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***Abstract*—We consider the observation problem for a particular class of bidimensional systems with scalar output which requires the consideration of an embedding in higher dimension for the usual high-gain observer synthesis. We propose a new approach that does not require any coordinates transformation. This approach is based on the design of a set of estimators running in parallel in the same dimension than the original system. Each estimator uses the knowledge of the first two derivatives of the output, and the further derivatives up to the m-th one (where m is the observability index over an invariant domain) are used to discriminate at any time among the different estimators. We give two examples showing the applicability of this approach with measurement noise. Biological systems used in batch bioprocess models are of particular motivation for this work.**

***Index Terms*—Nonlinear systems, Observability, State observer, Singularity, Biological systems.**

I. INTRODUCTION

A significant research has been devoted around the problem of observability and design of observers for finite-dimensional dynamical systems. The importance of this problem arises from many practical applications where the design of state estimators is needed [19]. One of the main and classical approach to obtain an observer in the original coordinates consists in transforming the original system into the so called *observability form* [10], determine an observer in this canonical form, and then expressing the estimation back in the original coordinates. Knowing that in general this change of coordinates is defined through an immersion in higher dimension space, and not simply a diffeomorphism, this approach has been widely investigated in the literature (see, e.g., [2], [4], [5], [6], [7], [14], [15], [20], [22]). The principal difficulty lies in the construction of an embedding and a Lipschitzian extension of the dynamics outside the set of its forward orbits. Another approach allowing the construction of local asymptotic observers without passing by the canonical form has been recently developed in [3].

In this paper, we propose another approach to deal with the problem of immersion in higher dimension space. We focus on the particular class of systems defined on a subset of $\mathbb{R}^2$

$$\dot{y} = f_1(y,s), \quad (1)$$

$$\dot{s} = f_2(y,s), \quad (2)$$

where f_1 is a rational function and f_2 is a sufficiently smooth function, along with the observation of y. Instead of a single observer in higher dimension, we propose a set of observers in the original coordinates and a test function which can discriminate between the observers the one that will give the right estimate. This test function is based on higher derivatives of the observation. More precisely, starting from equation (1) we build a set of estimators, each of them follows dynamically one and only one root (in s) of equation (1). Then, we use the further derivatives of the output together with equation (2) to select the right estimator at any time. This approach does not require any coordinates transformation, and by consequence no problem of transformation inversion of the observability map nor its Jacobian is posed. The technical task with this approach lies in the roots tracking method (in the spirit of continuation techniques [1]) which, as described in the preliminary result of this work (see [12]), needs to be robust near singularities (multiple roots); robust in the sense that it always distinguishes between different neighboring roots. Indeed, knowing that each estimator should have a time-varying root as a global attractor and knowing that such attractors intersect at multiple roots, then, close to singularities, two distinct estimators may exchange roles and even fuse together. In addition, a relevant roots tracking method should take into account the initialization problem. In fact, a computation of the different roots at initial time is needed to initialize properly the method. This can be possible if we deal with complex roots of equation (1). Here, based on a singular perturbation of equation (1), we propose a robust roots tracking method allowing the construction of its complex roots.

Knowing that the perfect knowledge of the output and its successive derivatives is hardly accessible in practice, the performance of this multi-observers approach in the case of measurement noise is shown through numerical simulations. Even in the case where the inverse of the observability map is available, which is not an easy task in general, we show by simulation the advantage of our approach.

For general class of systems of dimension $n > 2$ with scalar measurement y, observability singularities may occurs

This work is supported by a public grant overseen by the Research and Valorization Service (SRV) of ENSEA.

as functions of $(\dot{y}, \ddot{y}, \cdots, y^{(n-1)})$. Here, for sake of clearness, we restrict our study to bidimensional systems, so the singularity appears only as a function of $\dot{y}$. Nevertheless, several biological systems which are subject to the evoked singularity observability problem may encompassed by system (1)-(2) (see, e.g., [11], [20]).

The paper is organized as follows. In Section II we describe the new approach. In section III we describe our roots tracking methods and give the main results. Two examples showing the applicability of the proposed method are given in Section IV.

Notations: $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{C}$ denote the set of real, non-negative, and complex numbers, respectively. The imaginary unit number relative to $\mathbb{C}$ is denoted by i. By $(\mathbb{R}^n, \|\cdot\|_M)$ we denote the n-dimensional Euclidean space, where n is a positive integer and $\|\cdot\|_M$ is the norm induced by the inner product associated to a real symmetric positive definite $n \times n$ matrix M. By $\mathrm{dist}(x, A)$ we denote the distance between $x \in \mathbb{R}^n$ and a non-empty subset $A \subset \mathbb{R}^n$. The notations $\Re(x)$ and $\Im(x)$ stand, respectively, for the real and complex parts of $x \in \mathbb{C}$. For $f = (f_1, f_2)$ and $(y, s) \xrightarrow{h} y$, $L_f^i h$ stands for the i-th Lie derivative of h with respect to f. For $m \geq 2$, we define the map

$$(y, s) \xrightarrow{\Phi} \bar{z} := \Big(h(y, s), L_f h(y, s), \cdots, L_f^{m-1} h(y, s)\Big). \quad (3)$$

