# Positional effects in public good provision. Strategic interaction and inertia

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# Positional effects in public good provision. Strategic interaction and inertia

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**Abstract**

Consumption satisfaction depends on other factors apart from the inherent characteristic of commodities. Among them, positional concerns are central in behavioural economics. Individuals enjoy returns from the ranking occupied by the consumed item. In public good, agents obtain satisfaction from their relative contribution. We analyse how positional preferences for voluntary contribution to a public good favour players' contributions and could lead to social welfare improvements. A two-player public good game is analysed, first a one-shot game and later a simple dynamic game with inertia. Homogeneous and non-homogeneous individuals are considered and particular attention is given to the transition path.

**Keywords:** Public good, positional concerns, inertia, static and dynamic game.
**JEL code**: H41, D91, C72, C73.

## 1 Introduction

> Wealth: any income that is at least one hundred dollars more a year than the income of one's wife's sister's husband (H. L. Mencken).

One of the key elements on the quest to acknowledge fundamental aspects of human nature, previously neglected by the traditional economic theory, is the concern about status or the positional concerns.[1] For positional goods, consumer's utility depends not only on the absolute quantity consumed, but also on how this quantity compares with the quantities consumed by others. Due to this concern about status, each agent's effort to climb in the social ladder imposes a negative externality on all other agents. This can lead to "Red Queen" effects:[2] When all agents embark on a race on conspicuous consumption of a private good to signal wealth, the literature on positionality (commented in the following section) predicts

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[1]The status effect is a particular positional concern (see Wendner and Goulder 2008). The latter is a broader concept, according to which an agent is concerned about her relative economic status relative to the status of others, while the former focuses specifically on consumption. Nevertheless, we will use the terms status concern and positional concern indistinctly.

[2]This idea is taken from the Lewis Carrol book's *Through the Looking-Glass*. It takes all the running you can do to keep in the same place.

that too much is spent to maintain social status. The result is an inefficient situation and a loss in social welfare.

The contest for positional status can have a different effect when the positional good is represented by the private provision of a public good. In their quest for status, agents might be willing to contribute. The negative consumption externality persists for this type of good. However, the public good constitution represents a positive externality on all other agents, possibly enhancing social welfare. The main objective of the paper is twofold. On the one hand, we characterize the conditions under which positional concerns can lead standard selfish agents to contribute in a public good. On the other hand, we seek to understand under which assumptions the positive externality associated with the constitution of a public good exceeds the negative consumption externality. To address this question three different dimensions are analyzed. First, we study the effect of the positive contributions associated with positional concerns on the social welfare void of positional payoffs, denoted as intrinsic utility. Next, we study their effect on a broader measure of social welfare that introduces positional payoffs. Finally, we study how contributions and welfare evolve through time when agents take decisions strategically, but also based on theirs and other agents' past decisions.

When a two-player one-shot game is considered to analyze voluntary contributions to a public good, the free-rider problem involved typically leads to under-provision (or no provision at all). The public good provision is socially but not individually desirable. Our approach defines agent's preferences taking into account, not only the private cost she incurs and the benefits she obtains from the public good, but also that her utility depends on how her contribution relates to the contribution of her opponent. If the joy she obtains from contributing more than others is large enough, then the marginal private cost can be outweighed by the addition of her marginal private benefit plus her positional payoffs. Under this condition, the agent will be willing to privately provide some private good. Each player's contribution will be the strategic equilibrium in a game where players differ in their endowments, marginal benefits from public good consumption and status concerns. The amount of public good wished by each player differs depending on their utility from public good consumption and their positional concerns. If both players positional concerns are large, a full contribution equilibrium is possible. Otherwise, and if both players wish for the same amount of public good, then a continuum of equilibria exist where the contributions of the two agents add up to this amount. If they do not wish for the same amount three other equilibria are possible. The player who is more willing to contribute, contributes her wished amount of public good or her total endowment. In this last case, the other agent either contributes nothing or what is missing to constitute her wished amount of public good.

Positive contributions generate a public good which differently impacts the intrinsic utility that players obtain from absolute consumption. Their private costs differ because they typically contribute different amounts. Moreover, although due to non-excludability and non-rivalry both players have access to the same amount of public good, they do not equally enjoy it. On aggregate, the benefits from public good consumption exceed private costs, at least for small contributions. However, under the assumption of a diminishing marginal utility from public good consumption, social welfare gains slow down as the amount of public good rises. Indeed too much contribution can lead to a situation with less social welfare than in the case with no public good. We characterize the conditions for which the positive contributions enabled by positional concerns lead to an increment or a reduction in aggregate intrinsic utility. With low satiation from public good consumption, positional concerns enhance intrinsic utility almost everywhere, except when the two players show very high positional concerns.

By contrast, when agents satiate fast, positional concerns reduce intrinsic social welfare when the concern for status is strong for at least one player.

Social welfare is not only given by the intrinsic utility from private and public good absolute consumption. Agents also get utility from relative consumption. Thus social welfare encompasses the intrinsic aggregate utility plus the positional payoffs associated with the status concern. For this broader measure, and regardless of the speed of satiation from public good consumption, positional concerns typically increase social welfare. Social welfare only worsens when both agents' positional concerns are very large (assuming that the agent who values public good consumption the most also wishes the highest amount of public good), or when both agents' positional concerns are close to one another (assuming that the agent who wishes the highest amount of public good values public good consumption the least).

The static analysis is extended to a dynamic framework by virtue of two ideas. On the one hand, the first idea gives entrance to inertia, considering agents that are reluctant to modify past decisions. Changes in contribution decisions negatively affect utility. The idea of inertia can be related to the *status quo bias* in decision making explained by Samuelson and Zerkhauser (1988). In particular, when the default option is represented by one's previous action.[3] The *status quo bias* has been empirically analyzed in the literature (the more characteristic example is given by the agents' inclination to stick to their default option in saving for retirement), see Liu and Riyanto (2017) and references therein. In particular, these authors test the robustness of this hypothesis for public good games and find evidence of partial stickiness with respect to the default options. On the other hand, the second idea assumes that, when dealing with the strategic decision to constitute the public good, an agent bases her reaction on her opponent's current action, however, the status concern is built looking at her opponent's previous action.[4] This idea is deeply connected with the literature on conditional cooperation that makes cooperative decisions dependent on reciprocal behavior (see Figuières *et al.* 2011 and references therein). Although agents do not present status concern, they base contribution decisions on others' previously observed contribution (action-based reciprocity), or on beliefs about others' contributions (belief-based reciprocity). Our assumption aligns with the first option, although the second option is also explored in the appendix . The dynamic game generated by these two ideas is played by myopic agents. Although they take into account theirs and their opponents' past decisions, they disregard how current decisions affect the future choice set and/or utility. The dynamic equilibrium opens up the possibility for a interior solution, although in the long run, the process converges towards the static equilibrium. We characterize the equilibrium paths of individual and total contributions. In particular we prove that the trajectory eventually hits the boundary of feasible contributions (except, possibly, when both players wish for the same amount of public good) and converges henceforth monotonously to the static equilibrium.

A numerical analysis is carried out for the trajectories converging on each of the three type of Nash equilibria in the long-run. Interestingly, along the transitory path a hump-shaped contribution curve can occur for the player who wishes and contributes the least in the long run. Starting from a zero (or small) contribution, if she has less inertia than her opponent (adjusts faster), she will fast rise her contribution to constitute her desired public

[3]See, also, Kahneman *et al.* (1991), who explain *status quo* bias for loss adverse individuals, who perceive switching from the *status quo* as a loss.

[4]The idea of modeling interdependent preferences as dependent on what others consumed in the past (lagged interdependence) is presented in Pollak (1976). He based this hypothesis in the idea that the acquisition of preferences is part of the process of socialization.

good. However, these contributions pay less and less as the opponent more slowly increases her own contributed amount. When too much public good is provided, the agent will save costs by reducing her contribution, even at the expense of positional losses. If the overshooting behavior by this player is strong enough, then also this hump-shaped pattern is shown by the total contribution.

The trajectories for the intrinsic utility and for the social welfare are also analyzed. With no overshooting, intrinsic utility converges towards its long-run value from above (below) depending on whether the contribution in the long-run equilibrium is too small (large) with respect to its value in the social optimum. If overshooting occurs, the convergence from above reverses to convergence from below and vice versa. The social welfare lies above (below) its long-run value without (with) overshooting at almost every time, except possibly within a first sub-interval.

In the next section we present a brief review of the literature on the concern for status. Section 3 describes the conditions that characterize a public good game and it distinguishes between the intrinsic utility and the positional payoffs. The Nash equilibria are computed in Section 4. This section also describes how the intrinsic and the global utility are affected by the positive contributions to the public good that derives from the agents' concern for status. The dynamic extension is presented in Section 5, and Section 6 illustrates numerically the dynamic trajectories for different equilibria. Finally, Section 7 concludes.

## 2 What the literature says and how it relates to our assumptions and results

The standard assumption in economic theory that "each individual's preferences are independent of the behavior of other individuals" was called into question already in the mid twentieth century by Duesenberry (1949).[5] He argued that psychological and sociological reasons support the opposite view that preferences are interdependent. Duesenberry's relative income hypothesis states that an individual, in her search for status, seeks to signal wealth through her consumption decisions. Consumption (and saving) decisions depend on her relative income relative to the income of other agents. Individual status is thus measured through relative income. One year later, Leibenstein (1950) developed a theory of consumption behavior where income comparison is substituted by more subjective beliefs: how the agent's consumption depends on what she believes other agents are doing. Three situations are possible when social influences are considered. The bandwagon effect or herding occurs when the agent seeks to follow the consumption behavior of others; the snob effect, contrarily, refers to the agent's desire for exclusiveness; finally, the Veblen effect is the desire to buy expensive goods to signal wealth (conspicuous consumption). This theory does not necessarily disagree with the idea of rational agents in pursuit of their self-interest. However, it was generally ignored by the mainstream economic theory for most of the second half of this century. The interest for this type of ideas increased by the late $20^{th}$ and the early $21^{st}$ century, with works like Frank (1985) or Hopkins and Korninenko (2004) and the emergence of the behavioral economics.

[5]Previously, the influence of social context had been taken into account by Veblen (1909). His hypothesis, denoted in the literature as absolute status concern, suggested that individuals spend income in conspicuous good to signal wealth. However, agents do not worry on how their conspicuous consumption compare to that of others.

The role of positional concerns or interpersonal comparison has been widely studied in reference to people's income. How important is absolute income versus relative income? How important is the relative component to define poverty? The importance of relative standing has been also analyzed for conspicuous consumption of positional goods. An important recurrent question analyzed in the literature is the actual influence of positional concerns on income allocation or agents' spending behavior. The negative consumption externality associated with status concerns can lead to a race in wasteful efforts and too much consumption of the positional good, for example, too much work (Fisher and Holf, 2000), too much extraction in a common pool resource (see Benchekroun and Long, 2016), etc. Thus, this externality makes the Pareto efficiency of the competitive equilibrium unfeasible and hence can be the cause of reductions in agents' welfare (see, for example, Long and Wang, 2009 or Frank 2005).

We analyze whether status concerns can have a positive impact when dealing with the private provision of a public good. In this regard, one should first analyze whether positionality is strong for this type of goods. In fact, Galbraith (1958) suggested a weak positionality for public goods because "emulation operates mainly on behalf of privately produced goods and services". This is empirically refuted by Solnick and Hemenway (2005), who studied the strength of the positional concerns for different kind of goods and found that agents are more positional for public good than for private goods. While they focused on publicly provided public good (like national defense or space exploration), we center our attention on privately provided public good, for which we believe positional concerns should be even stronger. This idea that positional concerns can induce agents to privately provide a public good is supported by experimental studies (see, for example, references in Muñoz-García 2011).

The effect of the status concern on optimal taxes and redistributive taxation is studied, for example, in Ljungqvist and Uhlig (2000) and Boskin and Sheshinsky (1978). More specifically, its effect on the excess burden of taxation and the optimal provision of the public good is analyzed by Wendner and Goulder (2008). They focused on the government provision of the public good, while we center our attention on its private provision. We define a public good game where agents are endowed with a given amount of wealth that can be privately consumed or contributed to generate a public good. Endowments do not need to be exclusively defined in units of wealth, but can be defined in units of available time or effort. An example of this type of good is the compliance with environmental norms (or, in general, social norms) which provides the public good of a cleaner environment (incurring a private cost in terms of wealth, time or effort) and can be strongly motivated by the belief about the other individuals' behavior (see Nyborg *et al.* 2006, who also introduce moral motivations).

