An unified framework for measuring pollution-adjusted productivity change

Arnaud Abad

An unified framework for measuring pollution-adjusted productivity change

A. Abad[⊥]

[⊥]University of Lorraine, University of Strasbourg, AgroParisTech, CNRS, INRAE, BETA, 54000 Nancy, France.

Abstract

This paper aims to define an unified framework to analyse pollution-adjusted productivity change. Equivalence conditions for the additive and the multiplicative pollution-adjusted productivity measures (Abad and Ravelojaona, 2022, 2021) are established.

Keywords: Environmental Distance Functions, Pollution-adjusted Productivity Indices, Non Convexity, Pollution-generating Technology.

JEL: D21, D24

1 Introduction

Recently, Abad and Ravelojaona (2021, 2022) define both additive and multiplicative pollution-adjusted productivity measures. The multiplicative pollution-adjusted productivity indices inherit the structure of the Malmquist (Caves et al., 1982) and the Hicks-Moorsteen (Bjurek, 1996) productivity indices. Besides, the additive pollution-adjusted productivity indicators take the form of the Luenberger (Chambers et al., 1996) and the Luenberger-Hicks-Moorsteen (Briec and Kerstens, 2004) productivity indicators. Abad and Ravelojaona (2022) are the first to empirically implement additive and multiplicative pollution-adjusted productivity measures through convex neutral pollution-generating production model (Abad, 2018).

This contribution establishes equivalence conditions for the additive and the multiplicative pollution-adjusted productivity measures. These equivalence conditions extend the usual approximation results that link additive and multiplicative productivity measures (Briec and Kerstens, 2004; Boussemart et al., 2003). The first outcome defines specific conditions for which the pollution-adjusted Malmquist and Luenberger productivity measures are equal. Second, this paper introduces theoretically relations between the pollution-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures. Interestingly, this contribution also permits to present additive version of the pollution-adjusted Malmquist and Hicks-Moorsteen productivity indices. This result extends the widely applied Chung et al. (1997) methodology. Moreover, multiplicative version of the pollution-adjusted Luenberger and Luenberger-Hicks-Moorsteen productivity indicators are provided.

The remainder of this paper unfolds as follows. Section 2 introduces some theoretical preliminaries. The pollution-generating production process and the environmental distance functions are presented in this section. Section 3 displays multiplicative and additive pollution-adjusted productivity measures and further provides equivalence conditions for the additive and the multiplicative pollution-adjusted productivity measures. Finally, section 4 concludes.

2 Background

In this section, the properties of the pollution-generating production process are presented. Based upon this theoretical background, additive and multiplicative distance functions are displayed, and further equivalence condition between the proposed distance functions is revealed.

2.1 Technology definition and properties

Let $x_{\mathtt{t}} = (x_{\mathtt{t}}^{\mathtt{np}}, x_{\mathtt{t}}^{\mathtt{p}}) \in \mathbb{R}_+^n$ denotes the no polluting factors and the polluting inputs used to produce both no polluting and polluting outputs $y_{\mathtt{t}} = (y_{\mathtt{t}}^{\mathtt{np}}, y_{\mathtt{t}}^{\mathtt{p}}) \in \mathbb{R}_+^m$, such that $[n] = [n_{\mathtt{np}}] + [n_{\mathtt{p}}]$ and $[m] = [m_{\mathtt{np}}] + [m_{\mathtt{p}}]$. In addition, assume that $[n] = \mathrm{Card}(x_{\mathtt{t}})$ and $[m] = \mathrm{Card}(y_{\mathtt{t}})$. The pollution-generating technology is defined as follows:

$$T(x_{\mathtt{t}}, y_{\mathtt{t}}) := \left\{(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}_+^{n+m} : x_{\mathtt{t}} \,\text{can produce}\, y_{\mathtt{t}}\right\} \tag{2.1}$$

Assume that the production process (2.1) satisfies the following usual assumptions (Färe et al, 1985): no free lunch and inaction (*T1*); boundedness (*T2*); closedness (*T3*). Moreover, suppose that the production set satisfies (*T4*) the generalised B-disposal property (Abad, 2018). The axiomatic framework *T1-T4* follows the by production model of Murty et al. (2012) by defining the production process as an intersection of sub-technologies. Interestingly, the properties *T1-T4* are fairly weak and do not impose any convexity assumption to define non convex by-production model (Yuan et al., 2021).