II. A MULTI-OBSERVERS APPROACH

Let $\mathcal{D}$ be a relatively compact subset of $\mathbb{R}^2$ not containing the poles of f_1 and positively invariant by the dynamics (1)-(2). A sufficient condition for the construction of observers for system (1)-(2) on $\mathcal{D}$ is that the map (3) defines an injective immersion on $\mathcal{D}$, for some $m \geq 2$ (m is the observability index over $\mathcal{D}$). In this case, the extension of system (1)-(2) to $\mathbb{R}^m$ is possible (see, e.g., [8]), and a constructive method allowing the construction of an exponential observer for (1)-(2) on $\mathcal{D}$ in its original coordinates is proposed in [20]. In this paper, we suppose that system (1)-(2) along with the observation of y admits a finite observability index $m \geq 2$ over $\mathcal{D}$.

We propose a new approach based on the design of several estimators running in parallel. This approach is outlined in the sequel. Let N and D be the numerator and denominator of f_1, respectively. Since N and D are polynomial functions, then for each fixed $z \in \mathbb{R}^2$ (z will play the role of the first two components of $\bar{z}$) there exist at most p solutions $s_1, \cdots, s_p$ such that

$$F(z, s_i) := N(z_1, s_i) - z_2 D(z_1, s_i) = 0, \ \forall\, i = 1, \cdots, p, \quad (4)$$

where $p = \max\{\deg(N), \deg(D)\}$. The number of real solutions of (4) could depend on z. Observe that, since we suppose that $\mathcal{D}$ does not contain the poles of f_1 and that is positively invariant by the dynamics (1)-(2), then at any time $t \geq 0$, there exists at least one real root s_i such that $s(t) = s_i$. The multi-observers approach that we propose consists in computing dynamically p-parallel estimators $\hat{s}_1(\cdot), \cdots, \hat{s}_p(\cdot)$ of these roots; the way to construct these estimators will be addressed in the next section. Then, using the further derivatives of y and the injectivity of the map Φ, the test function $\mathcal{T}$ defined by

$$\mathcal{T}(\bar{z}, \hat{s}) := \|\,(\bar{z}_2 - \Phi_2(\bar{z}_1, \hat{s}), \cdots, \bar{z}_m - \Phi_m(\bar{z}_1, \hat{s}))\,\|_M, \quad (5)$$

will be used to provide, at each $t \geq 0$, an estimation $\hat{s}(t)$ of $s(t)$ among $\hat{s}_1(t), \cdots, \hat{s}_p(t)$. In fact, we choose $\hat{s}(t) = \hat{s}_{i^\star(t)}(t)$ for which $\hat{s}_{i^\star(t)}(t)$ minimizes the function $\mathcal{T}(\bar{z}(t), \hat{s}_i(t))$ among the estimators $\hat{s}_1(t), \cdots, \hat{s}_p(t)$, for $t \geq 0$. Any norm can be chosen for the test function. However, the performances of our method can be impacted by the choice of the norm when noisy output is considered as it will be discussed in Section IV.

III. A ROOTS TRACKING METHOD

For $\varepsilon > 0$, we introduce the polynomial function $F_\varepsilon : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{C}$, defined by

$$F_\varepsilon(z, \sigma) := F(z, \sigma) - \varepsilon \mathrm{i}. \quad (6)$$

For each $t \geq 0$, the polynomial $F_\varepsilon(z(t), \cdot)$ can be written as

$$F_\varepsilon(z(t), \sigma) = a_p(z(t))\sigma^p + \cdots + a_1(z(t))\sigma + a_0(z(t)) - \varepsilon \mathrm{i},$$

where $a_0(\cdot), \cdots, a_p(\cdot)$ are the time-dependent coefficients of F. The roots tracking method that we propose here consists first in estimating the different time-varying roots of $F_\varepsilon(z(t), \cdot)$, for $t \geq 0$. Knowing that these roots are arbitrarily close to those of $F(z(t), \cdot)$ for arbitrarily small values of ε (see, e.g., [17]), they will be used to closely track the different time-varying roots of $F(z(t), \cdot)$, for $t \geq 0$. The idea behind introducing the perturbation parameter ε will be clarified in the sequel. This roots tracking method requires the following two assumptions.

Assumption 1. *For all $z \in \Phi_1(\mathcal{D}) \times \Phi_2(\mathcal{D})$, the polynomial $\partial_s F(z, \sigma)$ does not admit complex roots.*

Remark that this assumption is related to the polynomial F in s only. If this assumption cannot be checked analytically (notably, when its degree is larger than 6), one may look for numerical verifications.

Assumption 2. *The time-varying coefficient $a_p(\cdot)$ is such that $a_p(z(t)) \neq 0$ for every $t \geq 0$.*

Under Assumption 2, the number of roots of $F_\varepsilon(z(t), \cdot)$ is constant and equal to p over $\mathbb{R}_+$. In addition, knowing that for $t \geq 0$ and $\varepsilon > 0$ the roots of $F_\varepsilon(z(t), \cdot)$ are always complex, and those of $\partial_s F_\varepsilon(z(t), \cdot) = \partial_s F(z(t), \cdot)$ are always reals (Assumption 1), the polynomial $F_\varepsilon(z(t), \cdot)$ cannot have multiple roots. By consequence, for $t \geq 0$ and $\varepsilon > 0$, $F_\varepsilon(z(t), \cdot)$ admits p-distinct time-varying complex roots, $s_{\varepsilon,1}(z(t)), \cdots, s_{\varepsilon,p}(z(t))$, which vary continuously with respect to time; this derives from the continuous dependency of these roots on the coefficients of F_ε together with the continuous dependency of these laters with respect to time.

