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Abstract

This work investigates two aspects linked to the nature of the feasible domain of anisotropic laminates. In the first part of the paper, proofs are given on the non-convexity of the feasible domain for full anisotropic and for membrane-orthotropic laminates, either in the lamination parameters space or in the polar parameters one. Then, adopting the polar formalism, some particular cases are studied, providing analytical expressions of new narrower bounds, in terms of polar parameters of the membrane stiffness tensor. For a particular case, the exact expression of the membrane stiffness tensor feasible domain is determined. In the second part of the paper, a discussion on the necessary and sufficient condition to get membrane/bending uncoupled and/or homogeneous laminates is presented. It is proved that, when the distinct orientations within the stack are two, quasi-triviality represents a necessary and sufficient condition to achieve uncoupling and/or quasi-homogeneity. This work disproves the common belief of the convexity of the feasible domain in the lamination parameters space and fosters new ideas to face the problem of the determination of the feasible domain of laminates.

Keywords: Anisotropy, Plane elasticity, Polar method, Composite materials, Laminates, Quasi-triviality, Convexity

1. Introduction

Two still-open problems in anisotropic plane elasticity for composite laminates are: (a) the definition of the general expression of the feasible domain; (b) the definition of sufficient and necessary conditions to obtain membrane/bending uncoupled

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laminates (*uncoupling* property), and/or the same group of symmetry for membrane and bending stiffness tensors (*homogeneity* property).

One can deal with the two problems in the general framework of the First-Order Shear Deformation Theory (FSDT) (Reddy (2003)). The FSDT is the generalization of the Classical Laminate Theory (CLT) (Jones (1975)) and constitutes a theoretical framework to describe the behaviour of composite laminates.

In the FSDT (Reddy (2003)), the analytic expression of the laminate stiffness tensor is

$$\mathbf{K}_{\mathrm{lam}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{O} \\ & \mathbf{D} & \mathbf{O} \\ \mathrm{sym} & & \mathbf{H} \end{bmatrix}, \tag{1}$$

where $\mathbf{A}$ is the membrane stiffness tensor of the laminate, $\mathbf{D}$ the bending stiffness tensor, $\mathbf{H}$ the out-of-plane shear stiffness tensor, $\mathbf{B}$ the membrane/bending coupling stiffness tensor. It is convenient to introduce also the following normalised tensors:

$$\mathbf{A}^* := \frac{1}{h}\mathbf{A}, \quad \mathbf{B}^* := \frac{2}{h^2}\mathbf{B}, \quad \mathbf{D}^* := \frac{12}{h^3}\mathbf{D}, \quad \mathbf{C}^* := \mathbf{A}^* - \mathbf{D}^*, \quad \mathbf{H}^* := \frac{1}{h}\mathbf{H}, \tag{2}$$

where h is the total thickness of the laminate.

In the special case of laminates made of identical plies, i.e. same material and thickness for the elementary layer, the expressions of the above tensors read:

$$\begin{aligned} &\mathbf{A}^* = \frac{1}{N}\sum_{k=1}^{N} a_k \mathbf{Q}(\theta_k), \quad \mathbf{B}^* = \frac{1}{N^2}\sum_{k=1}^{N} b_k \mathbf{Q}(\theta_k), \quad \mathbf{D}^* = \frac{1}{N^3}\sum_{k=1}^{N} d_k \mathbf{Q}(\theta_k), \\ &\mathbf{H}^* = \frac{1}{N}\sum_{k=1}^{N} h_k \hat{\mathbf{Q}}(\theta_k), \quad \mathbf{C}^* = \frac{1}{N^3}\sum_{k=1}^{N} c_k \mathbf{Q}(\theta_k). \end{aligned} \tag{3}$$

In Eq. (3), N is the number of plies of the laminate and θ_k is the orientation angle of the k-th ply. $\mathbf{Q}(\theta_k)$ is the in-plane reduced stiffness tensor of the k-th ply whose material frame is turned by an angle θ_k with respect to the global reference frame of the laminate. Analogously, $\hat{\mathbf{Q}}(\theta_k)$ is the out-of-plane reduced stiffness tensor of the k-th ply. Furthermore, coefficients a_k, b_k, d_k, h_k and c_k read:

$$\begin{aligned} &a_k = 1, \quad b_k = 2k - N - 1, \quad d_k = 12k(k - N - 1) + 4 + 3N(N + 2), \\ &h_k = 1, \qquad\qquad\qquad\qquad\;\; c_k = -2N^2 - 12k(k - N - 1) - 4 - 6N. \end{aligned} \tag{4}$$

The ordered sequence of orientations, from the bottom to the top of the laminate, is called stacking sequence (SS), or stack.

It is well known that there is no bijective relationship between arbitrary macroscopic elastic properties of the laminate (in terms of $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$ and $\mathbf{H}^*$ components) and SSs. This means that, given an arbitrary SS, a set of elastic properties is always uniquely determined, whilst the converse is generally false. This aspect is particularly important in practical applications, such as laminates design and SSs recovery. Without entering in the details, after the determination of the optimal stiffness distribution within a structure, with respect to a given merit function, SSs matching the target optimal elastic properties must be found out (Catapano and Montemurro (2014); Montemurro et al. (2019, 2013, 2012); Picchi Scardaoni and Montemurro (2020)). It is then of primary importance to determine the expressions of the feasible domain of the aforementioned elasticity tensors. However, given a SS, the feasible domains of $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$ must be correlated, since these tensors are associated to the same SS.

A first kind of bounds derives directly from thermodynamic considerations, since the elasticity tensor is positive-definite. Therefore, the elastic constants live in a feasible domain whose boundary constitutes the so-called *elastic-bound*.

However, a second kind of bounds can be introduced, which are often referred as *geometrical bounds*. To this purpose, it is important to adopt some mathematical descriptors of the anisotropy. The most common approach makes use of the well-known lamination parameters (LPs) (Jones (1975); M. (1982); Tsai and Hahn (1980); Tsai and Pagano (1968)). LPs unquestionably provide a compact representation of the stiffness tensor of the laminate; although, they are not all tensor invariants. A sound alternative for describing plane anisotropy is represented by the polar formalism introduced in (Verchery (1982)). Thanks to the polar formalism, it is possible to represent any plane tensor by means of tensor invariants, referred as polar parameters (PPs), which are related to the symmetries of the tensor. In particular, for a fourth-order elasticity-like plane tensor (i.e. a tensor having both major and minor symmetries), all possible elastic symmetries can be easily expressed in terms of conditions on the PPs. Moreover, the polar formalism offers a frame-invariant description of any plane tensor (Verchery (1982)).

In (Hammer et al. (1997)), geometrical bounds were derived as a consequence of the nature of the trigonometric functions involved in the definition of the laminate stiffness tensors. In (Vannucci (2012)), Vannucci derives the expressions of the geometrical bounds in the framework of the polar method, showing that these bounds are always stricter than the elastic counterpart: hence these bounds must be considered in practical design problems.

In (Grenestedt and Gudmundson (1993)) Grenestedt and Gudmundson claimed to provide a proof of the convexity of the feasible domain of general anisotropic laminates in the LPs space. However, as it will be discussed later in this paper, the proof provided in (Grenestedt and Gudmundson (1993)) proves only that the projection of the LPs on each axis (in the LPs space, which is of dimension 12) is convex. Indeed, this proof does not respect a fundamental lemma of convex analysis (Boyd and Vandenberghe (2019)), which states that if a set is convex then the projection over some of its coordinates is convex too, while the converse is not generally true. Based on this incorrect result, many works, dealing with the optimisation of laminates and making use of the formalism based on LPs, claim that this description endows the composite design problems with nice properties (Bloomfield et al. (2008); Diaconu et al. (2002); Hammer et al. (1997); Macquart et al. (2018); Raju et al. (2014); Setoodeh et al. (2006)). It is noteworthy that this is one of the main reasons at the basis of the wider diffusion of the approach based on LPs with respect to that based on PPs, which are intrinsic and frame-independent quantities of the stiffness tensor.

However, several works, based on LPs description, seem to provide evidences, both mathematical and graphical, of the convexity of the feasible domain. How is it possible? The answer is probably due to the overlapping of two concepts: the geometrical-admissible region (i.e. the region delimited through the geometrical bounds) and the *actual* feasible region, i.e. the set of all the mapped SSs, through LPS (or PPs), in the corresponding space. In fact, although the geometrical bounds define a region, possibly convex, it is not generally true that the actual feasible domain, which is a subset of the geometrical-admissible region, is a convex set. It is evident the conceptual overlapping between the geometrical-admissible region (which can be convex or not) and the actual feasible region, which should be the true object of investigation. In (Diaconu et al. (2002)) the authors formulate a variational problem to find the actual shape of the feasible domain in the LPs space. Moreover, they provide some plots of some 2D projections of the domain, which are convex set. Similar approaches are provided in (Bloomfield et al. (2008); Setoodeh et al. (2006)). All of these studies, as stated by the authors themselves, rely on the fact that the feasible region, in the LPs space, is a convex sets. In the light of the results of this work, it is possible to consider the findings of the existing literature as *convex hulls* (regardless of the number of layers) of the actual feasible region, which can be narrower by far.

The derivation of the laminate actual feasible domain in a closed form is of paramount importance when formulating the laminate design problem as a constrained non-linear programming problem. Keeping this in mind, a simple investigation led to the discovery of the analytic expression of a further bound, valid for any kind of symmetry of the laminate, which makes the feasible region a stricter non-convex region. These new bounds introduce the dependence from the number of plies. Indeed, up to now, it is odd that the feasible region of laminates having even very different number of plies is provided by the same equations, which in turn do not depend on the number of plies!

Starting from these considerations, and from some recent advances in the development of a global/local modelling approach for the multi-scale optimization of composite structures discussed in (Picchi Scardaoni and Montemurro (2020)), the research has been extended to the feasible domain determination problem for composite laminates. In particular, in this paper, the rigorous proof of the non-convexity of the laminate feasible domain is provided, regardless of the formalism used to describe the anisotropy (i.e. LPs or PPs).

A laminate is *uncoupled* if and only if $\mathbf{B}^* = \mathbf{O}$. This means that in-plane forces do not produce curvatures and, similarly, bending moments do not deform the laminate middle plane. A laminate is said *homogeneous* if and only if $\mathbf{C}^* = \mathbf{O}$. The homogeneity property is linked to the design of the pure bending tensor $\mathbf{D}^*$. The design of $\mathbf{D}^*$ is quite difficult because its properties depend not only on the plies orientation angles, but also on their position within the SS. In order to have the same properties both in membrane and bending in any direction, i.e. the same group of symmetry, the homogeneity property must be imposed.

The uncoupling property is often sought in many engineering applications. Since coefficients b_k assume antisymmetric values with respect to the laminate middle plane, a simple way to obtain $\mathbf{B}^* = \mathbf{O}$ consists of using a symmetric SSs, as commonly done in several works (Adams et al. (2004); Bloomfield et al. (2008); Macquart et al. (2016); Raju et al. (2014); Seresta et al. (2007)). Of course, this is only a sufficient condition, since asymmetric uncoupled SSs exist. In (Caprino and Crivelli Visconti (1982)), the existence of uncoupled anti-symmetric stacks was proven, while in (Verchery and Vong (1986)), the existence of completely asymmetric uncoupled SSs was shown. In (Vannucci and Verchery (2001)), a special class of uncoupled and possibly homogeneous laminates was found; the solutions belonging to this class are called *quasi-trivial* (QT) and represent a class of arithmetically-exact solutions. Furthermore, authors have shown that the number of independent QT solutions is by far larger than the number of symmetric ones. An efficient enumerating algorithm has been recently proposed in (Garulli et al. (2018)). In this work, the enumeration of all independent QT SSs, up to $N = 35$, is presented.

