Optimal control of a crop irrigation model under water scarcity

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Optimal control of a crop irrigation model underwater scarcity

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Contents

1 Introduction

In this internship, we are investigating optimal irrigation strategies in the context of water quotas with the help of a simplified crop model. This model was first introduced in the article **"Optimal control of a crop irrigation model under water scarcity"** presented by Boumaza, K., Kalboussi, N., Rapapor, A., Roux, S. and Carole, Sinfort. The model consists on considering respectively in two state variables $S(t)$ and $B(t)$ as relative soil humidity in the root zone and the crop biomass at time t in an interval $[0, T]$ representing the crop growth season, where 0 and T stand for sowing and harvesting dates. The equations of the model are:

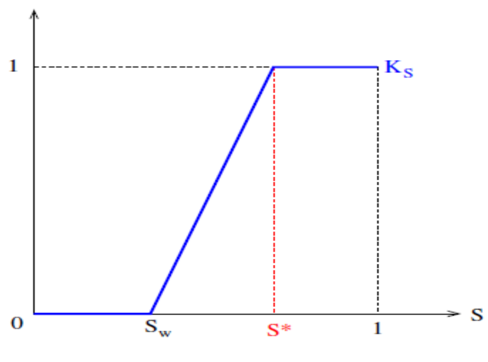
$$\dot{S} = k_1(-\phi(t)K_S(S) - (1 - \phi(t))K_R(S) + k_2 u(t))$$
$$\dot{B} = \phi(t)K_S(S)$$

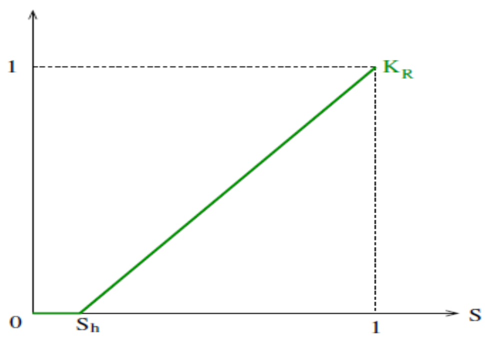
with the initial conditions:

$$S(0) = S_0 > S^*$$
$$B(0) = B_0 > 0$$

The functions K_S and K_R are assumed to be piecewise linear non decreasing from $[0, 1]$ to $[0, 1]$ given by the following expressions:

$$K_S(S) = \begin{cases} 0 \text{ if S} \in [0, S_w] \\ \frac{S - S_w}{S^* - S_w} \text{ if S} \in [S_w, S^*] \\ 1 \text{ if S} \in [S^*, 1] \end{cases} \qquad K_R(S) = \begin{cases} 0 \text{ if S} \in [0, S_h] \\ \frac{S - S_h}{1 - S_h} \text{ if S} \in [S_h, 1] \end{cases}$$





The constant value S_w represents the plant wilting point, usually higher than the hydroscopic point denoted by S_h.S^* is the minimal threshold on the soil humidity that gives the best biomass production.In the other hand, the function ϕ is C^1 increasing with $\phi(0) \geq 0$ and $\phi(T) \leq 1$ and k_1, k_2 are positive parameters with $k_2 \geq 1$.

2 A comparison between the best and the worst strategies

This optimal control problem can be adapted to both kind of needs: maximising the biomass B while respecting the water quota which is the best case that our crop model can reach, but also minimising this same biomass while wasting all of the available water to be informed about the significant difference between the two strategies and the key conclusions we can get out of it. Two kind of problems can be considered: the maximisation problem:

$$\max_{u(.)} \int_0^T \phi(t) K_s(S(t))\, dt$$

$$V(T) = \bar{V} \leq \frac{\bar{Q}}{F_{max}}$$

$$S(t) \leq 1$$

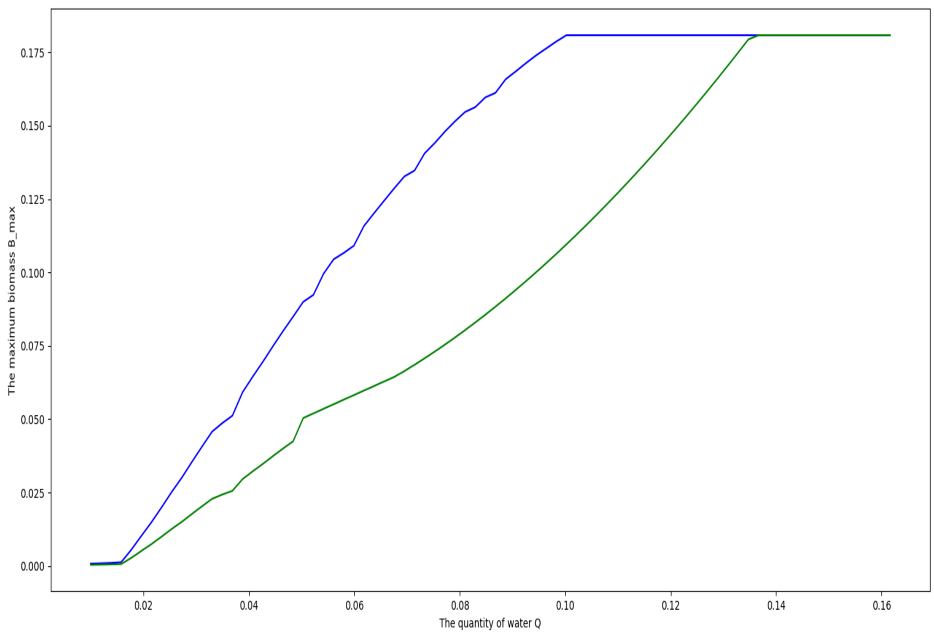
and the minimisation problem:

$$\min_{u(.)} \int_0^T \phi(t) K_s(S(t))\, dt \Leftrightarrow \max_{u(.)} \int_0^T -\phi(t) K_s(S(t))\, dt$$

$$V(T) = \bar{V} = \frac{\bar{Q}}{F_{max}}$$

$$S(t) \leq 1$$

A numerical resolution with the **BocopHJB** software of both problems in batch modes following different values of the water quota Q can be visualised in the following figure:



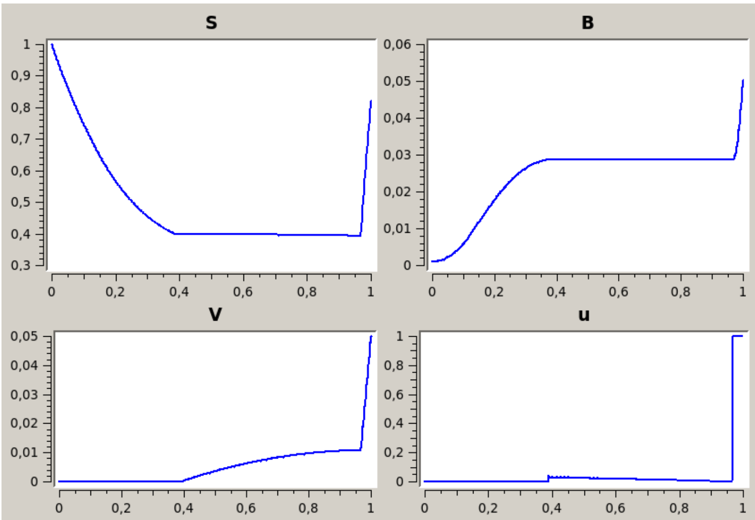
One could see that the maximisation is significantly different from the minimisation in terms of biomass production which means that an optimal strategy is worth it in most of the cases. Both strategies are the same starting from a certain value of the water quota Q which is quite logical given that the main difficulty of the problem is to respect water quotas and if the quotas are large enough the difficulty does not persist.