The following implicit dynamics

$$\begin{cases} \partial_s F_\varepsilon(z, \hat{s}_\varepsilon)\dot{\hat{s}}_\varepsilon = -\partial_z F_\varepsilon(z, \hat{s}_\varepsilon)\dot{z} - kF_\varepsilon(z, \hat{s}_\varepsilon), \\ \hat{s}_\varepsilon(0) \in \mathbb{C}\backslash\mathbb{R}, \end{cases} \quad (7)$$

with $k > 0$, will be used to estimate the different time-varying roots of $F_\varepsilon(z(t), \cdot)$, for $t \geq 0$. In order to define explicitly the dynamics of $\hat{s}_\varepsilon$ from (7), we prove the following lemma.

Lemma 1. *Suppose that Assumption 1 holds. Let $\varepsilon > 0$ and $\hat{s}_\varepsilon$ be a solution of (7) starting in $\mathbb{C}\backslash\mathbb{R}$. We have that $\partial_s F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) \neq 0$ for every $t \geq 0$.*

Proof. Suppose by contradiction that $\partial_s F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) = 0$, for some $t \geq 0$. Then, under Assumption 1, we have certainly that $\hat{s}_\varepsilon(t)$ is a real number. From equation (7), we have

$$0 = \partial_z F_\varepsilon(z(t), \hat{s}_\varepsilon(t))\dot{z}(t) + kF_\varepsilon(z(t), \hat{s}_\varepsilon(t)). \quad (8)$$

Equation (8) together with (6) lead to the following equality

$$k\varepsilon \mathrm{i} = \partial_z F(z(t), \hat{s}_\varepsilon(t))\dot{z}(t) + kF(z(t), \hat{s}_\varepsilon(t)), \quad (9)$$

which is not possible because the left and right hand sides of equation (9) are complex and real, respectively. □

Now, we introduce the map $\mathcal{F}_\varepsilon : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{C}$ given by

$$\mathcal{F}_\varepsilon(z, s) := -(\partial_s F_\varepsilon(z, s))^{-1} (\partial_z F_\varepsilon(z, s)\dot{z} + kF_\varepsilon(z, s)),$$

which, thanks to Lemma 1, is well defined. The dynamics (7) can be then equivalently written as

$$\begin{cases} \dot{\hat{s}}_\varepsilon(t) = \mathcal{F}_\varepsilon(z(t), \hat{s}_\varepsilon(t)), & \forall t \geq 0, \\ \hat{s}_\varepsilon(0) \in \mathbb{C}\backslash\mathbb{R}. \end{cases} \quad (10)$$

We introduce also, for $t \geq 0$, the set

$$S_\varepsilon(t) := \{s_{\varepsilon,i}(z(t)) : i = 1, \cdots, p\}. \quad (11)$$

The existence and uniqueness of solutions for (10) together with their asymptotical behavior are clarified by the following lemma.

Lemma 2. *Suppose that Assumption 1 and Assumption 2 hold. Then, for each $\varepsilon > 0$, system (10) with initial condition in $\mathbb{C}\backslash\mathbb{R}$, has a unique solution $\hat{s}_\varepsilon(\cdot)$ over $\mathbb{R}_+$. In addition, for every initial condition $\sigma_0 \in \mathbb{C}\backslash\mathbb{R}$, the corresponding solution satisfies the following property*

$$\mathrm{dist}(\hat{s}_\varepsilon(t), S_\varepsilon(t)) \leq \alpha(t)e^{-\frac{k}{p}t}, \quad \forall t \geq 0, \quad (12)$$

where

$$\alpha(t) = \max_{i=1,\cdots,p} |\sigma_0 - s_{\varepsilon,i}(z(0))| \left|\frac{a_p(z(0))}{a_p(z(t))}\right|^{1/p}. \quad (13)$$

Proof. Let $\varepsilon > 0$, $k > 0$ and $\sigma_0 \in \mathbb{C}\backslash\mathbb{R}$. Using the fact that the vector field defining the dynamics of $\hat{s}_\varepsilon$ is smooth, then, starting from σ_0, there exists $t_0 > 0$ such that system (10) has a unique solution over $[0, t_0)$ (see, e.g., [13]). For each $t \in [0, t_0)$, we have

$$F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) = a_p(z(t)) \prod_{i=1}^{p} (\hat{s}_\varepsilon(t) - s_{\varepsilon,i}(z(t))). \quad (14)$$

In addition, from (7), it derives straightforwardly that

$$F_\varepsilon(z(t), \hat{s}_\varepsilon(t)) = F_\varepsilon(z(0), \sigma_0)e^{-kt}, \quad \forall t \in [0, t_0). \quad (15)$$