Up to the present, it is not clear if quasi-triviality is a sufficient or even necessary condition to have uncoupled and/or homogeneous laminates. This work aims at clarifying this aspect too, by showing that quasi-triviality is, in general, only a sufficient condition. However, for a particular case, quasi-triviality becomes also a necessary condition. In other words, for that particular case, QT solutions constitute the full set of all possible uncoupled and/or homogeneous SSs.

The paper is organized as follows: a short recall of the polar formalism in the FSDT framework and of the LPs in the CLT framework is presented in Section 2. Section 3 provides the proofs of the non-convexity of the feasible domain in the PPs and LPs spaces for general anisotropic and membrane-orthotropic laminates. Sections 4 and 5 present the new analytical bounds for multilayer plates, together with some exact solutions. Section 6 provides the proof that quasi-triviality is not a necessary condition for uncoupling and/or homogeneity in the most general case. Finally, Section 7 ends the paper with some meaningful conclusions and prospects.

2. Fundamentals of the Polar Method and of the Lamination Parameters

2.1. Polar Method

The Polar Method allows representing any n-th order plane tensor in terms of invariants: this method was introduced, for the first time, in (Verchery (1982)). For a deeper insight in the matter, the reader is addressed to (Vannucci (2017)). In this context, a second-order symmetric plane tensor $\mathbf{Z}$ can be expressed in the local frame $\Gamma = \{\mathrm{O}, x_1, x_2, x_3\}$ as:

$$Z_{11} = T + R\cos 2\Phi, \quad Z_{12} = R\sin 2\Phi, \quad Z_{22} = T - R\cos 2\Phi, \tag{5}$$

where T is the isotropic modulus, R the deviatoric one and Φ the polar angle. Among them, only the moduli are invariants, whilst Φ is needed to set the reference frame. If $\mathbf{L}$ is a fourth-order plane elasticity-like tensor, i.e. with major and minors symmetries, its Cartesian components can be expressed by means of four moduli and two polars angles, namely T_0, T_1, R_0, R_1, Φ_0, Φ_1. The complete expression reads:

$$\begin{aligned}
&L_{1111} = T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1,\\
&L_{1112} = R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
&L_{1122} = -T_0 + 2T_1 - R_0\cos 4\Phi_0,\\
&L_{1212} = T_0 - R_0\cos 4\Phi_0,\\
&L_{2212} = -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1,\\
&L_{2222} = T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1.
\end{aligned} \tag{6}$$

In Eq. (6), T_0 and T_1 are the isotropic moduli, R_0 and R_1 the anisotropic ones, Φ_0 and Φ_1 the polar angles. Among them, only the four moduli and the difference $\Phi_0 - \Phi_1$ are tensor invariants.

One of the advantages of the polar method is that for a fourth-order elasticity-like plane tensor, the polar invariants are related to the elastic symmetries of the tensor. Indeed the polar formalism offers an algebraic characterization of the elastic symmetries (Catapano et al. (2012); Catapano and Montemurro (2018)). In particular, four different symmetries can be defined:

- *Orthotropy*: this symmetry corresponds to the condition $\Phi_0 - \Phi_1 = K\dfrac{\pi}{4}, \quad K = 0, 1$.
- R_0-*Orthotropy*: the algebraic condition to obtain this *special* orthotropy is $R_0 = 0$. This case has been studied in (Vannucci (2002)).
- *Square symmetry*: it can be achieved by imposing $R_1 = 0$. This case represents the 2D counterpart of the well-known 3D cubic syngony.
- *Isotropy*: the condition to be satisfied is $R_0 = R_1 = 0$.

In order to properly analyse the mechanical behaviour of a laminate, it is possible to express the stiffness tensors appearing in Eq. (3) in terms of their PPs. In particular, $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$, and thus $\mathbf{C}^*$, are fourth-order elasticity-like plane tensor, while $\mathbf{H}^*$ behaves like a second-order symmetric plane tensor (Montemurro (2015a,b)). The PPs of the laminate stiffness tensors can be expressed as functions of the PPs of the lamina reduced stiffness matrices and of the geometrical properties of the stack (i.e. the layers orientation, position and number). The polar representation of the normalised stiffness tensors of the laminate reads:

$$T_0^{A^*} = T_0, \quad T_1^{A^*} = T_1, \quad R_0^{A^*} \mathrm{e}^{\mathrm{i}4\Phi_0^{A^*}} = \frac{R_0}{N} \mathrm{e}^{\mathrm{i}4\Phi_0} \sum_{k=1}^{N} a_k \mathrm{e}^{\mathrm{i}4\theta_k}, \quad R_1^{A^*} \mathrm{e}^{\mathrm{i}2\Phi_1^{A^*}} = \frac{R_1}{N} \mathrm{e}^{\mathrm{i}2\Phi_1} \sum_{k=1}^{N} a_k \mathrm{e}^{\mathrm{i}2\theta_k}, \tag{7}$$

$$T_0^{B^*} = T_1^{B^*} = 0, \quad R_0^{B^*} \mathrm{e}^{\mathrm{i}4\Phi_0^{B^*}} = \frac{R_0}{N^2} \mathrm{e}^{\mathrm{i}4\Phi_0} \sum_{k=1}^{N} b_k \mathrm{e}^{\mathrm{i}4\theta_k}, \quad R_1^{B^*} \mathrm{e}^{\mathrm{i}2\Phi_1^{B^*}} = \frac{R_1}{N^2} \mathrm{e}^{\mathrm{i}2\Phi_1} \sum_{k=1}^{N} b_k \mathrm{e}^{\mathrm{i}2\theta_k}, \tag{8}$$

$$T_0^{D^*} = T_0, \quad T_1^{D^*} = T_1, \quad R_0^{D^*} \mathrm{e}^{\mathrm{i}4\Phi_0^{D^*}} = \frac{R_0}{N^3} \mathrm{e}^{\mathrm{i}4\Phi_0} \sum_{k=1}^{N} d_k \mathrm{e}^{\mathrm{i}4\theta_k}, \quad R_1^{D^*} \mathrm{e}^{\mathrm{i}2\Phi_1^{D^*}} = \frac{R_1}{N^3} \mathrm{e}^{\mathrm{i}2\Phi_1} \sum_{k=1}^{N} d_k \mathrm{e}^{\mathrm{i}2\theta_k}, \tag{9}$$

$$T^{H^*} = T, \quad R^{H^*} \mathrm{e}^{\mathrm{i}2\Phi_1^{H^*}} = \frac{R}{N} \mathrm{e}^{\mathrm{i}2\Phi} \sum_{k=1}^{N} h_k \mathrm{e}^{-i2\theta_k} = R_1^{A^*} \frac{R}{R_1} \mathrm{e}^{i2(\phi+\phi_1-\phi_1^{A^*})}, \tag{10}$$

$$T_0^{C^*} = 0, \quad T_1^{C^*} = 0, \quad R_0^{C^*} \mathrm{e}^{\mathrm{i}4\Phi_0^{C^*}} = \frac{R_0}{N^3} \mathrm{e}^{\mathrm{i}4\Phi_0} \sum_{k=1}^{N} c_k \mathrm{e}^{\mathrm{i}4\theta_k}, \quad R_1^{C^*} \mathrm{e}^{\mathrm{i}2\Phi_1^{C^*}} = \frac{R_1}{N^3} \mathrm{e}^{\mathrm{i}2\Phi_1} \sum_{k=1}^{N} c_k \mathrm{e}^{\mathrm{i}2\theta_k}, \tag{11}$$

In the above equations, T_0, T_1, R_0, R_1, Φ_0 and Φ_1 are the polar parameters of the in-plane reduced stiffness matrix $\mathbf{Q}$ of the lamina, while T, R, and Φ are those of the transverse shear stiffness matrix $\hat{\mathbf{Q}}$: all of these parameters solely depend upon the ply material properties. Since basic layers are actually orthotropic, without loss of generality it can be assumed that $\Phi_1 = 0$ ($\Phi_0 = K\frac{\pi}{4}$ with $K \in \{0, 1\}$) (Vannucci (2012)). For further details on the Polar Method in the FSDT framework, the reader is addressed to (Montemurro (2015a,b)).

2.2. Lamination Parameters

LPs are often used together with the parameters of Tsai and Pagano to describe the anisotropic behaviour of the laminate in the framework of the CLT (Grenestedt and Gudmundson (1993); Hammer et al. (1997); Jones (1975); Reddy (2003)). LPs express the properties of a laminate in terms of moments, relative to the plate mid-plane, of the trigonometric functions entering in the frame rotation formulas (Hammer et al. (1997)). For the sake of brevity, the complete expressions of the LPs in the CLT framework are not here reported. The interested reader is addressed to (Jones (1975)).

LPs are closely related to PPs: from a mathematical viewpoint, it is sufficient to report the following identities between PPs and LPs, using the same nomenclature of (Vannucci (2017)).

$$\xi_1 + \mathrm{i}\xi_2 = \frac{1}{N}\sum_{k=1}^{N} a_k \mathrm{e}^{\mathrm{i}4\theta_k}, \qquad \xi_3 + \mathrm{i}\xi_4 = \frac{1}{N}\sum_{k=1}^{N} a_k \mathrm{e}^{\mathrm{i}2\theta_k}, \qquad \xi_5 + \mathrm{i}\xi_6 = \frac{1}{N^2}\sum_{k=1}^{N} b_k \mathrm{e}^{\mathrm{i}4\theta_k},$$
$$\xi_7 + \mathrm{i}\xi_8 = \frac{1}{N^2}\sum_{k=1}^{N} b_k \mathrm{e}^{\mathrm{i}2\theta_k}, \quad \xi_9 + \mathrm{i}\xi_{10} = \frac{1}{N^3}\sum_{k=1}^{N} d_k \mathrm{e}^{\mathrm{i}4\theta_k}, \quad \xi_{11} + \mathrm{i}\xi_{12} = \frac{1}{N^3}\sum_{k=1}^{N} d_k \mathrm{e}^{\mathrm{i}2\theta_k}. \tag{12}$$

It is evident that 12 parameters are necessary to completely define the behaviour of the laminate in the CLT framework.