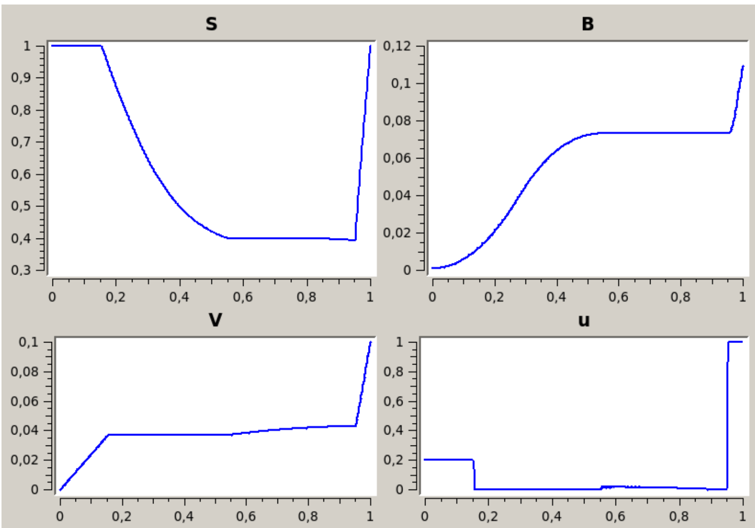
One could also see that some key values of Q exist in the figure which drives us to wonder about the form of the control in the minimisation process.

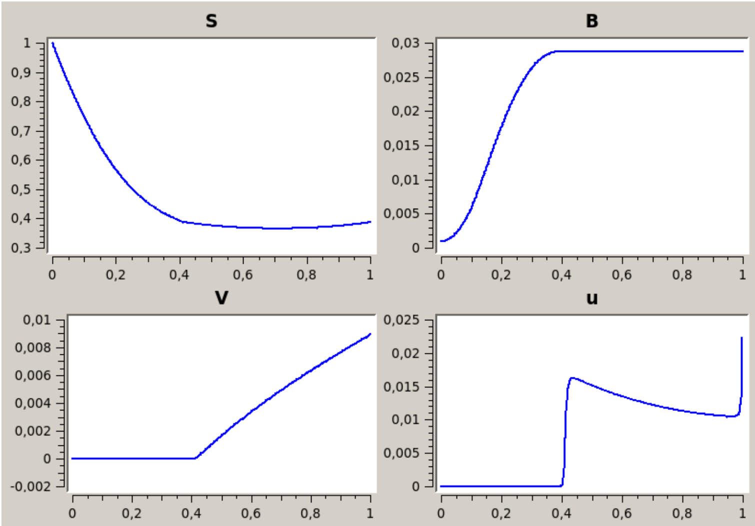
3 The control of the minimisation problem

3.1 Multiple possible forms of control

Using **BocopHJB**, three forms of the control are found according to three different value ranges of the quota Q:





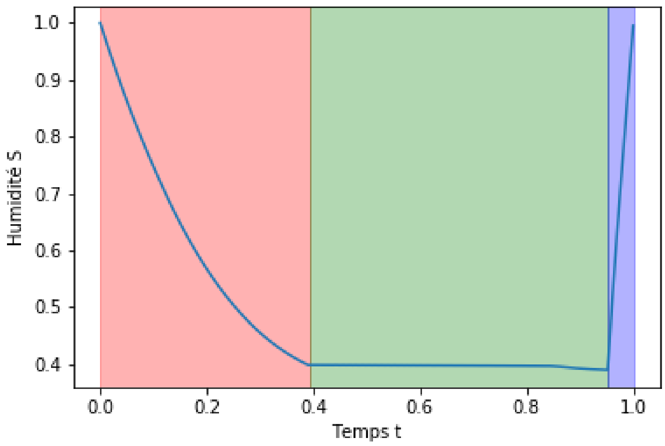


The goal is to find the range of Q values that triggers each form of the control and try to characterize it.

3.2 The first form

3.2.1 Range of Q values

We want to find the limiting quantity of water Q that allows the humidity S to decrease from the start with the control $u = 0$ to S_w and then to be stabilized around this value with the control u_{S_w} until it reaches the correct time to increase and reach $S = 1$ at $t = T$ and the control $u = 1$:



Using the definition of the water quota Q:

$$Q = F_{max} \times (\int_{t_1}^{t_2} u_{S_w}(t)dt + (T - t_2))$$

where t_1 and t_2 means respectively the instants where we reach for the first time S_w and when we surpass it for the first time (while increasing). We can find these key instants by:

$$1 - S_w = \int_{t_1}^{0} k_1(-\phi(t)K_s(S) - (1-\phi(t))K_r(S))dt$$

$$S_w - 1 = \int_{T}^{t_2} (k_1(-\phi(t)K_s(S) - (1-\phi(t))K_r(S)) + k_2)dt$$

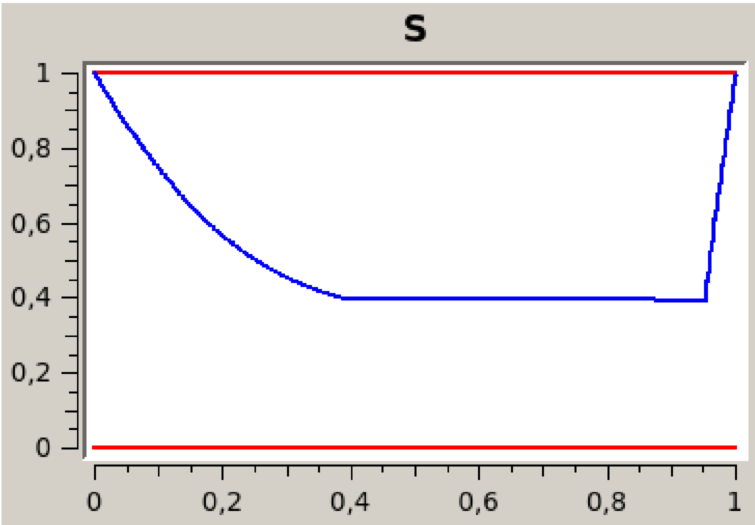
The rest of the work is purely numerical:

T	F_{max}	S_0	S^*	S_w	S_h	k_1	k_2	k_3	α
1	1.2	1	0.7	0.4	0.2	3	5	1	1

Numerically, we find that

$$Q_{lim} \approx 0.0673$$

and the humidity S graph is the following:

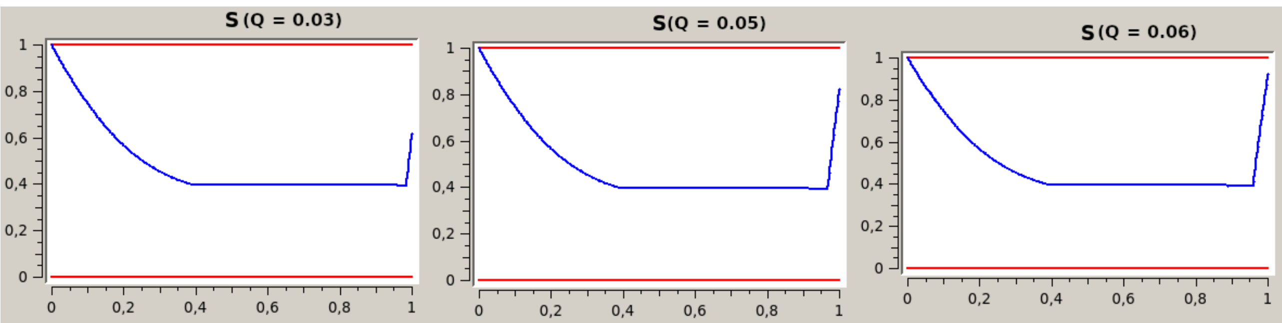


first case: $Q < Q_{lim}$:

we analyse the S figures with the following Q values:

0.03	0.05	0.06

The figures are :



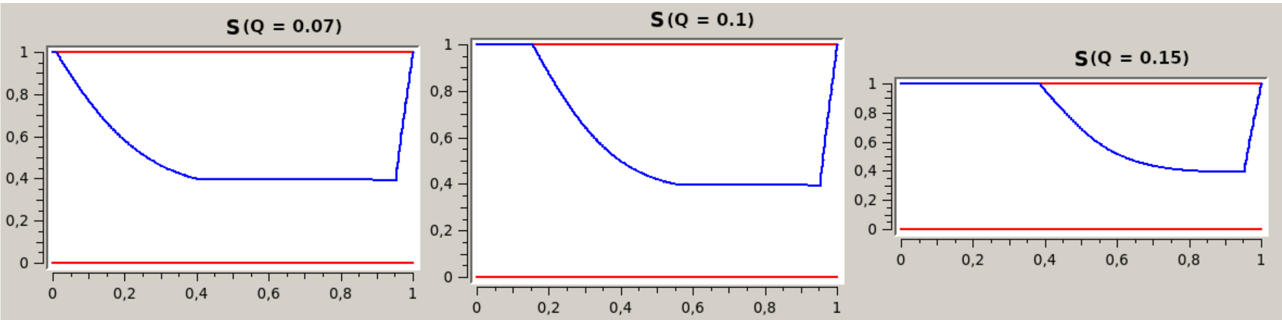
We can see that the humidity S behaves the same in the 3 situations: it decreases until reaching S_w, maintains itself at this value and increases in the end to reach $S = 1$ is the quota Q is big enough. In these 3 cases, the water quota was not enough to allow that.