By consequence, equation (14) together with (15) lead to the following property

$$|a_p(z(t))| \prod_{i=1}^{p} |\hat{s}_\varepsilon(t) - s_{\varepsilon,i}(z(t))| = |F_\varepsilon(z(0), \sigma_0)|e^{-kt}. \quad (16)$$

Knowing that $t \mapsto a_p(z(t))$ and $t \mapsto s_{\varepsilon,i}(z(t))$, $i = 1, \cdots, p$, are continuous over $\mathbb{R}_+$ (see, e.g., [17], for the continuity of the roots of F_ε), then, applying the limit when $t \to t_0$ on left and right hand sides of (16), we deduce that $\lim_{t\to t_0} |\hat{s}_\varepsilon(t)| < +\infty$. Then using a contradiction reasoning, one can prove that $t_0 = +\infty$ (see, e.g., [13]).
Concerning the asymptotical behavior of the solutions of (10), let $t \geq 0$ be fixed. From equation (16), we have the following inequality

$$\begin{aligned} &|a_p(z(t))| \min_{i=1,\cdots,p} |\hat{s}_\varepsilon(t) - s_{\varepsilon,i}(z(t))|^p \\ \leq\ &|a_p(z(0))| \max_{i=1,\cdots,p} |\sigma_0 - s_{\varepsilon,i}(z(0))|^p e^{-kt}, \end{aligned}$$

from which, using the fact that

$$\mathrm{dist}(\hat{s}_\varepsilon(t), S_\varepsilon(t)) = \min_{i=1,\cdots,p} |\hat{s}_\varepsilon(t) - s_{\varepsilon,i}(t)|, \quad (17)$$

we get (12). □

Remark 1. *Thanks to Lemma 2 we can affirm that, starting from a root of $F_\varepsilon(z(0), \cdot)$, the corresponding solution follows one and only one root of $F_\varepsilon(z(t), \cdot)$ over $\mathbb{R}_+$. In addition, in the case when $\min_{t\geq 0} |a_p(z(t))| = \alpha_0 > 0$, the distance between the corresponding solution and $S_\varepsilon(t)$ converges exponentially to 0.*

Lemma 3. *Suppose that Assumption 2 holds. Then, for every $\delta > 0$ there exists $\bar{\varepsilon} > 0$ such that the following inequality holds*

$$\sup_{t\geq 0} |s_{\varepsilon,i}(z(t)) - s_i(z(t))| < \delta, \ \forall \varepsilon \in [0, \bar{\varepsilon}), \ 1 \leq i \leq p. \quad (18)$$

Proof. Let $\delta > 0$ and $t \geq 0$. Knowing that the roots of a polynomial depends continuously on its coefficients (see, e.g, [17]), then there exists an $\bar{\varepsilon} = \bar{\varepsilon}(\delta, z(t)) > 0$ such that the p-distinct roots of $F_\varepsilon(z(t), \cdot)$ satisfy the following estimates

$$|s_{\varepsilon,i}(z(t)) - s_i(z(t))| < \delta, \quad \forall \varepsilon \in [0, \bar{\varepsilon}), \ 1 \leq i \leq p.$$

Knowing that $z(t)$ lies in a compact subset of $\mathbb{R}^2$, then, by compacity reasoning, there exists $\bar{\varepsilon} = \bar{\varepsilon}(\delta)$ such that inequality (18) holds. □

Theorem 1. *Suppose that Assumption 1 and Assumption 2 hold. Then, for every $\delta > 0$ and every $i \in \{1, \cdots, p\}$, there exists $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in (0, \bar{\varepsilon})$, the solution of (10) starting from $s_{\varepsilon,i}(z(0))$ satisfies the following inequality*

$$\sup_{t\geq 0} |\hat{s}_\varepsilon(t) - s_i(z(t))| < \delta. \quad (19)$$

Proof. The proof is a direct consequence of Lemma 2 and Lemma 3. □

Let us recall that the role of $\varepsilon > 0$ in Theorem 1 is essential to avoid singularities brought by multiple roots. Theorem 1 assumes the perfect knowledge of vector $z(\cdot)$, that is the first two components of vector $\bar{z}(\cdot)$. In practice, one can use a numerical differentiator to estimate $z(\cdot)$ allowing a short time interval $[0, \eta]$ for the differentiator to converge and then one can use the roots tracking method that we propose from time η (i.e. all the roots are computed once at time η and then tracked over time by continuation). This is illustrated on the examples detailed in the next section.

IV. Examples

In this section we give two examples showing the applicability of our method to deal with non-uniformly observable bidimensional systems.