3. The Non-Convexity of the Feasible Domain

For a generic anisotropic laminate, the geometrical bounds in the PPs space read (Vannucci (2012)):

$$\begin{cases} 0 \leq \rho_0 \leq 1, \\ 0 \leq \rho_1 \leq 1, \\ 2\rho_1^2 \leq \dfrac{1-\rho_0^2}{1-(-1)^K \rho_0 \cos 4\Phi_0^{A*}}, \end{cases} \tag{13}$$

where

$$\rho_0 := \frac{R_0^{A^*}}{R_0} = \frac{1}{N}\sqrt{\left(\sum_{j=1}^{N}\cos 4\theta_j\right)^2 + \left(\sum_{j=1}^{N}\sin 4\theta_j\right)^2}, \tag{14}$$

$$\rho_1 := \frac{R_1^{A^*}}{R_1} = \frac{1}{N}\sqrt{\left(\sum_{j=1}^{N}\cos 2\theta_j\right)^2 + \left(\sum_{j=1}^{N}\sin 2\theta_j\right)^2}. \tag{15}$$

In the LPs space, they read (Hammer et al. (1997))

$$\begin{cases} 2\xi_3^2(1-\xi_1) + 2\xi_4^2(1+\xi_1) + \xi_1^2 + \xi_2^2 - 4\xi_3\xi_2\xi_4 \leq 1 \\ \xi_3^2 + \xi_4^2 \leq 1 \\ -1 \leq \xi_1 \leq 1. \end{cases} \tag{16}$$

For a membrane-orthotropic laminate, geometrical bounds in the PPs space simplify to

$$\begin{cases} -1 \leq \rho_{0K} \leq 1 \\ 0 \leq \rho_1 \leq 1 \\ 2\rho_1^2 - 1 - (-1)^{-K}\rho_{0K} \leq 0 \end{cases} \tag{17}$$

where $\rho_{0K} := (-1)^{K^{A^*}}\rho_0$, $K^{A^*} \in \{0, 1\}$. In the LPs space, they read (M. (1982))

$$\begin{cases} -1 \leq \xi_3 \leq 1 \\ 2\xi_3^2 - 1 \leq \xi_1 \leq 1. \end{cases} \tag{18}$$

As discussed in (Montemurro (2015a,b); Vannucci (2017)), the design of a laminate lives in $\mathbb{R}^{12}$, since four PPs are needed to define the anisotropic part of tensors $\mathbf{A}^*$, $\mathbf{B}^*$ and $\mathbf{D}^*$ (the deviatoric part of tensor $\mathbf{H}^*$ being directly related, in general, to the anisotropic one of tensors $\mathbf{A}^*$ and $\mathbf{D}^*$). Regarding LPs, the same remark holds since 12 LPs are needed to fully describe the laminate behaviour in the CLT framework, as stated above. For both representations, the feasible domain is then a subset of $\mathbb{R}^{12}$. The determination of the feasible domain, in the most general case, is still an open problem. As stated in the introduction, in the literature some misleading results can be found about the convexity of the laminate feasible domain. For example, in (Grenestedt and Gudmundson (1993)), the feasible domain in the LPs space is claimed to be convex. Conversely, in (Vannucci (2017)), the feasible domain in the PPs space is claimed to be non-convex for anisotropic laminates and convex for orthotropic ones.

The main result of this Section is a rigorous proof of the non-convexity of the feasible domain in the LPs and PPs spaces, both for anisotropic and orthotopic laminates. To this purpose, some nomenclature must be introduced.

Let $z^* \in [0, 1]$ be the dimensionless coordinate defined through the laminate thickness, from the bottom to the top. The layup function is a combination of piecewise functions defined as

$$\theta(z^*) := \sum_{i=1}^{N} c_i \chi[\Delta z_i^*], \tag{19}$$

where $N \geq 1$, $N \in \mathbb{N}$, $c_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\chi[\Delta z_i^*]$ is the indicator function assuming unit value in the interval Δz_i^*. The function $\chi\,[\Delta z_i^*]$ is defined as:

$$\chi\,[\Delta z_i^*] := \begin{cases} 1 & \text{if } z^* \in \Delta z_i^*, \\ 0 & \text{otherwise} . \end{cases} \tag{20}$$

Intervals Δz_i^* are such that $\bigcup_{i=1}^{N} \Delta z_i^* = [0, 1]$, $\bigcap_{i=1}^{N} \Delta z_i^* = \emptyset$ and $\text{meas}(\Delta z_i^*) = 1/N$. It is clear that the range of $\theta(z^*)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The layup function can be seen as the SS counterpart when the integral description is used instead of summations.

Proposition 3.1. *The set of layup functions* $\Theta_N := \Big\{\theta(z^*) \;:\; \theta(z^*) = \sum_{i=1}^{N} c_i \chi[\Delta z_i^*],\, c_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]\Big\}$ *is convex.*

Proof. Let $\alpha \in [0, 1]$ and suppose that $\hat{\theta}(z^*)$ and $\check{\theta}(z^*)$ belong to Θ_N. Therefore, there exist some $\hat{c}_i$ and $\check{c}_i$ belonging to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\hat{\theta}(z^*) = \sum_{i=1}^{N} \hat{c}_i \chi[\Delta z_i^*]$ and $\check{\theta}(z^*) = \sum_{i=1}^{N} \check{c}_i \chi[\Delta z_i^*]$.

A convex combination of these two elements satisfies the following identity:

$$\begin{aligned} \alpha\hat{\theta} + (1-\alpha)\check{\theta} &= \alpha \sum_{i=1}^{N} \hat{c}_i \chi[\Delta z_i^*] + (1-\alpha) \sum_{i=1}^{N} \check{c}_i \chi[\Delta z_i^*] \\ &= \sum_{i=1}^{N} \left(\alpha\hat{c}_i + (1-\alpha)\check{c}_i\right) \chi[\Delta z_i^*] =: \sum_{i=1}^{N} \tilde{c}_i \chi[\Delta z_i^*], \end{aligned} \tag{21}$$

where $\tilde{c}_i$ are defined through the last equality. Therefore, the right-hand side member belongs to Θ_N for every α and for every $\hat{\theta}(z^*)$, $\check{\theta}(z^*)$. This claim is sufficient to conclude the proof. □

The following lemma (Boyd and Vandenberghe (2019)) is needed before introducing the main result of this Section.

Lemma 3.1. *Let $C \subset \mathbb{R}^{m\times n}$ be a convex set. Then, the projection over some of its coordinates $\mathcal{P}(C) = \{x_1 \in \mathbb{R}^m \,|\, (x_1, x_2) \in C$ for some $x_2 \in \mathbb{R}^n\}$ is convex.*

It is then immediate the following

Corollary 3.1. *If a projection $\mathcal{P}(S)$ (in the sense of Lemma 3.1) of a set S is non-convex, then S is a non-convex set.*

Of course, Lemma 3.1 has an important consequence: if a projection of the set S is convex, the set is not necessarily convex. Indeed, the proof provided in (Grenestedt and Gudmundson (1993)) about the convexity of the LPs space does not take into account for this aspect.

Let $\mathbf{p}[\theta]$ denote the vector consisting of twelve PPs (or LPs) obtained with the layup function $\theta(z^*)$. The following four Propositions express the non-convexity of the feasible domain regardless of the anisotropy representation. A more detailed proof is given only for full-anisotropic laminates in PPs space, the remaining four being conceptually identical.

Proposition 3.2. *The feasible region in the PPs space of all anisotropic laminates composed of N plies $\Pi_N := \{\mathbf{p}[\theta(z^*)]\,\forall\,\theta(z^*) \in \Theta_N\}$ is a non-convex bounded subset of $\mathbb{R}^{12}$ for every $N > 1$ ($N \in \mathbb{N}$).*

Proof. For the boundedness, it is sufficient to see that the twelve components of $\mathbf{p}$ are bounded. In fact, the six dimensionless anisotropic moduli (ρ_0, ρ_1 for tensor $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$) take values in the set $[0, 1]$, whilst the six dimensionless polar angles ($\phi_0 := \frac{\Phi_0}{\pi/4}$ and $\phi_1 := \frac{\Phi_1}{\pi/2}$ for tensors $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$) take values in the set $[-1, 1]$. Therefore, Π_N is a bounded subset of the 12D-parallelepiped $[0, 1]^6 \times [-1, 1]^6 \subset \mathbb{R}^{12}$.

Let $\hat{\mathbf{p}} := \mathbf{p}[\hat{\theta}(z^*)]$ and $\check{\mathbf{p}} := \mathbf{p}[\check{\theta}(z^*)]$ be two points of the feasible domain for the layup functions $\hat{\theta}(z^*)$ and $\check{\theta}(z^*)$ belonging to Θ_N. Moreover, consider the interval $[0, 1]$ subdivided into N disjoint intervals Δz_i^* of equal length, and let $\alpha \in [0,\,1]$.

A convex set, by definition, contains the whole line segment that joins any two points belonging to the set. For the characterisation of Π_N, its convexity would imply the existence of a layup function $\tilde{\theta} \in \Theta_N$ such that $\mathbf{p}[\tilde{\theta}] = \alpha\hat{\mathbf{p}} + (1 - \alpha)\check{\mathbf{p}}$ belongs to Π_N for every α and for every $\hat{\mathbf{p}}$, $\check{\mathbf{p}}$. Therefore, to prove the thesis of the proposition, it is sufficient to prove the violation of such definition at least for one case. Moreover, thanks to Proposition 3.1 and Corollary 3.1, it is sufficient to seek the violation in the projection of Π_N onto some of its hyper-planes. To this purpose, consider the projection of Π_N onto the plane $(\rho_1,\,\rho_0)$.

If layup functions are considered instead of SSs:

$$\begin{aligned}\rho_0[\theta(z^*)] &:= \sqrt{\left(\int_0^1 \cos 4\theta(z^*)\mathrm{d}z^*\right)^2 + \left(\int_0^1 \sin 4\theta(z^*)\,\mathrm{d}z^*\right)^2} \\ &= \sqrt{2\int_0^1\int_0^1 \cos^2 2(\theta(z^*) - \theta(t^*))\,\mathrm{d}z^*\,\mathrm{d}t^* - 1},\end{aligned} \tag{22}$$

and

$$\begin{aligned}\rho_1[\theta(z^*)] &:= \sqrt{\left(\int_0^1 \cos 2\theta(z^*)\mathrm{d}z^*\right)^2 + \left(\int_0^1 \sin 2\theta(z^*)\,\mathrm{d}z^*\right)^2} \\ &= \sqrt{\int_0^1\int_0^1 \cos 2(\theta(z^*) - \theta(t^*))\,\mathrm{d}z^*\,\mathrm{d}t^*},\end{aligned} \tag{23}$$

where the identities $\int f(x)\,\mathrm{d}x \int g(x)\,\mathrm{d}x = \int\int f(x)g(t)\,\mathrm{d}x\,\mathrm{d}t$, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\cos(4\alpha) = 2\cos^2 2\alpha - 1$ have been used. Consider two points of the (ρ_1, ρ_0) plane having coordinates $\check{P} = (1, 1)$ and $\hat{P} = (1 - \frac{2}{N}, 1)$. These two points belong to Π_N. In fact, $\check{P}$ corresponds to layup functions of the form $\sum_{i=1}^{N} c\chi[\Delta z_i^*]$ $(c \in [-\frac{\pi}{2}, \frac{\pi}{2}])$, whilst $\hat{P}$ corresponds to layup functions of the form $\sum_{i=1}^{N-1} c\chi[\Delta z_i^*] + \left(-\frac{\pi}{2} + c\right)\chi[\Delta z_N^*]$ $(c \in [-\frac{\pi}{2}, \frac{\pi}{2}])$, as it can be easily verified. Then, assuming that the convex combination of these two points belongs to Π_N, one has:

$$\alpha\begin{pmatrix}1 - \frac{2}{N}\\ 1\end{pmatrix} + (1-\alpha)\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}\rho_1[\tilde{\theta}(z^*)]\\ \rho_0[\tilde{\theta}(z^*)]\end{pmatrix}, \tag{24}$$