2.2 Environmental distance functions

In this section, additive and multiplicative distance functions are considered as functional representation of the production process (Chambers and Färe, 2020).

The next result presents the additive and the multiplicative environmental distance functions (Abad, 2018).

Definition 2.1 *Let $T(x_{\mathtt{t}}, y_{\mathtt{t}})$ be a production technology that satisfies properties T1-T4. For any $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}_+^{n+m}$,*

i. *The multiplicative environmental distance function is defined as follows,*

$$\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(x_{\mathtt{t}}, y_{\mathtt{t}}) := \inf_{\lambda} \left\{ \lambda \in]0,1] : \left(\lambda^{\alpha^{\mathtt{np}}} x_{\mathtt{t}}^{\mathtt{np}}, \lambda^{\alpha^{\mathtt{p}}} x_{\mathtt{t}}^{\mathtt{p}}, \lambda^{\beta^{\mathtt{np}}} y_{\mathtt{t}}^{\mathtt{np}}, \lambda^{\beta^{\mathtt{p}}} y_{\mathtt{t}}^{\mathtt{p}} \right) \in T(x_{\mathtt{t}}, y_{\mathtt{t}}) \right\} \tag{2.2}$$

where $\alpha = (\alpha^{\mathtt{np}}, \alpha^{\mathtt{p}})$ and $\beta = (\beta^{\mathtt{np}}, \beta^{\mathtt{p}})$, such that $\alpha^{\mathtt{p}} = \alpha^{\mathtt{np}} = \{0,1\}$, $\beta^{\mathtt{p}} = \{0,1\}$ and $\beta^{\mathtt{np}} = \{-1,0\}$.

ii. *The additive environmental distance function is defined as follows,*

$$\overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}(x_{\mathtt{t}}, y_{\mathtt{t}}) := \sup_{\delta} \left\{ \delta \geq 0 : \left((1-\delta\gamma^{\mathtt{np}}) x_{\mathtt{t}}^{\mathtt{np}}, (1-\delta\gamma^{\mathtt{p}}) x_{\mathtt{t}}^{\mathtt{p}}, (1+\delta\sigma^{\mathtt{np}}) y_{\mathtt{t}}^{\mathtt{np}}, (1+\delta\sigma^{\mathtt{p}}) y_{\mathtt{t}}^{\mathtt{p}} \right) \in T(x_{\mathtt{t}}, y_{\mathtt{t}}) \right\} \tag{2.3}$$

where $\gamma = (\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}})$ and $\sigma = (\sigma^{\mathtt{np}}, \gamma^{\mathtt{p}})$, such that $\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}} \in \mathbb{R}_+^n$, $\sigma^{\mathtt{np}} \in \mathbb{R}_+^{\mathtt{np}}$ and $\sigma^{\mathtt{p}} \in \mathbb{R}_-^{\mathtt{p}}$.

The additive (2.3) and the multiplicative (2.2) environmental distance functions fully characterise the production process such that:

$$\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(x_{\mathtt{t}}, y_{\mathtt{t}}) \in]0,1] \Leftrightarrow \overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}(x_{\mathtt{t}}, y_{\mathtt{t}}) \geq 0 \Leftrightarrow (x_{\mathtt{t}}, y_{\mathtt{t}}) \in T(x_{\mathtt{t}}, y_{\mathtt{t}}).$$

2.3 Environmental distance functions: equivalence condition

The next result defines equivalence condition for the additive and multiplicative environmental distance functions.

Proposition 2.2 *Let $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}_{++}^{n+m}$, equivalence condition for the additive and multiplicative environmental distance functions is defined as follows:*

$$\overrightarrow{\mathtt{D}}_{\mathtt{t}}^{\gamma;\sigma}(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}})) \equiv \mathtt{ln}\left(\mathtt{D}_{\mathtt{t}}^{\alpha;\beta}(x_{\mathtt{t}}, y_{\mathtt{t}})\right) \tag{2.4}$$

such that $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x_{\mathtt{t}}^{\mathtt{np}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x_{\mathtt{t}}^{\mathtt{p}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y_{\mathtt{t}}^{\mathtt{np}})}$ *and* $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y_{\mathtt{t}}^{\mathtt{p}})}$.