second case: $Q > Q_{lim}$:

we analyse the S figures with the following Q values:

0.07	0.1	0.15

The figures are :



We can see that going above the limit value Q_{lim} causes the control to behave the same as the first form. In order to explain that, we can compare the biomass production between t_1 and t_2:

$$B(t_1, t_2) = \int_{t1}^{t_2} \phi(t) \times K_s(S(t))dt$$

Given that K_S is the same on the two intervals $[0, t_{dec}]$ or $[t_{inc}, T]$ where t_{dec} is the instant where we leave $S = 1$ in the first case and t_{inc} is the moment we join $S = 1$ in the second case. As ϕ is increasing, we can see that it takes smaller values on the interval $[0, t_{dec}]$ than $[t_{inc}, T]$

3.2.2 Singular arcs and corner points

In this section, we shall proof that singular arcs can only occur on the corner points of K_R or K_S. The minimisation problem is:

$$min_{u(.)} \int_0^T \phi(t) K_s(S(t))\, dt \Longleftrightarrow max_{u(.)} \int_0^T -\phi(t) K_s(S(t))\, dt$$

$$V(T) = \bar{V} = \frac{\bar{Q}}{F_{max}}$$

The expression of the hamiltonian is:

$$H(t, S, \lambda_S, \lambda_V, u) = \lambda_S k_1(-\phi(t)K_s(S) - (1-\phi(t))K_R(S) + k_2 u(t)) + \lambda_V u + \lambda_0 \phi(t) K_S(S(t))$$

The adjoint equations:

$$\dot{\lambda}_S(t) = \phi(t) \frac{\partial K_S(S(t))}{\partial S}(\lambda_S k_1 - \lambda_0) + (1 - \phi(t))\lambda_S k_1 \frac{\partial K_R(S(t))}{\partial S}$$

$$\dot{\lambda}_V(t) = 0$$

With the maximisation criteria:

$$H(t, S(t), \lambda_S(t), \lambda_V(t), u(t)) = max_{v\in[0,1]} H(t, S(t), \lambda_S(t), \lambda_V(t), v(t))$$

the hamiltonian takes the form:

$$H(t, S, \lambda_S, \lambda_V, u) = (\lambda_S k_1 k_2 + \lambda_V)u + g$$

$$H(t, S, \lambda_S, \lambda_V, u) = \Phi u + g$$

with $\Phi(t) = (\lambda_S(t)k_1k_2 + \lambda_V(t))$. we have $u = 1$ if $\Phi > 0$, $u = 0$ if $\Phi < 0$ and u is a singular arc when $\Phi = 0$.

Let's prove that a singular arc with $\Phi = 0$ on an interval I can only occur on corner points of K_R or K_S.

if $\Phi = 0$ on an interval I then λ_S is constant equal to $\bar{\lambda}_S = -\frac{\lambda_V}{k_1k_2}$ on this same interval. Let's suppose that K_R et K_S are differentiables on $S_1 = S(t_1)$ where $t_1 \in I$. Then $\dot{K}_R(S_1)$ and $\dot{K}_S(S_1)$ are the constant values of the functions $\dot{K}_R(S(t))$ and $\dot{K}_S(S(t))$ on a neighborhood V at t_1.

According to the adjoint equation and for every t of the neighborhood V:

$$0 = \phi(t)\dot{K}_S(S_1)(\bar{\lambda}_S k_1 - \lambda_0) + (1 - \phi(t))\bar{\lambda}_S k_1 \dot{K}_R(S_1)$$

$$\phi(t)((\bar{\lambda}_S k_1 - 1)\dot{K}_S(S_1) - \bar{\lambda}_S k_1 \dot{K}_R(S_1)) = -\bar{\lambda}_S k_1 \dot{K}_R(S_1)$$

Given that $S_1 > S_h$ we have $\dot{K}_R(S_1) > 0$. On the other hand, we denote t_1 and t_2 as two instants of the neighborhood V such as $t_1 > t_2$ so:

$$\phi(t_1)((\bar{\lambda}_S k_1 - 1)\dot{K}_S(S_1) - \bar{\lambda}_S k_1 \dot{K}_R(S_1)) = -\bar{\lambda}_S k_1 \dot{K}_R(S_1)$$
$$\phi(t_2)((\bar{\lambda}_S k_1 - 1)\dot{K}_S(S_1) - \bar{\lambda}_S k_1 \dot{K}_R(S_1)) = -\bar{\lambda}_S k_1 \dot{K}_R(S_1)$$

Therefore, we can affirm that $\phi(t_1) = \phi(t_2)$ while $t_1 > t_2$ which is absurd because ϕ is a strictly increasing function. We conclude that K_R and K_S cannot be differentiable on S_1 therefore a singular arc can only occur on their corner points.

Now we will prove that a singular arc can only occur on S_w.

Let's prove first that we cannot have a singular arc on S^*. Let's suppose that there exists an interval $I = [t_1, t_2]$ where the control utilised is u_{S^*}. We will create a new control that gives a biomass below the biomass found on that interval using the same quantity of water:

$$u' = \begin{cases} 0 \text{ if } \mathrm{t}_1 \leq t \leq \tau \\ 1 \text{ if } \tau \leq t \leq t_2, \end{cases}$$

with $\tau = t_2 - \int_{t_1}^{t_2} u_{S^*} dt$, verifies $t_1 < \tau < t_2$ (u_{S^*} is positive so $\int_{t_1}^{t_2} u_{S^*} dt$ is positive, we have then $\tau < t_2$, and we have $\int_{t_1}^{t_2} u_{S^*} dt < t_2 - t_1$ so $t_1 < \tau$)

Let $Q_{u_{S^*}}(t_1, t_2)$ be the quantity of water used on the interval I with the control u_{S^*} and $Q_{u'}(t_1, t_2)$ the quantity of water used with the control u' on the same interval I, we have :

$$Q_{u_{S^*}}(t_1,t_2) = F_{max} \times \int_{t_1}^{t_2} u_{S^*} dt = F_{max} \times (t_2 - t_2 + \int_{t_1}^{t_2} u_{S^*} dt) = F_{max} \times (t_2 - \tau) = Q_{u'}(t_1, t_2)$$

In the other hand, the biomass created by this new control is inferior to the one created with the control u_{S^*} because $\forall t \in [t_1, t_2], S'(t) \leq S^*$ (S' is the soil humidity using u'). Let's consider:

$$u_1 = \begin{cases} \text{u(t) if t} \in [0, t_1[\cup]t_2, T] \\ 0 \text{ if } t_1 \leq t < \tau \\ 1 \text{ if } \tau \leq t \leq t_2 \end{cases}$$

This control perfomrs better than the former control, therefore a singular arc cannot occur on S^*.

3.2.3 Water consuming comparison

we denote $S(t)$ the soil humidity that follows the control observed numerically, and $S_*(t)$ the soil humidity in the case where we use a control that causes the humidity to decrease below the value S_w between the instants t_1 and t_2 and increases after with $S_*(t_1) = S_w$ and $S_*(t_2) = S_w$. We will show that the control which maintains the value of $S(t)$ constant and equal to S_w on the interval $[t_1, t_2]$ consumes more water on this interval that the control that produces $S_*(t)$.