for some $\tilde{\theta}(z^*) \in \Theta_N$. The second component of Eq. (24) is satisfied only if $\rho_0[\tilde{\theta}(z^*)] = 1$. Considering Eq. (22), it means that $\int_0^1\int_0^1 \cos^2 2(\theta(z^*) - \theta(t^*))\,\mathrm{d}z^*\,\mathrm{d}t^* = 1$. This condition is achieved if $\tilde{\theta}(z^*)$ is the constant function (the line segment represented by Eq. (24) would collapse in a point, so this case is disregarded) or if it is of the form

$$\tilde{\theta}(z^*) = \sum_{i=1}^{M} c\chi[\Delta z_i^*] + \sum_{i=M+1}^{N}\left(-\frac{\pi}{2} + c\right)\chi[\Delta z_i^*], \tag{25}$$

for some $M < N$, $M \in \mathbb{N}$. Assuming $\tilde{\theta}(z^*)$ as in Eq. (25), the value of $\rho_1\left[\tilde{\theta}\,(z^*)\right]$

can be calculated from Eq. (23) as:

$$
\begin{aligned}
\rho_1^2[\tilde{\theta}(z^*)] &= \int_0^1 \int_0^1 \cos 2(\tilde{\theta}(z^*) - \tilde{\theta}(t^*))\mathrm{d}t^*\mathrm{d}z^* \\
&= \int_0^1 \int_0^1 \cos 2\left(\sum_{i=1}^{M} c\chi[\Delta z_i^*] + \sum_{i=M+1}^{N}\left(-\frac{\pi}{2}+c\right)\chi[\Delta z_i^*] + \right.\\
&\left. - \sum_{i=1}^{M} c\chi[\Delta t_i^*] - \sum_{i=M+1}^{N}\left(-\frac{\pi}{2}+c\right)\chi[\Delta t_i^*]\right)\mathrm{d}t^*\mathrm{d}z^* \\
&= \int\limits_0^{M/N}\int\limits_0^{M/N} \cos 2\left(c\sum_{i=1}^{M}\chi[\Delta z_i^*] - \chi[\Delta t_i^*]\right)\mathrm{d}t^*\mathrm{d}z^* \\
&\quad + \int\limits_0^{M/N}\int\limits_{M/N}^{1} \cos 2\left(c\sum_{i=1}^{M}\chi[\Delta z_i^*] - \left(-\frac{\pi}{2}+c\right)\sum_{i=M+1}^{N}\chi[\Delta t_i^*]\right)\mathrm{d}t^*\mathrm{d}z^* \qquad (26)\\
&\quad + \int\limits_{M/N}^{1}\int\limits_0^{M/N} \cos 2\left(-c\sum_{i=1}^{M}\chi[\Delta t_i^*] + \left(-\frac{\pi}{2}+c\right)\sum_{i=M+1}^{N}\chi[\Delta z_i^*]\right)\mathrm{d}t^*\mathrm{d}z^* \\
&\quad + \int\limits_{M/N}^{1}\int\limits_{M/N}^{1} \cos 2\left(\left(-\frac{\pi}{2}+c\right)\sum_{i=M+1}^{N}\chi[\Delta z_i^*] - \chi[\Delta t_i^*]\right)\mathrm{d}t^*\mathrm{d}z^* \\
&= \left(\frac{M}{N}\right)^2 - \frac{M}{N}\left(1-\frac{M}{N}\right) - \frac{M}{N}\left(1-\frac{M}{N}\right) + \left(1-\frac{M}{N}\right)^2 \\
&= \left(1-2\frac{M}{N}\right)^2,
\end{aligned}
$$

and, hence,

$$\rho_1[\tilde{\theta}(z^*)] = \sqrt{\left(1-2\frac{M}{N}\right)^2}. \qquad (27)$$

Finally, from Eqs. (24) and (27), one obtains

$$1 - \frac{2\alpha}{N} = \sqrt{\left(1-2\frac{M}{N}\right)^2} = \begin{cases} 1-2\frac{M}{N}, & \text{if } 1 > 2\frac{M}{N}, \\ 2\frac{M}{N}-1, & \text{otherwise}. \end{cases} \qquad (28)$$

From Eq. (28), either $M = \alpha$ or $M = N - \alpha$. Both results are absurd, since M must be, by definition, an integer number (for all α). Therefore, no $\tilde{\theta}(z^*)$ exists satisfying Eq. (24), and the entire line segment between $\hat{P}$ and $\check{P}$ does not belong to the projection of Π_N onto the (ρ_1, ρ_0) plane. This counter-example proves the statement of the Proposition, remembering also Corollary 3.1. $\square$

It is noteworthy that Proposition 3.2 considers only geometrical aspects of laminate layups. With the same argument used to prove Proposition 3.2, one can prove that the feasible domain of laminates with an orthotropic membrane behaviour is a non-convex bounded set.

Proposition 3.3. *The feasible region in the PPs space of all membrane-orthotropic laminates composed of N plies $\Pi_N^{Ort} := \{\mathbf{p}[\theta(z^*)] \,|\, \Phi_0^{A^*} - \Phi_1^{A^*} = K^{A^*}\frac{\pi}{4}, K^{A^*} \in \{0,1\}, \forall\, \theta(z^*) \in \Theta_N\}$ is a non-convex bounded subset of $\mathbb{R}^{12}$ for every $N > 1$ ($N \in \mathbb{N}$).*

Proof. To show the boundedness, the argument is the same of the proof of Proposition 3.2. Adopting the integral description,

$$\Phi_0[\theta(z^*)] := \frac{1}{4}\arctan\left(\frac{\int_0^1 \sin 4\theta(z^*)\mathrm{d}z^*}{\int_0^1 \cos 4\theta(z^*)\mathrm{d}z^*}\right), \quad \Phi_1[\theta(z^*)] := \frac{1}{2}\arctan\left(\frac{\int_0^1 \sin 2\theta(z^*)\mathrm{d}z^*}{\int_0^1 \cos 2\theta(z^*)\mathrm{d}z^*}\right). \tag{29}$$

For the non-convexity, it is sufficient to notice that points $\hat{P}$ and $\check{P}$, used for the the proof of Proposition 3.2, actually belong to Π_N^{Ort} (with $K^{A^*} = 0$). $\square$

Proposition 3.4. *The feasible region in the LPs space of all anisotropic laminates composed of N plies $\Pi_N := \{\mathbf{p}[\theta(z^*)]\,\forall\,\theta(z^*) \in \Theta_N\}$ is a non-convex bounded subset of $\mathbb{R}^{12}$ for every $N > 1$ ($N \in \mathbb{N}$).*

Proof. To show the boundedness, the argument is the same as the proof of Proposition 3.2. For the non-convexity, the same argument of Proposition 3.2 is used, but settled in the (ξ_3, ξ_1) plane. Adopting the integral description,

$$\xi_1 = \int_0^1 \cos 4\theta(z^*)\mathrm{d}z^*, \quad \xi_3 = \int_0^1 \cos 2\theta(z^*)\mathrm{d}z^*. \tag{30}$$

Let $\check{P} = (1,\, 1)$ and $\hat{P} = (1 - \frac{2}{N},\, 1)$ be the same points used for Proposition 3.2. These two points belong to Π_N. In fact, $\check{P}$ corresponds to layup functions of the form $\sum_{i=1}^{N} 0\chi[\Delta z_i^*]$, whilst $\hat{P}$ corresponds to layup functions of the form $\sum_{i=1}^{N-1} 0\chi[\Delta z_i^*] + \left(-\frac{\pi}{2} + 0\right)\chi[\Delta z_N^*]$, as it can be easily verified. The convexity condition, to be confuted, reads:

$$\begin{pmatrix} \xi_3[\tilde{\theta}(z^*)] \\ \xi_1[\tilde{\theta}(z^*)] \end{pmatrix} = \alpha\hat{P} + (1-\alpha)\check{P} = \alpha \begin{pmatrix} 1-\frac{2}{N} \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{31}$$

for some $\tilde{\theta}(z^*) \in \Theta_N$ and for all $\alpha \in [0,1]$. Considering the second component of Eq. (31), $\tilde{\theta}(z^*)$ is necessarily of the form

$$\tilde{\theta}(z^*) = \sum_{i=1}^{M} c_1\chi[\Delta z_i^*] + \sum_{i=M+1}^{N} c_2\chi[\Delta z_i^*] \tag{32}$$

with $c_1, c_2 \in \{0, -\frac{\pi}{2}, \frac{\pi}{2}\}$ and $M < N$, $M \in \mathbb{N}$. As a consequence, it is easy to see that, for all combinations of c_1 and c_2, $\xi_3[\tilde{\theta}(z^*)] \in \{-1, 1, -1+2\frac{M}{N}, 1-2\frac{M}{N}\}$. The first component of Eq. (31) reads $\{-1, 1, -1+2\frac{M}{N}, 1-2\frac{M}{N}\} = 1 - \dfrac{2\alpha}{N}$, which either cannot be satisfied regardless of α or violates the condition $M \in \mathbb{N}$. Considerations similar to those of the proof of Proposition 3.2 can also be repeated for this case, allowing to conclude the proof. □

Proposition 3.5. *The feasible region in the LPs space of all membrane-orthotropic laminates composed of N plies $\Pi_N^{Ort} := \{\mathbf{p}[\theta(z^*)] \,|\, \Phi_0^{A^*} - \Phi_1^{A^*} = K^{A^*}\frac{\pi}{4},\ K^{A^*} \in \{0,1\},\ \forall\, \theta(z^*) \in \Theta_N\}$ is a non-convex bounded subset of $\mathbb{R}^{12}$ for every $N > 1$ ($N \in \mathbb{N}$).*

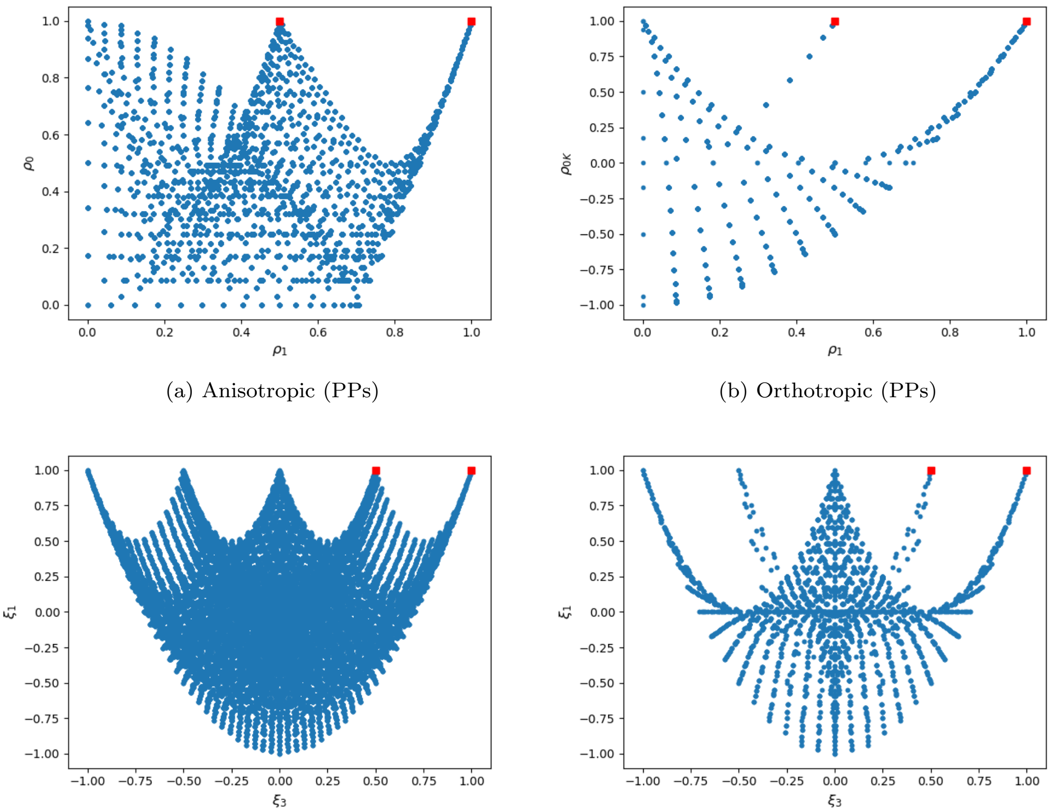
Proof. The proof follows straightforwardly from the previous ones. Indeed, points $\hat{P}$ and $\check{P}$ used in the proof of Proposition 3.4 correspond to membrane-orthotropic laminates (they are a particular case of the ones chosen in the proof of Proposition 3.2 and 3.3). Therefore, the proof can be considered concluded.