See Appendix I for the proof.

3 Pollution-adjusted productivity measures

This section displays equivalence condition for the additive and the multiplicative pollution-adjusted productivity measures (Abad and Ravelojaona, 2021, 2022).

3.1 Malmquist and Luenberger productivity measures: equivalence condition

The next result presents the pollution-adjusted Malmquist and Luenberger productivity measures (Abad and Ravelojaona, 2021).

Definition 3.1 *Assume that $T(x_{\mathtt{t}}, y_{\mathtt{t}})$ is a production process that satisfies properties T1-T4. For any $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{+}$,*

i. *The pollution-adjusted Malmquist productivity index is defined as follows,*

$$\mathtt{PM}^{\alpha;\beta}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}}) = \left[\frac{\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})}{\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \times \frac{\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \right. \tag{3.1}$$

$$\left. \times \frac{\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})} \times \frac{\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})}\right]^{\frac{1}{2}}$$

where $\alpha = (\alpha^{\mathtt{np}}, \alpha^{\mathtt{p}})$ and $\beta = (\beta^{\mathtt{np}}, \beta^{\mathtt{p}})$, such that $\alpha^{\mathtt{p}} = \alpha^{\mathtt{np}} = \{0, 1\}$, $\beta^{\mathtt{p}} = \{0, 1\}$ and $\beta^{\mathtt{np}} = \{-1, 0\}$.

ii. *The pollution-adjusted Luenberger productivity indicator is defined as follows,*

$$\mathtt{PL}^{\gamma;\sigma}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}}) = \frac{1}{2}\left[\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\right) \right. \tag{3.2}$$

$$- \left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right)$$

$$+ \left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})\right)$$

$$\left. - \left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right)\right]$$

where $\gamma = (\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}})$ and $\sigma = (\sigma^{\mathtt{np}}, \gamma^{\mathtt{p}})$, such that $\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}} \in \mathbb{R}^{n}_{+}$, $\sigma^{\mathtt{np}} \in \mathbb{R}^{\mathtt{np}}_{+}$ and $\sigma^{\mathtt{p}} \in \mathbb{R}^{\mathtt{p}}_{-}$.

The following proposition introduces equivalence condition for the pollution-adjusted Malmquist and Luenberger productivity measures.

Proposition 3.2 *Let $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$, equivalence condition for the pollution-adjusted Malmquist and Luenberger productivity measures is defined as follows:*

$$\mathtt{ln}\left(\mathtt{PM}^{\alpha;\beta}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}})\right) \equiv -\mathtt{PL}^{\gamma;\sigma}_{\mathtt{t,t+1}}\left(\mathtt{ln}(x_{\mathtt{t,t+1}}), \mathtt{ln}(y_{\mathtt{t,t+1}})\right), \tag{3.3}$$

such that $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}})}$ *and* $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}})}$.

In the next statement, additive (respectively, multiplicative) version of the pollution-adjusted Malmquist (respectively, Luenberger) productivity measure is proposed. The additive version of the pollution-adjusted Malmquist productivity index is defined through environmental additive distance functions. This productivity measure is named the pollution-adjusted Malmquist-Luenberger productivity index based on the initial work of Chung et al. (1997). Besides, the reciprocal pollution-adjusted Malmquist-Luenberger productivity measure proposes a multiplicative version of the pollution-adjusted Luenberger productivity indicator.

Corollary 3.3 *Let* $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$*, for any* $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}})}$ *and* $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}})}$*:*

i. *The pollution-adjusted Malmquist-Luenberger productivity index is defined as follows,*

$$
\begin{aligned}
&\mathtt{PML}^{\alpha;\beta}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}}) = \\
&\Bigg[\frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t}}(\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t+1}}), \mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}}), \mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t+1}}), \mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t}}(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t}}(\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t+1}}), \mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t}}(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}}))\right)} \\
&\times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(\mathtt{ln}(x_{\mathtt{t+1}}), \mathtt{ln}(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}};\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t+1}}), \mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t+1}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(\mathtt{ln}(x_{\mathtt{t+1}}), \mathtt{ln}(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}};\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t+1}}), \mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}}), \mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t+1}}), \mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}}))\right)}\Bigg]^{\frac{1}{2}} \\
&\mathtt{PML}^{\alpha;\beta}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}}) \equiv \left[\exp\left(\mathtt{PL}^{\gamma;\sigma}_{\mathtt{t,t+1}}\left(\mathtt{ln}(x_{\mathtt{t,t+1}}), \mathtt{ln}(y_{\mathtt{t,t+1}})\right)\right)\right]^{-1}
\end{aligned}
\tag{3.4}
$$

