes these elements (see Proposition 8). A set of transfers consistent with this property is referred hereafter as a *minimal set of transfers*.

**Definition 3** (Minimal set of transfers). *We say that the set of transfers  $\mathcal{T}$  is minimal if and only if it includes its irreducible elements, in the following sense:  $\mathcal{H}(\mathcal{C}(\mathcal{T})) \subseteq \mathcal{T}$ .<sup>12</sup>*

The last definition is easily understandable when we want to test whether one distribution is obtained (or not) from another by a sequence of transfers. As an illustration, consider again the notion of increment in the bidimensional case ( $\mathcal{S} \subset \mathbb{Z}_+^2$ ). First, suppose that the minimality requirement is not satisfied. For instance, consider that an increment is defined in a such way that it involves at least two individuals. We obtain the same definition as presented in Equations (3) and (4), but with 2 instead of 1 in Equation (4). The set of such transfers is denoted  $\mathcal{T}'_I$ . For any  $m \in \mathcal{T}'_I$ , the transfer  $m/2$  is equivalent to the increment  $m$  but involving only one individual. One remarks that  $m/2 \in \mathcal{C}(\mathcal{T}) \cap \mathbb{Z}^{|\mathcal{S}|}$ , but  $m/2 \notin \mathcal{T}'_I$ . If we reconsider the initial set of increments  $\mathcal{T}_I$  (as defined in (3) and (4)), then  $m/2 \in \mathcal{T}_I$ . More generally, any transfer in  $\mathcal{T}'_I$  can be decomposed by a sequence of transfers in  $\mathcal{T}_I$ , but the reverse implication is false. As the ethical meaning of  $\mathcal{T}_I$  and  $\mathcal{T}'_I$  is equivalent, it is interesting to use the ‘greater flexibility’ of  $\mathcal{T}_I$ . Here  $\mathcal{T}_I$  is defined as a minimal set (which is not true for  $\mathcal{T}'_I$ ).

We emphasize that a minimal set of transfers can be larger than a set of irreducible elements. For instance, the Hilbert basis of the convex cone generated by set of increments  $\mathcal{T}_I$  is such that

---

<sup>12</sup> One immediately observes that a set of transfers  $\mathcal{T}$  is minimal if and only if  $\mathcal{C}(\mathcal{T}) \cap \mathbb{Z}^{|\mathcal{S}|} = \mathcal{D}(\mathcal{T})$ . This last condition is related to the ‘Integral Carathéodory Property’ (see Firla and Ziegler (1999)).

$\mathcal{H}(\mathcal{C}(\mathcal{T}_I)) \subset \mathcal{T}_I$ . Precisely,  $\mathcal{H}(\mathcal{C}(\mathcal{T}_I))$  is the set of all transfers as defined in Equations (3) and (4) but such that the points  $(i, j), (k, l) \in \mathbb{Z}_+^2$  are ‘adjacent’, or equivalently such that  $(k, l) = (i+1, j)$  or  $(k, l) = (i, j+1)$ . Of course, the presentation of the results is simplified if the set of transfers coincides with its Hilbert basis, or equivalently if  $\mathcal{T} = \mathcal{H}(\mathcal{C}(\mathcal{T}))$ .

We now investigate the *duality relation* which exists between  $\mathcal{T}$  and  $\mathcal{U}(\mathcal{T})$ , the set of utility functions consistent with transfers in  $\mathcal{T}$ . This duality is the cornerstone of the proof of our main result. Moreover, as  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$  has been identified as a basis of  $\mathcal{T}$ , the duality will be used to identify the basis of  $\mathcal{U}(\mathcal{T})$ . We first need the following results.

**Lemma 1.** *Let  $\mathcal{M}$  be the set of all  $m \in \mathbb{R}^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} m_s = 0$ , and let  $b(m, u) = \sum_{s \in \mathcal{S}} m_s u_s$  for all  $m \in \mathbb{R}^{|\mathcal{S}|}$ . Then  $(\mathcal{M}, \mathcal{U}; b)$  is a strict dual pair.*

**Proof.**  $\mathcal{M}$  and  $\mathcal{U}$  are two vector spaces, and  $b : \mathcal{M} \times \mathcal{U} \rightarrow \mathbb{R}$  is a bilinear mapping. Hence,  $(\mathcal{M}, \mathcal{U}; b)$  is a dual pair. We still need to prove that it is also strict. Because  $\sum_{s \in \mathcal{S}} m_s = 0$  for all  $m \in \mathcal{M}$ , one observes that, if  $u = (\alpha, \dots, \alpha) \in \mathcal{U}$ , then  $b(m, u) = 0$  for all  $m \in \mathcal{M}$ . Nevertheless, as  $u_{(0, \dots, 0)} = 0$  (by normalization),  $u = (\alpha, \dots, \alpha)$  only if  $\alpha = 0$ . Then, one easily deduces that for each  $u \in \mathcal{U}$  (excluding the 0 vector) there is an  $m \in \mathcal{M}$  with  $b(m, u) \neq 0$ , and for each  $m \in \mathcal{M}$  (excluding the 0 vector) there is a  $u \in \mathcal{U}$  with  $b(m, u) \neq 0$ . One concludes that  $(\mathcal{M}, \mathcal{U}; b)$  is a strict dual pair. See Marshall *et al.* (2011), Page 99 or Muller (2013), Page 50, for related discussions.  $\square$

**Proposition 5.** *Under the duality  $(\mathcal{M}, \mathcal{U}; b)$ , the polar cone of  $\mathcal{T}$  is such that  $\mathcal{T}^\circ = \mathcal{D}(\mathcal{T})^\circ = \mathcal{U}(\mathcal{T})$ , and the bipolar cone of  $\mathcal{T}$  is such that  $\mathcal{T}^{\circ\circ} = \mathcal{U}(\mathcal{T})^\circ = \mathcal{C}(\mathcal{T})$ . We recall that  $\mathcal{D}(\mathcal{T}) \subset \mathcal{C}(\mathcal{T})$ .*

**Proof.** By definition,  $\mathcal{T} \subset \mathcal{M}$ . The polar cone of  $\mathcal{T}$ , under the duality  $(\mathcal{M}, \mathcal{U}; b)$ , is defined by:

$$\mathcal{T}^\circ = \{u \in \mathcal{U} \mid b(m, u) \geq 0, \forall m \in \mathcal{T}\} . \quad (10)$$

For any  $m \in \mathcal{T}$ , one can always find two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  with  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , such that  $m = n - n'$ . From the definition of  $\mathcal{U}(\mathcal{T})$  in (9) and  $b(n - n', u)$ , one immediately observes that  $\mathcal{U}(\mathcal{T}) = \mathcal{T}^\circ$ . Moreover, for any  $u \in \mathcal{U}$ , if  $b(m, u) \geq 0$  for all  $m \in \mathcal{T}$ , then  $b(m, u) \geq 0$  for all  $m \in \mathcal{D}(\mathcal{T})$  (see Definition 2). Because  $\mathcal{D}(\mathcal{T}) \subset \mathcal{M}$ , we also have  $\mathcal{T}^\circ = \mathcal{D}(\mathcal{T})^\circ$ . Then,  $\mathcal{T}^{\circ\circ} = \mathcal{U}(\mathcal{T})^\circ$  immediately follows from  $\mathcal{T}^\circ = \mathcal{U}(\mathcal{T})$ . Finally, by applying the bipolar theorem (Lemma 5), we know that  $\mathcal{T}^{\circ\circ}$  is the smallest closed set containing the convex cone generated by  $\mathcal{T}$ , which can be written as  $\mathcal{C}(\mathcal{T}) = \text{co}\{\lambda m \mid \lambda \in \mathbb{R}_+, m \in \mathcal{T}\}$ . Note that, as  $\mathcal{T}$  is a discrete and finite set,  $\mathcal{C}(\mathcal{T})$  is closed.  $\square$

A feature of our fully discrete framework is the strict inclusion  $\mathcal{T} \subset \mathcal{T}^{\circ\circ}$ , as deduced from Proposition 5 (which is the case in general for such duality relation). This is an important difference

with, for instance, Meyer and Strulovici (2013) or Muller (2013) who have, in their respective frameworks,  $\mathcal{T} = \mathcal{T}^{\circ\circ}$ . Hence the equivalence obtained in Theorem 1 below need more arguments, and the notion of Hilbert basis  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$  plays a central role.

We focus now on the characteristics of the set of utility functions  $\mathcal{U}(\mathcal{T})$ . The following proposition establishes that  $\mathcal{U}(\mathcal{T})$ , a convex cone which contains an infinite number of elements, has actually a finite structure. This property is necessary to obtain an implementation criterion, such as the Lorenz criterion in the standard HLP Theorem.

**Proposition 6.** *The set  $\mathcal{U}(\mathcal{T})$  of utility functions is finitely generated. Equivalently, there exists a finite set of independent utility vectors, denoted  $\mathcal{U}^*(\mathcal{T}) \subset \mathcal{U}(\mathcal{T})$ , such that any  $u \in \mathcal{U}(\mathcal{T})$  can be written as a (positive) linear combination of these vectors.*

**Proof.** As the convex cone  $\mathcal{U}(\mathcal{T})$  can be written  $\mathcal{U}(\mathcal{T}) = \{u \in \mathcal{U} \mid \sum_{s \in \mathcal{S}} m_s u_s \geq 0, \forall m \in \mathcal{T}\}$  and  $\mathcal{T}$  is a finite set,  $\mathcal{U}(\mathcal{T})$  is also polyhedral (it is the intersection of finitely many closed halfspaces). From the Farkas-Minkowski-Weyl Theorem (see Schrijver, 1986, Corollary 7.1.a, Page 87), we know that a polyhedral cone is finitely generated.  $\square$

We deduce from the previous result that if we want to compare two outcome distributions according to the entire class of utility functions  $\mathcal{U}(\mathcal{T})$ , it suffices to compare these distributions for the finite list of vectors which generate  $\mathcal{U}(\mathcal{T})$ . A strategy to characterize the generating system of this set is presented in Section 6. This strategy is then illustrated in the case of increments, and combinations of increments and Pigou-Dalton progressive transfers.

We conclude this section by noting that the sets  $\mathcal{C}(\mathcal{T})$  and  $\mathcal{U}(\mathcal{T})$ , which are convex cones related by duality, can have different geometrical aspect. Indeed, even if  $\mathcal{C}(\mathcal{T})$  is by definition pointed, this is not necessarily the case for  $\mathcal{U}(\mathcal{T})$ . Below we present the necessary and sufficient condition to have  $\mathcal{U}(\mathcal{T})$  pointed (See Bruns and Gubeladze, 2009, Proposition 1.19, Page 14).

**Proposition 7.** *The convex cone  $\mathcal{U}(\mathcal{T})$  is pointed if and only if  $\dim \mathcal{C}(\mathcal{T}) = \dim \mathcal{U}(\mathcal{T})$ .*

Because utility functions are such that  $u = (u_s)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$  with  $u_s = 0$  for  $s = (0, \dots, 0) \in \mathcal{S}$ , the dimension of  $\mathcal{U}(\mathcal{T})$  (over the field  $\mathbb{R}$ ) is  $\dim \mathcal{U}(\mathcal{T}) = \dim \mathcal{U} = |\mathcal{S}| - 1$ . In the examples provided in Section 6, the condition identified in Proposition 7 is satisfied. Thus the corresponding sets of utility functions are pointed. Actually, this is the case for most of the sets of transfers considered in the literature. The discussion in Section 6, related to the dimensions of  $\mathcal{C}(\mathcal{T})$  and  $\mathcal{U}(\mathcal{T})$ , will clarify this remark.