□

Propositions 3.2 and 3.4 claim that, for an anisotropic laminate, the feasible region Π_N, in terms of PPs or PPs, is a non-convex set. Propositions 3.3 and 3.5 state that the non-convexity of the feasible domain holds also in the case of laminates having an orthotropic membrane stiffness tensor. Specifically, the non-convexity is maintained in the projection of the 12-D feasible domain onto either the (ρ_1, ρ_{0K}) or the (ξ_3, ξ_1) plane[1].

[1] In the proof of Proposition 3.3, the non-convexity appears in the (ρ_1, ρ_0) plane. However, since the points $\hat{P}$ and $\check{P}$ correspond to orthotropic laminates with $K^{A^*} = 0$, the line segment delimited by the two points is still not included in the projection of the feasible domain onto the (ρ_1, ρ_{0K}) plane.

A way to visualise the non-convexity of the feasible domain is, trivially, to plot all the possible points of Π_N and Π_N^{Ort}, projected in the corresponding plane according to the anisotropy representation, for an arbitrary number of plies N. Inasmuch as the number of all the possible SSs grows exponentially with the number of layers N, a sampling step is assumed. It is convenient to introduce the notion of *number of groups*, denoted by m, which is is the number of distinct orientations within a SS. The number of plies associated to the i-th orientation is denoted by n_i (obviously $\sum_{i=1}^{m} n_i = N$). Figure 1 shows the feasible domain projections for a laminate having $N = 4$ plies ($K = 0$). Indeed, the projected feasible domains are non-convex sets.



(a) Anisotropic (PPs)

(b) Orthotropic (PPs)

(c) Anisotropic (LPs)

(d) Orthotropic (LPs)

Figure 1: Feasible domain projections for anisotropic (on the left) and membrane-orthotropic (on the right) laminates ($K = 0$) of $N = 4$ plies, on the top in terms of PPs, on the bottom in terms of LPs (discretisation step $= 5°$). In red the points $\hat{P}$ and $\check{P}$ used in the proofs of Section 3.

In (Grenestedt and Gudmundson (1993)), it was claimed that the feasible domain in LPs space is convex. The thesis and the proof of this claim are erroneous. The authors consider one of the twelve components of $\mathbf{p}$ at time. Each component of $\mathbf{p}$ is a continuous function whose range is a continuous bounded set: in $\mathbb{R}$ such a set (a segment) is necessary convex. Moreover, considering Lemma 3.1, one cannot conclude that a set is convex by knowing that some (not all) of its projections are convex sets. The point is that the components of $\mathbf{p}$ are not independent, since they rely on the same SS. This mutual influence is the actual reason of the shrinkage of the feasible domain from the 12D parallelepiped, as Fig. 1 clearly shows.

The feasible region of the limit infinite-ply laminate, i.e. Π_∞, is, actually, the convex hull of the feasible region of laminates regardless of the number of plies. However, a laminate has always a finite number of plies, which makes the determination of Π_N the true problem to address.

4. Some Exact Solutions for Geometrical Bounds

In this section, only anisotropic laminates, in the PPs space, will be considered. The non-convexity of the feasible region Π_N is preserved in the projection onto the (ρ_1, ρ_0) plane.

Fig. 2 shows some results for the cases $N = 2, \cdots, 7$, equivalent to the cases $m = 2, \cdots, 7$ with n_i mutually equal. It is evident the non-convexity of the feasible domain, which may degenerate to a curve or to a point. In fact, it is easy to see that, if $N = 1$ or $m = 1$, i.e. the case of a laminate where all plies share a single orientation, $\rho_0 = \rho_1 = 1$.

More interesting is the case $m = 2$. After few calculations, considering that $n_2 = N - n_1$, one achieves the following relation:

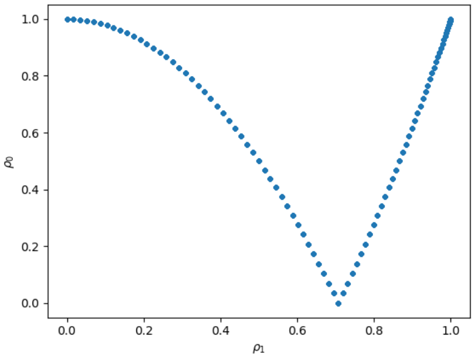
$$\rho_0 = \sqrt{\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\left\{\left[\frac{\rho_1^2 - \lambda^2 - (1-\lambda)^2}{2\lambda(1-\lambda)}\right]^2 - 1\right\}}, \tag{33}$$

or, in the implicit form:

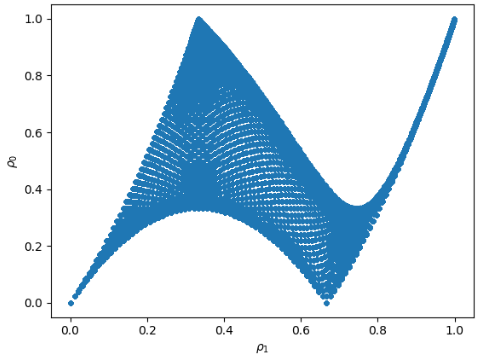
$$F_2(\rho_0, \rho_1, \lambda) := \rho_0 - \sqrt{\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\left\{\left[\frac{\rho_1^2 - \lambda^2 - (1-\lambda)^2}{2\lambda(1-\lambda)}\right]^2 - 1\right\}}, \tag{34}$$

where $\lambda := n_1/N$. Eq. (33) means that, for $m = 2$, the locus of lamination points, i.e. the pairs (ρ_1, ρ_0), is represented by a family of curves, parametrised with the relative number of plies of each orientation within the SS. Eq. (33) intersects the axis $\rho_0 = 1$ for the following values of ρ_1:

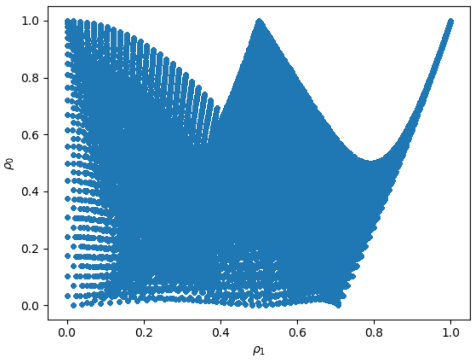
$$\begin{cases} 1 - 2\lambda, & \text{if } \lambda \leq 1/2, \\ 2\lambda - 1, & \text{if } \lambda \geq 1/2, \\ 1, \end{cases} \tag{35}$$



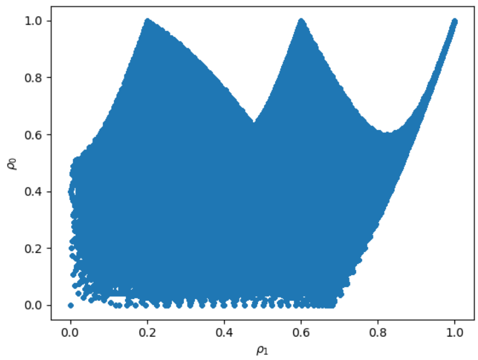
(a) $N = 2$ (discretisation step $= 1°$)



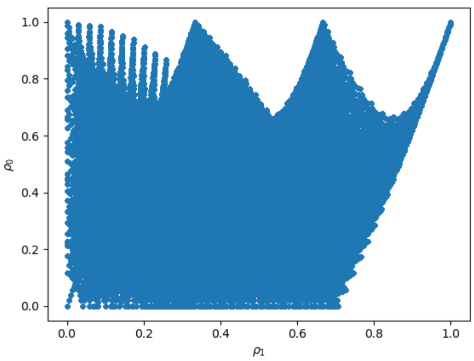
(b) $N = 3$ (discretisation step $= 1°$)



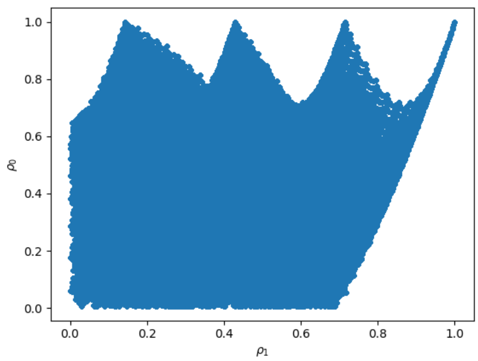
(c) $N = 4$ (discretisation step $= 2°$)



(d) $N = 5$ (discretisation step $= 3°$)



(e) $N = 6$ (discretisation step $= 5°$)



(f) $N = 7$ (discretisation step $= 6°$)

Figure 2: Feasible domain projection onto the plane (ρ_1, ρ_0) for anisotropic laminates of N plies (equivalent to consider laminates with m distinct orientations, all of them appearing the same number of times within the SS)

in perfect agreement with the proof of Proposition 3.2. The family of implicit curves, expressed by Eq.(34) in the form of $F_2(\rho_0, \rho_1, \lambda) = 0$, admits envelope, represented by the well-known geometrical-bound $\rho_0 = 2\rho_1^2 - 1$ (Vannucci (2012)). As an example, Fig. 3 shows the lamination points locus for a laminate having $N = 20$, $m = 2$, when n_1 varies in the range $[0, N]$ as a subset of $\mathbb{N}$. The analytic solution of Eq.

(33) is represented by the red curves. It is noteworthy the discrete nature of the locus, due to the discrete nature of the stack.

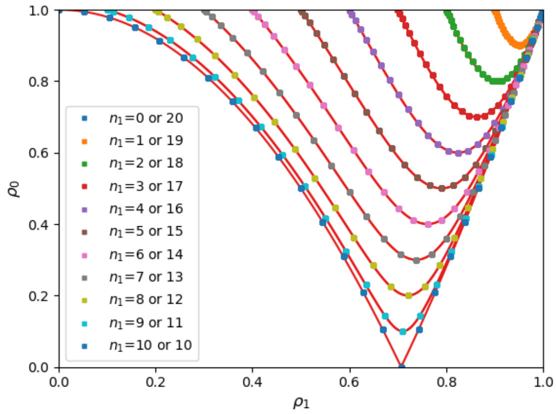


Figure 3: Lamination points for $N = 20$, $m = 2$ (discretisation step $= 3°$).