ii. *The reciprocal pollution-adjusted Malmquist-Luenberger productivity measure is defined as follows,*

$$
\begin{aligned}
&\overline{\mathtt{PML}}^{\gamma;\sigma}_{\mathtt{t,t+1}}\left(\mathtt{ln}(x_{\mathtt{t,t+1}}), \mathtt{ln}(y_{\mathtt{t,t+1}})\right) = \\
&\frac{1}{2}\Bigg[\left(\mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right) - \mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\right)\right) - \left(\mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})\right)\right. \\
&\quad \left.-\mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right)\right) + \left(\mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right) - \mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{np}};\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})\right)\right) \\
&\quad - \left(\mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\right) - \mathtt{ln}\left(\mathtt{D}^{\alpha^{\mathtt{p}};\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right)\right)\Bigg] \\
&\overline{\mathtt{PML}}^{\gamma;\sigma}_{\mathtt{t,t+1}}\left(\mathtt{ln}(x_{\mathtt{t,t+1}}), \mathtt{ln}(y_{\mathtt{t,t+1}})\right) \equiv -\mathtt{ln}\left(\mathtt{PM}^{\alpha;\beta}_{\mathtt{t,t+1}}(x_{\mathtt{t,t+1}}, y_{\mathtt{t,t+1}})\right)
\end{aligned}
\tag{3.5}
$$

3.2 Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures: equivalence condition

The upcoming statement displays the pollution-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures (Abad and Ravelojaona, 2022).

Definition 3.4 *Let* $T(x_{\mathtt{t}}, y_{\mathtt{t}})$ *be a production process that satisfies properties T1-T4. For any* $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{+}$*,*

i. *The pollution-adjusted Hicks-Moorsteen productivity index is defined as follows,*

$$\begin{aligned}\mathtt{PHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}}) = \Bigg[& \frac{\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})}{\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \times \frac{\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})}{\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \times \frac{\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t}})}{\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \\ & \times \frac{\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t}})}{\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})} \times \frac{\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})} \times \frac{\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})} \\ & \times \frac{\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t+1}})} \times \frac{\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})}{\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t+1}})} \Bigg]^{\frac{1}{2}}\end{aligned} \tag{3.6}$$

where $\alpha = (\alpha^{\mathtt{np}}, \alpha^{\mathtt{p}})$ *and* $\beta = (\beta^{\mathtt{np}}, \beta^{\mathtt{p}})$, *such that* $\alpha^{\mathtt{p}} = \alpha^{\mathtt{np}} = \{0, 1\}$, $\beta^{\mathtt{p}} = \{0, 1\}$ *and* $\beta^{\mathtt{np}} = \{-1, 0\}$.

ii. *The pollution-adjusted Luenberger-Hicks-Moorsteen productivity indicator is defined as follows,*

$$\begin{aligned}\mathtt{PLHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}}) = \frac{1}{2}\Bigg[& \Big(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\Big) - \Big(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\Big) \\ & - \Big(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\Big) - \Big(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\Big) \\ & + \Big(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\Big) - \Big(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}}) \\ & - \overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\Big) - \Big(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t+1}})\Big) \\ & - \Big(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}}) - \overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t+1}})\Big)\Bigg]\end{aligned} \tag{3.7}$$

where $\gamma = (\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}})$ *and* $\sigma = (\sigma^{\mathtt{np}}, \gamma^{\mathtt{p}})$, *such that* $\gamma^{\mathtt{np}}, \gamma^{\mathtt{p}} \in \mathbb{R}^n_+$, $\sigma^{\mathtt{np}} \in \mathbb{R}^{\mathtt{np}}_+$ *and* $\sigma^{\mathtt{p}} \in \mathbb{R}^{\mathtt{p}}_-$.

Equivalence condition for the pollution-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined in the next result.