The theoretical literature regarding the role of status concern as an incentive on the private provision of a public good is scarce. Muñoz-García (2011) analyzes this type of problem and following his approach, we assume that utility depends linearly on the consumption of a private good, while diminishing marginal utility is associated with public good consumption. We deviate from his assumption of non-separability between the public good consumption and the positional payoffs associated with an agent's relative contribution. Moreover, while he compares simultaneous versus sequential modes of play, we start with a one-shot game and later introduce a dynamic dimension stemming from the assumptions of inertia and positional concerns based on the comparison with the other agent's past action. Finally, he focuses on contributions but disregards welfare implications.

The competition for social status and its effect on the private provision of a public good as well as on social welfare is explored in Bougherara *et al.* (2019). We extend and more systematically analyze these two questions. In our formulation, the marginal utility from public good consumption is not only decreasing, but can even turn negative for a sufficiently large amount of public good. To compute the welfare implications of the status concern, we distinguish between it effect on the intrinsic utility, net of positional payoffs, and it effect on the aggregate social welfare. Likewise as Bougherara *et al.* (2019) we study the effect of a rise in all agents' positional concerns, but also the cases when only the positional concern of one agent changes.

# 3 Public good game

This section describes a static public good game between two individuals, $i \in \{1, 2\}$, focusing on their preferences. Each player is endowed with $w_i$ and contributes $x_i \in [0, w_i]$ to a public good. Thus $w_i - x_i$ is the numeraire which can be consumed in any other private good.

To characterize preferences, we distinguish the inner or intrinsic utility from the absolute level of consumption and the utility associated with relative consumption, that introduces the positional concerns. The former represents the utility an agent gets from the private and public good consumption, $u_i(x_i, X)$, with $X = x_i + x_j$. The latter refers to the effect on preferences of one player's relative consumption in comparison with the consumption of his opponent:

$$U_i(x_i, x_j) = u_i(x_i, X) + V_i(x_i - x_j). \tag{1}$$

The global utility or social welfare is defined as the addition: $U(x_i, x_j) = U_i(x_i, x_j) + U_j(x_i, x_j)$. It includes the intrinsic social welfare plus the positional payoffs for all players.

## 3.1 Intrinsic utility

The total intrinsic utility can be defined as $u(x_i, x_j) = u_i(x_i, X) + u_j(x_j, X)$. It is assumed that functions $u_i$ and $u_j$, hence $u$, are twice differentiable. Then, a public good game needs to satisfy the three following conditions.

C1 Individual provision always reduces own welfare:

$$\frac{\partial u_i}{\partial x_i}(x_i, X) < 0, \quad \forall (x_i, x_j) \in [0, w_i] \times [0, w_j].$$

C2 Individual provisions increase social welfare initially, but reduce it when the whole endowments are contributed:

$$C2a: \frac{\partial u}{\partial x_i}(0, 0) > 0, \qquad C2b: \frac{\partial u}{\partial x_i}(w_1, w_2) < 0.$$

Condition C2a) states that some contribution to the public good is socially desirable; while condition C2b) states that also some private consumption by each agent is socially desirable.

C3 Agents' contributions are subsitutes: an increment in $x_j$ reduces the marginal utility of $x_i$. Since a rise in $x_j$ implies a one-to-one increment in $X$, this condition can be stated as:

$$\frac{\partial^2 u_i}{\partial x_i \partial X}(x_i, X) < 0.$$

In this paper, the intrinsic utility is defined as an additively separable function:

$$u_i(x_i, X) = w_i - x_i + b_i(X). \tag{2}$$

Intrinsic utility for player $i$ comes from public good consumption, $b_i(X)$, and from the consumption of private goods. It is assumed that the contribution to the public good reduces the available amount that can be privately consumed. Because $w_i - x_i$ can be utilized in alternative private goods, we assume that marginal decaying utility has little effect in this part and approximate utility from this remaining endowment as a linear (one-to-one) function. Public good consumption increases utility (at least initially) at a decreasing rate, $b_i'(0) > 0$ and $b_i''(X) < 0$. A concave function $b_i$ is consistent with Assumption C3. This assumption is compatible with two possibilities. A positive marginal utility for any amount of public good, $b_i'(w_1 + w_2) \geq 0$ or an inverse-U-shaped function $b_i$, that reaches its maximum at some public good contribution $X_i^b$. Additional contributions above this quantity, directly reduce utility, $b_i'(X) < 0$, for $X > X_i^b$. Therefore, above this quantity the public good (PG, $b_i'(X) > 0$) turns into a public bad (PB, $b_i'(X) < 0$ ) for player $i$. Because utility linearly decreases with the private provision of the public good, one would typically expect that the amount of public good provided lays below $X_i^b$.

If one considers a non-cooperative game where players preferences are described by the intrinsic utility in (2), player $i$'s marginal utility would read:

$$\frac{\partial u_i}{\partial x_i} = -1 + b_i'(X).$$

From assumption C1, $-1 + b_i'(X) < 0$ for all $0 \leq X \leq w_1 + w_2$. The direct marginal cost from the private provision of a public good always exceeds the utility gains from the public good consumption. Consequently, player $i$ has no incentive to contribute.

On the other hand, the social optimum is obtained when one player's marginal disutility from private provision equates the marginal utility for the whole society from public good consumption. Since the intrinsic social welfare has been defined as the addition of the intrinsic utilities of the two players, it can be written as a function of the public good only:

$$u(X) = w_1 + w_2 - X + b_1(X) + b_2(X).$$

This function is concave since both $b_i$ and $b_j$ are concave. Under Assumption C2, it reaches its maximum at a public good contribution satisfying $X^{SO} < w_1 + w_2$. Moreover, a positive contribution, $X^{E0}$, can exist, for which $u(0) = w_1 + w_2 = u(X^{E0})$ (see Figure 1). Thus, when compared against the intrinsic social optimum, contributions can be in shortage $X < X^{SO}$, in excess $X > X^{SO}$, or at its efficient level, $X = X^{SO}$. Likewise, in comparison with the zero contribution case, positive contributions satisfying $X \in (0, X^{E0})$ are intrinsic welfare improving (IW-I), although too large contributions, $X > X^{E0}$, are intrinsic welfare reducing (IW-R).

### 3.2 Positional concerns

For a positional agent the second component in the agents' preferences in (1) are the positional concerns. It is assumed that the utility of agent $i$ increases linearly with his over-contribution above the contribution of the other player:

$$V_i(x_i - x_j) = v_i^{\mathrm{P}}(x_i - x_j). \tag{3}$$

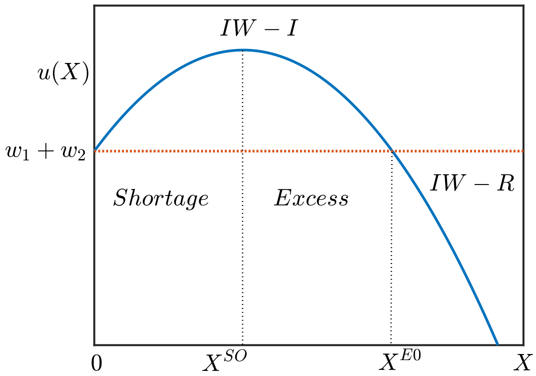


Figure 1: Global intrinsic utility as function of the public good

Parameter $v_i^{\mathrm{P}} \geq 0$ represents the marginal utility player $i$ gets from rising his contribution above that of player $j$, and denotes the positional concern of player $i$.

Adding the positional payoffs to the intrinsic utility, the problem for player $i \in \{1,2\}$ is:

$$\begin{aligned} &\max_{x_i} U_i(x_i, x_j) \\ &\text{s.t.: } 0 \leq x_i \leq w_i. \end{aligned} \tag{4}$$

The marginal utility of player $i$ now reads:

$$\frac{\partial U_i}{\partial x_i} = -1 + v_i^{\mathrm{P}} + b_i'(X). \tag{5}$$

Positionality reduces the marginal cost of private provision to $1 - v_i^{\mathrm{P}}$. Depending on the position of $b_i'(X)$ with respect to this value, a private provision of the public good can become individually rational for player $i$. We characterize the Nash equilibria of this game in the following section.

# 4 Nash Equilibria

In what follows, the Nash equilibria of the game (4) is characterized in Proposition 1. Section 4.1 also discusses the contributions of each player under a specific functional form for $b_i(X)$. Considering this particular functional form, a welfare analysis is performed in Section 4.2.

## 4.1 Contributions to the public good

The FOC of the problem (4) for player $i$ can be written as equation $b_i'(X) = 1 - v_i^{\mathrm{P}}$. Given that function $b_i'(X)$ is monotone and strictly decreasing for any $i \in \{1,2\}$, this equation has at most one solution. Therefore, the wished "uncoordinated" amount of public good for player $i$ can be written as follows.

**Definition 1 (Wished amount)** *The* wished amount *of Player $i$ is defined, for each $i \in \{1,2\}$, as:*

$$A_i = \begin{cases} 0 & \text{if } v_i^P \leq 1 - b_i'(0), \\ (b_i')^{-1}(1 - v_i^P) & \text{if } v_i^P \in (1 - b_i'(0), 1 - b_i'(w_1 + w_2)), \\ w_i + w_j & \text{if } v_i^P \geq 1 - b_i'(w_1 + w_2). \end{cases}$$

Observe that with this definition, the wished amount of Player $i$ may differ from the solution $(b_i')^{-1}(1 - v_i^{\mathrm{P}})$ of the first-order condition. When this value is negative, the player wishes for 0. When it is larger than the total endowment of both players, she wishes for this total wealth.

Note also that since $b_i$ is concave the wished amount increases with positionality, strictly in the interior case $A_i = (b_i')^{-1}(1 - v_i^{\mathrm{P}})$ since $\partial A_i / \partial v_i^{\mathrm{P}} > 0$. If $0 \leq v_{\mathrm{P}}^i < 1 - b_i'(0)$, the marginal cost from private provision is always greater than the marginal utility of the public good, and hence, no public good would be privately provided. However, if $v_{\mathrm{P}}^i > 1 - b_i'(0)$, then the marginal utility from the public good surpasses, at least initially, the marginal cost from private provision. Agents are willing to privately provide some public good and two situations can be distinguished:

1. $v_i^{\mathrm{P}} < 1$: private provision still represents a cost (standard problem). In equilibrium, the marginal utility from the public good (PG) is positive.
2. $v_i^{\mathrm{P}} > 1$: positionality is so strong that the player gets a positive reward from private provision. An equilibrium is only feasible for a negative marginal benefit (marginal damage) from the public good which, provided in excess, becomes a public bad (PB).

The next proposition characterizes players' contributions under several scenarios, crucially dependent on the amount of public good wished by each player.

**Proposition 1 (Nash equilibrium)** *Let $(x_1^N, x_2^N)$ denote a Nash equilibrium of the game (4).*

*a) If $w_1 + w_2 \leq \min\{A_1, A_2\}$, then $(x_1^N, x_2^N) = (w_1, w_2)$.*

*b) If $w_1 + w_2 > \min\{A_1, A_2\}$ and*

*bI) $A_1 = A_2 = A > 0$, the set of Nash equilibria is given by:*

$$\mathcal{N} = \{(x_1, x_2) | x_1 + x_2 = A \wedge (x_1, x_2) \in [0, \min\{A, w_1\}] \times [0, \min\{A, w_2\}].\}$$

*bII) If $A_i > A_j$ and $A_i > 0$, $i, j \in \{1, 2\}$, $i \neq j$,*

$$(x_i^N, x_j^N) = \begin{cases} (w_i, A_j - w_i) & w_i < A_j < w_i + w_j \quad [bII.1], \\ (w_i, 0) & A_j \leq w_i < A_i \quad [bII.2], \\ (A_i, 0) & A_i \leq w_i \quad [bII.3]. \end{cases} \tag{6}$$

*c) $A_1 = A_2 = 0$: $(x_1^N, x_2^N) = (0, 0)$.*

*The Nash equilibrium is unique, except in case bI.*

The proof is presented in Appendix A.

In a Nash equilibrium (NE), each positional player $i \in \{1, 2\}$ wishes for an uncoordinated total amount of public good, $A_i$. She will contribute the necessary amount, given the contribution from player $j$, and provided her endowment is sufficient.