The process may be conceptually extended to a higher number of groups. However, computations become intricate. For the sake of simplicity, consider the case $m = 3$. Eqs. (14) and (15) can be rearranged as (see Appendix A)

$$\rho_0 = \frac{1}{N}\sqrt{\sum_{j=1}^{3} n_i^2 - 2\sum_{i=1}^{2}\sum_{j=i+1}^{3} n_i n_j + 4\sum_{i=1}^{2}\sum_{j=i+1}^{3} n_i n_j \cos^2(2\Delta\theta_{ij})}, \tag{36}$$

$$\rho_1 = \frac{1}{N}\sqrt{\sum_{j=1}^{3} n_i^2 + 2\sum_{i=1}^{2}\sum_{j=i+1}^{3} n_i n_j \cos(2\Delta\theta_{ij})}. \tag{37}$$

Injecting, for example, the expression of $\cos(2\Delta\theta_{13})$ from Eq. (37) into Eq. (36), one obtains

$$F_3(\rho_0, \rho_1; \Delta\theta_{12}, \Delta\theta_{23}) := \rho_0 - \frac{1}{N}\left[\sum_{j=1}^{3} n_i^2 - 2\sum_{i=1}^{2}\sum_{j=i+1}^{3} n_i n_j + 4\sum_{i=1}^{2} n_i n_{(i+1)} \cos^2(2\Delta\theta_{i(i+1)}) + \frac{1}{n_1 n_3}\left(N^2\rho_1^2 - \sum_{j=1}^{3} n_i^2 - 2\sum_{i=1}^{2} n_i n_{(i+1)} \cos(2\Delta\theta_{i(i+1)})\right)^2\right]^{1/2}. \tag{38}$$

For the envelope, the following system must be solved:

$$\begin{cases} F_3(\rho_0,\rho_1;\Delta\theta_{12},\Delta\theta_{23})=0,\\ \dfrac{\partial F_3}{\partial\Delta\theta_{12}}(\rho_0,\rho_1;\Delta\theta_{12},\Delta\theta_{23})=0,\\ \dfrac{\partial F_3}{\partial\Delta\theta_{23}}(\rho_0,\rho_1;\Delta\theta_{12},\Delta\theta_{23})=0, \end{cases} \tag{39}$$

which simplifies to

$$\begin{cases} F_3(\rho_0,\rho_1;\Delta\theta_{12},\Delta\theta_{23})=0,\\ \sin(2\Delta\theta_{12})\left[N^2\rho_1^2-\sum\limits_{j=1}^{3}n_i^2-2n_1(n_2+n_3)\cos(2\Delta\theta_{12})-2n_2n_3\cos(2\Delta\theta_{23})\right]=0,\\ \sin(2\Delta\theta_{23})\left[N^2\rho_1^2-\sum\limits_{j=1}^{3}n_i^2-2n_3(n_1+n_2)\cos(2\Delta\theta_{23})-2n_1n_2\cos(2\Delta\theta_{12})\right]=0. \end{cases} \tag{40}$$

Four are the possibilities to annihilate the last two expressions of Eq. (40).

Case A: $2\Delta\theta_{12}=K_{12}\pi,\ 2\Delta\theta_{23}=K_{23}\pi,\ K_{12},K_{23}\in\mathbb{Z}.$. Therefore, $\cos(2\Delta\theta_{i(i+1)})=(-1)^{K_{i(i+1)}}, i=1,2$. This case corresponds to a SS composed by parallel laminæ. This condition corresponds to the point $\rho_0=\rho_1=1$.

Case B: $2\Delta\theta_{12}=K_{12}\pi,\ 2\Delta\theta_{23}\neq K_{23}\pi,\ K_{12},K_{23}\in\mathbb{Z}.$. From Eq.(40)$_3$, one obtains the expression of $\cos(2\Delta\theta_{23})$, and thus

$$\begin{cases} \cos(2\Delta\theta_{23})=\dfrac{N^2\rho_1^2-\sum\limits_{i=1}^{3}n_i^2-2n_1n_2(-1)^{K_{12}}}{2n_3(n_1+n_2)},\\ \rho_0=\dfrac{1}{N}\left[\sum\limits_{j=1}^{3}n_i^2-2\sum\limits_{i=1}^{2}\sum\limits_{j=i+1}^{3}n_in_j+4n_1n_2+4n_3(n_1+n_2)\cos^2(2\Delta\theta_{23})\right]^{1/2}. \end{cases} \tag{41}$$

Only the case $K_{12}=0$ is effective, corresponding to the case $m=2$, because orientations 1 and 2 coincide.

Case C: $2\Delta\theta_{12}\neq K_{12}\pi,\ 2\Delta\theta_{23}=K_{23}\pi,\ K_{12},K_{23}\in\mathbb{Z}.$. This case is analogous to case B, exchanging indices $1,2$ with $2,3$, respectively. Fig. 4 depicts the curves of cases B or C for a laminate having $N=6$, $m=3$ and $n_1=1$, $n_2=2$, $n_3=3$, obtained by permuting indices $1,2,3$ modulo 3. This is due to the arbitrary substitution of $\cos(2\Delta\theta_{13})$ in the derivation of Eq. (38). Because of the cyclic permutation, cases B and C coincide.

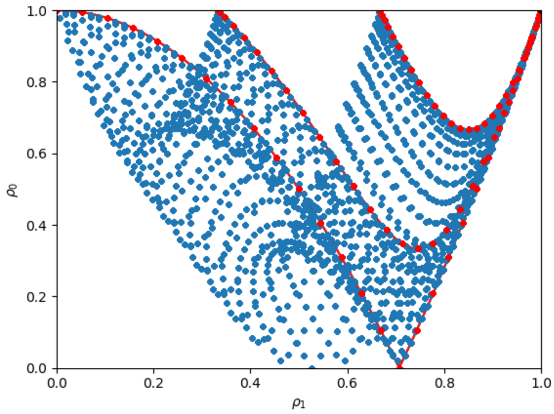


Figure 4: Cases B or C for $n_1 = 1$, $n_2 = 2$, $n_3 = 3$, (discretisation step = 3°) by permuting indices $1, 2, 3$ modulo 3.

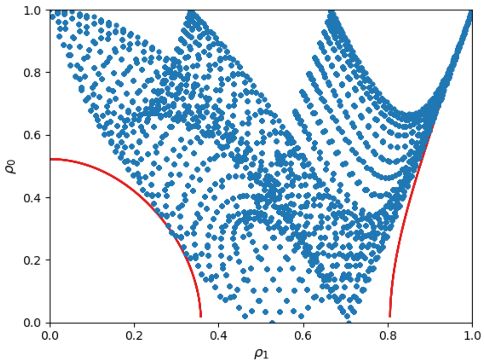
Case D: $2\Delta\theta_{12} \neq K_{12}\pi,\ 2\Delta\theta_{23} \neq K_{23}\pi,\ K_{12}, K_{23} \in \mathbb{Z}$.. Subtracting term by term the last two formulæ of Eq.(40), one gets the condition $\cos(2\Delta\theta_{12}) = \cos(2\Delta\theta_{23})$.

$$\begin{cases} \cos(2\Delta\theta_{12}) = \cos(2\Delta\theta_{23}) = \dfrac{N^2\rho_1^2 - \sum\limits_{j=1}^{3} n_i^2}{2\sum\limits_{i=1}^{2}\sum\limits_{j=i+1}^{3} n_i n_j}, \\ F_3(\rho_0, \rho_1; \Delta\theta_{12}, \Delta\theta_{23}) = 0. \end{cases} \tag{42}$$

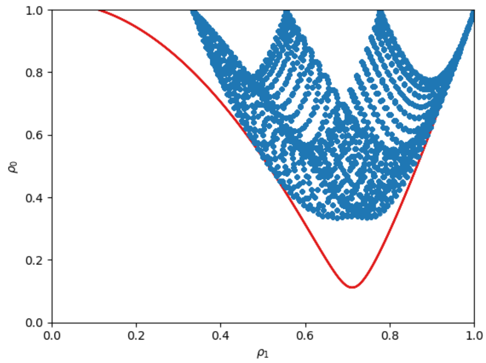
Fig. 5 depicts the bound. It can be shown that it defines a lower bound for the feasible region.

A further case, arising from a different change of variables, is worthy to be considered.

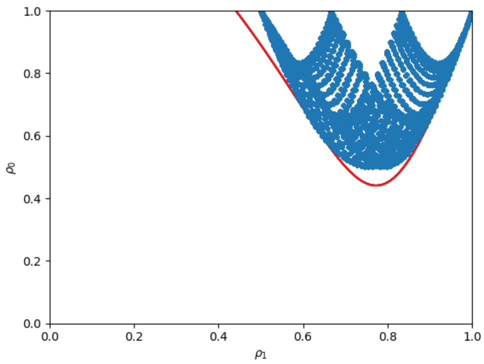
Case E: $2\Delta\theta_{12} \neq K_{12}\pi,\ 2\Delta\theta_{23} \neq K_{23}\pi,\ K_{12}, K_{23} \in \mathbb{Z}$.. Subtracting term by term the last two formulæ of Eq.(40), one gets the condition $\cos(2\Delta\theta_{12}) = \cos(2\Delta\theta_{23})$. Note that $\cos(2\Delta\theta_{13}) = \cos(2\Delta\theta_{12} + 2\Delta\theta_{23})$. From Eq. (37) one obtains the value



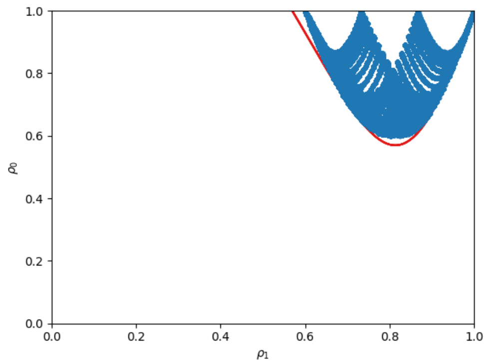
(a) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$



(b) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 6$



(c) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 9$



(d) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 12$

Figure 5: Case D for four different laminates (discretisation step = $3°$) .

of parameter $\cos(2\Delta\theta_{12})$.

$$\begin{cases} \cos(2\Delta\theta_{12}) = \cos(2\Delta\theta_{23}) = \dfrac{N^2\rho_1^2 - \sum\limits_{i=1}^{3} n_i^2 - 2n_1n_3}{2n_2(n_1+n_3)}, & \text{if } 2\Delta\theta_{12} = -2\Delta\theta_{23}, \\ \cos(2\Delta\theta_{12}) = \cos(2\Delta\theta_{23}) = \dfrac{-b \pm \sqrt{b^2-4c}}{2}, & \text{if } 2\Delta\theta_{12} = 2\Delta\theta_{23}, \\ F_3(\rho_0, \rho_1; \Delta\theta_{12}, \Delta\theta_{23}) = 0, & \end{cases} \tag{43}$$

where

$$b = \frac{n_2(n_1+n_3)}{2n_1n_3}, \quad c = -\frac{N^2\rho_1^2 - \sum\limits_{i=1}^{3} n_i^2 + 2n_1n_3}{4n_1n_3}. \tag{44}$$

Fig. 6 depicts the bound. The area it encloses is always inside the actual feasible region.