We consider the function $\delta(t) = S(t) - S_*(t)$ so $d\delta = S'(t)dt - S'_*(t)dt$. Using the model equations we find

$$d\delta = (-k1(\phi(t)K_s(S(t)) + (1-\phi(t))K_R(S(t)) + k_1k_2u(t) - k_1(\phi(t)K_s(S_*(t)) + (1-\phi(t))K_R(S_*(t)) + k_1k_2u_*(t))dt$$

$$d\delta = [-k_1(g(t, S(t)) - g(t, S_*(t)) + k_1k_2(u(t) - u_*(t))]dt$$

where $g(t, S(t)) = \phi(t)K_s(S(t)) + (1 - \phi(t))K_R(S(t))$

$$\int_{t_1}^{t_2} d\delta = \int_{t_1}^{t_2} [-k_1(g(t, S(t)) - g(t, S_*(t)) + k_1k_2(u(t) - u_*(t))]dt$$

$$\delta(t_1) - \delta(t_2) = \int_{t_1}^{t_2} -k_1(g(t, S(t)) - g(t, S_*(t))dt + \int_{t_1}^{t_2} k_1k_2(u(t) - u_*(t))dt$$

We know that on the interval $[t_1, t_2]$: $S(t) = S_w$ and $S_*(t) \leq S_w$ so $g(t, S(t)) = (1 - \phi(t))(\frac{S(t)-S_h}{1-S_h})$ and $g(t, S_*(t)) = (1 - \phi(t))(\frac{S_*(t)-S_h}{1-S_h})$. Therefore:

$$g(t, S(t)) - g(t, S_*(t)) = (1 - \phi(t))(\frac{S_w - S_*(t)}{1 - S_h})$$

Given that $S(t)$ remains above $S_*(t)$ on $[t_1, t_2]$ then $S(t) \geq S_*(t)$ sur $[t_1, t_2]$ therefore $g(t, S(t)) - g(t, S_*(t))$ is strictly positive, So:

$$\delta(t_1) - \delta(t_2) < \int_{t_1}^{t_2} k_1 k_2 (u(t) - u_*(t)) dt$$

$$\frac{\delta(t_1) - \delta(t_2)}{k_1 k_2} < \int_{t_1}^{t_2} (u(t) - u_*(t)) dt$$

$$\int_{t_1}^{t_2} (u(t) - u_*(t)) dt > 0$$

$$\int_{t_1}^{t_2} u(t) dt > \int_{t_1}^{t_2} u_*(t) dt$$

$$F_{max} \int_{t_1}^{t_2} u(t) dt > F_{max} \int_{t_1}^{t_2} u_*(t) dt$$

that is

$$\boxed{\mathrm{Q[u(.)]} > \mathrm{Q[u_*(.)]}}$$

3.2.4 Biomass production comparison

Using the same denotations as before, we shall prove that the control $u(t)$ produces less biomass than the control $u_*(t)$ on the interval $[t_1, T]$.

We define the instant τ_1 so that $S_*(\tau_1) = S*_{min}$ between t_1 and t_2 with $S_*(t_1) = S_w$ and $S_*(t_2) = S_w$. We also define the instant τ_2 where S_* stops increasing and becomes maximal $S_*(\tau_2) = 1$.

The water consumed quantity associated to the strategy u is:

$$Q[u(.)] = \int_{t_1}^{t_2} u_{S_w}(t)\, dt + \int_{t_2}^{T} 1\, dt$$

$$Q[u(.)] = \int_{t_1}^{t_2} u_{S_w}(t)\, dt + (T - t_2)$$

$$Q[u(.)] = \int_{t_1}^{t_2} u_{S_w}(t)\, dt + (T - \tau_2) + (\tau_2 - t_2)$$

and the water consumed quantity associated to the strategy u_*:

$$Q[u_*(.)] = \int_{\tau_1}^{\tau_2} 1\, dt + \int_{\tau_2}^{T} \frac{1}{k_2}\, dt$$

$$Q[u_*(.)] = (\tau_2 - \tau_1) + \frac{1}{k_2}(T_{-}\tau_2)$$

Therefore:

$$Q[u(.)] - Q[u_*(.)] = \int_{t_1}^{t_2} u_{S_w}(t)\, dt + (1 - \frac{1}{k_2})(T - \tau_2) + (\tau_1 - t_2)$$

Knowing that:

$$S_*(\tau_1) - S_*(t_1) = k_1 \int_{t_1}^{\tau_1} -(1-\phi(t))K_R(S_*(t))dt$$

$$S_*(t_2) - S_*(\tau_1) = k_1 \int_{\tau_1}^{t_2} (-(1-\phi(t))K_R(S_*(t)) + k_2)dt = k_1(\int_{\tau_1}^{t_2} -(1-\phi(t))K_R(S_*(t))dt) + k_1 k_2 (t_2 - \tau)$$

and using the fact that $S_*(t_1) = S_*(t_2) = S_w$ one could prove that:

$$\tau_1 = t_2 - \int_{t_1}^{t_2} \frac{K_R(S_*(t))}{k_2}(1-\phi(t))dt$$

Therefore:

$$Q[u(.)] - Q[u_*(.)] = \int_{t_1}^{t_2} u_{S_w}(t)\, dt + (1 - \frac{1}{k_2})(T - \tau_2) - \int_{t_1}^{t_2} \frac{K_R(S_*(t))}{k_2}(1-\phi(t))dt$$

$$Q[u(.)] - Q[u_*(.)] = \int_{t_1}^{t_2} (\frac{K_R(S(t)) - K_R(S_*(t))}{k_2})(1-\phi(t))\, dt + (1 - \frac{1}{k_2})(T - \tau_2)$$

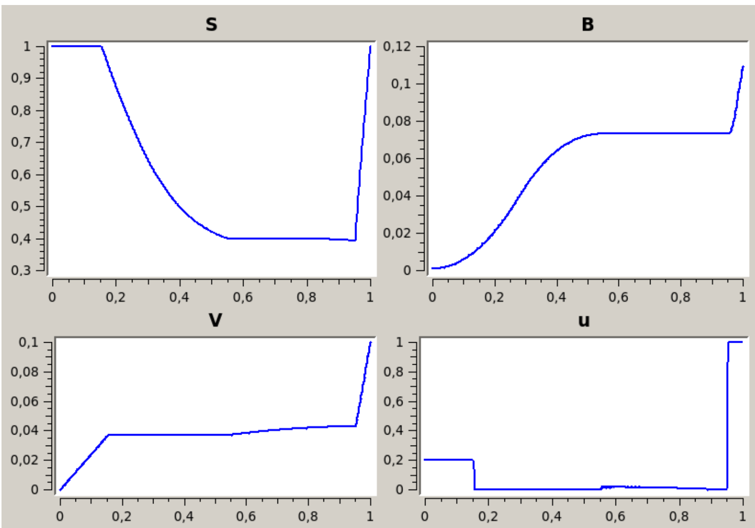
$$Q[u(.)] - Q[u_*(.)] > 0$$

Finally:

$$\boxed{\mathrm{Q[u(.)]>Q[u_*(.)]}}$$

and we conclude that the strategy u consumes more water, which means that this strategy is optimal in the minimisation case.

3.3 The second form



In the first case, the optimal strategy for minimisation is to irrigate with a control $u = \frac{1}{k_2}$ to maintain soil humidity at a maximum level for a duration T_1, ceasing the irrigation until the humidity reaches the threshold S_w and irrigate with a singular control to maintain the soil humidity at S_w until we reach the instant $T - \epsilon$ to irrigate with all of the remaining quota water. We denote:

- T_1 the period in which the soil humidity is maintained at its maximum value
- u_{S_w} the singular control that maintains the value of S constant and equal to S_w
- ϵ a small value where $T - \epsilon$ represents the instant in which we irrigate with all of the remaining water ($u = 1$)

The control takes the form:

$$u(t) = \begin{cases} \frac{1}{k_2} if t \leq T_1 \\ 0 \text{ if S} \geq S_w and t \geq T_1 \\ u_{S_w} \text{ if } S = S_w \text{ et } t \leq T - \epsilon \\ 1 \text{ if t} \geq T - \epsilon \end{cases}$$

3.3.1 The value of ϵ in this form

$$Q = F_{max} \times (\int_{t_{Sw}}^{t_1} u_{Sw}(t)dt + \int_{t_1}^{1} 1dt)$$

$$Q = F_{max} \times (\int_{t_{Sw}}^{t_1} \frac{K_R(S_w)}{k_2}(1 - \phi(t))dt + \int_{t_1}^{1} 1dt)$$