## 5. Main result

We are now ready to present our main result, which is a generalization of the Hardy-Littlewood-Polya Theorem. The equivalence between statements (a) and (b2) below is a full discretization of Theorem 2.4.1 provided by Muller (2013), page 51. Indeed, our distributions are discrete in the sense that we count the individuals (values in  $\mathbb{Z}$ ) in each possible discrete and finite outcome, whereas Muller (2013) considers probability measures (values in  $\mathbb{R}$ ). It follows that the sequence of transfers we consider is defined as a discrete cone rather than a convex cone. The core argument in the proof is an application of the bipolar theorem but with a novel feature, the introduction of the concept of Hilbert basis. Note that closely related results –comparable to the equivalence between (a) and (b2) below– can be found in Marshall (1991), Proposition 3.4, Page 237 and Meyer and Strulovici (2013), Theorem 1, Page 7. The equivalence between (a) and (b1) is an extension outside the utilitarian realm. Finally, (c) is an implementation criterion which generalizes Lorenz dominance in the usual HLP theorem. We recall that  $\mathcal{U}^*(\mathcal{T})$  is the finite set of independent utility functions which generates  $\mathcal{U}(\mathcal{T})$  (see Proposition 6).

**Theorem 1.** *Let  $\mathcal{T}$  be a minimal set of transfers. For all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , the following four statements are equivalent:*

$$(a) \ n \succeq_{\mathcal{T}} n',$$

$$(b1) \ W(n) \geq W(n'), \forall W \in \mathcal{W}(\mathcal{T}),$$

$$(b2) \ \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall u \in \mathcal{U}(\mathcal{T}),$$

$$(c) \ \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall u \in \mathcal{U}^*(\mathcal{T}).$$

**Proof.** We prove (a)  $\Leftrightarrow$  (b1), (a)  $\Leftrightarrow$  (b2) and (b2)  $\Leftrightarrow$  (c). The proof of (a)  $\Rightarrow$  (b1) is, actually, equivalent to the proof of (e)  $\Rightarrow$  (d) in Proposition 2. Equivalently, (a)  $\Rightarrow$  (b2) is obtained from (e)  $\Rightarrow$  (d) in Proposition 3. Then, (c)  $\Rightarrow$  (b2) is a direct consequence of Proposition 6. (b2)  $\Rightarrow$  (c) follows from the fact that  $\mathcal{U}^*(\mathcal{T}) \subset \mathcal{U}(\mathcal{T})$ . It remains to prove (b1)  $\Rightarrow$  (a) and (b2)  $\Rightarrow$  (a).

(b1)  $\Rightarrow$  (a). The argument exactly follows Marshall (1991), Proposition 2.8, Page 234. First, recall that  $n \succeq_{\mathcal{T}} n'$  if and only if  $n - n' \in \mathcal{D}(\mathcal{T})$ , where  $\mathcal{D}(\mathcal{T})$  is the discrete cone generated by  $\mathcal{T}$ . For all  $z \in \mathbb{Z}_+^{|\mathcal{S}|}$ , we choose  $W_n(z) = 0$  if  $n - z \in \mathcal{D}(\mathcal{T})$  and  $W_n(z) = 1$  otherwise. We first establish that  $W_n \in \mathcal{W}(\mathcal{T})$ . Consider  $u, v \in \mathbb{Z}_+^{|\mathcal{S}|}$  and assume that  $u \succeq_{\mathcal{T}} v$ . If  $W_n(u) = 1$ , then  $W_n(u) \geq W_n(v)$  because, by definition,  $W_n(v) \in \{0, 1\}$ . Now, suppose that  $W_n(u) = 0$ , or equivalently,  $n - u \in \mathcal{D}(\mathcal{T})$ . Because  $u \succeq_{\mathcal{T}} v$ , we also have  $u - v \in \mathcal{D}(\mathcal{T})$ . Because  $\mathcal{D}(\mathcal{T})$  is a discrete cone, it follows that  $(n - u) + (u - v) = n - v \in \mathcal{D}(\mathcal{T})$ . Hence,  $W_n(v) = 0$ , so that, again,  $W_n(u) \geq W_n(v)$ . We conclude that  $u \succeq_{\mathcal{T}} v$  implies  $W_n(u) \geq W_n(v)$ . Hence,  $W_n \in \mathcal{W}(\mathcal{T})$ .

Now, consider two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$  and assume that statement (b1) is true. It follows that  $W_n(n) \geq W_n(n')$ . By definition,  $n - n' = 0 \in \mathcal{D}(\mathcal{T})$ . Hence,  $W_n(n) = 0$ , which implies that  $W_n(n') = 0$  or equivalently,  $n - n' \in \mathcal{D}(\mathcal{T})$ . Thus,  $n \succeq_{\mathcal{T}} n'$ .

(b2)  $\Rightarrow$  (a). Let  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , assuming that (b2) is true. Recall that  $\mathcal{M}$  is the set of all  $m \in \mathbb{R}^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} m_s = 0$ , and  $b(m, u) = \sum_{s \in \mathcal{S}} m_s u_s$  for all  $m \in \mathbb{R}^{|\mathcal{S}|}$ . The polar cone of  $\mathcal{U}(\mathcal{T}) \subset \mathcal{U}$ , under the duality  $(\mathcal{M}, \mathcal{U}; b)$ , is defined by:

$$\mathcal{U}(\mathcal{T})^\circ = \{m \in \mathcal{M} \mid b(m, u) \geq 0, \forall u \in \mathcal{U}(\mathcal{T})\}. \quad (11)$$

By definition,  $n - n' \in \mathcal{M}$ . From statement (b2), we also have  $b(n - n', u) = \sum_{s \in \mathcal{S}} (n_s - n'_s) u_s \geq 0$  when  $u \in \mathcal{U}(\mathcal{T})$ . Thus  $n - n' \in \mathcal{U}(\mathcal{T})^\circ$ . From Lemma 4 we know that  $\mathcal{U}(\mathcal{T})^\circ = \mathcal{C}(\mathcal{T})$ , where  $\mathcal{C}(\mathcal{T})$  is the convex cone generated by  $\mathcal{T}$ , with  $\mathcal{T} \subset \mathbb{Z}^{|\mathcal{S}|}$ . Because integers are rational numbers,  $\mathcal{C}(\mathcal{T})$  is a rational cone. By applying Lemma 2, we know that  $\mathcal{C}(\mathcal{T})$  is generated by a (unique) Hilbert basis  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$ . Because the set  $\mathcal{T}$  is minimal (see Definition 3), we have  $\mathcal{H}(\mathcal{C}(\mathcal{T})) \subseteq \mathcal{T}$ . Then, because  $n - n' \in \mathcal{C}(\mathcal{T})$  and  $n - n' \in \mathbb{Z}^{|\mathcal{S}|}$ , one has  $n - n' \in \mathcal{C}(\mathcal{T}) \cap \mathbb{Z}^{|\mathcal{S}|}$ . By definition of an Hilbert basis, such  $n - n'$  can be expressed as a linear combination of elements in this basis (a subset of  $\mathcal{T}$ ) with coefficients in  $\mathbb{Z}_+$ . One deduces that  $n - n' \in \mathcal{D}(\mathcal{T})$  or, equivalently, that  $n \succeq_{\mathcal{T}} n'$ .  $\square$

First, we justify the initial statements of the theorem. In Remark 2, we have established that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$  is a necessary condition for  $n \succeq_{\mathcal{T}} n'$ . In the same way, the minimality of  $\mathcal{T}$  is required. Precisely, the minimality of  $\mathcal{T}$  is sufficient but also necessary to have (b2) implies (a), as established in Proposition 8 (an alternative theorem, without the minimality requirement, is presented in Appendix A).

**Proposition 8.** *The following two statements are equivalent:*

- (d) *For all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$  and all  $u \in \mathcal{U}(\mathcal{T})$ ,  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  implies  $n \succeq_{\mathcal{T}} n'$ ,*
- (e)  *$\mathcal{T}$  is a minimal set of transfers.*

**Proof.** (e)  $\Rightarrow$  (d). This implication, which establishes that the minimality of  $\mathcal{T}$  is sufficient to have (b2) implies (a) in Theorem 1, has been proved in the theorem.

(d)  $\Rightarrow$  (e). We argue a contrario and prove that if  $\mathcal{T}$  is not a minimal set of transfers, then there exist  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  with  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , and  $u \in \mathcal{U}(\mathcal{T})$ , such that  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  and  $\neg \{n \succeq_{\mathcal{T}} n'\}$ . First, suppose that  $u \in \mathcal{U}(\mathcal{T})$ . By construction, we have  $\sum_{s \in \mathcal{S}} m_s u_s \geq 0$  for any  $m \in \mathcal{C}(\mathcal{T})$ . Because  $\mathcal{H}(\mathcal{C}(\mathcal{T})) \subset \mathcal{C}(\mathcal{T})$ , we also have  $\sum_{s \in \mathcal{S}} m_s u_s \geq 0$  for any  $m \in \mathcal{H}(\mathcal{C}(\mathcal{T}))$ . Then, assume that  $\mathcal{T}$  is not minimal, so that there exist some  $m \in \mathcal{H}(\mathcal{C}(\mathcal{T}))$  with  $m \notin \mathcal{T}$ . Hence,

any element in  $\mathcal{T}$  can be obtained by a linear combination (with non-negative integer coefficients) of elements in  $\mathcal{H}(\mathcal{C}(\mathcal{T}))$ , but the converse is false. It follows that there exist some  $m \in \mathcal{H}(\mathcal{C}(\mathcal{T}))$  with  $m \notin \mathcal{D}(\mathcal{T})$ . Take such  $m$  and choose  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $m = n - n'$  (by definition,  $\sum_{s \in \mathcal{S}} (n_s - n'_s) = 0$ ). We have  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  and  $u \in \mathcal{U}(\mathcal{T})$ , but also  $n - n' \notin \mathcal{D}(\mathcal{T})$ .  $\square$

We now discuss the theorem. Considering two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$ , we know from Proposition 2 that  $W \in \mathcal{W}(\mathcal{T})$  and  $n \succeq_{\mathcal{T}} n'$  is sufficient to have  $W(n) \geq W(n')$ . The equivalence between (a) and (b1) in the previous theorem establishes that ‘ $n$  weakly better than  $n'$  according to all social welfare functions in  $\mathcal{W}(\mathcal{T})$ ’ is, actually, also sufficient to have  $n \succeq_{\mathcal{T}} n'$ . This equivalence allows us to go a step further to compare the distributions  $n$  and  $n'$  according to  $\succeq_{\mathcal{T}}$ , as it establishes the equivalence with an a priori distinct preorder based on the unanimity of the rankings among a group of social planners. Nevertheless,  $\mathcal{W}(\mathcal{T})$  encompasses a large class of social welfare functions, and the required unanimity cannot be checked. Indeed, the set  $\mathcal{W}(\mathcal{T})$  is (uncountably) infinite because, by definition, any strictly increasing transformation of a function  $W \in \mathcal{W}(\mathcal{T})$  is also in  $\mathcal{W}(\mathcal{T})$ .

We recall that  $\mathcal{W}(\mathcal{T})$  has the structure of a convex cone. Hence, (a)  $\Leftrightarrow$  (b1) is an equivalence between two preorders of a different nature. Whereas the preorder in statement (b1) is generated by a convex cone, the preorder in statement (a) is generated by a discrete cone.<sup>13</sup> Nevertheless one remarks that a common characteristic of these two preorders is that they satisfy a similar independence property (see Remark 1 and Equation 7). Because  $\mathcal{W}(\mathcal{T})$  has the structure of a convex cone, one objective can be to identify the minimal set of social welfare functions (namely, the basis), which characterizes  $\mathcal{W}(\mathcal{T})$ , if such a set exists. The equivalence between (a) and (b2) is a first step in this direction.

It was fairly immediate to establish that the unanimity of the ranking of  $n$  over  $n'$  among all  $W \in \mathcal{W}(\mathcal{T})$  is a necessary and sufficient condition to have  $n \succeq_{\mathcal{T}} n'$ . However, such a result is not so clear when the unanimity among only utilitarian social welfare functions is achieved. Indeed, the utilitarian class consistent with  $\succeq_{\mathcal{T}}$  is a strict subclass of  $\mathcal{W}(\mathcal{T})$ , and it is not obvious that such a unanimity, which, in some sense, is relatively small, is sufficient to have  $n \succeq_{\mathcal{T}} n'$ . This is what we prove here, as the main result of the paper.<sup>14</sup> Notice that  $\mathcal{U}(\mathcal{T})$  is also a convex cone, as expected. Notice also that constant utilities, such that  $u_s = c$  for all  $s \in \mathcal{S}$  and some  $c \in \mathbb{R}$ , belong to  $\mathcal{U}(\mathcal{T})$ .