A. Example 1

Let us consider the following dynamics in $\mathbb{R}_+^2$

$$\dot{y} = yr(s), \quad \dot{s} = -yr(s), \tag{20}$$

where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$r(s) = \frac{6s(s^2 - 2.5s + 2)}{s^2 - s + 3}. \tag{21}$$

along with the observation of y. It is easy to check that the positive orthant is invariant by (20) and that the poles of r does not belong there. In addition, we have

$$y(t) + s(t) = y(0) + s(0), \quad \forall t \geq 0, \tag{22}$$

which implies that, starting from the positive orthant, the trajectories of system (20) stay bounded over $\mathbb{R}_+$. The construction of s requires to solve the equation $r(s) = \dot{y}/y$, whose real solutions number varies in times, as one can observe from Figure 1-right. In order to construct the solutions of the latest equation, we consider its corresponding perturbed dynamics (7) with

$$F(z, s) = 6z_1 s(s^2 - 2.5s + 2) - (s^2 - s + 3)z_2. \tag{23}$$

The roots tracking method proposed in Section III consists in simulating dynamics (10) starting from the roots of $F_\varepsilon(z(0), \cdot)$, for some $\varepsilon > 0$. The three different solutions $\hat{s}_{\varepsilon,1}(t), \hat{s}_{\varepsilon,2}(t)$ and $\hat{s}_{\varepsilon,3}(t)$ (which are always complex) will follow the roots of $F_\varepsilon(z(t), \cdot)$, for $t \geq 0$. In order to construct an estimation of $s_\varepsilon(t)$, we have to determine among the three constructed solutions which one is the right one, at any $t \geq 0$. For this, we choose the solution $\hat{s}_\varepsilon(t) = \hat{s}_{\varepsilon,i(t)}(t)$ for which $\Re(\hat{s}_{\varepsilon,i(t)}(t))$ minimizes the function $\mathcal{T}(\bar{z}(t), \Re(\hat{s}_{\varepsilon,i}(t)))$ among the set $\{\hat{s}_{\varepsilon,i}(t)\}_{i=1,2,3}$. The choice of the norm in (5) plays an important role when dealing with numerical differentiators, especially when measurement noise is considered. In fact, when some a priori knowledge on the nature of the noise is known, one could determine numerically a covariance matrix of the estimation error (cf., e.g., [21]) of the time derivatives of the observation, whose inverse can be chosen for the norm in (5). Typically, one expects to have lower weight for high order derivatives whose estimation is more prone to be affected by measurement noise. In the absence of noise, the identity matrix can be simply used to define the norm. One can check, using the expression of the function r, the injectivity of the map Φ, given by (3), with $m = 3$. Observe also that

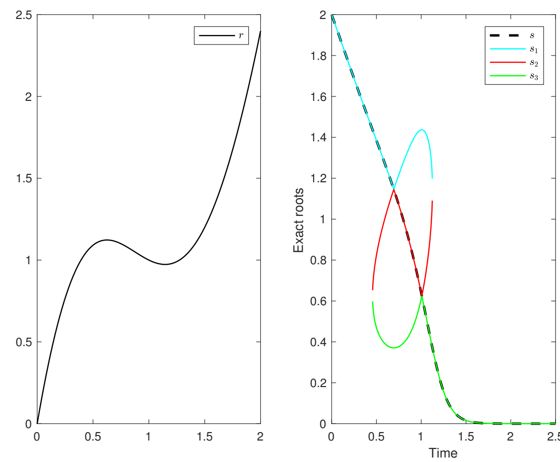

Fig. 1. Left: the function r given by (21). Right: the exact roots of (4).

Assumption 1 and Assumption 2 hold in this case. Indeed, we have

$$\partial_s F(z, s) = 18z_1 s^2 - 2(15z_1 + z_2)s + 12z_1 + z_2,$$

for which one can easily verify that, for each $z_1, z_2 \geq 0$ the solutions of $\partial_s F(z, s) = 0$ are always reals. In addition, $a_3(z(t)) = 6z_1(t) \neq 0$, for every $t \geq 0$. We simulate system (10) starting from the roots of $F_\varepsilon(z(0), \cdot) = 0$, for different values of ε. The initial conditions of the original system (20) are fixed at $(0.5, 2)$. The parameter k, relative to the estimator (10) has been fixed to $k = 150$. An explicit Euler scheme with a discretization step equal to 10^{-3} has been chosen for the simulation.

Assuming the perfect knowledge of $\bar{z}$, that is the perfect knowledge of y and $\dot{y}$, we show by Figure 2 (left) the different real and complex parts of the three roots, where the perturbation parameter ε is taken equal to 0.01 (top) and 0.1 (bottom). The constructed solution is shown by Figure 2 (right). We clearly observe the role of ε near the singularities, where the distance between neighboring solutions is somehow proportional to the magnitude of ε.

Here, we have assumed the perfect knowledge of the vector $\bar{z}(t)$ at any time $t \geq 0$, which is hardly accessible in practice. For this, we consider that the output measurements y_{obs} are randomly disturbed by a white noise proportional up to 5% of y. The first two derivatives of the output are estimated by the following high-gain differentiator

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 - \theta_1(\hat{z}_1 - y_f) \\ \dot{\hat{z}}_2 &= \hat{z}_3 - \theta_2(\hat{z}_1 - y_f) \\ \dot{\hat{z}}_3 &= -\theta_3(\hat{z}_1 - y_f) \end{aligned} \tag{24}$$

with $\theta_1 = 3 \times 10^2$, $\theta_2 = 3 \times 10^4$, $\theta_3 = 10^6$. The choice of θ_i is discussed in the literature, (see, for instance, [10], [13], [23]). Other numerical differentiators could be considered (see, e.g., [16]). The output as well as its numerical derivatives are filtered offline using moving average filters, and y_f denotes the filtered output. We also consider a short delay before using