We can choose any function ϕ that satisfies the conditions mentioned before to numerically find the value of . We take here $\phi(t) = t^4$

$$Q = F_{max} \times (\int_{t_{Sw}}^{t_1} \frac{S_w - S_h}{k_2(1 - S_h)}(1 - t^4)dt + \int_{t_1}^{1} 1dt)$$

$$Q = F_{max} \times (\frac{S_w - S_h}{k_2(1 - S_h)}(t_1 - \frac{t_1^5}{5} - t_{Sw} + \frac{t_{Sw}^5}{5}) + (1 - t_1))$$

We can find the instant t_1 by resolving the following equation:

$$\frac{S_w - S_h}{5k_2(1 - S_h)} t_1^5 + (\frac{S_w - S_h}{k_2(1 - S_h)} - 1)t_1 = \frac{Q}{F_{max}} - 1 - \frac{S_w - S_h}{k_2(1 - S_h)}(-t_{Sw} + \frac{t_{Sw}^5}{5})$$

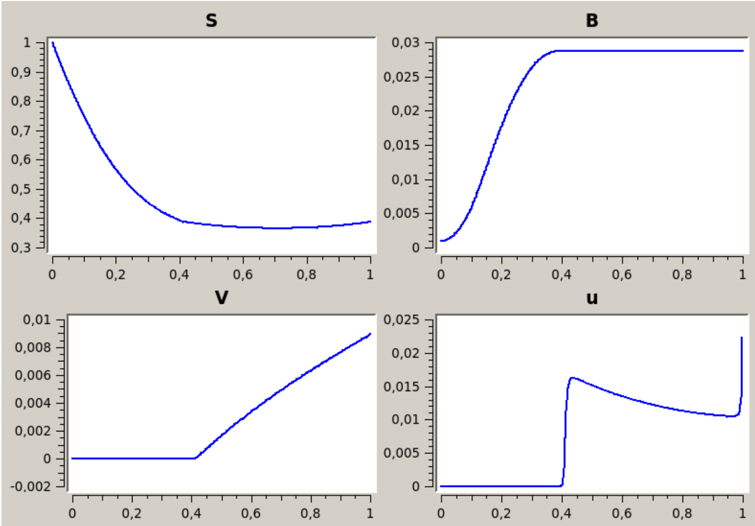
In this particular case, we are considering $Q = 0.1$.

With the constant values: $S_w = 0.4, S_h = 0.2, k_2 = 5, F_{max} = 1.2$, we have as an equation :

$$t_1^5 - 95t_1 + 90 \approx 0$$

This shows that $t_1 = 0.955764 \approx 0.95$ meaning that $\epsilon = 0.05$.

3.4 The third form



3.4.1 Q range values in the case

In this section, we shall find numerically the limit value of Q that characterizes the range values in this third form. This case corresponds to a value of quota that allows to have a singular arc on $[t_1, T]$:

$$Q = F_{max} \int_{t_1}^{T} u_{S_w}(t)dt$$

We calculate t_1 using the same method as before:

$$1 - S_w = \int_{t_1}^{0} k_1(-\phi(t)K_s(S) - (1 - \phi(t))K_r(S))dt$$

Numerically we find that

$$Q_{lim2} = 0.011$$

We can observe the behavior of the control with values below and above Q_{lim2} to validate our results:

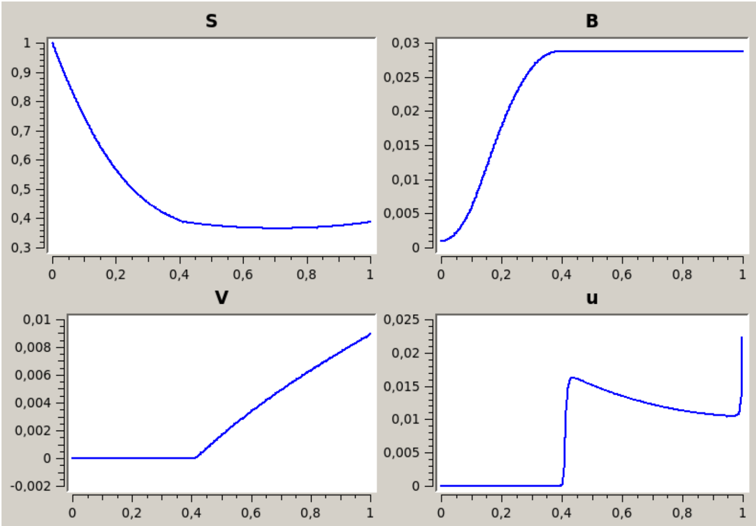


Figure 1: $Q_{lim2} = 0.09$

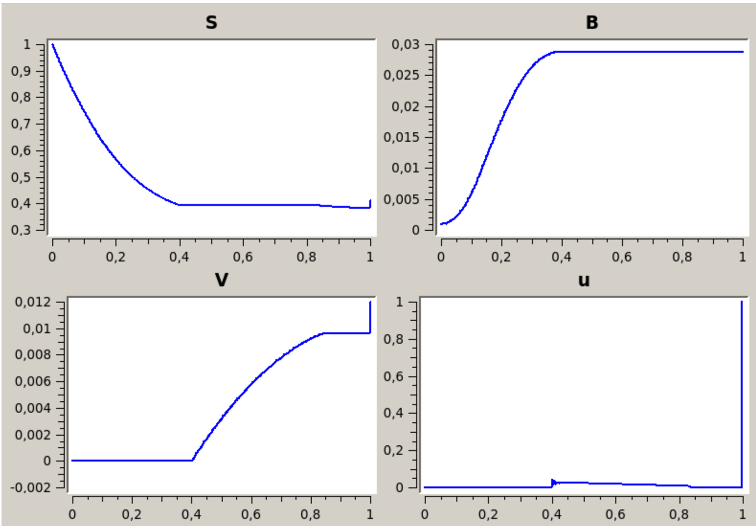


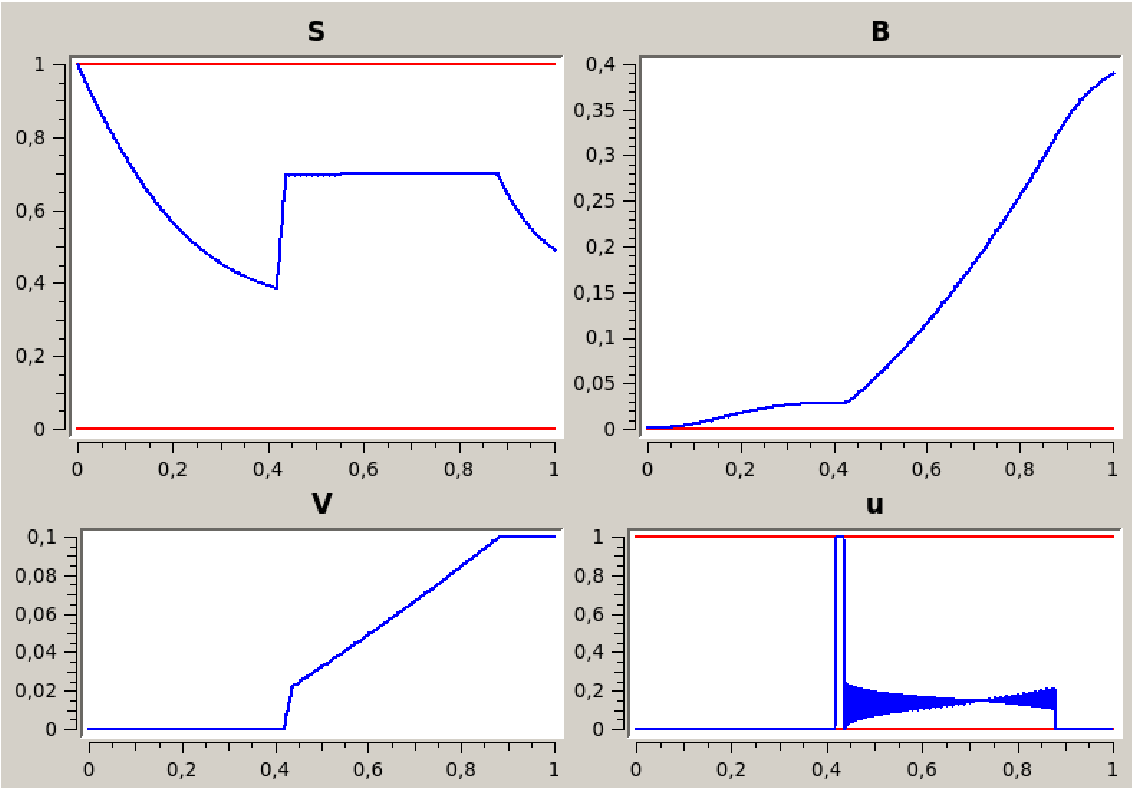
Figure 2: $Q_{lim2} = 0.012$

4 Problem with rain

At first, we'll study the problem of maximization of the biomass without rain with the following parameters:

T	F_{max}	S_0	S^*	S_w	S_h	k_1	k_2	k_3	α	Q
1	1.2	1	0.7	0.4	0.2	3	5	1	1	0.1

The maximum biomass in this case is : 0.39. Here are the figures of the variation of humidity S, the biomass B, V and the control u found by BocopHJB :



We will calculate the quantity of water Q_{max} that will allow us to have the maximum biomass using the code of the previous sections. We find that this value is equal to $Q_{max} = 0.2$, it will allow us to obtain a biomass $B = 0.501$.

