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Ihab Haidar[1], Jean-Pierre Barbot[1,2] and Alain Rapaport[3]

***Abstract*— Consider the observation problem of bidimensional systems which requires the construction of an embedding in higher dimension. A multi-observers approach has been recently introduced by the authors to deal with a class of such singularly observable systems. This approach does not require any coordinates transformation. In this work we illustrate through an academic example the applicability of this approach.**

Keywords Nonlinear systems, Observability, State observer, Singularity.

I. Introduction

Consider the dynamics in the plane

$$\dot{x}_1 = a x_2 - (x_1^2 + x_2^2 - a^2) \quad (1)$$
$$\dot{x}_2 = -a x_1 \quad (2)$$

where a is a positive real number. The objective is to reconstruct x_2 with the measurement of x_1. Observe that the construction of x_2 presents a singularity, which may recurs periodically over the time, at $x_2 = a/2$. This singularity problem requires an immersion in higher dimension for the observer design (see, e.g., [1], [2], [3], [4], [5], [10], [11], [12], [13]). A systematic approach to obtain an observer in the original coordinates with an exponential convergence consists in transforming the original system into the so called *observability form* [6], determine an observer in this canonical form, and then expressing the estimation back in the original coordinates. However, this approach presents several difficulties in the construction of the embedding and the lipschitzian extension of the dynamics outside the set of its forward orbits. Constructive methods have been proposed in [7], [12] to deal with such difficulties.

A new approach has been recently introduced in [9]. This is based on running in parallel a set of estimators in the original coordinates, each one follows dynamically one and only one root of equation (1), and then use the further derivatives of the output together with equation (2) to select at any time the right estimator. This approach does not require any coordinates transformation, and by consequence no problem of transformation inversion of the observability map nor its Jacobian is posed.

The paper is organized as follows. In Section II we recall the multi-observers approach. In section III we describe our roots tracking methods. Section IV is devoted to the illustration of the multi-observers approach through system (1)-(2).

[1] Laboratoire Quartz EA 7393, ENSEA, Cergy-Pontoise, France, {ihab.haidar, jean-pierre.barbot}@ensea.fr

[2] LS2N-CNRS UMR 6004, Nantes, France.

[3] MISTEA, Univ Montpellier, INRAE, France, alain.rapaport@inrae.fr

II. The multi-observers approach

Here we recall the multi-observers approach. Consider the dynamics

$$\dot{x}_1 = f_1(x_1, x_2) \quad (3)$$
$$\dot{x}_2 = f_2(x_1, x_2) \quad (4)$$

where f_1 is a rational function and f_2 is a sufficiently smooth function, along with the observation of $y = x_1$. Let $\mathcal{D}$ be a relatively compact subset of $\mathbb{R}^2$ not containing the poles of f_1 and positively invariant by (3)-(4). Suppose that the application

$$\bar{z} = \Phi(x) = \left[\ h(x), L_f h(x), \cdots, L_f^{m-1} h(x)\ \right]^T, \quad (5)$$

where $f = (f_1, f_2)^T$ and $h(x) = x_1$, defines an injective immersion on $\mathcal{D}$, for some $m \geq 2$. Let N and D be the numerator and denominator of f_1, respectively. Observe that, since we suppose that $\mathcal{D}$ does not contain the poles of f_1 and that is positively invariant by the dynamics (3)-(4), a solution $s(t)$ of equation (3) must satisfy the following equation

$$N(y(t), s(t)) - \dot{y}(t) D(y(t), s(t)) = 0, \quad \forall\, t \geq 0. \quad (6)$$

The solution s of (6) is, in general, not uniquely determined. But, since N and D are polynomial functions, then, for each fixed z in $\mathbb{R}^2$ (z will play the role of the first two components of $\bar{z}$), there exist at most p solutions $s_1, \cdots, s_p$ such that