Finally, statement (c) is useful for empirical applications on real data. As the sets  $\mathcal{W}(\mathcal{T})$

---

<sup>13</sup>The fact that two cone orderings of different natures appear to be equivalent does not come as a surprise. By using the terminology of Marshall (1991), the cone ordering induced by  $\mathcal{W}(\mathcal{T})$  is called the *completion* of  $\succeq_{\mathcal{T}}$ .

<sup>14</sup>As illustrated in Section 9, one notices that this result is not true when  $\succeq_{\mathcal{T}}$  is not a discrete (or convex) cone. This result confirms, again, that the definition of the sequence of transfers as a cone is a crucial assumption in our framework.

and  $\mathcal{U}(\mathcal{T})$  contain an infinite number of elements, dominance according to statement (b1) or (b2) cannot be applied in practice on real distributions. The last statement says that there exists a finite and discrete subset of  $\mathcal{U}(\mathcal{T})$  which ensures that if one distribution dominates another according to this subset, then the first distribution dominates the other according to the larger sets  $\mathcal{W}(\mathcal{T})$  and  $\mathcal{U}(\mathcal{T})$ . Of course, at this level of abstraction, we only have an existence result. We propose, in the following section, a concrete strategy to identify this subset for particular sets  $\mathcal{U}(\mathcal{T})$ .

We conclude this section by briefly addressing the issue of distributions of different population sizes. All our results can be easily extended by appealing to the Dalton principle of population, according to which an identical replication of the population is distributionally equivalent to the initial one. We denote by  $n(r) = (n_s(r))_{s \in \mathcal{S}}$  with  $r \in \mathbb{Z}_{++}$  the  $r$ -times replication of distribution  $n$ , such that  $n_s(r) = r n_s$ . Applied to a class of social welfare functions, this principle requires that  $W(n) = W(n(r))$ . Now consider two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = r$  and  $\sum_{s \in \mathcal{S}} n'_s = r'$  for some  $r, r'$  with  $r \neq r'$ . In order to apply our results, we compare the distributions  $n(r')$  and  $n'(r)$ , noting that  $\sum_{s \in \mathcal{S}} n_s(r') = \sum_{s \in \mathcal{S}} n'_s(r) = r r'$ . Then, the comparison of  $n$  and  $n'$  is obtained by applying the principle of population, noting that:

$$W(n(r')) \geq W(n'(r)) \iff W(n) \geq W(n'). \quad (12)$$

As an illustration in the utilitarian realm, one has to consider the average of the utilities (instead of the sum).

The following section is focused on the equivalence between (b2) and (c) in Theorem 1. Precisely, we present a strategy to identify the finite subset  $\mathcal{U}^*(\mathcal{T}) \subset \mathcal{U}(\mathcal{T})$ , when  $\mathcal{U}(\mathcal{T})$  is pointed.<sup>15</sup> Then, we discuss the related literature. The applications in the following sections are focused on the equivalence between (a), (b1) and (b2).

## 6. Looking for an implementation criterion: a strategy for pointed convex cones

Assume that  $\mathcal{U}(\mathcal{T})$  is a pointed convex cone. Because  $\mathcal{U}(\mathcal{T})$  is a cone, we know that any non-negative linear transformation of a utility function  $u \in \mathcal{U}(\mathcal{T})$  is also in  $\mathcal{U}(\mathcal{T})$ . The set of non-negative linear transformations of  $u \in \mathcal{U}(\mathcal{T})$ , denoted by  $\langle u \rangle = \{v \in \mathcal{U}(\mathcal{T}) \mid v = \lambda u, \lambda \in \mathbb{R}_+\}$ , is called a *ray* of  $\mathcal{U}(\mathcal{T})$ . Moreover, if  $u$  is one of the functions which generate  $\mathcal{U}(\mathcal{T})$  (see Proposition 6), then  $u$  is called an *extreme point* of  $\mathcal{U}(\mathcal{T})$ , and  $\langle u \rangle$  is an *extreme ray*. The following result is a corollary of Proposition 3.2.2, Page 46, in Florenzano and Le Van (2001).

---

<sup>15</sup> Actually, almost all the sets of transfers  $\mathcal{T}$  considered in the literature lead to a pointed convex cone  $\mathcal{U}(\mathcal{T})$ .

**Proposition 9.** *Let  $\mathcal{U}(\mathcal{T}) = \{u \in \mathcal{U} \mid \sum_{s \in \mathcal{S}} m_s u_s \geq 0, m \in \mathcal{T}\}$  be a pointed convex cone. If the system  $\sum_{s \in \mathcal{S}} m_s u_s = 0$ , for all  $m \in \mathcal{T}$ , has a subsystem of rank  $|\mathcal{S}| - 2$ ,<sup>16</sup> then the solution of this subsystem is an extreme ray of  $\mathcal{U}(\mathcal{T})$  if this solution is also in  $\mathcal{U}(\mathcal{T})$  (hence satisfying the other inequalities characterizing this set).*

As a consequence, the set  $\mathcal{U}^*(\mathcal{T})$  we are looking for can be found by taking one point (one utility function) in each of all the possible extreme rays of  $\mathcal{U}(\mathcal{T})$ .

We illustrate this strategy in the following framework. Consider a unidimensional and cardinal variable, for instance income, taking the possible values  $\mathcal{S} = \{0, 1, \dots, K\}$ , with  $K \geq 2$ . A distribution is a list  $n = (n_0, \dots, n_K) \in \mathbb{Z}_+^{K+1}$  and the mean of  $n$  is  $\mu(n) = \sum_{s \in \mathcal{S}} n_s s$ . We will consider two particular sets  $\mathcal{U}(\mathcal{T})$ , precisely the set of increasing utility functions and the set of increasing and concave utility ones. First we recall that the set of increments  $\mathcal{T}_I$ , as introduced in Section 2, can be written here as the set of all vectors  $m \in \mathbb{Z}^{K+1}$  such that there exist  $i < K - 1$  with  $m_i = -1$ ,  $m_{i+1} = 1$  and  $m_s = 0$  for all  $s \neq i, i + 1$ .<sup>17</sup> As well-known,  $\mathcal{U}(\mathcal{T}_I)$  is the set of increasing utility functions.

Then, we say that distribution  $n$  is obtained from distribution  $n'$  by means of a *Pigou-Dalton progressive income transfer (PT)* if some income is taken from an individual and is given to a poorer individual (it is a mean-preserving transfer), without reversing their positions on the income scale. It follows that the set of progressive income transfers, denoted by  $\mathcal{T}_{PT}$ , is the set of all vectors  $m \in \mathbb{Z}^{K+1}$  such that there exist  $i, j \in \mathcal{S}$  with  $i + 1 < j$  and:

$$m_s = 0, \text{ for all } s \in \mathcal{S} \setminus \{i, i + 1, j - 1, j\}, \quad (13)$$

$$m_i = m_j = -1, \quad (14)$$

$$m_{i+1} = m_{j-1} = 1 \text{ if } i + 1 \neq j - 1, \quad \text{and} \quad m_{i+1} = 2 \text{ if } i + 1 = j - 1. \quad (15)$$

Again, we know that  $\mathcal{U}(\mathcal{T}_{PT})$  is the set of concave utility functions. In what follows we will not be interested in  $\mathcal{U}(\mathcal{T}_{PT})$  by itself, but by the set of utility functions consistent both with increments and progressive transfers (hence the set of increasing and concave utility functions) defined by  $\mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT}) = \mathcal{U}(\mathcal{T}_I) \cap \mathcal{U}(\mathcal{T}_{PT})$ . Notice that the sets  $\mathcal{T}_I$  and  $\mathcal{T}_I \cup \mathcal{T}_{PT}$  are consistent with the framework described in Section 2. Thus all our previous results apply.

For the sake of clarity, we illustrate Proposition 9 when  $K = 3$  (it is easily generalizable for any fixed  $K$ ), and we start with  $\mathcal{T}_I$ . Because  $\mathcal{S} = \{0, 1, 2, 3\}$ , we have  $\mathcal{T}_I = \{m_{.1}, m_{.2}, m_{.3}\}$ , where  $m_{.1} = (-1, 1, 0, 0)$ ,  $m_{.2} = (0, -1, 1, 0)$  and  $m_{.3} = (0, 0, -1, 1)$ . Hence,  $\dim \mathcal{C}(\mathcal{T}_I) = 3$ .

<sup>16</sup> We recall that  $\dim(\mathcal{U}) = |\mathcal{S}| - 1$ .

<sup>17</sup> Notice that  $\mathcal{T}_I$  written this way reduces to the Hilbert basis of the set of increments.

Then, a utility function  $u \in \mathcal{U}(\mathcal{T}_I)$  is written  $u = (u_0, u_1, u_2, u_3)$  with  $u_0 = 0$ . It follows that  $\dim \mathcal{U}(\mathcal{T}_I) = \dim \mathcal{C}(\mathcal{T}_I) = 3$ . One deduces from Proposition 7 that  $\mathcal{U}(\mathcal{T}_I)$  is a pointed convex cone.

Then, the set  $\mathcal{U}(\mathcal{T}_I)$  is characterized by the inequalities  $\sum_{s \in \mathcal{S}} m_{st} u_s \geq 0$ ,  $t = 1, 2, 3$ . From Proposition 9, an extreme ray is obtained as follows. First we replace  $|\mathcal{S}| - 2 = 2$  inequalities (among the three) by equalities. Then we have to find the solution of this system, taking care that the other restrictions of  $\mathcal{U}(\mathcal{T}_I)$  are satisfied (including  $u_0 = 0$ ). We have three possibilities, each of them leading to one extreme ray. The possibilities are:

$$0 = u_0 = u_1 = u_2 < u_3; \tag{16a}$$

$$0 = u_0 = u_1 < u_2 = u_3; \tag{16b}$$

$$0 = u_0 < u_1 = u_2 = u_3. \tag{16c}$$

One deduces the following three extreme rays, respectively:  $\langle u_{.1} \rangle = \langle (0, 0, 0, 1) \rangle$ ,  $\langle u_{.2} \rangle = \langle (0, 0, 1, 1) \rangle$  and  $\langle u_{.3} \rangle = \langle (0, 1, 1, 1) \rangle$ . Thus, the set which generates  $\mathcal{U}(\mathcal{T}_I)$  can be written  $\mathcal{U}^*(\mathcal{T}_I) = \{u_{.1}, u_{.2}, u_{.3}\}$ . Aside from  $u_0 = 0$ , the set  $\mathcal{U}^*(\mathcal{T}_I)$  (only three points) and the whole pointed convex cone  $\mathcal{U}(\mathcal{T}_I)$  are illustrated in Figure 3.A.