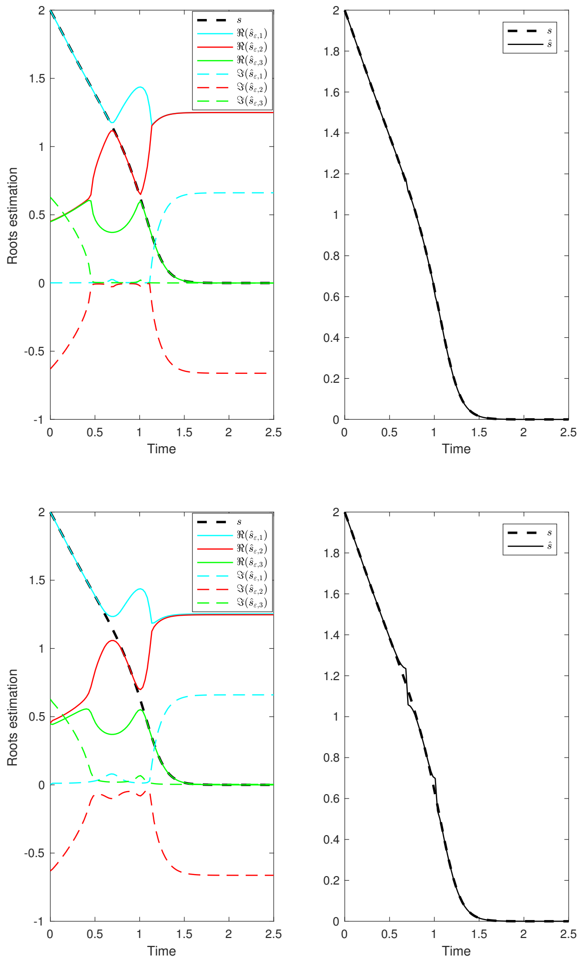

Fig. 2. Left: illustration of the estimated roots of (23) with perfect knowledge of $\bar{z}$ with $\varepsilon = 0.01$ (top) and $\varepsilon = 0.1$ (bottom). Right: the exact solution together with the constructed one.

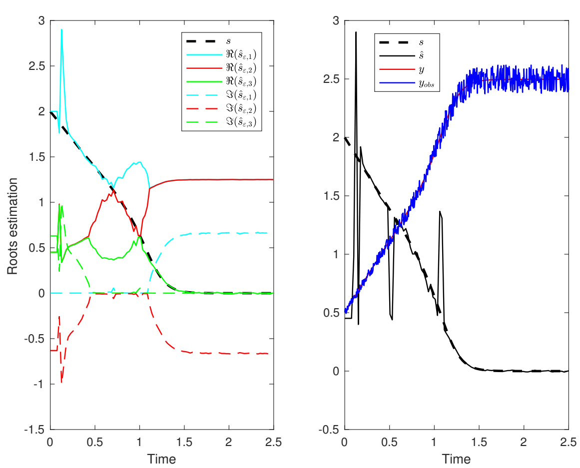

Fig. 3. Left: illustration of the estimated roots of (23) with $\varepsilon = 0.01$, in the case of estimated output's derivatives with white noise proportional up to 5% of y. Right: the exact solution together with the constructed one.

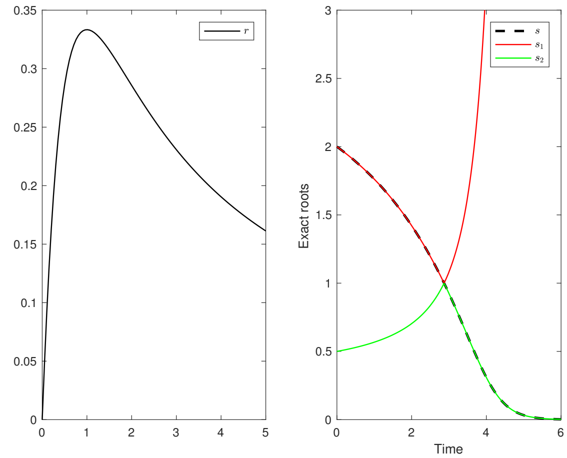

Fig. 4. Left: the function r given by (25). Right: the exact roots of (4).

the estimated derivatives, the time for the differentiator to converge. In order to check the robustness of the proposed approach to estimate the roots of (23), we simulate system (10) in the same conditions as before. As it is already mentioned, the perturbation parameter ε plays a crucial role in separating neighboring solutions. This is particularly interesting in the case of measurement noise, where a sufficiently large ε allows the separation of the disturbed estimators $\hat{s}_{\varepsilon,i}(\cdot)$ around the singularities. Of course, a large value for ε acts against the precision of the estimator (10), and this should be fixed depending on the amplitude of the noise. Concerning the test function, as $\bar{z}_3$ is a further derivative of $\bar{z}_2$, it is more affected by the noise measurement; consequently we have simply chosen $M = \text{diag}(1, 0.1)$ to define the norm in (5). By Figure 3 we show the different real and complex parts of the estimated roots of (23) together with the estimation of the solution of s where ε is fixed at 0.01.