In the particular case where the addition of the two players' endowments is not enough to provide the amount of public good of the player who wishes the least (case [a]), the two players would contribute their total endowments. The opposite extreme of zero contributions occurs when no player wishes for a positive amount of public good (case [c]). This occurs when their positional concerns are small, $0 \le v_i^{\mathrm{P}} < 1 - b_i'(0)$, $i \in \{1,2\}$, and in particular in the standard case with no positional concerns:

When at least one player wishes some public good and endowments are enough to satisfy the player who wishes the minimum amount of public good (cases [b]), then several cases can be distinguished depending on the amount of public good each player wishes for.

If the two players wish for the same amount of public good, $A_1 = A_2 = A$, then the game presents multiple solutions. This would be the case with symmetric players regarding both their valuation of the public good, $b_1(A) = b_2(A)$, and their degree of positionality, $v_1^{\mathrm{P}} = v_2^{\mathrm{P}} > 0$.[6] Any solution satisfying $x_1 + x_2 = A$ is a NE, regardless of the actual amount paid by each player. The only limit to each player's contribution is the player endowment, $w_i$, or his desired amount of public good, $A_i$.

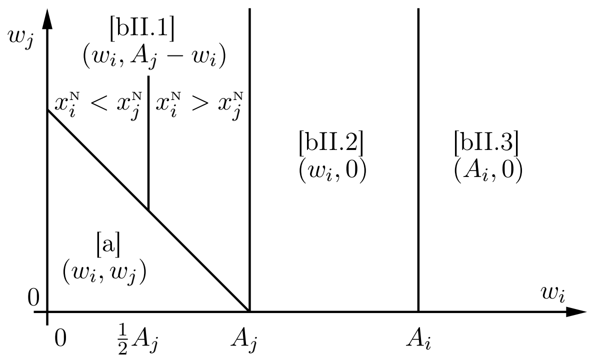


Figure 2: Location of the Nash equilibrium as a function of endowments, when $A_i > A_j > 0$

Indeterminacy is not an issue when the two players wish for a different amount of public good. Case [bII], depicted in Figure 2, assumes that player $i$ wishes for a positive greater amount of public good than player $j$'. Then, three situations are possible for $(x_i^{\mathrm{N}}, x_j^{\mathrm{N}})$:

bII.1 $(w_i, A_j - w_i)$: The endowment of player $i$ is not enough to constitute either the amount of public good wished by player $j$ nor his own wished amount, $A_i$. He will contribute all his endowment, $w_i$, and player $j$ contributes what is missing, $A_j - w_i$. Interestingly, this is the only case where even if player $i$ wishes more public good than player $j$, thes latter can contribute more than the former. Player $i$ contributes above or below player $j$ $(x_i^{\mathrm{N}} \gtrless x_j^{\mathrm{N}})$ iff $w_i \gtrless A_j/2$, or equivalently, $v_j^{\mathrm{P}} \lessgtr 1 - (b_j')^{-1}(2w_i)$.

bII.2 $(w_i, 0)$: Because the endowment of player $i$ is not enough to constitute his wished public good, $A_i$, he will contribute all his endowment. Moreover, since $w_i$ is greater than the amount of public good player $j$ wishes, this latter contributes nothing.

bII.3 $(A_i, 0)$: The endowment of player $i$ is enough to constitute his wished public good, $A_i$. But since $A_i > A_j$, this amount is also more than enough from the viewpoint of player $j$, who consequently provides nothing.

[6] It will also be the case with asymmetric valuation of the public good, $b_1(A) \neq b_2(A)$, if the positional concerns, $v_1^{\mathrm{P}}$ and $v_2^{\mathrm{P}}$, satisfy the knife-edge condition $A_1 = A_2$.

In the rest of the paper and with the intention to have more precise results the following specific functional form for the utility of public good consumption is assumed:

$$b_i(X) \;=\; \alpha_i\left(X - \frac{\varepsilon}{2}X^2\right). \tag{7}$$

This particular function must fulfill conditions $C1$–$C3$, together with a positive marginal utility of the first unit of public good, and a diminishing marginal utility from public good consumption. The marginal utility from the first unit contributed is $b_i'(0) = \alpha_i > 0$. Moreover, a diminishing marginal utility, $b_i''(X) = -\alpha_i\varepsilon < 0$, also requires $\varepsilon > 0$. This parameter represents the agents degree of satiation with public good consumption, common for both players. That is, the speed of decay in the marginal utility of the public good. Under assumptions:

**Assumption 1** $0 \le \alpha_i < 1,\ i = 1, 2.$

**Assumption 2** $\varepsilon > 0.$

**Assumption 3** $\alpha_1 + \alpha_2 > 1.$

**Assumption 4** $w_1 + w_2 > \dfrac{\alpha_1 + \alpha_2 - 1}{\varepsilon(\alpha_1 + \alpha_2)} \equiv \left(X^{SO} = \dfrac{X^{E0}}{2}\right).$

$C_1, C_2$ and $C_3$ are verified. Observe that the quantity appearing in the right-hand side in Assumption 4 is $X^{SO} = X^{EO}/2$, see Figure 1. Moreover, from Definition 1, the amount of public good wished by player $i$ for the explicit expression of $b_i(X)$ given in (7) reads:

$$A_i \;=\; \begin{cases} 0 & \text{if } v_i^{\mathrm{P}} \le 1 - \alpha_i, \\ \dfrac{v_i^{\mathrm{P}} - (1 - \alpha_i)}{\alpha_i\varepsilon} & \text{if } v_i^{\mathrm{P}} \in (1 - \alpha_i, 1 - \alpha_i + \alpha_i\varepsilon(w_1 + w_2)), \\ w_i + w_j & \text{if } v_i^{\mathrm{P}} \ge 1 - \alpha_i + \alpha_i\varepsilon(w_1 + w_2) \end{cases} \tag{8}$$

which lies within $[0, w_1 + w_2]$.

In what follows we ignore the two extreme cases of full (case[a]) and zero (case[c]) contribution. We focus on the case where the agents' endowments are enough to provide some amount of public good. In particular, we assume $A_i > 0$, $A_j < w_i + w_j$, that is $v_i^{\mathrm{P}} > 1 - \alpha_i$ and $v_j^{\mathrm{P}} < 1 - \alpha_j + \alpha_j\varepsilon(w_i + w_j)$. Moreover, for the simplicity of the exposition from now on it will also be assumed that the amount of public good constituted never turns the PG into a PB, i.e. $v_1^{\mathrm{P}}, v_2^{\mathrm{P}} \in [0, 1]$.

In this case, the total amount of public good easily follows from Proposition 1:

$$X^{\mathrm{N}} = \begin{cases} A_j & w_i < A_j < w_i + w_j \quad [bII.1], \\ w_i & A_j \le w_i < A_i \quad [bII.2], \\ A_i & A_i \le w_i \quad [bII.3]. \end{cases} \tag{9}$$

Note that Proposition 1 and expression (9) characterize the contributions under the assumption that player $i$ wishes for more public good than player $j$, i.e. $A_i > A_j$. To characterize which player is willing to privately provide the greatest amount of PG, one needs to take into account each player's positional concern as well as her valuation of the PG consumption, $\alpha_i$. Thus,

$$A_1 \gtrless A_2 \quad\Longleftrightarrow\quad v_1^{\mathrm{P}} \gtrless v_2^{\mathrm{P}}\frac{\alpha_1}{\alpha_2} - \frac{\alpha_1 - \alpha_2}{\alpha_2} \equiv h(v_2^{\mathrm{P}}). \tag{10}$$

Assume that the weight given to the utility from the PG is highest for player 1, $\alpha_1 > \alpha_2$. Then, as shown in Figure 3, the line $v_1^{\mathrm{P}} = h(v_2^{\mathrm{P}})$ delimits two regions, where player 1 wishes more public good than player 2 ($A_1 > A_2$, shaded cyan), or vice versa ($A_1 < A_2$, shaded light-red). In the up-left region player 1 not only values the consumption of public good the most but is also the player with the greatest positional concern, then undoubtedly $A_1 > A_2$. By contrast, if $v_2^{\mathrm{P}} > v_1^{\mathrm{P}}$ player 1 still wishes more PG if positional concerns are not too distant, although player 2 would wish more PG if his positional concern is much larger than player 1's. As shown in the figure no player provides public good in the bottom-left white rectangle characterized by $(v_1^{\mathrm{P}}, v_2^{\mathrm{P}}) \in [0, 1-\alpha_1] \times [0, 1-\alpha_2]$. Opposite reasoning applies if conversely $\alpha_2 > \alpha_1$. Finally, if $\alpha_1 = \alpha_2$, lines $v_1^{\mathrm{P}} = h(v_2^{\mathrm{P}})$ and $v_1^{\mathrm{P}} = v_2^{\mathrm{P}}$ coincide and $v_1^{\mathrm{P}} \gtrless v_2^{\mathrm{P}}$ would be equivalent to $A_1 \gtrless A_2$.

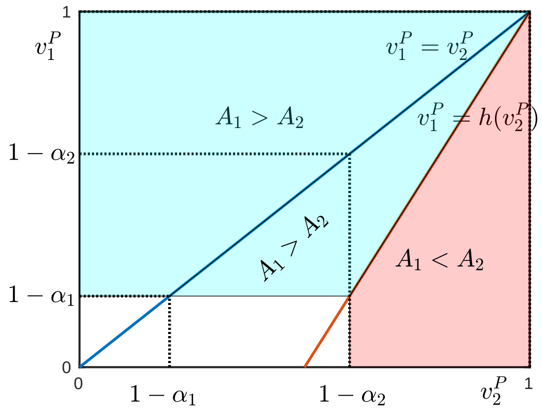


Figure 3: Regions when $\alpha_1 > \alpha_2$

Proposition 1 states that positional concerns allow for positive contributions. The following section answers two main questions. How do the positional concerns affect the social welfare? And more specifically, under which conditions the positive contribution allowed by the existence of positional concerns implies greater/lower social welfare than the zero contribution equilibrium when no positional concerns exist?

## 4.2 Welfare analysis

Global utility or social welfare is defined as the aggregation of both players' utilities. Two parts can be distinguished: the intrinsic social welfare, $u(X)$, and the aggregate positional payoffs, $V(x_i - x_j) = V_i(x_i - x_j) + V_j(x_i - x_j)$. For clearness of the exposition the intrinsic utility is written between brackets:

$$U(x_i, x_j) = [u(X)] + V(x_i - x_j) = \left[w_i + w_j - X + (\alpha_i + \alpha_j)\left(X - \frac{\varepsilon}{2}X^2\right)\right] + (v_i^{\mathrm{P}} - v_j^{\mathrm{P}})(x_i - x_j). \quad (11)$$

Positional concerns have a double effect on social welfare. On the one hand, by allowing the constitution of the public good, $X$, they indirectly influence the intrinsic utility. On the other hand positional concerns also directly determine positional payoffs. Moreover, these payoffs crucially depend on the players' relative contribution, $x_i - x_j$, which, in turn, is also affected by relative positional concerns. We analyze first how each player's positional concern affects both intrinsic utility and aggregate positional payoff. Next, we characterize the conditions under which the positive contributions made possible by the existence of positional concerns can improve the intrinsic utility (net of positional payoffs) and the social welfare (introducing positional payoffs).

### 4.2.1 Marginal effect of positional concerns on social welfare

Utility in (11) can be rewritten for the three different equilibria in case [bII] as:

$$U^{\mathrm{N}} = \begin{cases} [u(A_j)] + (v_i^{\mathrm{P}} - v_j^{\mathrm{P}})(2w_i - A_j) & [bII.1], \\ [u(w_i)] + (v_i^{\mathrm{P}} - v_j^{\mathrm{P}})w_i & [bII.2], \\ [u(A_i)] + (v_i^{\mathrm{P}} - v_j^{\mathrm{P}})A_i & [bII.3]. \end{cases}$$

Note that the aggregate positional payoff is positive if $v_i^{\mathrm{P}} > v_j^{\mathrm{P}}$ and $2w_i > A_j$, in case [bII.1], i.e., if the player who contributes the most also has the highest positional concern.