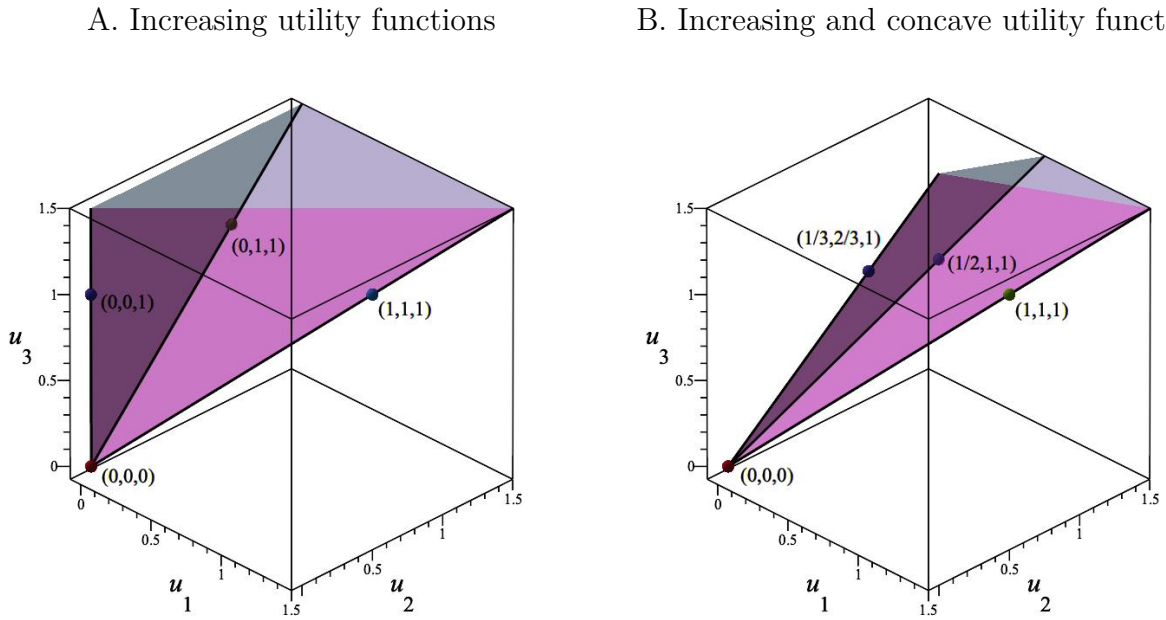


Figure 3: Utility cones for first-order and second-order stochastic dominance

Now, we are ready to apply Theorem 1. Consider two distributions  $n = (n_0, \dots, n_3)$  and  $n' = (n'_0, \dots, n'_3)$ , with  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ . We know from Theorem 1 that  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  for all  $u \in \mathcal{U}(\mathcal{T}_I)$  if and only if this inequality is true for all  $u \in \mathcal{U}^*(\mathcal{T}_I)$  or, equivalently, if and only

if  $\sum_{s \in \mathcal{S}} n_s u_{st} \geq \sum_{s \in \mathcal{S}} n'_s u_{st}$  for all  $t = 1, 2, 3$ . One obtains the following inequalities:  $n_3 \geq n'_3$ ,  $n_2 + n_3 \geq n'_2 + n'_3$  and  $n_1 + n_2 + n_3 \geq n'_1 + n'_2 + n'_3$ . Recalling that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , these three inequalities are respectively equivalent (in reverse order) to:

$$n_0 \leq n'_0, \quad (17a)$$

$$n_0 + n_1 \leq n'_0 + n'_1, \quad (17b)$$

$$n_0 + n_1 + n_2 \leq n'_0 + n'_1 + n'_2. \quad (17c)$$

Hence, the implementation criterion resulting from the statement ' $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  for all  $u \in \mathcal{U}^*(\mathcal{T}_I)$ ' is first-order stochastic dominance, as described in Equation (17).

The same strategy can be applied to  $\mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT})$ . The details are left to the reader. When  $K = 3$ , there are three extreme rays,  $\langle v_{.1} \rangle = \langle (0, 1/3, 2/3, 1) \rangle$ ,  $\langle v_{.2} \rangle = \langle (0, 1/2, 1, 1) \rangle$  and  $\langle v_{.3} \rangle = \langle (0, 1, 1, 1) \rangle$ . The set of utility functions which generates  $\mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT})$  is  $\mathcal{U}^*(\mathcal{T}_I \cup \mathcal{T}_{PT}) = \{v_{.1}, v_{.2}, v_{.3}\}$ . Again, for two distributions  $n = (n_0, \dots, n_3)$  and  $n' = (n'_0, \dots, n'_3)$  with  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , we know from Theorem 1 that  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  for all  $u \in \mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT})$  if and only if this inequality is true for all  $u \in \mathcal{U}^*(\mathcal{T}_I \cup \mathcal{T}_{PT})$  or, equivalently, if and only if  $\sum_{s \in \mathcal{S}} n_s v_{st} \geq \sum_{s \in \mathcal{S}} n'_s v_{st}$  for all  $t = 1, 2, 3$ . Because  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$  we have, for all  $t = 1, 2, 3$ :

$$\sum_{s \in \mathcal{S}} (n_s - n'_s) v_{st} \geq 0 \iff t \sum_{s \in \mathcal{S}} (n_s - n'_s) - t \sum_{s \in \mathcal{S}} (n_s - n'_s) v_{st} \leq 0. \quad (18)$$

By simplifying the last expression for all  $t = 1, 2, 3$ , one obtains:

$$n_0 \leq n'_0, \quad (19a)$$

$$n_0 + (n_0 + n_1) \leq n'_0 + (n'_0 + n'_1), \quad (19b)$$

$$n_0 + (n_0 + n_1) + (n_0 + n_1 + n_2) \leq n'_0 + (n'_0 + n'_1) + (n'_0 + n'_1 + n'_2). \quad (19c)$$

These three conditions are equivalent to second-order stochastic dominance, applied to discrete variables. The set  $\mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT})$  and  $\mathcal{U}^*(\mathcal{T}_I \cup \mathcal{T}_{PT})$  (which are, respectively, the whole cone and three points in the graph), are illustrated in Figure 6.B. We note that, as expected,  $\mathcal{U}(\mathcal{T}_I \cup \mathcal{T}_{PT}) \subset \mathcal{U}(\mathcal{T}_I)$ .

We conclude this section with some comments on the related stochastic dominance literature.<sup>18</sup> Given a set of utility functions, say  $\tilde{\mathcal{U}} \subset \mathcal{U}$ , a statement like ' $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$  for all  $u \in \tilde{\mathcal{U}}$ ' is called a stochastic dominance relation. Consider first that  $\tilde{\mathcal{U}}$  is a convex cone (which is not necessarily the case). It has been noticed, a long time ago, that this property can be used to identify a subset  $\tilde{\tilde{\mathcal{U}}} \subset \tilde{\mathcal{U}}$  such that (i) the dominance relation over  $\tilde{\tilde{\mathcal{U}}}$  is equivalent to the dominance

<sup>18</sup>I thank a referee for inviting me to discuss this literature.

relation over  $\tilde{\mathcal{U}}$ , and (ii)  $\tilde{\mathcal{U}}$  is the convex cone generated by  $\tilde{\mathcal{U}}$ . It has also been pointed out that, by application of the Krein-Milman theorem, the best strategy consists in defining  $\tilde{\mathcal{U}}$  as the set of extreme points of  $\tilde{\mathcal{U}}$  (see Meyer, 1966; Brumelle and Vickson, 1975). More recently, Athey (2000) has investigated a parametrized version of the expected utility model, where  $\tilde{\mathcal{U}}$  is not necessarily a convex cone. Nevertheless, as established in this paper, the convex cone property is still relevant: the dominance relation over  $\tilde{\mathcal{U}} \subset \tilde{\mathcal{U}}$  is equivalent to the dominance relation over  $\tilde{\mathcal{U}}$  if the convex cone generated by  $\tilde{\mathcal{U}}$  is equal to the convex cone generated by  $\tilde{\mathcal{U}}$ .

As this literature is focused on continuous (probability) distributions, the dominance relation over  $\tilde{\mathcal{U}} \subset \tilde{\mathcal{U}}$  may not be an implementable criterion for the dominance relation over  $\tilde{\mathcal{U}}$ , as  $\tilde{\mathcal{U}}$  is usually an infinite set (even if it is defined as the set of extreme points of  $\tilde{\mathcal{U}}$ ). In this paper, we exploit the discrete structure of our framework (distributions, transfers) by establishing that the extreme points of  $\mathcal{U}(\mathcal{T})$  can be obtained by solving a linear system of equations.<sup>19</sup> We have provided two simple and well-known examples, leading to first and second stochastic dominance. Nevertheless, Proposition 9 can be difficult to apply in practice, and the complexity depends on the specific set of transfers we consider. In general, finding an analytic solution for the extreme points is far from trivial.

## 7. Application to a unidimensional ordinal variable

The discrete framework developed in this paper is required for ordinal variables defined using an ordered categorical scale, which is the case for most of the dimensions characterizing individuals' social welfare, except for income (for instance, self-reported health, happiness, access to basic services, or educational outcomes).

The question of inequality and social welfare measurement for ordered categorical variables is investigated by, among others, Gravel *et al.* (2021).<sup>20</sup> Their objective is to provide a result comparable to the HLP theorem, initially developed for a cardinal (and continuous) variable. Consider as an example health, self-reported on a scale with five categories ('very bad', 'bad', 'so-so', 'good' and 'very good') and assume that this variable is interpersonally comparable. On the one hand, one can say that the social welfare experienced by an individual is higher for one category than another one, lower in the scale. On the other hand, it is quite impossible to provide, with compelling arguments, an unambiguous measure of the magnitude of social welfare one applies to each category and, even more troublesome, of the distance between the social welfare of the two categories. Gravel *et al.* (2021) take the route that, for such a variable, the ranking of the

---

<sup>19</sup>This approach has been initiated (to the best of our knowledge) by Chateauneuf *et al.* (2020) within the Yaari dual framework. I am particularly grateful to Alain Chateauneuf for detailed discussions on this point.

<sup>20</sup>An extension for ordinal but continuous variables is developed in Gravel *et al.* (2019).



categories is the only relevant information that can be used to assess social welfare. In that case, we face a purely ordinal variable –the scale is defined up to an increasing transformation– which cannot be, by definition, ‘summed’ or ‘averaged’ across individuals.

Consider an outcome scale  $\mathcal{S} = \{0, 1, \dots, K\}$  with  $K \geq 2$  fixed as in the previous section ( $K$  is equal to 4 in the self-reported health example), but now the scale is defined up to an increasing transformation. A distribution is a list  $n = (n_0, \dots, n_K) \in \mathbb{Z}_+^{K+1}$ . The equalizing transfer proposed by Gravel *et al.* (2021) refers to the *equity principle* introduced by Hammond (1976) in social choice theory: A distribution  $n$  is obtained from distribution  $n'$  by means of a Hammond transfer if there exist  $i, j, k, l \in \mathcal{S}$  such that  $i < j \leq k < l$  and:

$$n_s = n'_s, \text{ for all } s \in \mathcal{S} \setminus \{i, j, k, l\}, \quad (20)$$

$$n_i - n'_i = n_l - n'_l = -1, \quad (21)$$

$$n_j - n'_j = n_k - n'_k = 1 \text{ if } j \neq k, \quad \text{and} \quad n_j - n'_j = 2 \text{ if } j = k. \quad (22)$$

The authors argue that ‘Hammond transfers’ are the only relevant way to define an equalizing process in an ordinal setting. Because the scale is defined up to an increasing transformation, the equalizing process has to be invariant to such transformation. This condition is satisfied by Hammond transfers but violated by the standard Pigou-Dalton progressive income transfer (which is well-suited for a cardinal variable; see Section 6). They also establish the necessary and sufficient condition that is applied to the utilitarian class of social welfare functions to ensure consistency with Hammond transfers. They obtain the following result.

**Proposition 10** (Gravel *et al.* (2021), Prop. 4, Page 15). *The following two statements are equivalent:*

(d) *For all  $n, n' \in \mathbb{Z}_+^{K+1}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ ,  $n$  being obtained from  $n'$  by means of a finite sequence of Hammond transfers implies  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$ ,*

(e)  $u \in \left\{ (u_0, \dots, u_K) \in \mathbb{R}^{K+1} \mid u_j - u_i \geq u_l - u_k, \forall i < j \leq k < l \right\}$ .

The methodology developed in this paper is well-suited for completing the previous result to obtain, without any other proof, the equivalence between dominance according to a sequence of Hammond transfers and dominance according to the utilitarian class of social welfare functions consistent with these transfers. The set  $\mathcal{T}_H$  of Hammond transfers is defined as follows:

**Definition 4** (Set of Hammond transfers). *The set  $\mathcal{T}_H$  of Hammond transfers is the set of all  $m \in \mathbb{Z}^{K+1}$  such that there exist  $i, j, k, l \in \mathcal{S}$  with  $i < j \leq k < l$  and:*

$$m_s = 0, \text{ for all } s \in \mathcal{S} \setminus \{i, j, k, l\}, \quad (23)$$

$$m_i = m_l = -1, \quad (24)$$

$$m_j = m_k = 1 \text{ if } j \neq k, \quad \text{and} \quad m_j = 2 \text{ if } j = k. \quad (25)$$

As in Definition 2, we can formally define the notion of the sequence of transfers in  $\mathcal{T}_H$ : For all  $m, m' \in \mathbb{Z}^{K+1}$ , we write  $m \succeq_{\mathcal{T}_H} m'$  if and only if  $\succeq_{\mathcal{T}_H}$  is induced by the discrete cone generated by  $\mathcal{T}_H$ .