B. Example 2

An important class of bioprocesses, which is mainly used in food and pharmaceutical industry, is the *batch* bioreactor [18]. This bioprocesses is characterized by the fact that after the initial charge of the substrate in the bioreactor, there is no inflow or outflow of the medium. The typical model characterizing the substrate biodegradation in a batch culture is given by the same system (20), where in this case the variables y and s represent the biomass and substrate concentration, respectively. Without loss of generality, we assume here that the yield coefficient of the transformation of the substrate into biomass is equal to one. When the microbial growth rate function is non-monotonic, usually related to some inhibition effect of the reaction, the function r is typically given by

$$r(s) = \frac{k_1 s}{k_2 + s + k_3 s^2}, \tag{25}$$

where k_1, k_2 and k_3 are positive constants. In this case, when only the biomass concentration is measured, the observability problem becomes particularly difficult. In fact, in this case, a

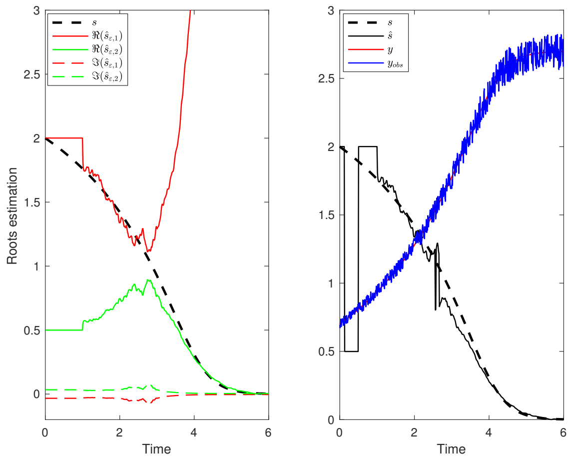

Fig. 5. Left: illustration of the estimated roots corresponding to Example 2 with $\varepsilon = 0.01$. Right: the exact solution together with the constructed one. Case of estimated $\bar{z}$ with noise measurement proportional up to 5% of y.

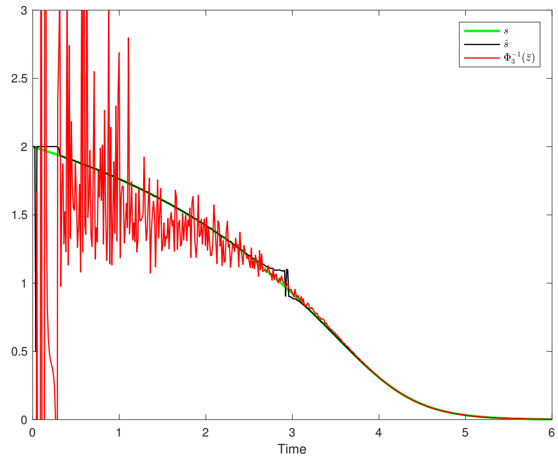

Fig. 6. The exact solution together with the constructed one (with $\varepsilon = 0.01$) and that obtained with Φ^{-1}. Case of estimated $\bar{z}$ with noise measurement proportional up to 0.001% of y and without any filtering strategy.

singularity observability problem appears in the state space, more precisely over the set $\{(y, s) \in \mathbb{R}^2_+ : s = \sqrt{k_2 k_3}\}$. However, the system is observable but with an injectivity index $m = 3$. This problem is treated in [20] with the usual immersion approach described in the introduction. In order to show the applicability of our method in this case, we repeat the same analysis as in Example 1, where the function r is given by (25), with $k_1 = k_2 = k_3 = 1$. By Figure 4 (right) we show the exact roots of (4) relative to the case of this example. By Figure 5 we show the different real and complex parts of the estimated roots together with the estimation of s where ε is fixed at 0.01, and a noise measurement proportional up to 5% of y is considered. The matrix $M = \text{diag}(1, 0.1)$ is chosen to define the test function in (5). It is worth noting that the estimators become quite sensitive to noise near the equilibrium point. This is explained by the fact that the exact derivatives become practically null close to the equilibrium point (this difficulty was already present with the approach proposed in [20]).

We end this section by underlying the advantage of the proposed approach to deal with singularities. For this, we compare the constructed solution together with the one obtained directly by inverting the observability map Φ. A straightforward computation gives

$$\Phi^{-1}(\bar{z}) = \left(\bar{z}_1, \frac{2h_1^3(\bar{z}) + h_1(\bar{z})h_2(\bar{z})}{(1 - h_1(\bar{z}))(h_1^2(\bar{z}) + h_2(\bar{z}))}\right)^T,$$

where

$$h_1(\bar{z}) = \frac{\bar{z}_2}{\bar{z}_1}, \quad h_2(\bar{z}) = \frac{\bar{z}_2^2 - \bar{z}_1\bar{z}_3}{\bar{z}_2\bar{z}_1^2}.$$

By Figure 6 we compare the solution constructed by our method, with $\varepsilon = 0.01$, together with that obtained by Φ^{-1}, in the case of estimated $\bar{z}$, with white noise proportional up to 0.001% of y and without any filtering strategy. We see clearly the performance of our approach in this case.