Proposition 3.5 *Let* $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$, *equivalence condition for the pollution-adjusted Hicks-Moorsteen and Luenberger-Hicks-Moorsteen productivity measures is defined as follows:*

$$\mathtt{ln}\Big(\mathtt{PHM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}})\Big) \equiv -\mathtt{PLHM}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\Big(\mathtt{ln}(x_{\mathtt{t},\mathtt{t+1}}), \mathtt{ln}(y_{\mathtt{t},\mathtt{t+1}})\Big), \tag{3.8}$$

such that $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}})}$ *and* $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}})}$.

Additive version of the pollution-adjusted Hicks-Moorsteen productivity index and it reciprocal are presented in the statement below. These productivity measures are named the Hicks-Moorsteen-Luenberger index and the reciprocal Hicks-Moorsteen-Luenberger indicator, respectively.

Corollary 3.6 *Let* $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$, *for any* $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}})}$ *and* $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}})}$*:*

i. *The pollution-adjusted Hicks-Moorsteen-Luenberger productivity index is defined as follows,*

$$
\begin{aligned}
&\mathtt{PHML}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}}) = \\
&\Bigg[\frac{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}, \ln(y^{\mathtt{np}}_{\mathtt{t+1}}), \ln(y^{\mathtt{p}}_{\mathtt{t}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}), \ln(y_{\mathtt{t}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}, \ln(y^{\mathtt{np}}_{\mathtt{t}}), \ln(y^{\mathtt{p}}_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}), \ln(y_{\mathtt{t}}))\right)} \\
&\quad\times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t}}(\ln(x^{\mathtt{np}}_{\mathtt{t+1}}), \ln(x^{\mathtt{p}}_{\mathtt{t}}), \ln(y_{\mathtt{t}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}), \ln(y_{\mathtt{t}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t}}(\ln(x^{\mathtt{np}}_{\mathtt{t}}), \ln(x^{\mathtt{p}}_{\mathtt{t+1}}), \ln(y_{\mathtt{t}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t}}(\ln(x_{\mathtt{t}}), \ln(y_{\mathtt{t}}))\right)} \\
&\quad\times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}), \ln(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{np}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}, \ln(y^{\mathtt{np}}_{\mathtt{t}}), \ln(y^{\mathtt{p}}_{\mathtt{t+1}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}), \ln(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\sigma^{\mathtt{p}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}, \ln(y^{\mathtt{np}}_{\mathtt{t+1}}), \ln(y^{\mathtt{p}}_{\mathtt{t}}))\right)} \\
&\quad\times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}), \ln(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{np}}}_{\mathtt{t+1}}(\ln(x^{\mathtt{np}}_{\mathtt{t}}), \ln(x^{\mathtt{p}}_{\mathtt{t+1}}), \ln(y_{\mathtt{t+1}}))\right)} \times \frac{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t+1}}(\ln(x_{\mathtt{t+1}}), \ln(y_{\mathtt{t+1}}))\right)}{\exp\left(\overrightarrow{\mathtt{D}}^{\gamma^{\mathtt{p}}}_{\mathtt{t+1}}(\ln(x^{\mathtt{np}}_{\mathtt{t+1}}), \ln(x^{\mathtt{p}}_{\mathtt{t}}), \ln(y_{\mathtt{t+1}}))\right)}\Bigg]^{\frac{1}{2}} \\
&\mathtt{PHML}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}}) \equiv \left[\exp\left(\mathtt{PL}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\left(\ln(x_{\mathtt{t},\mathtt{t+1}}), \ln(y_{\mathtt{t},\mathtt{t+1}})\right)\right)\right]^{-1}
\end{aligned}
\tag{3.9}
$$

ii. *The reciprocal pollution-adjusted Hicks-Moorsteen-Luenberger productivity indicator is defined as follows,*