$$F(z, s_i) := N(z_1, s_i) - z_2 D(z_1, s_i) = 0, \quad (7)$$

for $i = 1, \cdots, p$, where $p = \max\{\deg(N), \deg(D)\}$. Observe that the number of real solutions of (7) could depend on z. At any time $t \geq 0$, there exists at least one root s_i such that $s(t) = s_i$. The multi-observers approach consists in computing dynamically p-parallel estimators $\hat{s}_1, \cdots, \hat{s}_p$ of these roots. The way to determine these estimators will be addressed in the next section. The final task is to provide, at each $t \geq 0$, an estimation of $x_2(t)$ among $\hat{s}_1(t), \cdots, \hat{s}_p(t)$. For this purpose, using the further derivatives of y and the injectivity of the map Φ, we consider the test $\mathcal{T}(\bar{z}, s) = 0$, where the function $\mathcal{T}$ is defined as follows

$$\mathcal{T}(\bar{z}, s) := \| (\bar{z}_2 - \Phi_2(\bar{z}_1, s), \cdots, \bar{z}_m - \Phi_m(\bar{z}_1, s)) \|_M,$$

and $\|\cdot\|_M$ denotes the norm associated to a real symmetric positive definite $(m-1) \times (m-1)$ matrix M. We choose $\hat{s}(t) = \hat{s}_{i^\star(t)}(t)$ for which $\hat{s}_{i^\star(t)}(t)$ minimizes the function $\mathcal{T}(\bar{z}(t), \hat{s}_i(t))$ among the estimators $\{\hat{s}_i(t)\}_{i=1,\cdots,p}$, for $t \geq 0$. The choice of the norm plays a role when noisy output is considered.

III. THE ROOTS TRACKING METHOD

Consider, for $k > 0$, the following implicit dynamics with complex variable

$$\begin{cases} \frac{d}{dt} F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) = -k F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) & \forall\, t \geq 0, \\ \hat{s}(0) \in \mathbb{C} \backslash \mathbb{R}, \end{cases} \tag{8}$$

where the function $F_\varepsilon : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{C}$ is defined, for $\varepsilon > 0$, by $F_\varepsilon(z, s) := F(z, s) - \varepsilon \mathrm{i}$, and i denotes the imaginary unit number. The roots tracking method proposed in [9] requires the following two assumptions.

Assumption 1: For all $z \in \Phi_1(\mathcal{D}) \times \Phi_2(\mathcal{D})$, the polynomial $\partial_s F(z, s)$ does not admit complex roots.

Remark that Assumption 1 is related to the polynomial F in s only. If this assumption cannot be checked analytically (notably, when its degree is larger than 6), one may look for numerical verifications.

Assumption 2: The number of roots of $F_\varepsilon(z(t), \cdot)$ is constant and equal to p over $\mathbb{R}_+$.

The idea behind introducing the perturbation parameter ε is clarified by the following observation. Knowing that for $t \geq 0$ and $\varepsilon > 0$ the roots of $F_\varepsilon(z(t), \cdot)$ are always complex, and those of $\partial_s F_\varepsilon(z(t), \cdot) = \partial_s F(z(t), \cdot)$ are always reals (Assumption 1), the polynomial $F_\varepsilon(z(t), \cdot)$ cannot have multiple roots. By consequence, for $t \geq 0$ and $\varepsilon > 0$, $F_\varepsilon(z(t), \cdot)$ admits p-distinct time-varying complex roots, $s_{\varepsilon,1}(z(t)), \cdots, s_{\varepsilon,p}(z(t))$, which vary continuously with respect to time.

Let us introduce the map $\mathcal{F}_\varepsilon : \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}$, given by

$$\mathcal{F}_\varepsilon(t, \sigma) = -(\partial_s F_\varepsilon(z, \sigma))^{-1} \left(\partial_z F_\varepsilon(z, \sigma) \dot{z} + k F_\varepsilon(z, \sigma) \right),$$

which, thanks to Assumption 1 and the perturbation parameter ε, is well defined. Using $\mathcal{F}_\varepsilon$, the dynamics (8) can be explicitly written as

$$\begin{cases} \dot{\hat{s}}_\varepsilon(t) = \mathcal{F}_\varepsilon(t, \hat{s}_\varepsilon(t)), & \forall\, t \geq 0, \\ \hat{s}_\varepsilon(0) \in \mathbb{C} \backslash \mathbb{R}. \end{cases} \tag{9}$$

The following theorem shows the uniform convergence of the solutions of (9) to the roots of (7).