It is not difficult to verify that  $\mathcal{T}_H \subset \mathcal{T}$ . Indeed,  $\mathcal{T}_H$  satisfies all the properties of a set of transfers as described in Definition 1, including the independence requirement (see Remark 1). It follows that Theorem 1, with  $\mathcal{T}_H$  instead of the more general set  $\mathcal{T}$ , also applies in that context. One obtains the following result:

**Corollary 1.** *For all  $n, n' \in \mathbb{Z}_+^{K+1}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , the following three statements are equivalent:*

$$(a) \ n \succeq_{\mathcal{T}_H} n',$$

$$(b1) \ W(n) \geq W(n'), \ \forall W \in \mathcal{W}(\mathcal{T}_H),$$

$$(b2) \ \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \ \forall u \in \mathcal{U}(\mathcal{T}_H).$$

This equivalence is helpful for completing the HLP theorem for ordered categorical variables, as attempted by Gravel *et al.* (2021).

## 8. Application to a bidimensional variable

To show that the set of possible applications of our model is large, we consider, as a second example, the case of a bidimensional variable, consisting of a first dimension that is cardinally measurable and transferable between individuals and a second dimension that is assumed to be ordinal and nontransferable (respectively, income and health, for instance). This framework is investigated by Gravel and Moyes (2012) and Faure and Gravel (2017). We assume here that both dimensions take a finite number of values (euro cents for income),<sup>21</sup> such that an outcome is a point  $s = (i, j) \in \mathbb{Z}_+^2$ , with  $0 \leq i \leq I$  and  $0 \leq j \leq H$ , with  $I$  and  $H$  fixed. The set of outcomes is thus a finite and fixed set  $\mathcal{S} \subset \mathbb{Z}_+^2$ . We use  $n = (n_s)_{s \in \mathcal{S}} \in \mathbb{Z}_+^{|\mathcal{S}|}$  to denote a distribution. Gravel and Moyes (2012) propose two inequality-reducing transfers, as described below.

<sup>21</sup> As emphasized by the authors (Gravel and Moyes, 2012, Page 1392), the fact that both dimensions take a finite number of values is not a restriction, even if they assume in their paper that the cardinally measurable variable (income) is continuous.

They first introduce the notion of *favorable income permutation*, which consists of exchanging the income of an individual with that of another individual, who is initially worse-off in both dimensions. Consider, for instance, an initial distribution  $n'$  and two individuals, the first (the worse-off) with outcome  $(i, j)$  and the second (the best-off) with outcome  $(k, l)$ , such that  $(i, j) < (k, l)$ . After permutation, one obtains distribution  $n$  such that the outcomes of all are not changed, apart for the individual who is worse-off (of the two individuals mentioned above), who reaches outcome  $(k, j)$ , and the individual who is best-off, who, in return, obtains  $(i, l)$ . An illustration is provided in Figure 4. Now, we define the set of favorable income permutations.

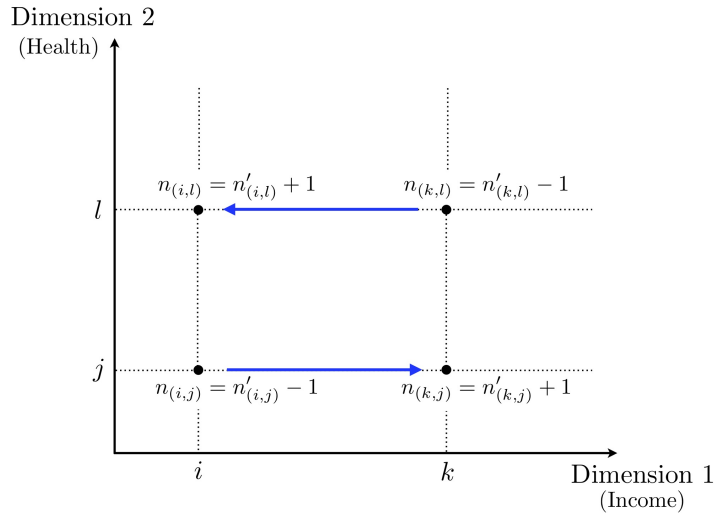


Figure 4: A favorable income permutation

**Definition 5** (Set of favorable income permutations). *The set  $\mathcal{T}_1$  of favorable income permutations is the set of all  $m \in \mathbb{Z}^{|\mathcal{S}|}$  such that there exist  $i, j, k, l$  with  $0 \leq i, k \leq I$  and  $0 \leq j, l \leq H$  and  $(i, j) \ll (k, l)$ , and such that:*

$$m_s = 0, \text{ for all } s \in \mathcal{S} \setminus \{(i, j), (k, j), (i, l), (k, l)\}, \quad (26)$$

$$m_{(i,j)} = m_{(k,l)} = -1, \quad \text{and} \quad m_{(i,l)} = m_{(k,j)} = 1. \quad (27)$$

Then, Gravel and Moyes (2012) discuss the notion of *within-type progressive income transfer*, which consists of a progressive income transfer between two individuals having the same health status. Consider an initial distribution  $n'$  and two individuals, the first (the worse-off) with outcome  $(i, j)$  and the second (the best-off) with outcome  $(k, j)$ , such that  $i < k$ . After the transfer, one obtains distribution  $n$  such that their outcomes are, respectively  $(i+1, j)$  and  $(k-1, j)$ , without any modification for the individuals who are not affected by the transfer. This transfer is illustrated in Figure 5.

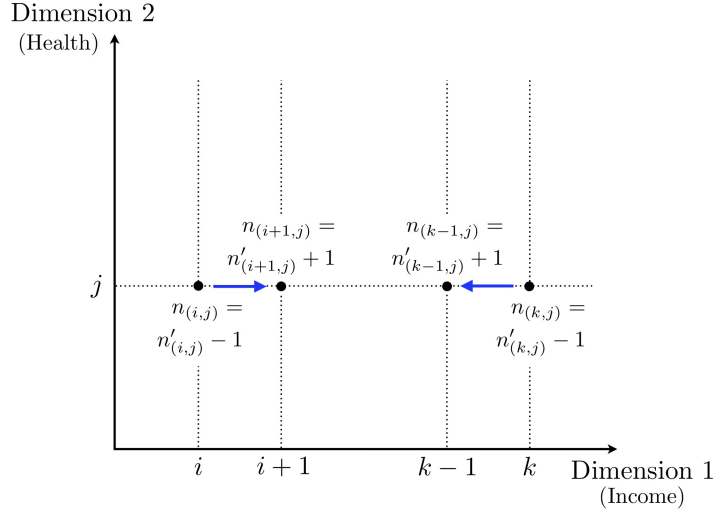


Figure 5: A within-type progressive income transfer

**Definition 6** (Set of within-type progressive income transfers). *The set  $\mathcal{T}_2$  of within-type progressive income transfers is the set of all  $m \in \mathbb{Z}^{|\mathcal{S}|}$  such that there exist  $i, j, k$  with  $0 \leq i < k \leq I$  and  $0 \leq j \leq H$ , and such that:*

$$m_s = 0, \text{ for all } s \in \mathcal{S} \setminus \{(i, j), (i+1, j), (k-1, j), (k, j)\}, \quad (28)$$

$$m_{(i,j)} = m_{(k,j)} = -1, \quad (29)$$

$$m_{(i+1,j)} = m_{(k-1,j)} = 1 \text{ if } i+1 \neq k-1, \quad \text{and} \quad m_{(i+1,j)} = 2 \text{ if } i+1 = k-1. \quad (30)$$

Because it is assumed that both favorable income permutations and within-type progressive income transfers are inequality-reducing transfers, it could be interesting to pool them together into a larger set of ‘admissible’ transfers. For instance, one could directly investigate the set  $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$ . For each of these three types of transfers – permutation, income transfer or the combination of both – Gravel and Moyes (2012) obtain a result comparable to Proposition 10 in the previous Section (Lemmas 4.1, 5.1 and 5.2 in their paper). On the basis of these results, one can deduce that the class of utilitarian social welfare functions consistent with transfers in  $\theta = \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ , characterized by the sets  $\mathcal{U}(\theta)$ , as defined in Equation (9), which can be respectively written as:<sup>22</sup>

$$\mathcal{U}(\mathcal{T}_1) = \{u \in \mathcal{U} \mid u_{(i+1,j)} - u_{(i,j)} \geq u_{(i+1,l)} - u_{(i,l)}, \forall i, \forall j < l\}, \quad (31)$$

<sup>22</sup> Note that the restrictions imposed on  $u$  in  $\mathcal{U}_{\mathcal{T}_1}$  and  $\mathcal{U}_{\mathcal{T}_2}$  correspond to discrete versions of, respectively, *submodularity* and *concavity in the first dimension*.

$$\mathcal{U}(\mathcal{T}_2) = \{u \in \mathcal{U} \mid u_{(i+1,j)} - u_{(i,j)} \geq u_{(k+1,j)} - u_{(k,j)}, \forall i < k, \forall j\}, \quad (32)$$

$$\mathcal{U}(\mathcal{T}_3) = \mathcal{U}(\mathcal{T}_1) \cap \mathcal{U}(\mathcal{T}_2). \quad (33)$$

Then, the sequences of transfers in  $\theta = \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$  can be written through preorder  $\succeq_\theta$  (see Definition 2) such that for all  $m, m' \in \mathbb{Z}^{|\mathcal{S}|}$ ,  $m \succeq_\theta m'$  if and only if  $\succeq_\theta$  is induced by the discrete cone generated by  $\theta$ . We also let  $\mathcal{W}(\theta)$  be the set of social welfare functions consistent with  $\succeq_\theta$ . Because  $\theta \subset \mathcal{T}$  (for  $\theta = \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ ) our Theorem 1 –with  $\theta$  instead of the more general set  $\mathcal{T}$ – directly applies. One obtains:

**Corollary 2.** *Let  $\theta = \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ . For all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , the following three statements are equivalent:*

$$(a) \ n \succeq_\theta n',$$

$$(b1) \ W(n) \geq W(n'), \forall W \in \mathcal{W}(\theta),$$

$$(b2) \ \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall u \in \mathcal{U}(\theta).$$

The equivalence between statements (a) and (b2) for  $\theta = \mathcal{T}_1, \mathcal{T}_3$  is established in Gravel and Moyes (2012) –respectively in Theorems 4.1 and 5.1– but on the basis of further proof arguments. Our general methodology based on discrete cones immediately ensures that the results are true.

## 9. Social deprivation: transfers outside the framework

Some sets of transfers are not consistent with our model. In this section we consider equalization processes investigated by Chateauneuf and Moyes (2006) and Magdalou and Moyes (2009) for (unidimensional) income distributions. The framework is equivalent to the one used in the illustrations of Section 6. Income can take possible values  $\mathcal{S} = \{0, 1, \dots, K\}$  with a fixed  $K \geq 2$ , and the distributions are written as lists  $n = (n_0, \dots, n_K) \in \mathbb{Z}_+^{K+1}$ .<sup>23</sup>

As emphasized in Chateauneuf and Moyes (2006) and Magdalou and Moyes (2009), a number of experimental studies have established that the inequality-reducing impact of a Pigou-Dalton progressive transfer (the cornerstone of inequality measurement, see Section 6) is not unanimously accepted (Amiel and Cowell, 1992). Based on social deprivation theory (Runciman, 1966), they propose alternative principle of transfers that impose solidarity among the individuals involved in the equalizing process. We present here one example, called *uniform of the right progressive*

<sup>23</sup> The difference from Section 7 relates to the interpretation of  $\mathcal{S}$ . Because income is a cardinal variable, the distance between two incomes is relevant for measurement. It follows that we can compute, for instance, the mean income.

income transfer (*URPT*). According to this principle, if some income is taken from an individual, the same amount has to be taken from every individual who is as rich or richer than that individual. Then, the total amount collected is transferred to a poorer individual. This is therefore a (mean-preserving) progressive income transfer that maintains solidarity among the richer individuals.