$$
\begin{aligned}
&\overline{\mathtt{PHML}}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\left(\ln(x_{\mathtt{t},\mathtt{t+1}}), \ln(y_{\mathtt{t},\mathtt{t+1}})\right) = \\
&\frac{1}{2}\Bigg[\left(\ln\left(\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\right)\right) - \left(\ln\left(\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right)\right) \\
&\quad - \left(\ln\left(\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right)\right) - \left(\ln\left(\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t}})\right) - \ln\left(\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\right)\right) \\
&\quad + \left(\ln\left(\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t}}, y^{\mathtt{p}}_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\beta^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right)\right) - \left(\ln\left(\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right)\right. \\
&\quad \left. - \ln\left(\mathtt{D}^{\beta^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y^{\mathtt{np}}_{\mathtt{t+1}}, y^{\mathtt{p}}_{\mathtt{t}})\right)\right) - \left(\ln\left(\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\alpha^{\mathtt{np}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t}}, x^{\mathtt{p}}_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right)\right) \\
&\quad - \left(\ln\left(\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t+1}}(x_{\mathtt{t+1}}, y_{\mathtt{t+1}})\right) - \ln\left(\mathtt{D}^{\alpha^{\mathtt{p}}}_{\mathtt{t+1}}(x^{\mathtt{np}}_{\mathtt{t+1}}, x^{\mathtt{p}}_{\mathtt{t}}, y_{\mathtt{t+1}})\right)\right)\Bigg] \\
&\overline{\mathtt{PHML}}^{\gamma;\sigma}_{\mathtt{t},\mathtt{t+1}}\left(\ln(x_{\mathtt{t},\mathtt{t+1}}), \ln(y_{\mathtt{t},\mathtt{t+1}})\right) \equiv -\ln\left(\mathtt{PM}^{\alpha;\beta}_{\mathtt{t},\mathtt{t+1}}(x_{\mathtt{t},\mathtt{t+1}}, y_{\mathtt{t},\mathtt{t+1}})\right)
\end{aligned}
\tag{3.10}
$$

4 Concluding Comments

This contribution presents equivalence conditions for the additive and multiplicative pollution-adjusted productivity measures. Therefore, an unified framework allowing to compare additive and multiplicative pollution-adjusted productivity measures is provided.

In further work, theoretical comparisons between the additive and the multiplicative pollution-adjusted productivity measures could be empirically illustrated. The empirical outcomes could be estimated within both convex and non convex pollution-generating production models considering either parametric or non parametric approaches.

References

[1] Abad, A., P. Ravelojaona (2022) A Generalization of Environmental Productivity Analysis, *Journal of Productivity Analysis*, 57, 61-78.

[2] Abad, A., P. Ravelojaona (2021) Pollution-adjusted Productivity Analysis: The Use of Malmquist and Luenberger Productivity Measures, *Managerial and Decision Economics*, 42(3), 635-648.

[3] Abad, A. (2018) *Les Enseignements de la Micro-économie de la Production face aux Enjeux Environnementaux: Etude des Productions Jointes. Théorie et Applications*, Ph.D dissertation, University of Perpignan.

[4] Bjurek, H. (1996) The Malmquist Total Factor Productivity Index, *Scandinavian Journal of Economics*, 98, 303-313.

[5] Boussemart, J.P., W. Briec, K. Kerstens, J.-C. Poutineau (2003) Luenberger and Malmquist Productivity Indices: Theoretical Comparisons and Empirical Illustration, *Bulletin of Economic Research*, 55(4), 391-405.

[6] Briec, W., K. Kerstens (2004) A Luenberger-Hicks-Moorsteen Productivity Indicator: Its Relation to the Hicks-Moorsteen Productivity Index and theLuenberger Productivity Indicator, *Economic Theory*, 23(4), 925-939.

[7] Caves, D.W., L.R. Christensen, W.E. Diewert (1982) The Economic Theory of Index Numbers and the Measurement of Inputs, Outputs and Productivity, *Econometrica*, 50, 1393-1414.

[8] Chambers, R.G., R. Färe (2020) Distance Functions in Production Economics, in Ray S.C., Chambers R., Kumbhakar S. (eds) *Handbook of Production Economics*, Springer, Singapore.

[9] Chambers, R.G., Färe, R., S. Grosskopf (1996) Productivity Growth in APEC Countries, *Pacific Economic Review*, 1, 181-190.

[10] Chung, Y.H., Färe, R., S. Grosskopf (1997) Productivity and undesirable outputs: A directional distance function approach, *Journal of Environmental Management*, 51, 229-240.

[11] Färe, R., Grosskopf, S., C.A.K. Lovell (1985) *The Measurement of Efficiency of Production*, Springer.

[12] Murty, S. , R. R. Russell, S. B. Levkoff (2012) On Modeling Pollution-Generating Technologies, *Journal of Environmental Economics and Management*, 64, 117-135.