Theorem 1 ([9]): Suppose that Assumption 1 and Assumption 2 hold. Then, for every $\delta > 0$ and every $i \in \{1, \cdots, p\}$ there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ the solution of (8) starting from $s_{\varepsilon,i}(z(0))$ satisfies the following inequality

$$\sup_{t \geq 0} |\hat{s}_\varepsilon(t) - s_i(z(t))| < \delta. \tag{10}$$

Theorem 1 assumes the perfect knowledge of z, that is the first two components of $\bar{z}$. In practice, one can use a numerical differentiator to estimate z allowing a short time interval $[0, \eta]$ for the differentiator to converge and then one can use the roots tracking method that we propose from time η (i.e. all the roots are computed once at time η and then tracked over time by continuation).

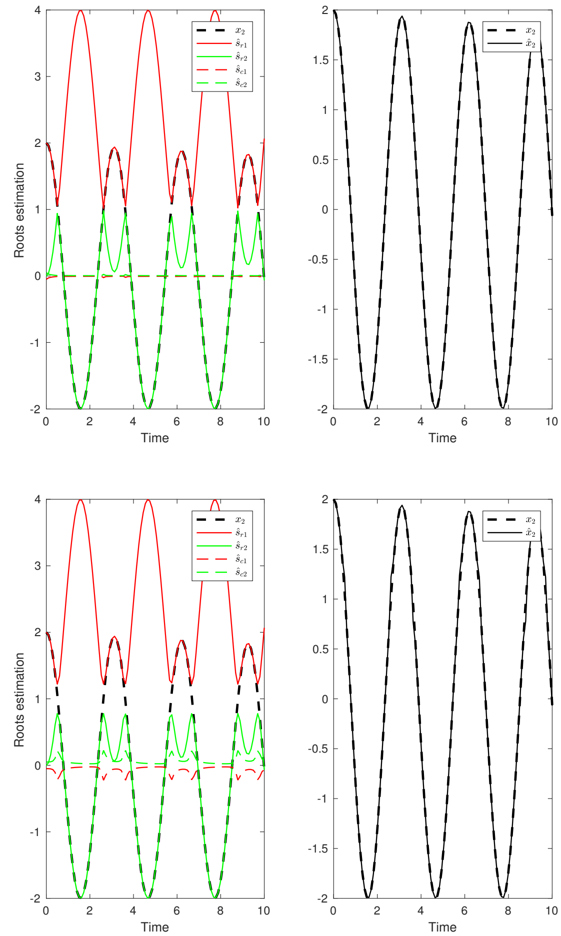

Fig. 1. Illustration of the estimated roots (left) with perfect knowledge of z together with the exact and the constructed solution (right) with $\varepsilon = 0.001$ (top) and $\varepsilon = 0.1$ (bottom).

IV. APPLICATION TO EXAMPLE (1)-(2)

Consider system (1)-(2) along with the observation of x_1. It is easy to see that the ball $B(0, a)$ is invariant by the dynamics (1)–(2). For the construction of x_2 we define firstly the function $F(z, s) = -s^2 + as - z_1^2 - z_2 + 1$. We can easily verify that Assumption 1 and 2 are satisfied. In order to show the applicability of our approach in this case of periodic singularity, we simulate system (9) starting from the initial conditions of $F_\varepsilon(z(0), s_{\varepsilon,i0}) = 0$, for $i = 1, 2$ and for different values of ε. The value of a is fixed to $a = 2$ and the initial condition of the original system (1)–(2) is fixed at $(0, 2)$. The parameter k, relative to (8) has been fixed to $k = 150$. An explicit Euler scheme with a discretization step equal to $h = 10^{-3}$ has been chosen for the simulation. Assuming the perfect knowledge of z, by Figure 1 we show the different real and complex parts of the estimated roots (left) together with the constructed solution (right), where the perturbation parameter ε is taken equal to 0.001 (top) and 0.1 (bottom). The test function defined in Section II is used to select at any time the right estimated root. Now, we consider that the output measurements are randomly disturbed by a white noise proportional up to 50% of y. The first two derivatives of the output are estimated using a