4.1. Exact analytic expression of the feasible domain for the case $n_1 = n_2 = n_3 = n$

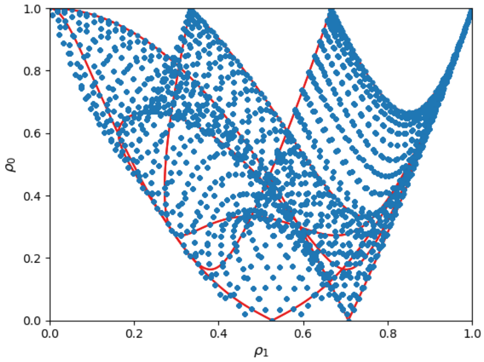
In the case $n_1 = n_2 = n_3 = n$, for every $N = 3n$, the aforementioned relationships can be further simplified. It is noteworthy that for this case, Eqs. (41) and (43) provide the analytic expression of the actual projection of the feasible domain, i.e.

$$\begin{cases} 0 \leq \rho_0 \leq 1, \\ 0 \leq \rho_1 \leq 1, \\ \rho_0 \leq \rho_1 \left|3\rho_1 + 2\right|, \\ \rho_0 \geq \rho_1 \left|3\rho_1 - 2\right|, \\ \rho_0 \leq \dfrac{1}{3}\sqrt{1 + \dfrac{(9\rho_1^2-5)^2}{2}}, \end{cases} \tag{45}$$

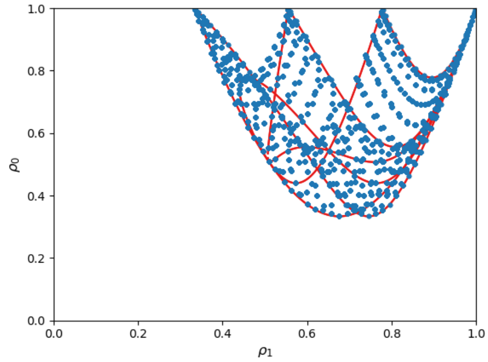
which, of course, does not depend on N. Fig. 7 shows the result for this case. To the best of the authors' knowledge, it is the first time that an exact closed-form expression for the projection of the feasible domain of a laminate is given.

5. New Geometrical Bounds

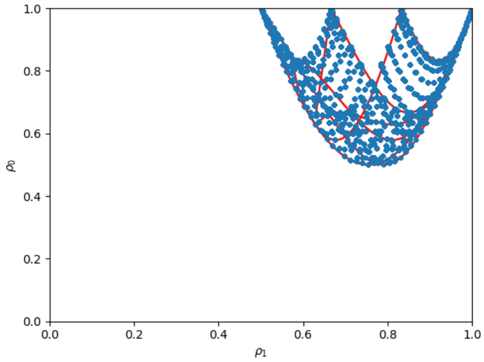
It is possible to infer that Eq. (34) evaluated for $\lambda = 1/N$, i.e. $F_2(\rho_0, \rho_1, 1/N) = 0$, is, in any case, a geometrical bound for anisotropic laminates. Similarly,



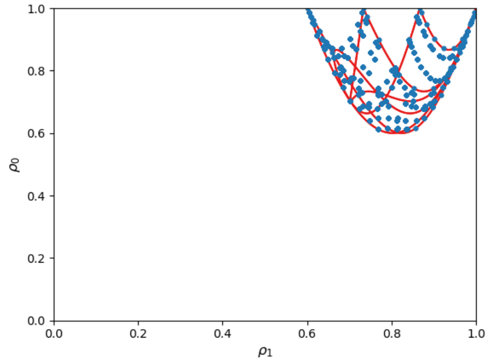
(a) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 3$



(b) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 6$



(c) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 9$



(d) Lamination points and envelope, $n_1 = 1$, $n_2 = 2$, $n_3 = 12$

Figure 6: Case E for four different laminates.

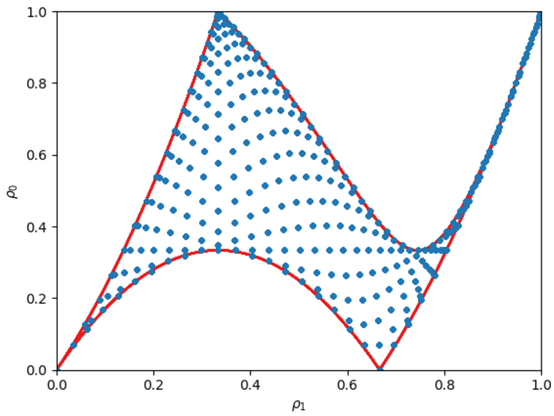


Figure 7: Lamination points and envelope for the case $n_1 = n_2 = n_3 = n$ (discretisation step $= 3°$).

$F_2((-1)^K \rho_{0K}, \rho_1, 1/N) = 0$ is a geometrical bound for membrane-orthotropic laminates.

From the proofs of Propositions 3.2 and 3.3, there exists a region in the (ρ_1, ρ_0)

plane that is excluded from the feasible domain projection. This results has a physical interpretation. For a SS made of N plies, the case $m = 2$, $n_1 = 1$, $n_2 = N - 1$ is the minimal condition to make the feasible domain not to degenerate into a point (case $n_1 = 0$). Therefore, no lamination points can lie in the epigraph of $F_2(\rho_0, \rho_1, 1/N) = 0$. If so, the amount of plies would be a non-integer number, leading, thus, to a contradiction. The excluded area is more important for relative small N, whilst it can be negligible as N increases.

Therefore, new geometrical bounds can be proposed for a general anisotropic laminate:

$$\begin{cases} 0 \leq \rho_0 \leq 1, \\ 0 \leq \rho_1 \leq 1, \\ 2\rho_1^2 \leq \dfrac{1-\rho_0^2}{1-(-1)^K \rho_0 \cos 4\Phi_0^{A*}}, \\ F_2(\rho_0, \rho_1, 1/N) \leq 0. \end{cases} \tag{46}$$

Moreover, for a laminate having an orthotropic membrane stiffness tensor, the above bounds simplifies to

$$\begin{cases} -1 \leq \rho_{0K} \leq 1, \\ 0 \leq \rho_1 \leq 1, \\ 2\rho_1^2 - 1 - (-1)^K \rho_{0K} \leq 0 \\ F_2((-1)^K \rho_{0K}, \rho_1, 1/N) \leq 0. \end{cases} \tag{47}$$

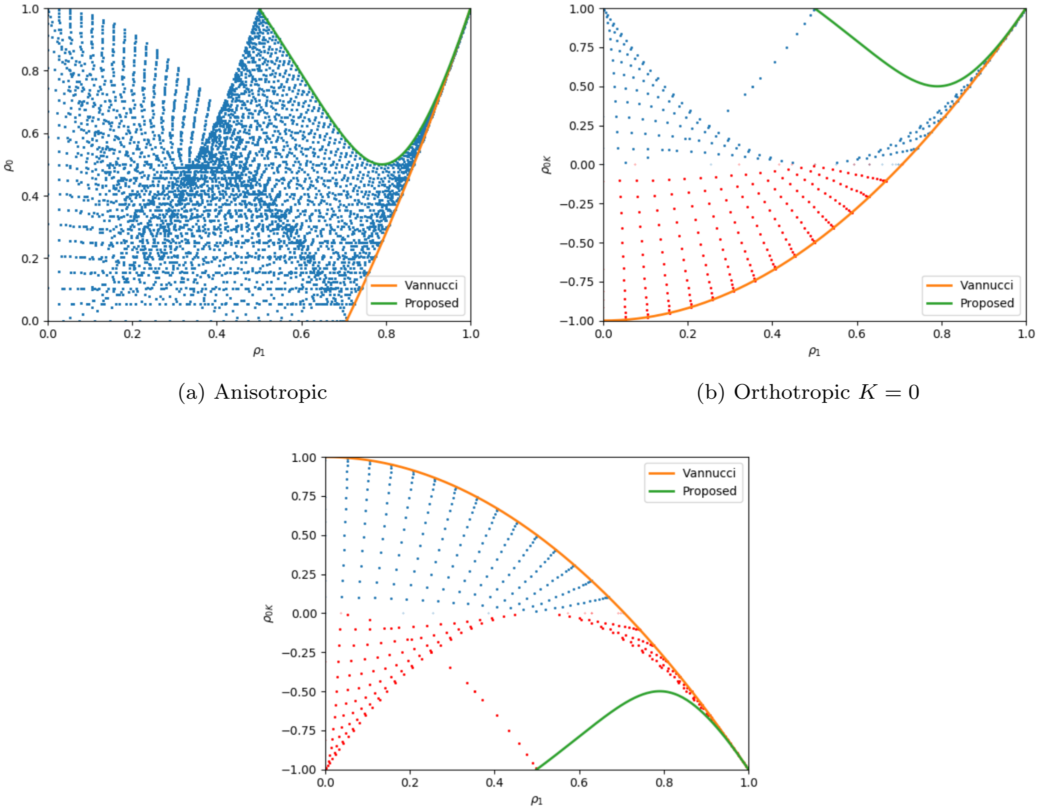
It is noteworthy that, for a quasi-homogeneous laminate, Eqs. (46) and (47) offer a description of the laminate feasible domain richer than those available in the literature (M. (1982); Vannucci (2012)).

Figure 8 shows the Vannucci's geometrical bounds (Vannucci (2012)) and and the proposed bound for anisotropic (Eq. (46)) and membrane-orthotropic (Eq. (47)) laminates with $N = 4$.

It is remarkable that the proposed bounds formulation depends also on the number of plies of the laminate.

6. On the Necessary and Sufficient Conditions for Uncoupling and Homogeneity

QT SSs are characterised by an interesting and very useful property: membrane/bending uncoupling and/or homogeneity requirements can be exactly met



(a) Anisotropic (b) Orthotropic $K = 0$

(c) Orthotropic $K = 1$

Figure 8: Vannucci's geometrical bound (Vannucci (2012)) and proposed bounds for a laminate having $N = 4$ (discretisation step $= 3°$)

regardless of the values of the orientation angles. In particular, these requirements can be fulfilled by acting only on the position of the layers into the stack (Garulli et al. (2018); Vannucci and Verchery (2001)). QT SSs have been efficiently used in many practical problems (Montemurro and Catapano (2017); Montemurro et al. (2016, 2019, 2018)). Specifically, a QT stack represents an equivalence class for all possible orientations that each group of plies can assume. As an example, $\{90°, -26°, 90°, 90°, 90°, -26°, 90°\}$ and $\{1°, 42°, 1°, 1°, 1°, 42°, 1°\}$ are elements of the same equivalence class $[\{0, 1, 0, 0, 0, 1, 0\}]$, where 0 and 1 are just labels identifying two possibly distinct orientations. Of course, the choice of the orientations depends upon the desired elastic behaviour of the laminate.