From this expression, the effect of each player's positional concern on social welfare reads:

$$\left(\frac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}}, \frac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}}\right) = \begin{cases} \left(2w_i - A_j, \left[u'(A_j)\frac{\partial A_j}{\partial v_j^{\mathrm{P}}}\right] + \frac{v_j^{\mathrm{P}} - v_i^{\mathrm{P}}}{\alpha_j \varepsilon} - (2w_i - A_j)\right) & [bII.1], \\ (w_i, -w_i) & [bII.2], \\ \left(\left[u'(A_i)\frac{\partial A_i}{\partial v_i^{\mathrm{P}}}\right] + \frac{v_i^{\mathrm{P}} - v_j^{\mathrm{P}}}{\alpha_i \varepsilon} + A_i, -A_i\right) & [bII.3]. \end{cases} \tag{12}$$

Positional concerns affect the intrinsic utility when they directly determine the total contribution. From Proposition 1, contributions directly depend on the positional concern of player $j$ in [bII.1] (where $X^{\mathrm{N}} = A_j$), or on player $i$'s in [bII.3] (where $X^{\mathrm{N}} = A_i$). In these two cases:

$$\frac{\partial A_j}{\partial v_j^{\mathrm{P}}} = \frac{1}{\alpha_j \varepsilon} > 0, \quad [bII.1]; \qquad \frac{\partial A_i}{\partial v_i^{\mathrm{P}}} = \frac{1}{\alpha_i \varepsilon} > 0, \quad [bII.3].$$

A rise in the positional concern of player $j$ in [bII.1] or player $i$ in [bII.3] induces a "quantity effect" increasing this player's global contribution. This "quantity effect" raises the total amount of public good affecting intrinsic utility. Positional concerns also affect the positional payoffs through two channels:

- A "quantity effect":
  A higher positional concern of the player who contributes the most (least) in case [bII.1] ($x_i^{\mathrm{N}} \gtrless x_j^{\mathrm{N}}$ if $2w_i \gtrless A_j$) widens (narrows) the contribution gap, $x_i - x_j$. This gap widens with $v_i^{\mathrm{P}}$ in case [bII.3] and is unaffected by positional concerns in case [bII.2]. A wider contribution gap benefits the player who contributes the most and harms the player who contributes the least. The opposite is true for a narrower contribution gap.

- A "price effect":
  One player's positional payoff can be interpreted as the price that this player gets from contributing more than his opponent. Under asymmetric contributions both players face the same contribution gap but of opposite sign. Thus, a rise in the positional concern of the player who contributes the most improves aggregate positional payoffs. Conversely, aggregate positional payoffs shrinks with the positional concern of the player who contributes the least. Player $i$ contributes the most in all three cases bII, except in case bII.1 with $2w_i < A_j$.

Social welfare encompasses the utility stemming from the private provision of the public good, $u(X)$, plus the positional payoffs, $V(x_i - x_j)$. The two players' positional payoffs have opposite sign, and hence positional gains by one player are necessarily linked to positional losses by the other. By contrast, social welfare net of positional payoffs, or intrinsic utility, is a utility measure more evenly distributed among agents. In what follows we analyze to what extent the contributions made possible by the existence of positional concerns can improve/worsen the utility of society with respect to the utility under zero contribution. This analysis focuses first on the intrinsic utility and later on the broader measure of social welfare.

#### 4.2.2 Intrinsic social welfare and positional concerns

Contributions associated with the existence of positional concerns can be in shortage, equal or in excess with respect to the amount that maximizes the intrinsic social welfare, $X^{SO}$. Moreover, the positional concerns could be strong enough to lead the public good above $X^{E0}$ (the level which provides the same intrinsic utility than zero contribution, see Figure 1). An equilibrium with positive contributions below (above) this level would be intrinsic welfare improving (reducing). Table 1 summarizes the results for the three equilibria in [bII] (note that $\hat{\alpha}_i = \alpha_i/(\alpha_i + \alpha_j)$).[7]

| | Intrinsic Welfare improving | | Intrinsic Welfare reducing |
|---|---|---|---|
| | Shortage | Excess | |
| bII.1 | $v_j^{\mathrm{P}} \in (1 - \alpha_j, \hat{\alpha}_i)$ | $v_j^{\mathrm{P}} \in (\hat{\alpha}_i, \hat{\alpha}_i + \alpha_j - \hat{\alpha}_j)$ | $v_j^{\mathrm{P}} > \hat{\alpha}_i + \alpha_j - \hat{\alpha}_j$ |
| bII.2 | $w_i < X^{SO}$ | $X^{SO} < w_i \leq X^{E0}$ | $w_i > X^{E0}$ |
| bII.3 | $v_i^{\mathrm{P}} \in (1 - \alpha_i, \hat{\alpha}_j)$ | $v_i^{\mathrm{P}} \in (\hat{\alpha}_j, \hat{\alpha}_j + \alpha_i - \hat{\alpha}_i)$ | $v_i^{\mathrm{P}} > \hat{\alpha}_j + \alpha_i - \hat{\alpha}_i$ |

Table 1: Positional concerns and intrinsic utility

In cases [bII.1] and [bII.3], the public good matches the amount wished by one of the agents, $A_j$ and $A_i$, respectively. Thus, the condition under which contributions are intrinsic welfare improving, $X < X^{E0}$, trivially turns into an upper bound on $v_j^{\mathrm{P}}$ and $v_i^{\mathrm{P}}$ respectively. Moderate positional concerns leading to moderate contributions (in shortage or moderately in excess) improve the intrinsic utility above that in the case without positional concerns, characterized by zero contributions. However, too large positional concerns will push contributions too high (above $X^{E0}$) leading the intrinsic utility below $u^0 = w_1 + w_2$. In case [bII.2] only player $i$ contributes, and he contributes his total endowment. Hence the contribution is not directly dependent on positional concerns. Still, sufficient conditions[8] characterize whether contributions are in shortage, $v_i^{\mathrm{P}} < \hat{\alpha}_j$, or they are welfare reducing, $v_j^{\mathrm{P}} > \hat{\alpha}_i + \hat{\alpha}_j(\alpha_i + \alpha_j - 1)$.

The results stated in Table 1 are depicted in Figure 4 for the particular case where players share the same intrinsic utility, $w_i = w$ and $\alpha_i = \alpha$ for all $i \in \{1, 2\}$. In this symmetric case, Assumptions 1 and 3 imply $\alpha \in (1/2, 1)$. Moreover, $A_i > A_j \Leftrightarrow v_i^{\mathrm{P}} > v_j^{\mathrm{P}}$ and only the region above the bisector $v_i^{\mathrm{P}} = v_j^{\mathrm{P}}$ is drawn. Five regions can be distinguished. Case [a]

[7]Provided that $\alpha_i, \alpha_j > 0$ then $\alpha_i - \hat{\alpha}_i, \alpha_j - \hat{\alpha}_j > 0$, and therefore $\hat{\alpha}_i < \hat{\alpha}_i + \alpha_j - \hat{\alpha}_j$ and $\hat{\alpha}_j < \hat{\alpha}_j + \alpha_i - \hat{\alpha}_i$.

[8][bII.2] is characterized by $A_j \leq X^{\mathrm{N}} \equiv w_i \leq A_i$. Then

- A sufficient condition for a shortage in provision is $A_i \leq X^{SO}$ or equivalently $v_i^{\mathrm{P}} < \hat{\alpha}_j$.
- A sufficient condition for a welfare reducing provision is $A_j > X^{E0}$ or equivalently $v_j^{\mathrm{P}} > \hat{\alpha}_i + \hat{\alpha}_j(\alpha_i + \alpha_j - 1)$.

with $v_i^{\mathrm{P}} > v_j^{\mathrm{P}} \in (\hat{\Theta}, 1)$, with $\hat{\Theta} = 1 - \alpha + 2\alpha\varepsilon w$, in the up-right triangle. Case [bII.1] with $v_i^{\mathrm{P}} > v_j^{\mathrm{P}} \in (\Theta, \hat{\Theta})$, with $\Theta = 1 - \alpha + \alpha\varepsilon w$, in the up-mid trapezoid. Case [bII.2] with $v_i^{\mathrm{P}} \in (\Theta, 1)$ and $v_j^{\mathrm{P}} \in (0, \Theta)$ in the up-left squre. Case [bII.3] with $v_j^{\mathrm{P}} < v_i^{\mathrm{P}} \in (1 - \alpha, \Theta)$ in the mid-left trapezoid. And case [c] with $v_j^{\mathrm{P}} < v_i^{\mathrm{P}} \in (0, 1 - \alpha)$ in the bottom-left triangle. Figure 4 depicts the level curves for the intrinsic utility, where arrows indicate the direction of growth. The intrinsic utility in case of zero contribution is labeled as $u^0$ and its maximum value as $u^{\max}$. Assumptions on parameters are $\alpha = 2/3$ and $w = 1$. The crucial difference between the two graphs in Figure 4 is whether $\alpha < \Theta$ (left) or $\alpha > \Theta$ (right). Define the value of $\varepsilon$ at which $\Theta = \alpha$ as $\hat{\varepsilon} = (2\alpha - 1)/(w\alpha)$, with $\hat{\varepsilon} = 1/2$ for the chosen parameters. In Figure 4 (left) $\varepsilon = 0.7 > \hat{\varepsilon}$, while $\varepsilon = 0.4 < \hat{\varepsilon}$ in Figure 4 (right).

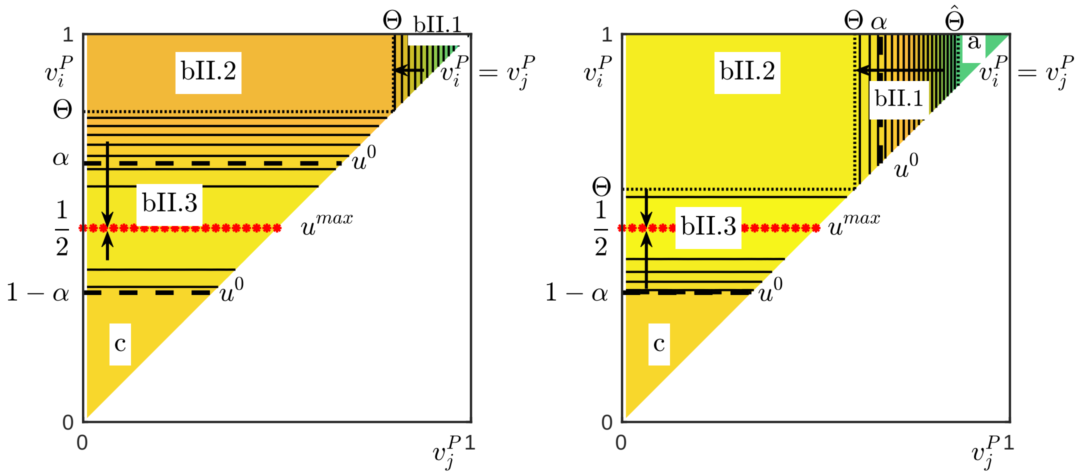


Figure 4: Level curves for intrinsic utility: $\varepsilon = 0.7$ (left), $\varepsilon = 0.4$ (right).

Figure 4 (left) depicts the case of a high degree of satiation, $\varepsilon = 0.7$. In region [bII.3] only player $i$ contributes and his contribution, $A_i$, increases with his positional concern. Starting at $v_i^{\mathrm{P}} = 1 - \alpha$, a higher positional concern for player $i$ increases his contribution and the intrinsic welfare, up until the red-dotted line with $v_i^{\mathrm{P}} = \hat{\alpha} = 1/2$, where the intrinsic utility reaches its maximum. Further increments in $v_i^{\mathrm{P}}$ reduce utility. At the bold line[9] $v_i^{\mathrm{P}} = \alpha$ players would get the same intrinsic utility as in the case with zero contribution, $u^0$. In the region where $v_i^{\mathrm{P}} \in (1 - \alpha, \alpha)$ positional concerns are intrinsic welfare improving. Conversely, above $v_i^{\mathrm{P}} = \alpha$ greater positional concerns reduce intrinsic welfare further below $u^0$. In regions [bII.1] and [bII.2] contribution is excessively high and the intrinsic utility is smaller than $u^0$. In case [bII.2] player $i$ is again the only contributor, who contributes his total endowment, independently of players' positional concerns. Finally, in region [bII.1] both players contribute a total amount of $A_j$. Global contribution increases with the positional concern of player $j$, and since players over-contribute, this causes a reduction in the intrinsic utility.

In the case of a low degree of satiation, $\varepsilon = 0.4$ in Figure 4 (right), player $i$ never contributes enough to place $X$ above $X^{E0}$ in cases [bII.3] and [bII.2], always intrinsic welfare improving. The level curve $u^0$ is now placed in region [bII.1]. To the right of this vertical line global contribution is too high, while to the left, including case [bII.2], positive contributions improve intrinsic social welfare.

[9]In the symmetric case where players share the same intrinsic utility, $\hat{\alpha}_i + \alpha_j - \hat{\alpha}_j = \hat{\alpha}_j + \alpha_i - \hat{\alpha}_i = \alpha$.