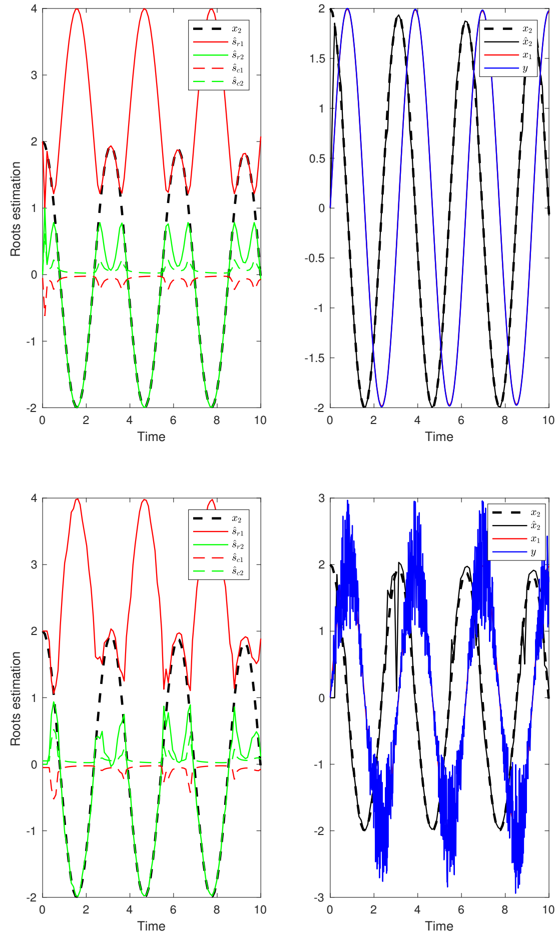

Fig. 2. Illustration of the estimated roots (left) without assuming the perfect knowledge of z together with the exact and the constructed solution (right) for $\varepsilon = 0.1$. Case of estimated z without noise (top) and with noise

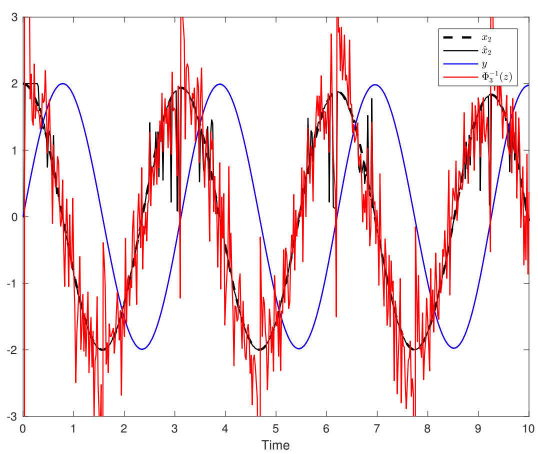

Fig. 3. The exact solution together with the constructed one (with $\varepsilon = 0.1$) and that obtained with Φ^{-1}. Case of estimated z with noise measurement proportional up to 0.2% of x_1 and without any filtering strategy.

high-gain differentiator. The output as well as its numerical derivatives are filtered offline using moving average filters. We also consider a short delay before using the estimated derivatives, the time for the differentiator to converge. We simulate system (9) in the same conditions as before, where ε is fixed at 0.1. By Figure 2 we show the different real and complex parts of the estimated roots (left) together with the estimation of x_2 (right) in two cases: without noise measurement (top) and with noise measurement proportional up to 50% of x_1 (bottom). The identity matrix $M = I_2$ is simply chosen to define the test function.

We end this section by underlying the advantage of the proposed approach to deal with singularities. For this, we compare the constructed solution together with the one obtained directly by inverting the observability map Φ. A straightforward computation gives

$$\Phi^{-1}(z) = \left(z_1, \frac{a^2 + 2z_2 + z_3/z_1}{2a}\right)^T.$$

By Figure 3 we show the solution constructed by our method, with $\varepsilon = 0.1$, together with the one obtained by Φ^{-1}, in the case of estimated z, with white noise proportional up to 0.2% of x_1. This comparison is done without using any filtering strategy. The performance of our approach is once again shown by this comparison.

V. Conclusion

We illustrate through an example the multi-observers approach introduced recently in the literature to deal with a class of singularly observable bidimensional systems. Future works will attempt to extend this approach to more general class of higher dimensional autonomous systems.

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