[13] Yuan, Q., Balezentis, T., Shen, Z., D. Streimikiene (2021) Economic and environmental performance of the belt and road countries under convex and nonconvex production technologies, *Journal of Asian Economics*, 75, 101321.

Appendix I

Proof of Proposition 2.2: For any $(x_{\mathtt{t}}, y_{\mathtt{t}}) \in \mathbb{R}^{n+m}_{++}$, the environmental additive distance function (4.2) is defined as follows,

$$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}}) := \sup_{\delta} \left\{ \delta \geq 0 : \left((1-\delta\gamma^{\mathtt{np}})x^{\mathtt{np}}_{\mathtt{t}}, (1-\delta\gamma^{\mathtt{p}})x^{\mathtt{p}}_{\mathtt{t}}, (1+\delta\sigma^{\mathtt{np}})y^{\mathtt{np}}_{\mathtt{t}}, (1+\delta\sigma^{\mathtt{p}})y^{\mathtt{p}}_{\mathtt{t}} \right) \in T(x_{\mathtt{t}}, y_{\mathtt{t}}) \right\} \quad (4.1)$$

Therefore,

$$\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}})) := \sup_{\delta} \left\{ \delta \geq 0 : \left((1-\delta\gamma^{\mathtt{np}})\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}}), (1-\delta\gamma^{\mathtt{p}})\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}}), (1+\delta\sigma^{\mathtt{np}})\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}}), (1+\delta\sigma^{\mathtt{p}})\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}}) \right) \in T(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}})) \right\} \quad (4.2)$$

Following the definition of the environmental multiplicative efficiency measure (2.2),

$$\begin{aligned} \mathtt{ln}\big(\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\big) &:= \inf_{\lambda} \left\{ \lambda \in]0,1] : \left(\mathtt{ln}(\lambda^{\alpha^{\mathtt{np}}} x^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(\lambda^{\alpha^{\mathtt{p}}} x^{\mathtt{p}}_{\mathtt{t}}), \mathtt{ln}(\lambda^{\beta^{\mathtt{np}}} y^{\mathtt{np}}_{\mathtt{t}}), \mathtt{ln}(\lambda^{\beta^{p}} y^{p}_{\mathtt{t}}) \right) \in \mathtt{ln}\left(T(x_{\mathtt{t}}, y_{\mathtt{t}})\right) \right\} \\ &:= \inf_{\lambda} \left\{ \lambda \in]0,1] : \left(\mathtt{ln}(x^{np}_{\mathtt{t}}) + \alpha^{np}\mathtt{ln}(\lambda), \mathtt{ln}(x^{p}_{\mathtt{t}}) + \alpha^{p}\mathtt{ln}(\lambda), \mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}}) + \beta^{\mathtt{np}}\mathtt{ln}(\lambda), \mathtt{ln}(y^{p}_{\mathtt{t}}) + \beta^{p}\mathtt{ln}(\lambda) \right) \in \mathtt{ln}\left(T(x_{\mathtt{t}}, y_{\mathtt{t}})\right) \right\}. \end{aligned} \quad (4.3)$$

Hence, $\overrightarrow{\mathtt{D}}^{\gamma;\sigma}_{\mathtt{t}}(\mathtt{ln}(x_{\mathtt{t}}), \mathtt{ln}(y_{\mathtt{t}})) \equiv \mathtt{ln}\big(\mathtt{D}^{\alpha;\beta}_{\mathtt{t}}(x_{\mathtt{t}}, y_{\mathtt{t}})\big)$ with $\gamma^{\mathtt{np}} = \dfrac{\alpha^{\mathtt{np}}}{\mathtt{ln}(x^{\mathtt{np}}_{\mathtt{t}})}$, $\gamma^{\mathtt{p}} = \dfrac{\alpha^{\mathtt{p}}}{\mathtt{ln}(x^{\mathtt{p}}_{\mathtt{t}})}$, $\sigma^{\mathtt{np}} = -\dfrac{\beta^{\mathtt{np}}}{\mathtt{ln}(y^{\mathtt{np}}_{\mathtt{t}})}$ and $\sigma^{\mathtt{p}} = -\dfrac{\beta^{\mathtt{p}}}{\mathtt{ln}(y^{\mathtt{p}}_{\mathtt{t}})}$.$\square$