We will introduce now the rain that is pre-defined with a quantity of water Q_{rain}. At first, we will study the case where the sum of the quantity of water Q and the quantity of water brought by rain Q_{rain} is equal to Q_{max}

4.1 Near Q_{max}

We will use a quantity of water $Q = 0.1$ and we will change the quantity of water brought by rain. The values Q_{rain}that we'll use are:

Q_{rain}	0.01	0.05	0.1

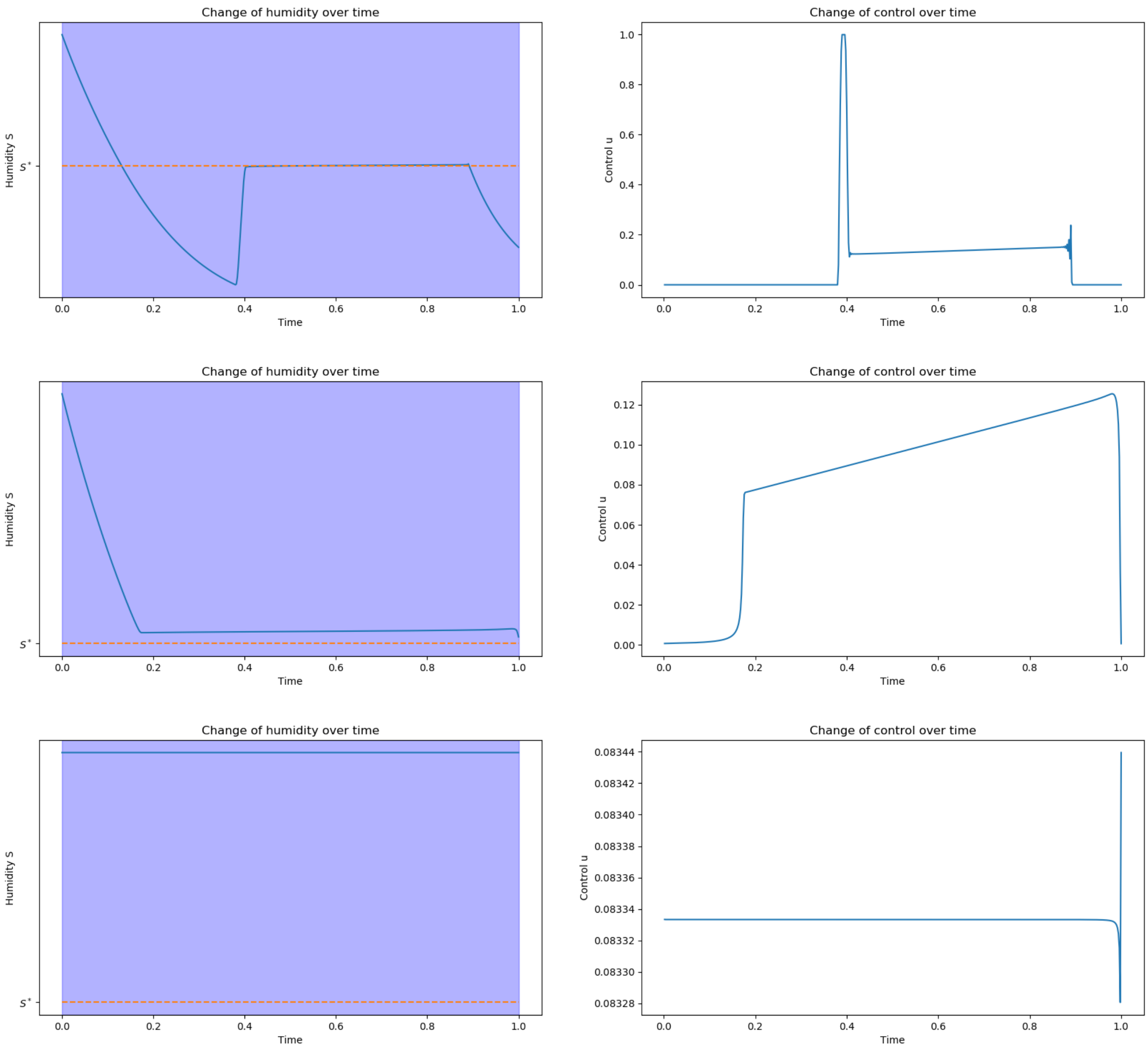
We will define the rain as a piecewise constant function on the interval $[0, T]$ with an amplitude of:

$$p = \frac{k_2}{T} * Q_{rain}$$

The corresponding values of p are:

p	0.05	0.25	0.5

The results of humidity S and control u are:



The final biomass is:

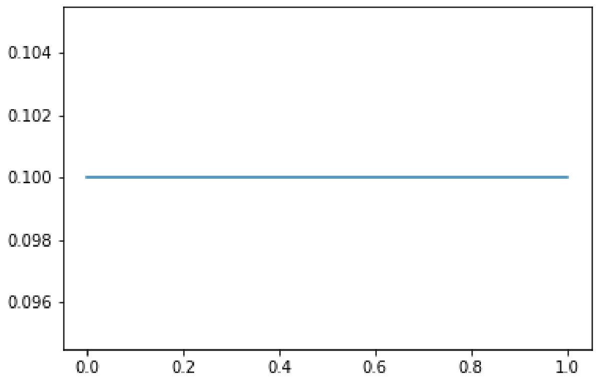
Q_{rain}	B
0.01	0.418
0.05	0.500
0.1	0.501

zz In the first example, one could see that the variation of humidity is similar to the case where we have an initial quantity of water $Q = 0.1 + Q_{rain}$ because the rain intensity is weak during the totality of $[0, T]$. The precipitations brought by rain in the second example allow us to always maintain the humidity above S^* and to have an almost maximum biomass. In the last example, the rain brings a quantity of water Q_{rain} that allows the system to have a maximum biomass because $Q_{max} = Q + Q_{rain}$ and the humidity is maintained at $S = 1$.

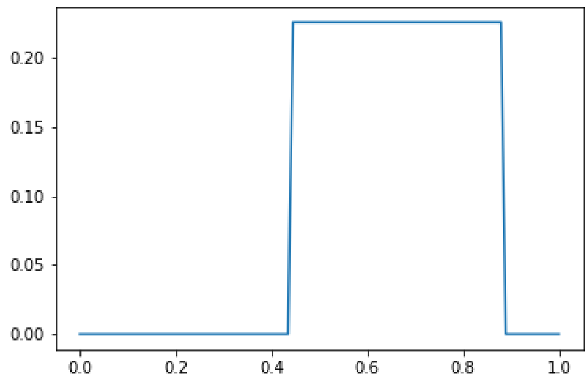
The goal of the next section is to find out how we shall distribute the quantity of water brought by rain Q_{rain} that doesn't allow us to get to Q_{max} and to maximaze the biomass B.