The following proposition characterizes the regions where positional concerns are IW-I or IW-R for the symmetric case where players share the same intrinsic utility. It shows that when the marginal utility from public good consumption decreases slowly (rapidly), i.e. $\varepsilon$, is small (large), positional concerns are widely IW-I (IW-R).

**Proposition 2** *Assume $w_i = w$ and $\alpha_i = \alpha$ for all $i \in \{1,2\}$ (and, from Assumption 3, $\alpha > 1/2$). Define $\hat{\varepsilon} = (2\alpha - 1)/(w\alpha)$, then:*

1. *If $\varepsilon < \hat{\varepsilon}$, positional concerns are IW-I (resp. IW-R) if $v_j^P < \alpha$ ($v_j^P > \alpha$). The IW-I area is wider than the IW-R area.*

2. *If $\varepsilon > \hat{\varepsilon}$, positional concerns are IW-I (resp. IW-R) if $1 - \alpha < v_i^P < \alpha$ ($v_i^P > \alpha$). The IW-I area is wider (narrower) than the IW-R area if $\alpha \in (\sqrt{3} - 1, 1)$ ($\alpha \in (1/2, \sqrt{3} - 1)$).*

The proof is presented in Appendix B.

As shown in Figure 4, positive contributions from the existence of positional concerns improve intrinsic social welfare when $\varepsilon > \hat{\varepsilon}$ and $v_j^{\mathrm{P}} \leq v_i^{\mathrm{P}} \in (1 - \alpha, \alpha)$, or in the wider area with $v_i^{\mathrm{P}} \in (1 - \alpha, 1)$ and $v_j^{\mathrm{P}} \leq \alpha$ when $\varepsilon < \hat{\varepsilon}$. Moreover, comparing the areas in the $(v_i^{\mathrm{P}}, v_j^{\mathrm{P}})$ plane of the IW-I and IW-R regions one can estimate the likelihood that positive contributions increase intrinsic utility. An increment in intrinsic utility is more likely than a decrement except under a strong satiation and a small utility from the public good (large $\varepsilon$ and small $\alpha$).

Figure 4 and Proposition 2 refer to the symmetric case where both players have the same endowment and equally enjoy public good consumption. Asymmetric situations are presented in Figure 5[10] under the assumption $\varepsilon = 0.4 < \hat{\varepsilon}$. In Figure 5 (left) agent $i$ values public good consumption more than agent $j$, $\alpha_i > \alpha_j$. Hence , it is possible that she wishes more public good than player $j$ even though she has less positional concern (below the $v_i^{\mathrm{P}} = v_j^{\mathrm{P}}$ line). Under this assumption the IW-I area becomes larger relatively to the IW-R area. The opposite is true when agent $i$ values public good consumption less than player $j$ in Figure 5 (right). Her positional concern needs to be much higher than $j$'s for $A_i > A_j$ to occur. In consequence positional concerns are relatively less likely to improve intrinsic utility (the IW-I area decreases in relation to the IW-R area).

### 4.2.3 Social welfare and positional concerns

In this subsection, social welfare is analyzed as a function of the two players' positional concerns. The analysis is carried out numerically taking into account the same parameters used in the previous subsection. First, we study the symmetric case, where players share the same intrinsic utility, and this assumption is relaxed later on.

Figure 6 represents the level curves of the global utility for the three equilibria in [b.II], in the symmetric case. A rise in the positional concern of this player, $v_i^{\mathrm{P}}$, reduces the intrinsic utility in region [bII.3] (unless $v_i^{\mathrm{P}} < \hat{\alpha} \equiv 1/2$), and does not affect it in regions [bII.1] and [bII.2]. However, taking also into account the positional payoffs, global utility increases with $v_i^{\mathrm{P}}$ almost everywhere in [bII].[11] On the other hand, a rise in the positional concern of player $j$ (with the lowest positional concern), reduces the intrinsic utility in region [bII.1] (unless

[10] In this Figure, $\Theta_i(w) = 1 - \alpha_i + \alpha_i \varepsilon w$.

[11] Only when $\varepsilon$ is large (Figure 6 left) and only if $v_i^{\mathrm{P}}$ is below but close to $\Theta$ (i.e. $A_i$ is below but close to $w$) and $v_j^{\mathrm{P}}$ is close behind, a greater positional concern of player $i$ can reduce social welfare.

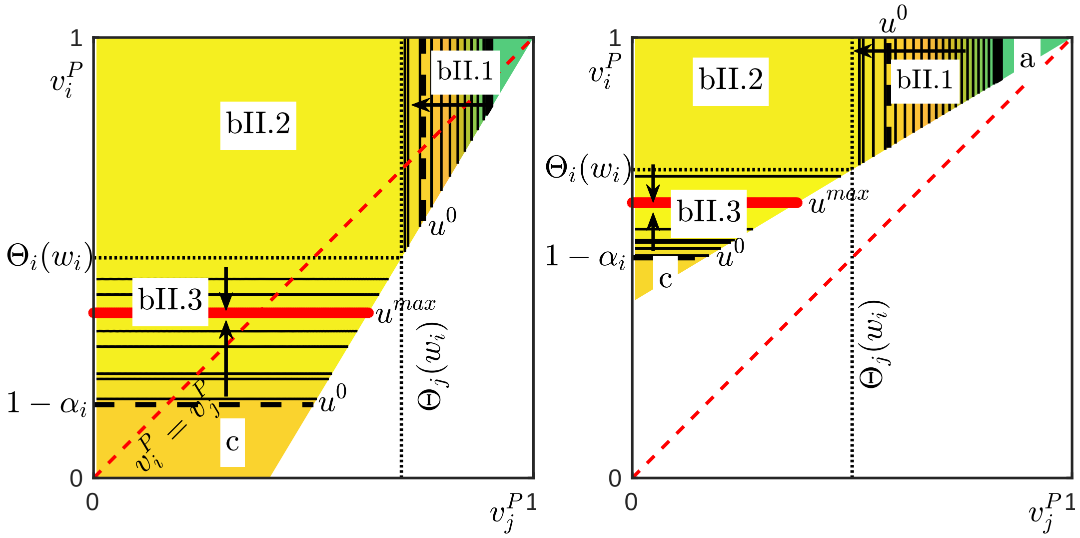


Figure 5: Level curves for intrinsic utility: $\alpha_i = \frac{5}{6}$, $\alpha_j = \frac{1}{2}$ (left), $\alpha_i = \frac{1}{2}$, $\alpha_j = \frac{5}{6}$ (right).

$v_j^{\mathrm{P}} < \hat{\alpha} \equiv 1/2$), and does not affect it in regions [bII.2] and [bII.3]. Moreover, for a broader measure of utility that also includes positional concerns, the greater this player's positional concern, the greater the losses in terms of social welfare in all three regions in [bII].

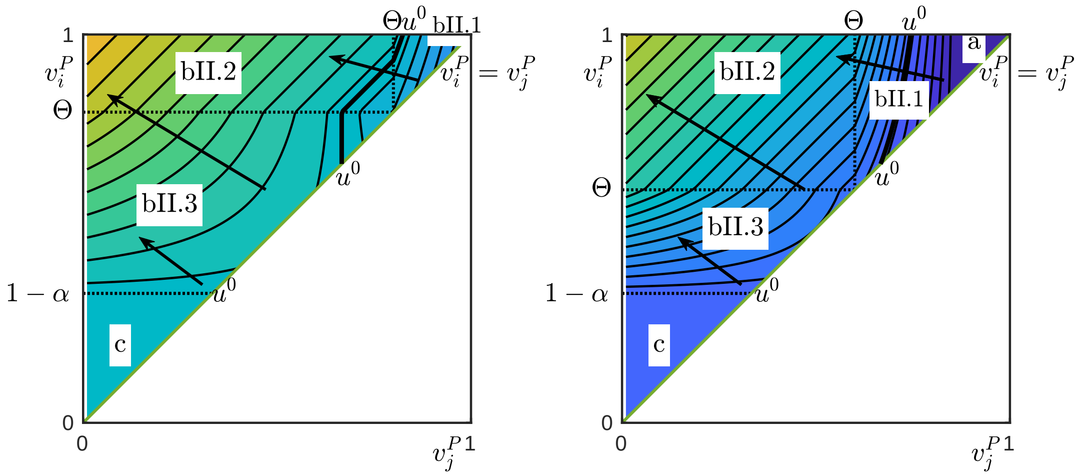


Figure 6: Level curves for utility: $\varepsilon = 0.7$ (left), $\varepsilon = 0.4$ (right).

The more unequal is the distribution of positional concerns (i.e. the wider the gap $v_i^{\mathrm{P}} - v_j^{\mathrm{P}}$) the stronger is the role played by the positional payoffs and the greater the global utility. Conversely, the more even these concerns, the more important is the role played by the intrinsic utility and the smaller the global utility.

Figure 6 also draws the $u^0$ level curves, where the global utility equates the utility of zero contribution. This occurs when contributions are actually equal to zero, $v_j^{\mathrm{P}} \leq v_i^{\mathrm{P}} \leq 1 - \alpha$ (case[c]), but also for positive contributions in the $u^0 - u^0$ line. Up-left of this latter, contributions improve global utility above the $u^0$ level. Conversely, down-right of the curve, positive contributions drive global utility below the utility of zero contribution.

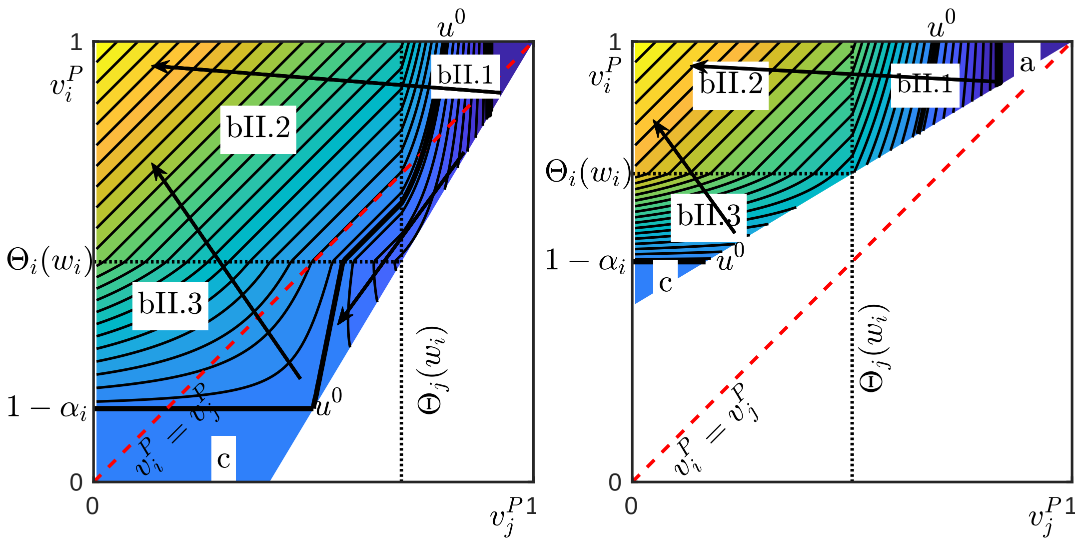


Figure 7: Level curves for utility: $\alpha_i = \frac{5}{6}$, $\alpha_j = \frac{1}{2}$ (left), $\alpha_i = \frac{1}{2}$, $\alpha_j = \frac{5}{6}$ (right).

The analysis is generalized to the asymmetric case in Figure 7. In the symmetric case the existence of positional concerns improves social welfare with respect to the zero contributions utility level almost everywhere, except when the positional concerns are large (close to one) and sufficiently even. This result remains valid in the asymmetric case when player $i$ values public good consumption the least although, having a strong positional concern wishes the highest amount of public good (Figure 7-right). However, in Figure 7 (left), when the player who wishes the most public good is also the one who values its consumption the most, then contributions can reduce social welfare even for small positional concerns, as long as they are sufficiently close. Too large contributions (above $X^{E0}$) induce losses in terms of intrinsic utility. These losses are not counteracted by positional payoffs because $v_i^{\mathrm{P}}$ and $v_j^{\mathrm{P}}$ are close together and the positional gains by one player are similar to the positional losses by the other and hence, positional payoffs have a limited impact on social welfare.