**Proposition 11** (Chateauneuf and Moyes (2006)). *Let  $\theta = PT, URPT$ . The following two statements are equivalent:*<sup>24</sup>

- (d) For all  $n, n' \in \mathbb{Z}_+^{K+1}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ ,  $n$  being obtained from  $n'$  by means of a finite sequence of transfers  $\theta$  implies  $\sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s$ ,
- (e)  $u \in \left\{ (u_0, \dots, u_K) \in \mathbb{R}^{K+1} \mid u_{i+1} - u_i \geq u_{l+1} - u_l, \forall i < l \right\}$ .

The previous result highlights the lack of flexibility of the utilitarian model to distinguish the inequality views captured by *PT* and *URPT* transfers. In both cases, the condition to be imposed on the utility is concavity (statement (e)). Our model sheds new light on this surprising result. First, one immediately observes that  $\mathcal{T}_{PT} \subset \mathcal{T}$  (see Section 6), hence our framework perfectly fits for sequences of *PT* transfers. It follows that Theorem 1 also applies for  $\mathcal{T}_{PT}$ . One deduces from Proposition 11 that  $\mathcal{U}(\mathcal{T}_{PT})$  is the set of concave utilities. Because Proposition 11 also concerns *URPT* transfers, a question arises: Can one obtain an equivalent result for these transfers? The answer is no, as expected.<sup>25</sup> By using a simple example, we actually demonstrate that a sequence of *URPT* transfers cannot be described as a discrete cone.

Consider an income scale  $\mathcal{S} = \{0, 1, 2, 3\}$  and a distribution  $n' = (1, 1, 1, 0)$  for a society consisting of 3 individuals. Assume that one unit of income is taken from the individual having initially 2 units and given to the individual having initially 0 unit. This transformation is a *URPT* transfer, and we obtain, after the transfer, distribution  $n = (0, 3, 0, 0)$ . If the set of *URPT* transfers is a subset of  $\mathcal{T}$  (Definition 1), we know from Remark 1 that the following statement is true: if  $n, n' \in \mathbb{Z}_+^{K+1}$  and  $n$  is obtained from  $n'$  by means of a *URPT* transfer, then  $n'' = n + \epsilon$  is also obtained from  $n''' = n' + \epsilon$  by means of a *URPT* transfer (as soon as  $\epsilon$  is such that  $n'', n''' \in \mathbb{Z}_+^{K+1}$ ). Consider the admissible  $\epsilon = (0, -1, 0, 1)$ , so that  $n'' = (1, 0, 1, 1)$  and  $n''' = (0, 2, 0, 1)$ . Obviously,  $n''$  cannot be obtained from  $n'''$  by means of a *URPT* transfer or a sequence of such transfers. Because the independence requirement described in Remark 1 is not satisfied, the set of *URPT*

<sup>24</sup> This result is not directly proved in Chateauneuf and Moyes (2006). It results from Propositions 1, 3 and 5, respectively in Pages 28, 43 and 52.

<sup>25</sup> Indeed, a *URPT* transfer can be obtained by means of a finite sequence of *PT* transfers, but the reverse is not true. Hence, if the set  $\succeq_{\mathcal{T}_{URPT}}$  exists, we would have  $\succeq_{\mathcal{T}_{URPT}} \subset \succeq_{\mathcal{T}_{PT}}$ . Moreover, dominance according to the class of utilitarian social welfare functions with concave utilities cannot be equivalent to dominance according to  $\succeq_{\mathcal{T}_{PT}}$  and, at the same time, dominance according to  $\succeq_{\mathcal{T}_{URPT}}$ . Only the first equivalence is true, as is well-known in the literature.

transfers cannot be a subset of  $\mathcal{T}$ . It follows that a sequence of such transfers cannot be written as a discrete cone, in line with Definition 2.

As presented in Section 5, our model puts forward the close relationship between the sequences of transfers that can be written as discrete cones and the utilitarian class of social welfare functions. The inability of a sequence of *URPT* transfers to be written as a discrete cone explains why this class of social welfare functions is not well-suited to capture the inequality views of such transfers.

## 10. Extension to decision-making under risk: the key role of the EU model

The appropriate way to represent individual preferences in situations of decision-making under risk remains an important research issue. The expected utility model has, for many reasons – among them, its simplicity and tractability– a prominent place in the literature. But at the same time, some of the underlying assumptions are not consistent with behavior observed in practice, for instance, the independence axiom. Since Machina (1982), many studies have put forward the key role of the expected utility model for modeling risk attitudes, even for nonexpected utility preferences. In this Section, we provide new arguments in this tradition.

A (unidimensional) *lottery* is usually represented by a cumulative distribution, denoted  $F$  or  $G$ , which is defined on a closed interval  $I$  of  $\mathbb{R}$ . The standard approach consists of assuming an expected utility representation of the decision maker’s preferences. In that case, letting  $u$  be a continuous utility function defined on  $I$ ,  $F$  is considered as weakly better than  $G$  if and only if  $\int_I u(x)dF(x) \geq \int_I u(x)dG(x)$ . The unanimity of the ranking within a class of utility functions, usually called an (*integral*) *stochastic order*, is often used to distinguish distributions in terms of risk. For instance, usually,  $F$  is considered less risky than  $G$  when  $G$  is a *mean-preserving spread* of  $F$ . And it is well-known that, for any distributions  $F$  and  $G$  such that  $\int_I xdF(x) = \int_I xdG(x)$ ,  $G$  is a mean-preserving spread of  $F$  if and only if  $\int_I u(x)dF(x) \geq \int_I u(x)dG(x)$  for all  $u$  that are weakly concave.

A first question is how nonexpected utility preferences can be addressed. By assuming that the decision maker’s preferences can be represented by a continuous and *smooth functional* –that is, a (Gâteaux) differentiable functional  $V$ , which associates with each cumulative distribution a value in  $\mathbb{R}$ – Machina (1982) has established that any nonexpected utility decision maker acts locally as an expected utility maximizer. This argument is based on the following observation:

$$V(G) - V(F) \approx \int_I v_F(x)dG(x) - \int_I v_F(x)dF(x), \quad (34)$$

where  $v_F$  is the derivative of  $V$  at  $F$ , a continuous function. This derivative is usually called *local utility*, as the difference  $V(G) - V(F)$  can be approximated by the difference in the expected value

of  $v_F$  with respect to  $G$  and  $F$ . We emphasize that this observation relies on the differentiability assumption required for  $V$ .

A second issue deals with the consistency of nonexpected utility preferences with stochastic orders, as these criteria are usually related to transparent notions of *risk reduction* through simple probability mass transfers (such as mean-preserving spreads). By considering, again, a continuous and smooth functional  $V$  representing the decision maker's preferences, Cerreia-Vioglio *et al.* (2016) establish the following general result.

**Proposition 12** (Cerreia-Vioglio *et al.* (2016), Prop. 1, Page 1102). *Let  $\mathcal{U}^*$  be a set of bounded and continuous functions on  $I$ . The following two statements are equivalent:*

- (d) *For all distributions  $F$  and  $G$  defined on  $I$ ,  $\int_I u(x)dF(x) \geq \int_I u(x)dG(x)$  for all  $u \in \mathcal{U}^*$  implies  $V(F) \geq V(G)$ ,*
- (e) *The set of all derivatives of  $V$  (namely, all the local utilities) is included in the closed convex cone generated by  $\mathcal{U}^*$  and all the constant functions.*

This result establishes the necessary and sufficient condition for the functional  $V$  to make this preference representation consistent with the stochastic order characterized by the set of utilities  $\mathcal{U}^*$ .

As established by Muller (2013), most of the stochastic orders, even for multivariate distributions, can be associated to sequence of probability mass transfers, which can be written as convex cones. On this basis, we can provide some answers to the two questions previously raised. We first slightly modify our framework. We still consider a partially ordered, finite and fixed set of outcomes  $\mathcal{S} \subset \mathbb{Z}_+^d$ , where each dimension is, for instance, monetary variables written in euro cents. A (probability) distribution is a list of  $p = (p_s)_{s \in \mathcal{S}}$ , where  $p_s$  indicates the probability of having ( $d$ -dimensional) outcome  $s \in \mathcal{S}$ , and  $\mathcal{P} = \{p \in \mathbb{R}_+^{|\mathcal{S}|} \mid \sum_{s \in \mathcal{S}} p_s = 1\}$  indicates the set of distributions. The set  $\mathcal{T}$  of (probability mass) transfers, which is assumed to be compact, is the set of all  $m \in \mathbb{R}^{|\mathcal{S}|}$ , which can be written as  $m = p - p'$  for some  $p, p' \in \mathcal{P}$ , and  $m \in \mathcal{T}$  implies  $(-m) \notin \mathcal{T}$ . A sequence of transfers is described by the relation  $\succeq_{\mathcal{T}}$ , such that we write  $p \succeq_{\mathcal{T}} p'$  if and only if  $\succeq_{\mathcal{T}}$  is induced by the convex cone generated by  $\mathcal{T}$ . The decision maker's preferences are represented by the function  $W : \mathcal{P} \rightarrow \mathbb{R}$ . If this function has an expected utility form, then it can be written as  $W(p) = \sum_{s \in \mathcal{S}} p_s u_s$ . Finally, let  $\mathcal{U}(\mathcal{T})$  be a set of utility vectors  $u = (u_s)_{s \in \mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$  consistent with  $\succeq_{\mathcal{T}}$ .

By adopting this revisited framework our Theorem 1, when applied to probability distributions  $p, p' \in \mathcal{P}$ , remains true. The proof arguments used to establish the equivalence between statements (a) and (b1) in our theorem are still valid, and the equivalence between (a) and (b2) results



from Theorem 2.4.1 in Muller (2013). This result provides an answer (partial and in a specific context, of course) to the first question raised above, namely, how to represent nonexpected utility preferences. Indeed, if the objective is to reach a unanimity of ranking between two probability distributions for any decision-makers –even with nonexpected utility preferences– who agree with the risk-reducing impact of transfers in  $\mathcal{T}$  (mean-preserving spread, for instance), it is sufficient to check the unanimity among the expected utility maximizers. This observation highlights the core aspect of expected utility preferences in a situation where the definition of ‘risk reduction’ is unanimously approved. Note that we are in a multivariate realm and that no ‘smoothness’ assumptions are required for the decision maker’s preferences.

To complement Proposition 12, which is related to the second question raised, we have the following result. It is based on our Theorem 1 and is almost equivalent to Theorem 2 in Marshall *et al.* (1967). We emphasize that, again, we deal with multivariate variables, without any smoothness assumption.

**Proposition 13.** *The following two statements are equivalent:*

- (d) For all  $p, p' \in \mathcal{P}$  and all  $\mathcal{U}(\mathcal{T})$ ,  $\sum_{s \in \mathcal{S}} p_s u_s \geq \sum_{s \in \mathcal{S}} p'_s u_s$  implies  $W(p) \geq W(p')$ ,
- (e)  $W(p + \lambda m) - W(p) \geq 0$  for all  $p \in \mathcal{P}$ , all  $m \in \mathcal{T}$  and all  $\lambda > 0$  such that  $p + \lambda m \in \mathcal{P}$ .

**Proof.** (d)  $\Rightarrow$  (e). Consider  $p \in \mathcal{P}$ ,  $m \in \mathcal{T}$ , and  $\lambda > 0$  such that  $p + \lambda m \in \mathcal{P}$ . By definition, we have  $(p + \lambda m) \succeq_{\mathcal{T}} p$  or, equivalently, we have  $\sum_{s \in \mathcal{S}} (p_s + \lambda m_s) u_s \geq \sum_{s \in \mathcal{S}} p_s u_s$  for all  $\mathcal{U}(\mathcal{T})$  (from Theorem 1). Now, assume that (a) is true. It follows that  $W(p + \lambda m) - W(p) \geq 0$ .