4.2 Distribution of rain

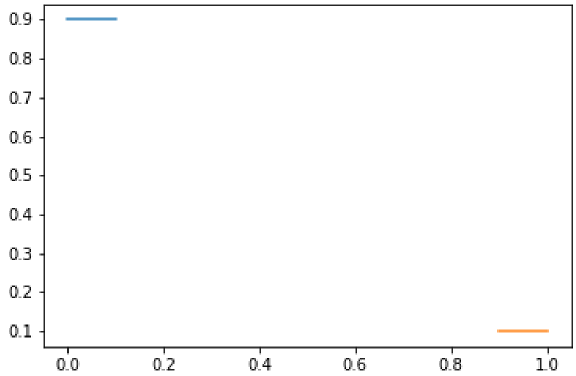
At first, we will see that the change of the distribution of Q_{rain} ($= 0.01$ in this case) on the interval $[0, T]$ allows us to have different values of biomass.
Furthermore, because it seems impossible to study all the possible scenarios of rain, we will proceed by studying 4 scenarios. The first one is to bring constant precipitations on the interval $[0, T]$. The second one is to bring constant precipitations on the interval $[t_1, t_2]$ where the humidity S is equal to S^*. The third one is to bring strong precipitations on the interval $[0, T/10]$ and weak ones on the interval $[9T/10, T]$. The last one is to bring weak precipitations on the interval $[0, T/10]$ and strong ones on the interval $[9T/10, T]$.
The graph of rain on each scenario is:



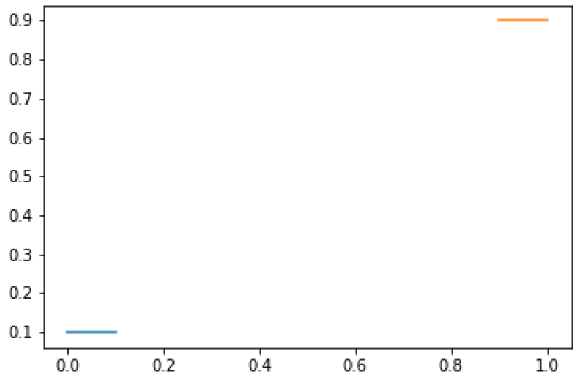
First scenario



Second scenario



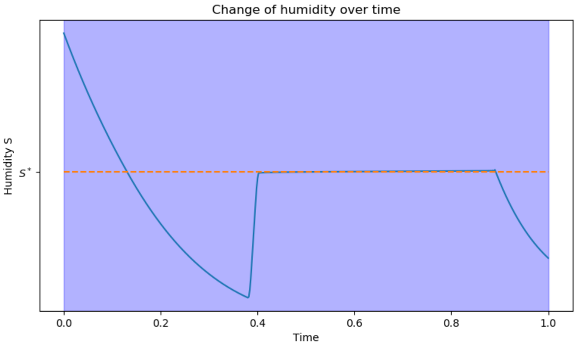
Third scenario

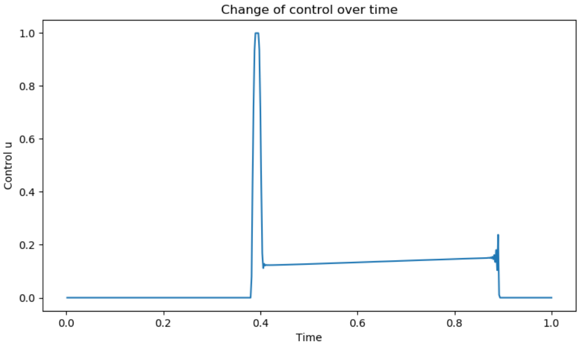


Fourth scenario

4.2.1 First scenario

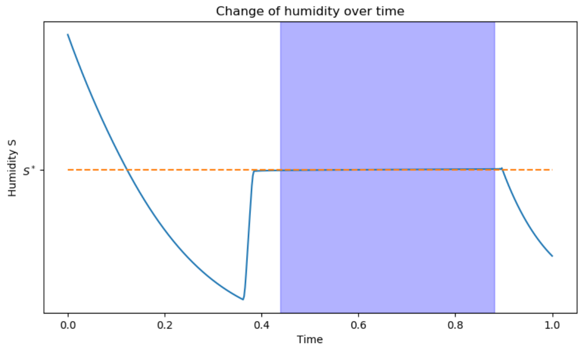
$B \approx 0.418746996901586$

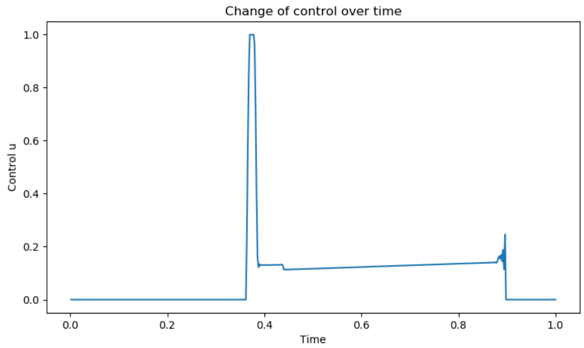




4.2.2 Second scenario

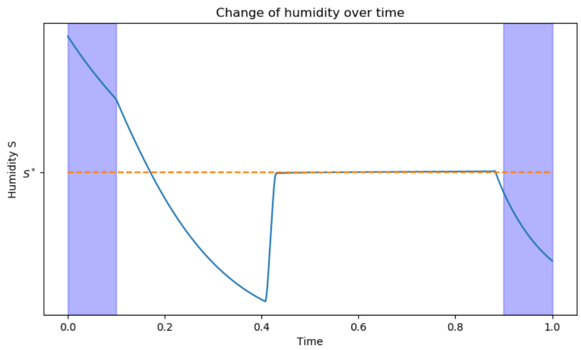
$B \approx 0.422153152386759$

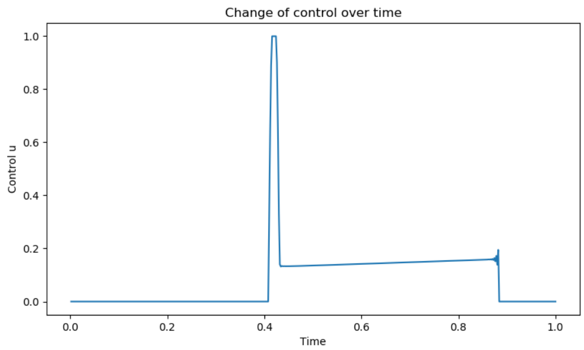




4.2.3 Third scenario

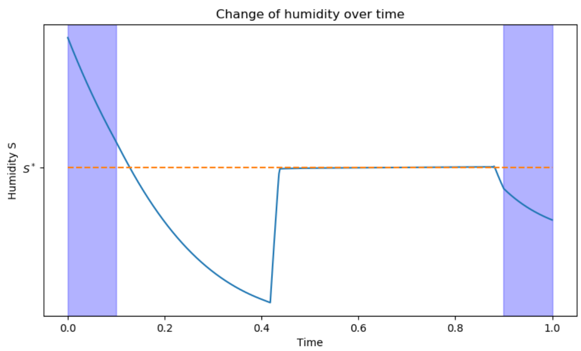
$B \approx 0.40865427948806$

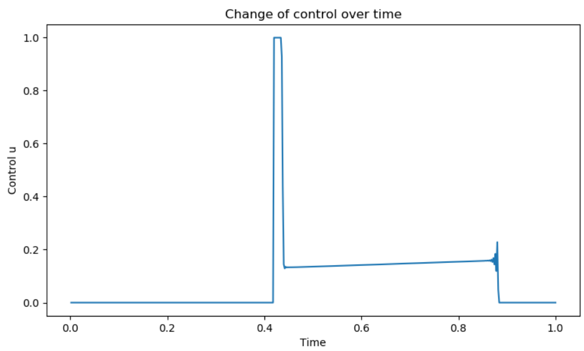




4.2.4 Fourth scenario

$B \approx 0.408235738195503$



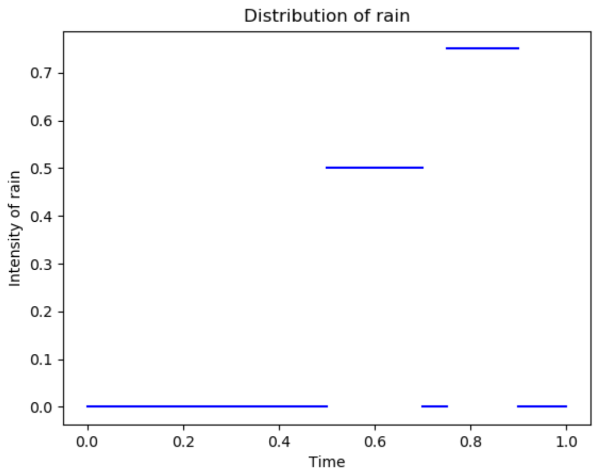


We found out that the best scenario is the second one where the precipitations are present on the interval $[t_1, t_2]$, if we compare this graph with the one without rain, we can see that the humidity is longer maintained at S^* and that explains why we have a better final biomass. Same for the first scenario but with less biomass produced as the humidity is also maintained at S^* but for a shorter duration than on the second scenario. For the third and fourth scenarios, we can spot that the duration of the singular arc on S^* is shorter than on the previous scenarios, the rain brought isn't well distributed to help maintaining S on S.