Expression (12) and Figures 6 and 7 allow us to answer another relevant question: Under which conditions a rise in the positional concerns by both players increases, reduces or leave social welfare unchanged? To answer this question one needs to characterize the points in the $(v_j^{\mathrm{P}}, v_i^{\mathrm{P}})$ plane for which the slope of the isocline is smaller, larger[12] or equal to one. In the symmetric case, the following proposition shows that social welfare increases, remain unchanged or shrinks when the addition of the players' positional concerns is below, equal or above one.

**Proposition 3** *Assume that $\alpha_i = \alpha$ for all $i \in \{1,2\}$. The effect of a joint increment in the positional concerns of both players is:*

$$\text{[bII.1] and [bII.3]:} \qquad \frac{dU^N}{dv_i^P} + \frac{dU^N}{dv_j^P} \gtreqless 0 \Leftrightarrow v_i^P + v_j^P \lesseqgtr 1.$$

$$\text{[bII.2]:} \qquad \frac{dU^N}{dv_i^P} + \frac{dU^N}{dv_j^P} = 0.$$

[12] As explained in Proposition 3, for the isoclines with negative slope, in region [bII.3], stronger positional concerns by both players worsens social welfare.

The proof is presented in Appendix C.

# 5 The dynamic model with myopic players

As in the static game we consider two players $i \in \{1, 2\}$. Each player is endowed with the same $w_i$ at every time $t$ and contributes $x_{it}$ to a non-durable public good. Thus her utility at this instant of time is given by:

$$u_i(x_{it}, X_t) = w_i - x_{it} + \alpha_i \left[ X_t - \frac{\varepsilon}{2} X_t^2 \right], \quad \text{with} \quad X_t = x_{it} + x_{jt}.$$

Assumptions 1-4 are fulfilled and we further assume that:

1. Agents show **Inertia** from previous actions. People in general are reluctant to changes and have a tendency to maintain the same behaviour. This captures the *status quo* bias, assuming that the default option is the agent's previous decision. This idea is introduced in the model by adding a disutility to deviations from the previous action: $-v_i^{\mathrm{I}}(x_{it} - x_{it-1})^2/2$, with $v_i^{\mathrm{I}} \geq 0$. Agents not having this tendency to inertia can be modeled by taking $v_i^{\mathrm{I}} = 0$.

2. A **positional** agent gets joy from contributing above others. This status concern is built looking at the opponent's previous contribution. Therefore, positional payoffs are introduced in the model as a gain (loss) for contributions above (below) the other player's previous contribution: $+v_i^{\mathrm{P}}(x_{it} - x_{jt-1})$, with $v_i^{\mathrm{P}} \geq 0$.

In consequence, the utility at each time $t$ reads[13]:

$$U_i(x_{it}, x_{jt}, x_{it-1}, x_{jt-1}) = w_i - x_{it} + \alpha_i \left[ X_t - \frac{\varepsilon}{2} X_t^2 \right] - \frac{v_i^{\mathrm{I}}}{2}(x_{it} - x_{it-1})^2 + v_i^{\mathrm{P}}(x_{it} - x_{jt-1}). \quad (13)$$

## 5.1 Dynamics of the system with myopic agents. General case.

At each specific time $t$, an agent with inertia is reluctant to change her previous decision, $x_{it-1}$. Moreover, she builds her positional payoff by looking at her opponent's previous decisions $x_{jt-1}$. This behavior characterizes a dynamic model. However, the players are myopic because they maximize their instantaneous utility at every time $t$, solving the problem:

$$\begin{aligned} &\max_{x_{it}} U_i(x_{it}, x_{jt}, \hat{x}_{it}, \hat{x}_{jt}), \\ &\text{s.t.: } \hat{x}_{it} = x_{it-1}, \quad i \in \{1, 2\}. \end{aligned} \quad (14)$$

The first-order conditions for an interior equilibrium are:

$$-1 + \alpha_i[1 - \varepsilon(x_{it} + x_{jt})] - v_i^{\mathrm{I}}(x_{it} - x_{it-1}) + v_i^{\mathrm{P}} = 0 \quad \forall i \in \{1, 2\}. \quad (15)$$

Solving this linear equation, we obtain the unconstrained best reaction as a function of the opponent's current action and the player's action in the previous step:

$$r_i(x_{jt}; x_{it-1}) = \frac{\Phi_i + v_i^{\mathrm{I}} x_{it-1} - \alpha_i \varepsilon x_{jt}}{v_i^{\mathrm{I}} + \alpha_i \varepsilon}, \quad (16)$$

[13]An analysis of the dynamics when the contribution of the other player is based on the past is given in E.2.4.

where

$$\Phi_i := \alpha_i - 1 + v_i^{\mathrm{P}}. \tag{17}$$

Taking the constraints into account, we deduce the dynamic reaction function as:

$$r_i^D(x_{jt}; x_{it-1}) = \begin{cases} w_i & \text{if } r_i(x_{jt}; x_{it-1}) \geq w_i \\ r_i(x_{jt}; x_{it-1}) & \text{if } 0 \leq r_i(x_{jt}; x_{it-1}) \leq w_i \\ 0 & \text{if } r_i(x_{jt}; x_{it-1}) \leq 0. \end{cases} \tag{18}$$

It readily follows that a Nash equilibrium at step $t$ must satisfy $x_{it} = r_i^D(x_{jt}; x_{it-1})$ for $i, j \in \{1, 2\}$, $i \neq j$. Provided the Nash equilibrium exists and is unique,[14] this procedure defines a dynamical process $\{x_t = (x_{it}, x_{jt}); t = 0, 1, \ldots\}$.

### 5.2 Properties of the dynamical system: summary

When discussing the behavior of the dynamic system, we are particularly interested in: a) the asymptotic behavior of the sequence of contributions, and b) the monotonicity (or lack of it) in the sequence of individual contributions, total contributions and utilities. The statements of the propositions are given for $X_0 = 0$. They will be valid for $X_0$ small enough but they can be extended for any $X_0$. We chose $X_0 = 0$ as benchmark because it is the static Nash equilibrium when players have no positional concerns. Without loss of generality, we assume in the remainder of this section that Player $i$ wishes no less public good than Player $j$: $A_i \geq A_j$.

In the following statements, "the interior" refers to those contributions satisfying $(x_i, x_j) \in (0, w_i) \times (0, w_j)$. "The boundary" of feasible contributions is reached when either $x_i$ or $x_j$ vanishes or reaches its maximum at $w_i$ or $w_j$.

**Proposition 4 (Convergence)** *Assume $A_i \geq A_j$.*

*[R0] The static equilibrium is a fixed point of the dynamical system;*

*[R1a] The trajectory of the contributions eventually hits the boundary of feasible contributions, in the part where the static equilibrium lies;*

*[R1b] After hiting the boundary where the static equilibrium lies, the trajectory converges monotonously to this static equilibrium;*

*[R1c] Without inertia, the system arrives to the static equilibrium at the first step.*

Once the convergence of the transition to the static equilibrium has been established, next we analyze the monotonicity of each player's and the total contribution. Proposition 5 presents the properties that are valid for all types of equilibria. Proposition 6 proves monotonicity for some particular cases. Later we discuss the scenarios under which initial over-contribution by one player or globally are feasible.

**Proposition 5 (General behaviour of contributions)** *Assume $X_0 = 0$ and $A_i \geq A_j$.*

*[R2] The contribution of Player i is monotonously increasing, both in the interior and the boundary;*

[14] Uniqueness follows here from the fact that reaction functions are decreasing and Lipschitz-continuous with a Lipschitz constant strictly less than 1.

*[R3] The total contribution is monotonously increasing in the interior.*

**Proposition 6 (Monotonicity in particular cases)** *Assume $X_0 = 0$ and $A_i \geq A_j$.*

*[R4] In cases [a] ($w_1 + w_2 < A_j$) and [bI] ($A_i = A_j = A$), the contribution of the two players and the total contribution are both monotonously increasing;*

*[R5] In case [bII.3] ($A_i < w_i$), the contribution of Player $i$ and the total contribution are both monotonously increasing.*

The proof of [R0] is given in Appendix D. [R1a] and [R3] are proved in Section 5.3.2, [R1b] follows from Proposition 7 and Proposition 8 in Appendix F. For [R1c] see Section 5.3.1. Statement [R2] is proved in Section 5.3.2 for the interior part, and Appendix F.2 or Appendix F.3 for the boundary part. Property [R4] follows from [R2] (for the contribution of Player $i$) and from the behaviour of the contribution of Player $j$ in the interior (Appendix E.2.3) and on the boundary $x_i = w_i$ (Appendix F.1). The proof of [R5] is a consequence of [R2] (for the contribution of Player $i$) and from [R1a] and [R1b] for the total contribution.

The general behavior of the trajectories can be described in the case where the wished amounts are different ($A_i > A_j$) and the initial contribution is $x_i = x_j = 0$. In a first phase, the contributions of player $i$ (with the greatest wished amount) is positive monotonously increasing. The contribution of player $j$ may possibly increase in a first phase and decrease thereafter, a phenomenon we call *overshooting*.[15]

Eventually, the trajectory hits the boundary of feasible contributions, with one of the players constrained to contribute her whole endowment or nothing. In this second phase, the trajectory converges monotonously to the limiting state, which is precisely the equilibrium of the static game described in Proposition 1. In this second phase, numerical simulations will show that total contribution can converge from above or below. The speed at which this convergence occurs inversely depends on the inertia parameter of the player whose contribution is not constrained.

**Remark 1** *From propositions 5 and 6 overshooting is only feasible for player $j$ and in case [bII]. As we will show numerically, overshooting for player $j$ can lead to overshooting of total contribution when the player who wishes the most contributes her total endowment (cases [bII.1] and [bII.2])*

## 5.3 Properties of the dynamical system: analysis

We proceed with the analysis of the properties stated in the previous section. First, we consider the case where both players have no inertia and later the case where at least one player has inertia.

### 5.3.1 Players without inertia ($v_i^{\mathrm{I}} = 0$ and $v_j^{\mathrm{I}} = 0$)

In this particular case, the optimization problem does not depend on the previous state $(x_{it-1}, x_{jt-1})$ (see (15)). Therefore, the agents play a static game at each time $t$. Indeed, the comparison of (29) and (26) confirms that the reaction functions are the same. This proves

[15]Overshooting can be so strong as to exceed $w_j$: in this case, player $j$'s contribution remains blocked at this level for some time (illustration in Section 6.2 and Figure 10).

the result [R1c]. Moreover, starting with the initial contribution $(0,0)$, the trajectory jumps to the Nash equilibrium with $x_i^N \geq 0$ and $x_j^N \geq 0$, and then stays there. This proves the statements [R2], [R4] and [R5] in this case, albeit in a degenerate way.

### 5.3.2 Players with inertia ($v_i^{\mathrm{I}} > 0$ or $v_j^{\mathrm{I}} > 0$)

We turn to the analysis of the *interior* intersection of the two reaction curves (see (15)). The current contribution of Player $i$ can be written as a function of her own and her opponent's past contributions as:

$$x_{it} = V_i^0 + V_i^{\mathrm{I}}\, x_{it-1} + V_i\, x_{jt-1} \tag{19}$$

with:

$$V_i^0 = \frac{v_j^{\mathrm{I}}\Phi_i + \varepsilon(\alpha_j\Phi_i - \alpha_i\Phi_j)}{D}, \quad V_i^{\mathrm{I}} = \frac{v_i^{\mathrm{I}}v_j^{\mathrm{I}} + \varepsilon\alpha_j v_i^{\mathrm{I}}}{D}, \quad V_i = -\frac{\varepsilon\alpha_j v_i^{\mathrm{I}}}{D},$$

and

$$D = v_i^{\mathrm{I}}v_j^{\mathrm{I}} + \varepsilon(\alpha_j v_i^{\mathrm{I}} + \alpha_i v_j^{\mathrm{I}}). \tag{20}$$

Since either $v_i^{\mathrm{I}} > 0$ or $v_j^{\mathrm{I}} > 0$, then $D > 0$. Written in matrix/vector form, the dynamics (19) reads:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M \begin{pmatrix} x_{it-1} \\ x_{jt-1} \end{pmatrix} + \begin{pmatrix} V_i^0 \\ V_j^0 \end{pmatrix} \qquad \text{where} \qquad M = \begin{pmatrix} V_i^{\mathrm{I}} & V_i \\ V_j & V_j^{\mathrm{I}} \end{pmatrix}. \tag{21}$$

It is shown in Appendix E.2 that the solution to the recurrence is:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = \delta t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \left(1 - \lambda_2^t\right) \frac{X^* - X_0}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_i v_j^{\mathrm{I}} \\ \alpha_j v_i^{\mathrm{I}} \end{pmatrix}, \tag{22}$$

where, according to (42), (43) and (44),

$$\delta = \frac{\varepsilon\alpha_i\alpha_j(A_i - A_j)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \qquad \lambda_2 = \frac{v_i^{\mathrm{I}}v_j^{\mathrm{I}}}{v_i^{\mathrm{I}}v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \qquad X^* = \frac{v_j^{\mathrm{I}}\alpha_i A_i + v_i^{\mathrm{I}}\alpha_j A_j}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}}.$$

The definitions of $\delta$ and $X^*$ using wished amounts $A_i$ and $A_j$ are valid when they lie in the interior of expression (8). Otherwise, when they lie in the boundary, they must be replaced by $A_i = \Phi_i/(\alpha_i\varepsilon)$, $\Phi_i$ being defined in (17). Since we have assumed that $A_i \geq A_j$, we have $\delta \geq 0$.