(e)  $\Rightarrow$  (d). Assume that (e) is true and that  $\sum_{s \in \mathcal{S}} p_s u_s \geq \sum_{s \in \mathcal{S}} p'_s u_s$  for all  $\mathcal{U}(\mathcal{T})$  or, equivalently,  $p \succeq_{\mathcal{T}} p'$  (from Theorem 1). We then use the same arguments as in the proof of Theorem 2 in Marshall *et al.* (1967), which can be sketched as follows. Because  $p \succeq_{\mathcal{T}} p'$ , there exists  $T$  such that we can write  $p - p' = \sum_{t=1}^T \lambda_t m_{.t}$  with  $\lambda_t > 0$  and  $m_{.t} \in \mathcal{T}$ . From statement (e), the transitivity of  $W$  and because  $p = p' + \sum_{t=1}^T \lambda_t m_{.t}$ , one obtains  $W(p) - W(p') \geq 0$ .  $\square$

This result is restricted to stochastic orders that can be associated with a sequence of mass transfers written as convex cones. In practice, this restriction is not so demanding (see Muller, 2013). This result establishes the necessary and sufficient condition to be placed on  $W$ , to make this preference representation consistent with such a stochastic order. Statement (b) can be interpreted –loosely speaking because here,  $W$  is not assumed differentiable– as a set of restrictions on the *directional derivatives* of  $W$ . It is thus a simple and helpful solution for answering the second question raised above, namely, how to characterize the consistency of nonexpected preferences with stochastic orders.

## 11. Conclusion

In this paper, we present a general model for assessing social welfare in a discrete and multi-dimensional framework. On the basis of a normative definition of what can be called a ‘social welfare improvement’, we present a result in the vein of the HLP theorem (Hardy *et al.*, 1934), establishing the equivalence between four preorders. The first is a dominance criterion based on a sequence of transfers, reflecting our normative views on social welfare. The third is a unanimous ranking of the distributions among utilitarian social planners who agree with these welfare views, whereas the second preorder extends this unanimity to the largest class of social welfare functions sharing these views. The last preorder is a dominance criterion, empirically implementable and comparable to the Lorenz criterion in the standard HLP theorem.

This result, provided at an abstract level, can be applied to various contexts. It can be used for uni- or multidimensional variables with cardinal or ordinal dimensions. The only requirement is that the considered set of transfers can be written as a discrete cone. As an illustration, this process has been used to complete the HLP-type theorem for a discrete ordinal variable, partially obtained in Gravel *et al.* (2021). It is worth noting that, due to its additive structure, the utilitarian class of social welfare functions is closely related to the sequence of transfers we consider, which are defined as preorders induced by a discrete cone. The utilitarian class consistent with the welfare views provided by the transfers appears to be a core set of preferences, as the unanimity of ranking within this class is necessary but also sufficient to reach unanimity in the largest class of social welfare functions (also consistent with these views).

Although most of the transfers studied in the literature fall within this framework, others do not. This is the case of the progressive transfers with solidarity among the individuals involved in the equalizing process (Chateauneuf and Moyes, 2006; Magdalou and Moyes, 2009), which can not be written as discrete cones, unlike the Pigou-Dalton progressive transfers. This particular feature explains why the utilitarian class cannot distinguish these transfers from the standard Pigou-Dalton ones.

Even if it is applied in this paper to social welfare and inequality measurement, our generalized HLP theorem can be applied in various economic fields such that decision-making under risk, finance or econometrics. An application of this theorem in these fields, as illustrated in the last section, might be the object of future studies.

### A. Alternative theorem, without the minimality requirement for the set of transfers

As established in Proposition 8, the minimality of the set of transfers  $\mathcal{T}$  is required to have an equivalence, in Theorem 1, between statement (a) and the other statements. Nevertheless, if

one accepts to slightly weaken the preorder defined in (a), a comparable result can be reached without the minimality requirement, by a direct application of the well-known rational-integer Farkas Lemma.<sup>26</sup> First, I introduce the class of integral-valued utility functions consistent with transfers in  $\mathcal{T}$ , which is denoted by  $\mathcal{U}_{\mathbb{Z}}(\mathcal{T}) = \mathcal{U}(\mathcal{T}) \cap \mathbb{Z}^{|\mathcal{S}|}$ .

**Theorem 2.** *For all  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  such that  $\sum_{s \in \mathcal{S}} n_s = \sum_{s \in \mathcal{S}} n'_s$ , the following three statements are equivalent:*

$$(a) \exists \theta \in \mathbb{Z}_+ : \theta n \succeq_{\mathcal{T}} \theta n',$$

$$(b1) \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall u \in \mathcal{U}_{\mathbb{Z}}(\mathcal{T}),$$

$$(b2) \sum_{s \in \mathcal{S}} n_s u_s \geq \sum_{s \in \mathcal{S}} n'_s u_s, \forall u \in \mathcal{U}(\mathcal{T}).$$

**Proof.** (b1)  $\Rightarrow$  (a). Assume that (b2) is true, and let  $m = n - n'$ . We know from (2)  $\Rightarrow$  (1) in Lemma 3 (Appendix B) that there exist some  $x_t \in \mathbb{Q}_+$  such that  $m = \sum_{t=1}^{|\mathcal{T}|} x_t m_{.t}$ , with  $m_{.t} \in \mathcal{T}$ . Equivalently, there exist  $\theta \in \mathbb{Z}_+$  and some  $\lambda_t \in \mathbb{Z}_+$  such that  $\theta m = \sum_{t \in \mathcal{T}} \lambda_t m_{.t}$ .

(a)  $\Rightarrow$  (b2). Assume that (a) is true, and let  $m = n - n'$ . Hence, there exist  $\theta \in \mathbb{Z}_+$  and some  $\lambda_t \in \mathbb{Z}_+$  such that  $\theta m = \sum_{t=1}^{|\mathcal{T}|} \lambda_t m_{.t}$ , with  $m_{.t} \in \mathcal{T}$ . Then consider any  $u \in \mathcal{U}(\mathcal{T})$ , such that  $\sum_{s \in \mathcal{S}} m_{st} u_s \geq 0$  for all  $m_{.t} \in \mathcal{T}$ . One deduces that:

$$\theta \sum_{s \in \mathcal{S}} (n_s - n'_s) u_s = \sum_{s \in \mathcal{S}} \theta m_s u_s = \sum_{s \in \mathcal{S}} \left( \sum_{t=1}^{|\mathcal{T}|} \lambda_t m_{st} \right) u_s = \sum_{t=1}^{|\mathcal{T}|} \lambda_t \left( \sum_{s \in \mathcal{S}} m_{st} u_s \right) \geq 0. \quad (35)$$

Finally, the proof (b2)  $\Rightarrow$  (b1) directly follows from the definition of  $\mathcal{U}_{\mathbb{Z}}(\mathcal{T})$ .  $\square$

On the basis of the Farkas Lemma, this alternative theorem is very intuitive and the proof quite immediate. It also completes Proposition 8, by providing new information on the role of the minimality requirement in Theorem 1. The price to pay is to accept a weaker version of the preorder in statement (a). Indeed, we know from Proposition 1 that the ranking of two distributions  $n, n' \in \mathbb{Z}_+^{|\mathcal{S}|}$  (with equal population size) according to  $\succeq_{\mathcal{T}}$  is preserved after an identical replication of the populations in  $n$  and  $n'$ . But we also know that the reverse implication is not necessarily true. In statement (a) above, we know that there exists an identical replication of the populations in  $n$  and  $n'$  such that the ‘fictive distributions’  $\theta n$  and  $\theta n'$  can be compared according to  $\succeq_{\mathcal{T}}$ . But there is no guarantee that, in turn, the ‘real distributions’  $n$  and  $n'$  are comparable. Nevertheless, this result deserves a important place in this paper as it is very close to the main theorem, and obtained with a different and effective proof strategy.

<sup>26</sup> The rational-integer Farkas Lemma is presented in Appendix B. This theorem has been proposed by the Associate Editor of the Journal. I am very grateful to him for bringing to my attention this alternative approach and its relevance in the framework of the present paper.

## B. Mathematical background

A **convex cone** is a nonempty set  $\mathcal{C} \subset \mathbb{R}^d$  such that  $c_1, c_2 \in \mathcal{C}$  implies  $\lambda_1 c_1 + \lambda_2 c_2 \in \mathcal{C}$  for all  $\lambda_1, \lambda_2 \in \mathbb{R}_+$ . We say that a convex cone is *generated by* a set  $\mathcal{C}_1 \subset \mathcal{C}$  if  $\mathcal{C} = \text{co}\{\lambda c \mid \lambda \in \mathbb{R}_+, c \in \mathcal{C}_1\}$ , where  $\text{co}$  indicates the convex hull of the set. The convex cone  $\mathcal{C}$  is *finitely generated* if it is generated by a finite set  $\mathcal{C}_1 = \{c_1, \dots, c_T\}$  or, equivalently, if it can be written as  $\mathcal{C} = \{\sum_{t=1}^T \lambda_t c_t \mid \lambda_t \in \mathbb{R}_+, c_t \in \mathcal{C}_1\}$ . A *rational cone* is a convex cone  $\mathcal{C} \subset \mathbb{R}^d$  finitely generated by a set  $\{c_1, \dots, c_T\}$  of elements in  $\mathbb{Q}^d$ . A *discrete cone* is the integer analogue of a convex cone. It is a nonempty set  $\mathcal{C} \subset \mathbb{Z}^d$  such that  $c_1, c_2 \in \mathcal{C}$  implies  $\lambda_1 c_1 + \lambda_2 c_2 \in \mathcal{C}$  for all  $\lambda_1, \lambda_2 \in \mathbb{Z}_+$ . Finally, a cone is said to be *pointed* if  $c \in \mathcal{C}$  and  $(-c) \in \mathcal{C}$  imply  $c = 0$ .

For a convex cone  $\mathcal{C} \subset \mathbb{R}^d$ , a finite set  $\mathcal{H}(\mathcal{C}) = \{c_1, \dots, c_T\}$  of elements in  $\mathcal{C} \cap \mathbb{Z}^d$  is called a **Hilbert basis** if every  $c \in \mathcal{C} \cap \mathbb{Z}^d$  can be expressed as  $c = \sum_{t=1}^T \lambda_t c_t$  with  $c_t \in \mathcal{H}(\mathcal{C})$  and  $\lambda_t \in \mathbb{Z}_+$ , and if every  $c_t \in \mathcal{H}(\mathcal{C})$  are irreducible elements (which means that there don't exist  $c_1, c_2 \in \mathcal{H}(\mathcal{C})$  with  $c_1, c_2 \neq 0$  and  $c_t = c_1 + c_2$ ).

**Lemma 2.** *If  $\mathcal{C}$  is a rational cone, then it is generated by a Hilbert basis  $\mathcal{H}(\mathcal{C})$ . Moreover if  $\mathcal{C}$  is pointed, then  $\mathcal{H}(\mathcal{C})$  is unique.*

The existence part of the result was first established by Gordan (1873) and Hilbert (1890). The unicity of  $\mathcal{H}(\mathcal{C})$  has been shown by van der Corput (1931a,b) (see also Schrijver, 1986, Theorem 16.4, Page 233). We illustrate the notion of Hilbert basis in Figure 6. We present a (pointed) convex cone  $\mathcal{C} \subset \mathbb{R}_+^2$  generated by two elements (the polygons). The elements  $\mathcal{C} \cap \mathbb{Z}^2$  are indicated by black dots, and the Hilbert basis  $\mathcal{H}(\mathcal{C})$  by the circled dots.

Another well-known result in linear inequalities is called the **rational-integer Farkas Lemma** (Farkas, 1902), which can be stated as follows.

**Lemma 3.** *Consider the matrix  $A \in \mathbb{Q}^{m \times n}$  and let  $b \in \mathbb{Q}^m$ . The following two statements are equivalent:*

- (1)  $\exists x \in \mathbb{Q}_+^n$  such that  $Ax = b$ ,
- (2)  $\forall y \in \mathbb{Z}^m$  such that  $A^\top y \geq 0$ , we have  $b^\top y \geq 0$ .