From these four scenarios, we can conclude that having precipitations on the interval $[t_1, t_2]$ is the most favourable thing to do to have a maximum biomass.

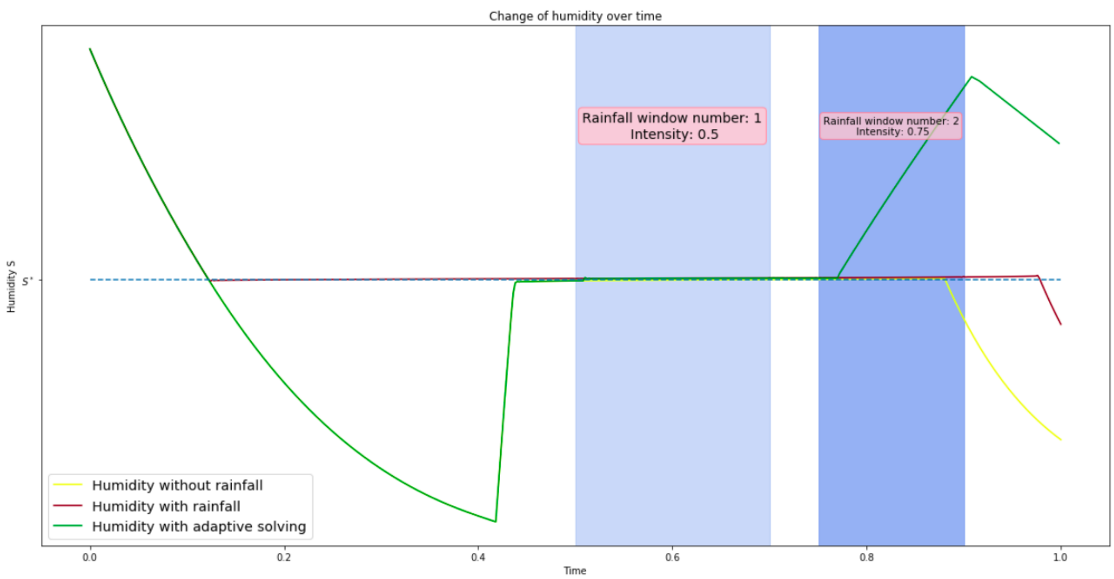
4.3 Adaptive Solving

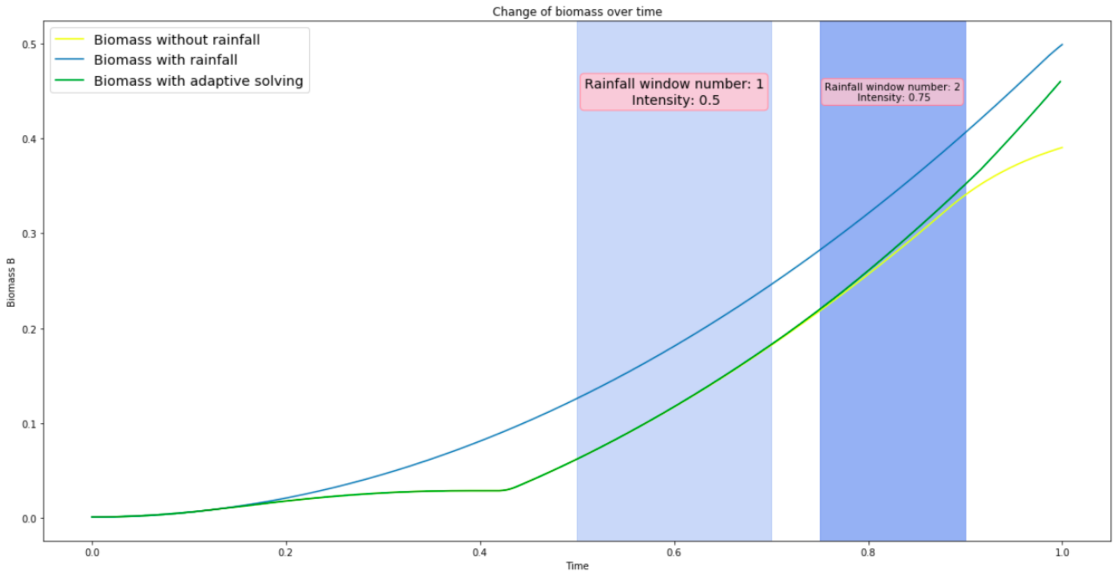
In reality, it's difficult to predict the distribution of rainfall on the whole season, that's why we will use adaptive control to solve the problem with rain, this type of scenario is closer to the real scenario than solving knowing at first the distribution of rain.
The program will adapt its dynamic on each step taking on consideration if it rains at that specific moment or not. Let's take an example where the real distribution of rain is as follows (here not known in advance):

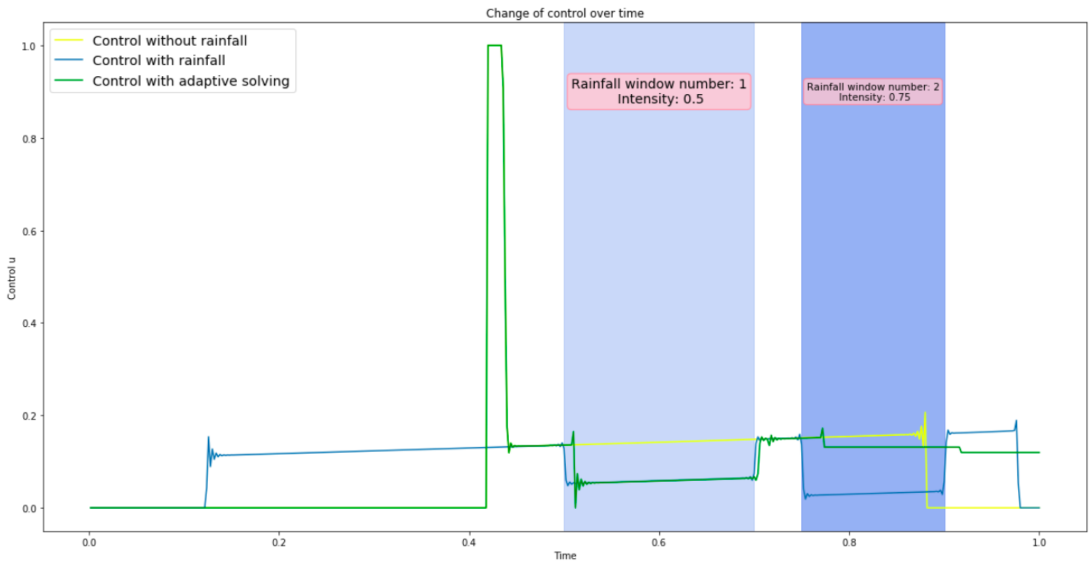


The final biomass is:

Problem without rain	Problem with rainfall known at first	Problem with adaptive solving
0.390	0.498	0.459







As we can see, solving the problem with rainfall known at first gives us the maximum biomass because the solving mechanism can decide even before starting on how to distribute the quantity of water brought to get a maximum biomass, whereas in the adaptive solving, the solution of the problem is to be adapted to the real time rainfall.

5 References

[1] Kenza Boumaza, Nesrine Kalboussi, Alain Rapaport, Sébastien Roux, and Carol Sinfort. Optimal control of a crop irrigation model under water scarcity. May 2021.

[2] Bocop HJB the optimal control solver.