V. Conclusion

In this work we propose a new approach to deal with a family of observability problems which, in general, requires the construction of embedding in higher dimensional space. This approach does not require any change of coordinates, and consequently there is no need to inverse the observability map nor its Jacobian. This relies on the following idea: instead of single observer in higher dimension, a set of estimators can be constructed in the original space together with a test function, based on higher derivative of the observation, to discriminate between the different estimators at any time; the right one is the one minimizing the test function. The choice of the norm for the test function gives flexibility to suit to the nature of the noise in the case of noisy measurements. Although this study is reduced to a particular class of bidimensional systems, non-uniform observability encountered in some biological systems can be overcome by our approach. Questions about robustness in the coupling of a differentiator to estimate $z(\cdot)$ with the roots tracking method, which gives good results in numerical simulations, has not been addressed in this note by lack of room, but will be the matter of future investigations. Future works will also attempt to extend this work to more general class of higher dimensional autonomous systems.

References

[1] E. L. Allgower, and K. Georg, *Introduction to Numerical Continuation Methods*, Society for Industrial and Applied Mathematics, 2003.

[2] A. Astolfi, R. Ortega and A. Venkatraman, *A globally exponentially convergent immersion and invariance speed observer for mechanical systems with non-holonomic constraints*, Automatica, 46(1), 182-189, 2010.

[3] D. Astolfi and C. Possieri, *Design of local observers for autonomous nonlinear systems not in observability canonical form*, Automatica, 103, 443-449, 2019.

[4] J. Back and J. Seo, *An algorithm for system immersion into nonlinear observer form: SISO case*, Automatica, 42(2), 321-328, 2006.

[5] P. Bernard, V. Andrieu and L. Praly, *Expressing an observer in preferred coordinates by transforming an injective immersion into a surjective diffeomorphism*, SIAM J. Control Optim., 56(3), 2327-2352, 2018.

[6] P. Bernard, L. Praly, V. Andrieu and H. Hammouri, *On the triangular canonical form for uniformly observable controlled systems*, Automatica, 85, 293-300, 2017.

[7] G. Besancon and A. Ticlea, *An immersion-based observer design for rank-observable nonlinear Systems*, IEEE Transactions on Automatic Control, 52, 83-88, 2007.

[8] J.-P. Gauthier, H. Hammouri and I. Kupka, *Observers for nonlinear systems*, IEEE Conference on Decision and Control (CDC), Brighton, England, 1991.

[9] J.-P. Gauthier, H. Hammouri and S. Othman, *A simple observer for nonlinear systems: Applications to bioreactors*, IEEE Transactions on Automatic Control, 37(6), 875-880, 1992.

[10] J.-P. Gauthier and I. Kupka, *Deterministic observation theory and applications*. Cambridge University Press, 2001.

[11] I. Haidar, J.-P. Barbot, A. Rapaport and M. Ghanes, *Observability singularity of batch bioreactors: A solution based on high order sliding mode differentiator approach*, 11th Asian Control Conference (ASCC), 336-338, 2017.

[12] I. Haidar, J.-P. Barbot and A. Rapaport, *A multi observers approach when observability index is higher than the state dimension-a case study-*, 58th IEEE Conference on Decision and Control (CDC), 2019.

[13] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, 2002.

[14] J. Levine and R. Marino, *Nonlinear system immersion, observers and finite-dimensional filters*, Systems & Control Letters, 7(2), 133-142, 1986.

[15] K.A.A. Langueh, O. Datcu, J.-P. Barbot, G. Zheng and K. Busawon, *Observabilty singularities and observer design: dual immersion approach*, IFAC-PapersOnLine, 49(18), 511-516, 2016.

[16] M.-R. Mojallizadeh, B. Brogliato, A. Polyakov, S. Selvarajan, L. Michel, F. Plestan, M. Ghanes, J.-P. Barbot and Y. Aoustin, *Discrete-time differentiators in closed-loop control systems: experiments on electro-pneumatic system and rotary inverted pendulum*, Preprint hal-03125960.

[17] R. Naulin and C. Pabst, *The roots of a polynomial depend continuously on its coefficients*, Revista Colombiana de Matematicas, 28(1), 35-37, 1994.

[18] J. Pirt, *Principles of microbe and cell cultivation*. John Wiley and Sons, 1976.

[19] A. Rapaport and D. Dochain, *A robust asymptotic observer for systems that converge to unobservable states. A batch reactor case study*, IEEE Transactions on Automatic Control, 65(6), 2693-2699, 2020.

[20] A. Rapaport and M. Maloum, *Design of exponential observers for nonlinear systems by embedding*, International Journal of Robust and Nonlinear Control, 14, 273-288, 2004.

[21] G. Seber and C. Wild, *Nonlinear regression*, Wiley, 2003.

[22] A. Ticlea and G. Besancon, *Immersion-based observer design*, Nonlinear Observers and Applications, Lecture Notes in Control and Information Sciences 363, (Ed. G. Besancon), Springer, 2007.

[23] A. Tornambè, *High-gain observers for non-linear systems*, International Journal of Systems Science, 23(9), 1475-1489, 1992.

[24] X. Xia and M. Zeitz, *On nonlinear continuous observers*, International Journal of Control, 66(6), 943-954, 1997.