We then introduce some definitions regarding **binary relations**. A binary relation  $\succeq$  defined on a set  $\mathcal{A}$  is a subset of  $\mathcal{A} \times \mathcal{A}$ . For all  $c_1, c_2 \in \mathcal{A}$ , we write  $c_1 \succ c_2$  if and only if  $c_1 \succeq c_2$  and not  $c_2 \succeq c_1$ . We let  $c_1 \sim c_2$  if and only if  $c_1 \succeq c_2$  and  $c_2 \succeq c_1$ . The relation  $\succeq$  is *reflexive* if  $c \succeq c$  for all  $c \in \mathcal{A}$ . It is *transitive* if  $c_1 \succeq c_2$  and  $c_2 \succeq c_3$  imply  $c_1 \succeq c_3$  for all  $c_1, c_2, c_3 \in \mathcal{A}$ . A reflexive and transitive relation is a *preorder*. A binary relation is *antisymmetric* if  $c_1 \sim c_2$  implies  $c_1 = c_2$

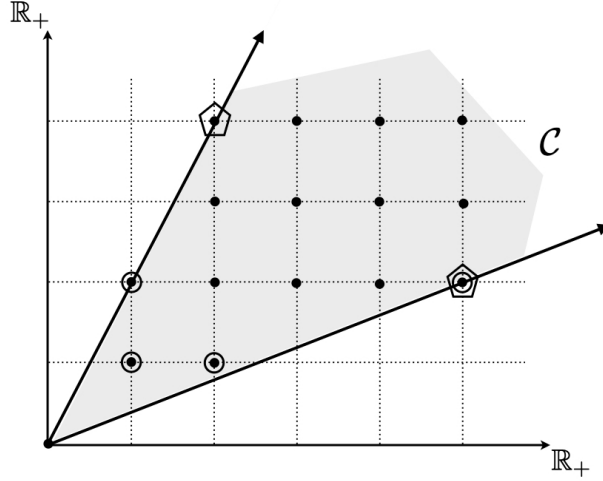


Figure 6: Example of an Hilbert basis

for all  $c_1, c_2 \in \mathcal{A}$ . In this paper, a binary relation defined on  $\mathcal{A}$  is said *additive* if  $c_1 \succeq c_2$  implies  $c_1 + c_3 \succeq c_2 + c_3$  for all  $c_1, c_2, c_3 \in \mathcal{A}$ . A binary relation is said *scale-invariant* if  $c_1 \succeq c_2$  implies  $\lambda c_1 \succeq \lambda c_2$  for all  $c_1, c_2 \in \mathcal{A}$  and all  $\lambda \in \mathbb{R}_+$ . Finally, a relation  $\succeq$  on  $\mathcal{A}$  is *induced by the set  $\mathcal{C}$*  when  $c_1 \succeq c_2$  if and only if  $c_1 - c_2 \in \mathcal{C}$ . A relation  $\succeq$  induced by a cone  $\mathcal{C}$  is called a *cone ordering*. The following result is established in Marshall *et al.* (1967).

**Lemma 4.** *A convex cone ordering, defined on  $\mathbb{R}^d$ , is an additive and scale-invariant preorder. Conversely, a preorder on  $\mathbb{R}^d$  must be additive and scale-invariant to be a convex cone ordering. A convex cone is pointed if and only if the induced binary relation is antisymmetric.*

We now discuss the concept of **duality between convex cones**. Let  $\mathcal{V}, \mathcal{W}$  be two vector spaces. A function  $b : \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{R}$  is called a bilinear mapping if it is linear in each argument separately, namely, with  $\lambda \in \mathbb{R}_{++}$ :

$$b(v + z, w) = b(v, w) + b(z, w) \quad \text{and} \quad b(\lambda v, w) = \lambda b(v, w), \quad (36)$$

$$b(v, w + z) = b(v, w) + b(v, z) \quad \text{and} \quad b(v, \lambda w) = \lambda b(v, w). \quad (37)$$

The pair  $(\mathcal{V}, \mathcal{W})$  of vector spaces is said to be in duality if there exists a bilinear mapping  $b : \mathcal{V} \times \mathcal{W} \rightarrow \mathbb{R}$ . We use  $(\mathcal{V}, \mathcal{W}; b)$  to denote such a *dual pair*. The duality is said to be *strict*, if for each  $0 \neq v \in \mathcal{V}$ , there is a  $w \in \mathcal{W}$  with  $b(v, w) \neq 0$  and for each  $0 \neq w \in \mathcal{W}$ , there is a  $v \in \mathcal{V}$  with  $b(v, w) \neq 0$ . The (positive) *polar cone* of  $\mathcal{V}_1 \subset \mathcal{V}$  under the duality  $(\mathcal{V}, \mathcal{W}; b)$  is defined by:

$$\mathcal{V}_1^\circ = \{w \in \mathcal{W} \mid b(v, w) \geq 0, \forall v \in \mathcal{V}_1\}. \quad (38)$$

Note that  $\mathcal{V}_1^\circ$  is always a convex cone even if  $\mathcal{V}_1$  is neither convex nor a cone. The *bipolar cone* of  $\mathcal{V}_1$  is denoted by  $\mathcal{V}_1^{\circ\circ}$  and defined by  $\mathcal{V}_1^{\circ\circ} = (\mathcal{V}_1^\circ)^\circ$ . The **bipolar theorem** can be stated as follows (see Rockafellar, 1970; Aliprantis and Border, 2006):

**Lemma 5.** *Assume that  $\mathcal{V}, \mathcal{W}$  are two vector spaces in strict duality and let  $\mathcal{V}_1 \subset \mathcal{V}$ . Then,  $\mathcal{V}_1^{\circ\circ}$  is the smallest closed set containing the convex cone generated by  $\mathcal{V}_1$ .*

As a particular case of the previous theorem, note that  $\mathcal{V}_1 \subset \mathcal{V}$  is a nonempty closed convex cone if and only if  $\mathcal{V}_1^{\circ\circ} = \mathcal{V}_1$ .

## References

- ALIPRANTIS, C. and BORDER, K. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Berlin: Springer-Verlag, Third Edition.
- AMIEL, Y. and COWELL, F. (1992). Measurement of income inequality: Experimental test by questionnaire. *Journal of Public Economics*, **47**, 3–26.
- ATHEY, S. (2000). Characterizing properties of stochastic objective functions, revised version of MIT Working Paper No. 96-1, 1995.
- ATKINSON, A. (1970). On the measurement of inequality. *Journal of Economic Theory*, **2**, 244–263.
- BLACKORBY, C., BOSSERT, W. and DONALDSON, D. (2002). Utilitarianism and the theory of justice. In K. Arrow, A. Sen and K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 1, Amsterdam: Elsevier, pp. 543–596.
- BRUMELLE, S. and VICKSON, R. (1975). A unified approach to stochastic dominance. In W. Ziemba and R. Vickson (eds.), *Stochastic Optimization Models in Finance*, New York: Academic Press, pp. 101–113.
- BRUNS, W. and GUBELADZE, J. (2009). *Polytopes, Rings, and K-Theory*. Dordrecht Heidelberg London New York: Springer, Springer Monographs in Mathematics.
- CERREIA-VIOGLIO, S., MACCHERONI, F. and MARINACCI, M. (2016). Stochastic dominance analysis without the independence axiom. *Management Science*, **63**, 1097–1109.
- CHATEAUNEUF, A., MAGDALOU, B. and MOYES, P. (2020). Giving up Lorenz consistency or another look at inequality measurement, mimeo.
- and MOYES, P. (2006). A non-welfarist approach to inequality measurement. In M. McGillivray (ed.), *Inequality, Poverty and Well-Being*, Basingstoke and New York: Palgrave Macmillan, pp. 22–65.
- DASGUPTA, P., SEN, A. K. and STARRETT, D. (1973). Notes on the measurement of inequality. *Journal of Economic Theory*, **6**, 180–187.

- DECANCO, K. (2012). Elementary multivariate rearrangements and stochastic dominance on a Fréchet class. *Journal of Economic Theory*, **147**, 1450–1459.
- FARKAS, J. (1902). Über die theorie der einfachen ungleichungen. *Journal für die Reine und Angewandte Mathematik*, **124**, 1–27.
- FAURE, M. and GRAVEL, N. (2017). Equality among unequals, AMSE Working Paper 2017-02, Aix-Marseille School of Economics.
- FIELDS, G. and FEI, J. (1978). On inequality comparisons. *Econometrica*, **46**, 305–316.
- FIRLA, R. and ZIEGLER, G. (1999). Hilbert bases, unimodular triangulations, and binary covers of rational polyhedral cones. *Discrete and Computational Geometry*, **21**, 205–216.
- FLORENZANO, M. and LE VAN, C. (2001). *Finite Dimensional Convexity and Optimization*. Berlin, Heidelberg, New York: Springer, Studies in Economic Theory.
- GORDAN, P. (1873). Ueber die auflösung linearer gleichungen mit reellen coefficienten. *Mathematische Annalen*, **6**, 23–28.
- GRAVEL, N., MAGDALOU, B. and MOYES, P. (2019). Inequality measurement with an ordinal and continuous variable. *Social Choice and Welfare*, **52(3)**, 453–475.
- , — and — (2021). Ranking distributions of an ordinal variable. *Economic Theory*, **71(1)**, 33–88.
- and MOYES, P. (2012). Ethically robust comparisons of bidimensional distributions with an ordinal attribute. *Journal of Economic Theory*, **147**, 1384–1426.
- HAMMOND, P. J. (1976). Equity, Arrow’s conditions and Rawls’s difference principle. *Econometrica*, **44**, 793–803.
- HARDY, G. H., LITTLEWOOD, J. E. and POLYA, G. (1929). Some simple inequalities satisfied by convex functions. *Messenger of Mathematics*, **58**, 145–152.
- , — and — (1934). *Inequalities, 1st edition*. Cambridge, UK: Cambridge University Press.
- HILBERT, D. (1890). Über die theorie der algebraischen formen. *Mathematische Annalen*, **36**, 473–534.
- KOLM, S.-C. (1969). The optimal production of social justice. In J. Margolis and H. Guitton (eds.), *Public Economics*, London: Macmillan, pp. 145–200.
- MACCHERONI, F., MULIERE, P. and ZOLI, C. (2005). Inverse stochastic orders and generalized Gini functionals. *Metron*, **63**, 529–559.
- MACHINA, M. (1982). “Expected utility” analysis without the independence axiom. *Econometrica*, **50**, 277–323.
- MAGDALOU, B. and MOYES, P. (2009). Deprivation, welfare and inequality. *Social Choice and Welfare*, **32**, 253–273.
- MARSHALL, A. (1991). Multivariate stochastic orderings and generating cones of functions. In K. Mosler and M. Scarsini (eds.), *IMS Lecture Notes Monograph Series 19*, Hayward, CA: Inst.

- Math. Stat.
- , OLKIN, I. and ARNOLD, B. (2011). *Inequalities: Theory of Majorization and its Applications*. New York: Springer, Springer Series in Statistics, Second Edition.
- , WALKUP, D. and WETS, R.-B. (1967). Order-preserving functions: Applications to majorization and order statistics. *Pacific Journal of Mathematics*, **23**, 569–584.
- MEYER, M. and STRULOVICI, B. (2013). The supermodular stochastic ordering, Discussion Paper 1563, Center for Mathematical Studies in Economics and Management Science, Kellogg School of Management.
- MEYER, P. (1966). *Probability and Potentials*. Boston, Massachusetts: Ginn (Blaisdell).
- MULLER, A. (1997). Stochastic orders generated by integrals: A unified study. *Advances in Applied Probability*, **29**, 414–428.
- (2013). Duality theory and transfers for stochastic order relations. In H. Li and X. Li (eds.), *Stochastic Orders in Reliability and Risk*, New York: Lectures Notes in Statistics 208, Springer, pp. 137–155.
- and SCARSINI, M. (2012). Fear of loss, inframodularity, and transfers. *Journal of Economic Theory*, **147**, 1490–1500.
- ROCKAFELLAR, R. (1970). *Convex Analysis*. Princeton: Princeton University Press.
- ROTHSCHILD, M. and STIGLITZ, J. (1970). Increasing risk: I. a definition. *Journal of Economic Theory*, **2**, 225–243.
- RUNCIMAN, W. (1966). *Relative Deprivation and Social Justice*. London: Routledge and Kegan Paul.
- SCHRIJVER, A. (1986). *Theory of Linear and Integer Programming*. Chichester: Wiley-Interscience.
- SEN, A. (1973). *On Economic Inequality*. Oxford: Clarendon Press.
- VAN DER CORPUT, J. (1931a). Konstruktion der minimalbasis für spezielle diophantische systeme von linear-homogenen gleichungen und ungleichungen. *Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam*, **34**, 515–523.
- (1931b). Ueber diophantische systeme von linear-homogenen gleichungen und ungleichungen. *Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam*, **34**, 372–382.