We first state general results about the contribution of Player $i$ and the total contribution, then we turn to a more detailed analysis of the solution (22). To determine the asymptotic properties, we must distinguish two cases, depending on whether $A_i = A_j$ or $A_i > A_j$ ($\delta$ is 0 or not).

**Contribution of Player $i$.** Expression (22) describes the trajectory of individual contributions to the public good. Assume that $A_i > 0$ and $A_j > 0$, which implies $X^* > 0$. Note that under the assumption that some inertia is present, $\lambda_2 \in [0,1)$ and hence the sequence $(1 - \lambda_2^t)$ is decreasing to 0. This implies that the sequence $x_{it}$ of Player $i$'s contributions is increasing. This proves [R2] in the interior.

**Total contribution.** Expression (22) also provides information on the total contribution. We have:

$$X_t \;=\; X_0 + (1-\lambda_2^t)(X^* - X_0). \tag{23}$$

From (23), the value $X^*$ is the limit of $X_t$ when $t \to \infty$. It is also a fixed point: if $X_0 = X^*$, then $X_t = X^*$ for all $t$. It is the convex combination of individual preferred amounts $A_i$ and $A_j$ with relative coefficients $\alpha_i v_j^{\mathrm{I}}$ and $\alpha_j v_i^{\mathrm{I}}$. Moreover, from (23), it is clear that $X_t$ increases with $t$ if $X_0 < X^*$ and decreases with $t$ if $X_0 > X^*$. This proves result [R3]. Observe also that, while in the interior, a trajectory starting from $X_0 < X^*$ cannot exceed $X^* \le \max\{A_i, A_j\}$. This observation is relevant for [R5]. Convergence to $X^*$ requires the solution to remain within the interior of the feasible set.

**Players with inertia and the same wished amount.** In the very specific case where $\delta = 0 \iff A_i = A_j = A$, the leading term in (22) vanishes. Besides, the value of $X^*$ is simply $A$. The solution of the free dynamic system simplifies as:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + (1-\lambda_2^t)\frac{A - X_0}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_j v_i^{\mathrm{I}} \\ \alpha_i v_j^{\mathrm{I}} \end{pmatrix} . \tag{24}$$

A first consequence is that both sequences $x_{it}$ and $x_{jt}$ are monotonous: they are increasing if $X_0 < A$ and decreasing if $X_0 > A$. A second consequence is that these sequences converge. The limit is then:

$$x_i^* := \lim_{t\to\infty} x_{it} = x_{i0} + \frac{\alpha_j v_i^{\mathrm{I}}(A - X_0)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}}.$$

This limit point depends both on the initial condition and on the inertia parameters. However, we have observed in (23) that the sum of its components converges to $X^* = A$. The point therefore lies in the line $x_i + x_j = A$, which is the set of Nash equilibria of the static game (Proposition 1). From (24), it also follows that all points in the trajectory lie in the line which joins $x_0$ and $x^*$. Moreover, from the definition of $x^*$ it follows that the vector $x^* - x_0$ is proportional to the vector

$$v^* = \begin{pmatrix} \alpha_i v_j^{\mathrm{I}} \\ \alpha_j v_i^{\mathrm{I}} \end{pmatrix} .$$

The limit point $x^*$ is therefore at the intersection of the line passing through $x_0$ with direction $v^*$, and the line $x_i + x_j = A$. In general, the limit point $x^*$ does not need to lie inside the rectangle of constraints. When it does, the sequences of individual contributions are monotonously increasing, as argued above. When it does not, the trajectory hits the boundary and it stays on the boundary converging monotonously to the equilibrium: see Proposition 7 and Proposition 8 in Appendix F. This proves [R4] in that case.

**Players with inertia and different wished amounts.** Consider now the case $A_i > A_j$ and hence $\delta > 0$. As a consequence:

$$\lim_{t\to\infty} x_{it} = -\lim_{t\to\infty} x_{jt} = +\infty.$$

However, the player's optimization problem is constrained. The dynamics of the controls cannot obey (19) indefinitely, since then either $x_{it}$ or $x_{jt}$ or both will eventually exit its domain of validity. Since the contribution of Player $i$ increases in the interior, when the boundary is hit at some step $t$, then either $x_{it} = w_i$ or $x_{jt} = 0$ or $x_{jt} = w_j$. We argue now that the trajectory eventually joins one of these boundaries where the static equilibrium lies, which is statement [R1a].

Consider first Case [a]. The static equilibrium lies on both boundaries $x_i = w_i$ and $x_j = w_j$, so only the boundary $x_j = 0$ should be excluded. Indeed, from the equations of the dynamic best-reply (58), it follows that if $x_{jt} = 0$ then $x_{jt+1} > 0$. The trajectory therefore cannot remain on this boundary. Since it cannot stay in the interior either, it necessarily ends up with either $x_{it} = w_i$ or $x_{jt} = w_j$.

Consider next Case [bII.1]. Since the static equilibrium lies on the boundary $x_i = w_i$, we have to exclude the boundaries $x_j = 0$ and $x_j = w_j$. This case is characterized by the condition $A_j > w_i$, which implies, as in Case [a], that the trajectory cannot stay on the boundary $x_j = 0$. Focusing then on the boundary $x_j = w_j$, Proposition 9 implies that, as long as the trajectory stays on this boundary, the contribution of Player $i$ continues to increase. If the trajectory were to stay forever on this boundary, the dynamic system would converge to some point $(x_i^*, x_j^*)$ satisfying $x_j^* = w_j$. But this is a contradiction because, according to [R0] in Proposition 4, convergence occurs only to Nash equilibria of the static game. We have then proved that the trajectory cannot remain indefinitely in the interior nor on the boundaries where the equilibrium does not lie.

The argument that the trajectory cannot stay forever on boundary $x_j = w_j$ applies also to case [bII.2]. So statement [R1a] also holds in this case, in which the equilibrium is on both boundaries $x_i = w_i$ and $x_j = 0$.

Finally, consider Case [bII.3]. It is still impossible that the trajectory stays on the boundary $x_j = w_j$. It is also impossible to stay on $x_j = 0$: in this case $A_i \leq w_i$ and using the dyamic best-response (52), we conclude that if $x_{it} = w_i$, $x_{it+1} < w_i$. This completes the proof of [R1a].

The dynamics described in (22) can be interpreted as the combination of two phenomena. The first one is that of a geometric convergence of $x$ in the direction of $x^*$, as in Section 5.3.2. The point $x^*$ is located at the intersection of the line $x_i + x_j = X^*$, where $X^*$ is given in (44) and the line passing through $x_0$ with direction $v^*$. The geometric factor of this convergence is $\lambda_2$.

The second phenomenon, represented by the linear part in (22), is a $\delta$ transfer of contribution effort from Player $j$ to Player $i$, with the total contribution remaining the same. This repeated transfer eventually causes the point $x_t$ to hit either zero for $j$ or its budget constraint for player $i$. This will be illustrated in the simulations of Section 6.

# 6 Simulations

This section presents numerical simulations corresponding to the different static Nash equilibria described in Proposition 1. In order to ease the reading of figures, we switch from the "$(x_i, x_j)$" convention for naming players, to the cartesian "$(x_1, x_2)$" notation. Unless otherwise specified, players' budgets are $w_1 = w_2 = 1$. Moreover, for the simplicity of the exposition, we assume in this section that the two players equally value public good consumption, $\alpha_1 = \alpha_2$.

## 6.1 Players with inertia; equilibrium of type [bI]

In equilibria of type [bI] players have the same wished amount. Figure 8 shows the trajectories for the two players with inertia, when $A_1 = A_2 = A$. The parameters are:

$$\varepsilon = \frac{4}{10}, \alpha_1 = \frac{2}{3}, \alpha_2 = \frac{2}{3}, v_1^{\mathrm{P}} = \frac{11}{15}, v_2^{\mathrm{P}} = \frac{11}{15}, v_1^{\mathrm{I}} = 2, v_2^{\mathrm{I}} = 3.$$

The common value for $A$ is $3/2$. The figure displays one trajectory starting at $(0,0)$, and another one starting at $(1/2,0)$. The limit point of the free dynamic system (21), lies in both cases on the line $\{x_1 + x_2 = A\}$ which is the set of Nash equilibria. The trajectory for contributions, $x_t$, follows the direction $v^* = (3,2)'$ starting from the initial point $x_0$. For the trajectory starting at $(1/2,0)$, the limit point lies outside the rectangle of constraints: the trajectory first hits the boundary and continues towards the Nash equilibrium $(1,1/2)$.

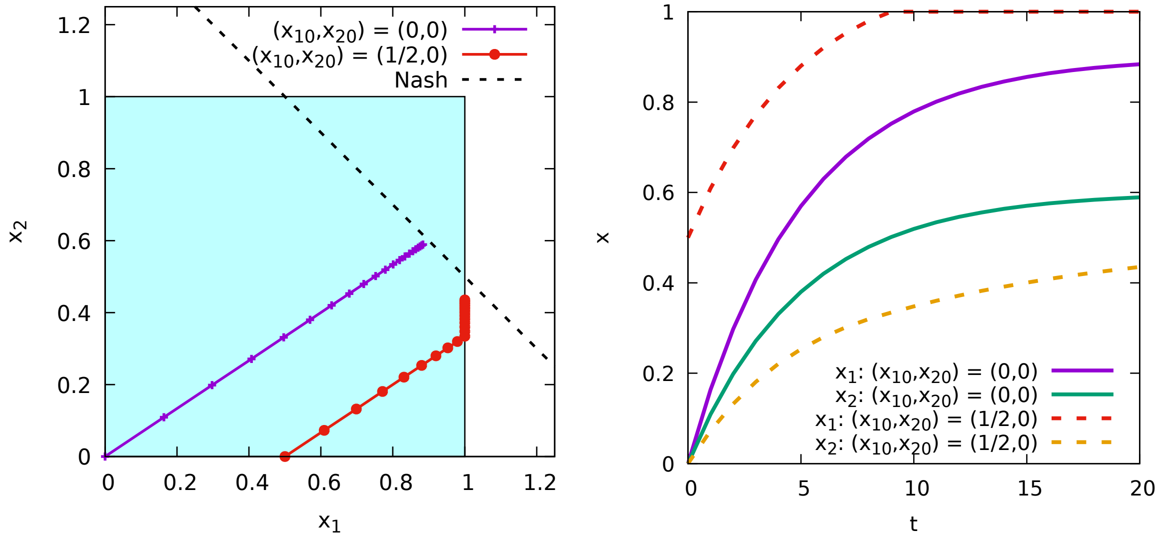


Figure 8: Trajectories for two players with inertia and same $A$ (type [bI])

### 6.2 Players with inertia; equilibrium of type [bII.1] and [bII.2]

The behavior observed in cases [bII.1] and [bII.2] is similar. To avoid repetition, we present the insights from the numerical analysis in the Equilibria of type [bII.1].

In order to demonstrate the influence of inertia we present the numerical results for different values of the parameters which define inertia. To analyze the case with **excess** contribution at the equilibrium, we consider the following parameters values:

$$\varepsilon = \frac{4}{10}, \alpha_1 = \frac{2}{3}, \alpha_2 = \frac{2}{3}, v_1^{\mathrm{P}} = \frac{13}{15}, v_2^{\mathrm{P}} = \frac{33}{50},$$

with the initial condition $x_0 = (0.0)$. The wished amounts and the static equilibrium are then given by:

$$A_1 = 2, \quad A_2 = \frac{49}{40}, \quad x^N = \left(1, \frac{9}{40}\right).$$

The equilibrium is indeed of type [bII.1]. The total contribution at the static equilibrium is $A_2$. It lies between $X^{SO} = 5/8$ and $X^{EO} = 5/4$, and therefore it corresponds to the case of a public good with excess contribution but still IW-I.