To explain clearly the concept of QT solutions, consider a laminate with N plies and $m \leq N$ different orientations and define

$$G_j := \{k : \theta_k = \theta_j\}, \tag{48}$$

the set of indices within the SS sharing the same orientation θ_j. Conditions for uncoupling and homogeneity can be than split as multiple sums over the different sets G_j, $j = 1, \ldots, m$ (Garulli et al. (2018)). Therefore, the uncoupling condition reads:

$$\sum_{k=1}^{N} b_k e^{i\beta\theta_k} = \sum_{j=1}^{m} e^{i\beta\theta_j} \sum_{k\in G_j} b_k = 0, \quad \beta = 2, 4, \tag{49}$$

while the homogeneity requirement can be expressed as:

$$\sum_{k=1}^{N} c_k e^{i\beta\theta_k} = \sum_{j=1}^{m} e^{i\beta\theta_j} \sum_{k\in G_j} c_k = 0, \quad \beta = 2, 4. \tag{50}$$

In this context, a group of plies oriented at θ_j, for which

$$\sum_{k\in G_j} b_k = 0, \quad \sum_{k\in G_j} c_k = 0, \tag{51}$$

is called *saturated group* with respect to coefficient b_k and c_k, respectively. QT SSs are entirely composed of saturated groups. For more details on this topic, the reader is addressed to (Garulli et al. (2018); Vannucci (2017)).

In this Section, some theorems of linear algebra are used to prove that quasi-triviality is only a sufficient condition for uncoupling and/or homogeneity for $m \geq 3$. For $m = 2$, quasi-triviality is also a necessary condition. The results are based on the following two theorems discussed in (Green (1916)), which are reported here for the sake of completeness.

Theorem 6.1. *Consider the p-dimensional cube $A := (-\pi/2, \pi/2)^p$. Let y_1 and y_2 be functions of the p independent variables u_1, u_2,...,u_p for which all partial derivatives of the first order, $\partial y_1/\partial u_k$, $\partial y_2/\partial u_k$, ($k = 1, 2, \ldots, p$) exist throughout the region A. Furthermore, suppose that one of the functions, i.e. y_i, does not vanish in A. Then, if all the two-rowed determinants in the matrix*

$$\begin{bmatrix} y_1 & y_2 \\ \dfrac{\partial y_1}{u_1} & \dfrac{\partial y_2}{u_1} \\ \dfrac{\partial y_1}{u_2} & \dfrac{\partial y_2}{u_2} \\ \vdots & \vdots \\ \dfrac{\partial y_1}{u_p} & \dfrac{\partial y_2}{u_p} \end{bmatrix}, \tag{52}$$

vanish identically in A, y_1 and y_2 are linearly dependent in A.

Theorem 6.2. *Consider the p-dimensional cube $A := (-\pi/2, \pi/2)^p$. Moreover, consider the matrix $\boldsymbol{M}_s(y_1, y_2, \dots, y_r)$ in which the first row consists of the functions y_1, y_2 up to y_r (for some r) and the other s rows of derivatives of these functions:*

$$\boldsymbol{M}_s = \begin{bmatrix} y_1 & y_2 & \cdots & y_r \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_r^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(s)} & y_2^{(s)} & \cdots & y_r^{(s)} \end{bmatrix}. \tag{53}$$

Let the set of n functions y_1, y_2,…,y_n of the p independent variables u_1, u_2,…,u_p possess enough partial derivatives, of any orders whatever, to form a matrix $\boldsymbol{M} = \boldsymbol{M}_{(n-2)}(y_1, y_2, \dots, y_n)$, of n columns and $n-1$ rows, in which at least one of the $(n-1)$-rowed determinants, i.e.

$$\boldsymbol{W}_n = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_{n-1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} \end{bmatrix}, \tag{54}$$

vanishes nowhere in A. Moreover, suppose that all of the first derivatives of each of the elements of the above matrix $\boldsymbol{M}$ exist, and adjoin to the matrix $\boldsymbol{M}$ such of these derivatives as do not already appear in $\boldsymbol{M}$, to form the new matrix $\boldsymbol{M}' = \boldsymbol{M}_q(y_1, y_2, \dots, y_n)$, which has n columns and at least n rows, so that $q \geq n-1$. Then if all the n-rowed determinants of the matrix $\boldsymbol{M}'$ in which the determinant $\boldsymbol{W}_n$ is a first minor vanish identically in A, the functions y_1, y_2,…,y_n are linearly dependent in A.

Thanks to the above Theorems, the following Proposition can be introduced.

Proposition 6.1. *Quasi-trivial stacking sequences represent a necessary and sufficient condition to satisfy membrane/bending uncoupling and/or homogeneity requirements when the number of saturated groups m is equal to two.*

Proof. Consider a laminate with a SS composed of m different orientations and apply Theorem 6.1 to the set of functions $\mathrm{e}^{\mathrm{i}\beta\theta_j}$, $j = 1, \dots, m$, $\beta = 2, 4$. When $m = 2$, Eq. (52) reads

$$\begin{bmatrix} e^{i\beta\theta_1} & e^{i\beta\theta_2} \\ i\beta e^{i\beta\theta_1} & 0 \\ 0 & i\beta e^{i\beta\theta_2} \end{bmatrix}. \tag{55}$$

One can easily check that none of the two-rowed determinants of Eq.(55) is singular in $A = (-\pi/2, \pi/2)^2$. Therefore, it follows that, for $m = 2$, the combination of functions $e^{i\beta\theta_j}$, $j = 1, 2$, $\beta = 2, 4$ is linearly independent. Accordingly, the proof can be considered concluded. □

Remark 6.1. *Proposition 6.1 means that uncoupled and/or homogeneous laminates imply the annealing of the sum of coefficients b_k and/or c_k associated to each orientation. In other words, when $m = 2$, uncoupled and/or homogeneous laminates can be composed only of QT stacks and vice versa.*

Remark 6.2. *In the light of Theorem 6.1 and Proposition 6.1, it is not possible to obtain membrane-isotropic laminates with only two different orientations, since the sum of coefficients a_k cannot be zero. This result is in agreement with the evidence that no isotropic SS is known having only two distinct orientation angles (Vannucci (2017); Warren and Norris (1953); Wu and Avery (1992)).*

Remark 6.3. *The application of Theorem 6.2 for $m \geq 3$ is cumbersome, and it is not reported here for the sake of simplicity. The application of such a Theorem proves that functions $e^{i\beta\theta_j}$, $j = 1, \ldots, m$, $\beta = 2, 4$ are linearly dependent for $m \geq 3$. Therefore, for $m \geq 3$, one can find uncoupled and/or homogeneous laminates which are not QT solutions. Moreover, in this case, membrane-isotropic laminates exist. For example, the laminate $[0/-60/60/0/60_2/-60_3/0_2/60]$, from (Vannucci (2017)), is membrane-isotropic (the isotropic behaviour in membrane has been obtained by using the well-known Warren and Norris rule (Warren and Norris (1953))). For this laminate, the isotropy condition is achieved because of the linearly dependence of the complex exponential functions appearing in the definition of the anisotropic moduli, since $a_k = 1$, $\forall k$.*

7. Conclusions

In this paper two aspects linked to the nature of the feasible domain of composite laminates have been studied. Proofs of the non-convexity of the feasible domain in polar parameters and lamination parameters spaces have been provided for both anisotropic and orthotropic-membrane laminates. The results correct the erroneous common belief of the convexity of the feasible domain in lamination parameters space. In the light of this finding, the *old* feasible domain can be interpreted as the convex hull of the projected feasible domain, regardless of the number of plies composing the laminate.

This work clarifies some preliminary aspects of the feasible domain determination problem. Although tackling the problem in full generality is probably impractical, a closed form of stricter feasibility bounds has been derived in terms of the membrane stiffness tensor polar parameters. This is an aspect of paramount importance in the optimisation of composite laminates, where the anisotropy is tailored to satisfy some merit functions and constraints.

Furthermore, the problem of retrieving sufficient and necessary conditions for membrane / bending uncoupling and homogeneity has been addressed in this study. It has been shown that, in general, quasi-triviality is only a sufficient condition. It becomes also a necessary one if the stacking sequence has only two distinct orientations.

As prospects of this study are concerned, the derivation of the analytic expressions of the feasible domain of laminates (in both lamination parameters and polar parameters spaces), at least for the membrane stiffness tensor, should be derived in more general cases, maintaining the dependence from the number of plies. These expressions must be included in the formulation of the optimisation problem of the composite in order to get true feasible optimal solutions. Moreover, to support the composite design, necessary and sufficient conditions to achieve uncoupled and/or homogenous and/or isotropic laminates should be derived. This could represent a step-forward to the definition of the feasible domain considering membrane and bending stiffness tensors at once.

Of course, the main (and most difficult) problem to address still remains the derivation of the actual feasible domain of a laminate when the full stiffness matrix is considered either in the fully anisotropic case or when introducing some hypotheses on the laminate stiffness tensors elastic symmetries.

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A. A note on the square of the sum of n terms

Consider the square of the sum of terms $g_1, g_2, \ldots, g_n$, $n \in \mathbb{N}$. Hence, it easy to see that

$$\left(\sum_{k=1}^{n} g_k\right)^2 = \sum_{k=1}^{n} g_k^2 + 2\sum_{k=1}^{n-1} g_k \sum_{j=k+1}^{n} g_j = \sum_{k=1}^{n} g_k^2 + 2\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} g_k g_j, \tag{A.1}$$

where the distributivity property of sum and product operators has been used.

Therefore, for a laminate having N plies and m distinct orientations, Eq. (14) reads

$$\rho_0 = \frac{1}{N}\sqrt{\left(\sum_{j=1}^{N} \cos 4\theta_j\right)^2 + \left(\sum_{j=1}^{N} \sin 4\theta_j\right)^2} = \frac{1}{N}\sqrt{\left(\sum_{j=1}^{m} n_j \cos 4\theta_j\right)^2 + \left(\sum_{j=1}^{m} n_j \sin 4\theta_j\right)^2}. \tag{A.2}$$

Applying Eq. (A.1) to Eq. (A.2), one has

$$\begin{aligned}
\rho_0 &= \frac{1}{N}\sqrt{\left(\sum_{j=1}^{m} n_j \cos 4\theta_j\right)^2 + \left(\sum_{j=1}^{m} n_j \sin 4\theta_j\right)^2} \\
&= \frac{1}{N}\left[\sum_{j=1}^{m} n_j^2 \cos 4^2\theta_j + \sum_{j=1}^{m} n_j^2 \sin^2 4\theta_j + \right. \\
&\left. +2\sum_{k=1}^{m-1}\sum_{j=k+1}^{m} n_k n_j \cos 4\theta_k \cos 4\theta_j + 2\sum_{k=1}^{m-1}\sum_{j=k+1}^{m} n_k n_j \sin 4\theta_k \sin 4\theta_j\right]^{\frac{1}{2}} \\
&= \frac{1}{N}\sqrt{\sum_{j=1}^{m} n_j^2 + 2\sum_{k=1}^{m-1}\sum_{j=k+1}^{m} n_k n_j \cos 4\left(\theta_k - \theta_j\right)} \\
&= \frac{1}{N}\sqrt{\sum_{j=1}^{m} n_j^2 + 2\sum_{k=1}^{m-1}\sum_{j=k+1}^{m} n_k n_j \left[2\cos^2 2\left(\theta_k - \theta_j\right) - 1\right]},
\end{aligned} \tag{A.3}$$

where the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$ has been used. In a similar manner, the expression of ρ_1 of Eq. (15) can be rearranged as Eq. (37).

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