Figure 9 displays the trajectories in the phase diagram for values of $v_1^{\mathrm{I}}$ within $\{0.1, 1, 5, 20\}$, while $v_2^{\mathrm{I}}$ is kept constant at 1. It also depicts the straight lines at which the total contribution is $X^N$, $X^{SO}$ and $X^{E0}$. Figure 10 displays the individual contributions (left) and the total contributions (right), as time functions. When Player 1 shows lower inertia than Player 2, the former rapidly increases contributions while the latter raises hers very slowly. The trajectory quickly reaches the $x_1 = w_1$ boundary and $x_2$ monotonously rises to reach the Nash equilibrium $x_2^{\mathrm{N}} = A_2 - w_1$. Conversely, if inertia is larger for Player 1 than for Player 2, this latter increases her contribution more rapidly (as in case $(v_1^{\mathrm{I}}, v_2^{\mathrm{I}}) = (5, 1)$) and can even reach her total endowment (in (20,1)). Although more slowly, Player 1 also steadily raises her contribution. As total contribution rises, contributing is less and less attractive for Player 2, who wishes less public good than Player 1. From a certain level of total contribution Player 2 starts reducing her contribution to free ride on the contribution of Player 1. Figures 9 and 10 (left) depict overshooting in Player 2's contribution, initially rising above her long-run value and decaying towards this level later on. Interestingly, the overshooting in Player 2's contribution can lead to overshooting also in total contribution. This occurs for cases (5,1) and (20,1) as shown in Figure 10 (right).

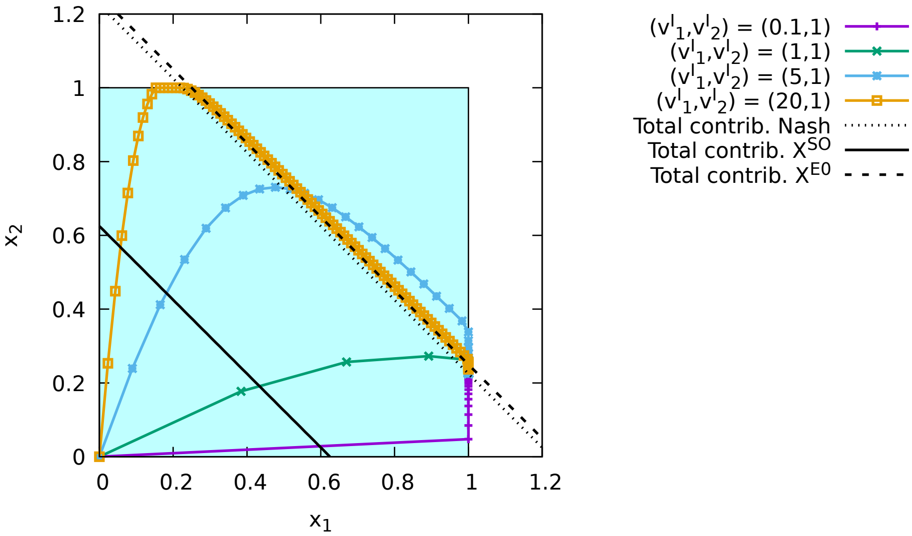


Figure 9: Phase diagram of contributions; case [bII.1] with welfare reduction then excess; varying $v_1^{\mathrm{I}}$

Figure 11 focuses on the total utility as a time functions. On the left-hand side of this figure, we display the total *intrinsic* utility relative to $u_0 = w_1 + w_2$, which is the utility resulting from no contribution. On the right-hand side, we display the total utility also relative to $u_0$. Two types of curves are plotted in Figure 11 (left). In case (0.1,1) Player 1 contributes her total endowment very rapidly and total contribution surpasses $X^{SO}$ already in the first step. From this moment on Player 1 contributes $w_1$ while Player 2 continuously raises her contribution. Thus, intrinsic social welfare decays towards it long-run value. In the other two cases (with overshooting in total contribution) intrinsic utility rises until the moment when total contribution reaches $X^{SO}$. Further contributions above this level worsen intrinsic utility. Through the transition total contribution surpasses its long-run value, and hence, intrinsic utility falls below its long-run value and converges towards it from below. Moreover it is also

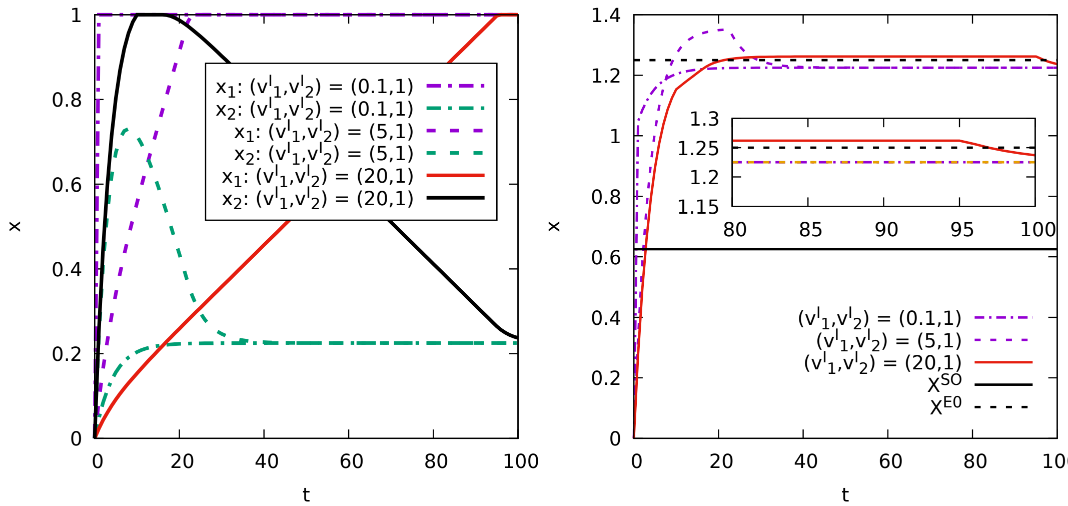


Figure 10: Individual (left) and total (right) contributions; case [bII.1] with welfare reduction then shortage; varying $v_1^{\mathrm{I}}$

possible (and it occurs for the chosen parameters) that overshooting in total contribution is so strong to overpass $X^{E0}$. Then, although the long-run equilibrium belongs to the IW-I region, total contribution pass to the IW-R region for some time through the transition, where the intrinsic utility is lower than with no contribution. Figure 11 (left) shows that this effect is more acute when Player 1 has a moderate inertia but it last longer when Player 1 has a strong inertia.

Figure 11 (right) focuses on the global utility, which adds the positional payoffs to the intrinsic utility. Indeed, the observed behavior is greatly driven by these positional payoffs. With no overshooting, Player 1 immediately reaches her equilibrium contribution, while Player 2 slowly increases hers. Thus, a positive gap in contributions is initially opened in favor of the player who values positionality the most, giving rise to positive positional payoffs for society as a whole. This gap is wider than the gap in the long run and so are the positional payoffs, implying greater social welfare through the transition than at the equilibrium. By contrast, when overshooting happens, it is the player who values positionality the least who increases initial contributions faster, giving rise to a gap in contributions with positional “losses” for society. This player eventually starts reducing her contributions. Thus, the gap between players’ contribution, initially in favor of Player 2, first shrinks and later turns into a gap in favor of Player 1, hence rising the positional payoff towards the long-run value. When Player 1 shows a strong inertia, even if the contributions to the public good are welfare enhancing in the long run, the initial positional losses can be so strong as to lead social welfare temporally below its value without contributions.

The previous example corresponds to the case of excess contribution in the long run, $X^{\mathrm{N}} > X^{\mathrm{SO}}$. This can be reversed by lowering the positional concern by Player 2 and/or raising the valuation of the public good by Player 1. The case of shortage in the equilibrium contributions (possibly the most realistic) is obtained for the alternative parameters:

$$\varepsilon = 0.8, \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{3}{4}, v_1^{\mathrm{P}} = \frac{13}{16}, v_2^{\mathrm{P}} = \frac{31}{64}.$$

Moreover, total endowments are modified with respect to the standard situation: $w_1 = w_2 =$

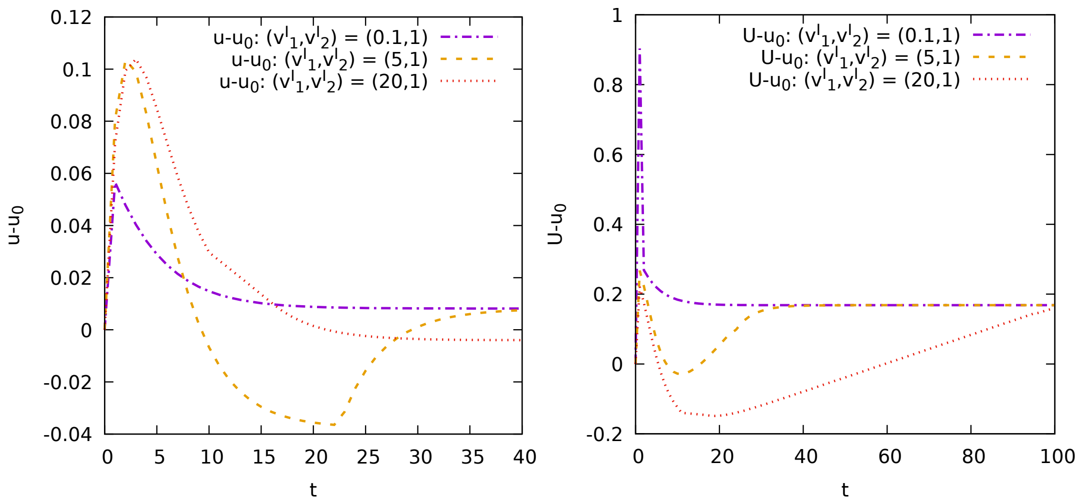


Figure 11: Total intrinsic utility $u$ (left) and total utility $U$ (right), relative to utility without consumption ($u_0 = w_1 + w_2$); case [bII.1] with welfare reduction then shortage; varying $v_1^{\mathrm{I}}$

0.35.

Three cases are studied depending on the players' inertia. In case 1, with $(v_1^{\mathrm{I}}, v_2^{\mathrm{I}}) = (2, 10)$ there is no overshooting. As shown in Figure 12 (right), the total contribution monotonously increases towards the contribution under the Nash equilibrium. Correspondingly, Figure 13 (left) shows that the intrinsic utility also converges to its long-run value from below. Cases 2 and 3, with $(v_1^{\mathrm{I}}, v_2^{\mathrm{I}})$ equal to (10,10) and (10,2), are characterized by overshooting: total contribution exceeds $X^N$ at a given time and decays towards this value thereon. Similarly, intrinsic utility surpasses its long-run value at this same time and converges towards it from above. Moreover, if overshooting is so strong as to surpass $X^{\mathrm{SO}}$ for a given interval, then the total intrinsic utility that reaches its maximum at the beginning of this interval falls and rises to reach again the social optimum at the end of this interval. From this moment on intrinsic utility again decreases converging to its long-run value from above. Figures 11 and 13 (right) show that the utility, $U$, in the case of shortage follows a similar behavior as in the case of excess contribution (both with or without overshooting).

### 6.3 Players with inertia; equilibrium of type [bII.3]

We now show an example of an equilibrium of type [bII.3] (that is, with $x^N = (A_1, 0)$), where overshooting occurs for Player 2. The parameters are:

$$\varepsilon = \frac{4}{10}, \alpha_1 = \frac{2}{3}, \alpha_2 = \frac{2}{3}, v_1^{\mathrm{P}} = \frac{8}{15}, v_2^{\mathrm{P}} = \frac{5}{12}, v_1^{\mathrm{I}} = 5, v_2^{\mathrm{I}} = 1.$$

With these values, we have:

$$A_1 = \frac{3}{4} \qquad A_2 = \frac{5}{16} \qquad x^N = \left(\frac{3}{4}, 0\right), \qquad X^{SO} = \frac{5}{8}.$$

The trajectory of individual and total contributions are shown on Figure 14. We see that the contribution of Player 2 indeed becomes initially positive and it turns down to 0 in a second phase. Player 1's contribution slowly converges to the static Nash equilibrium, $x_1^N = 3/4$, due

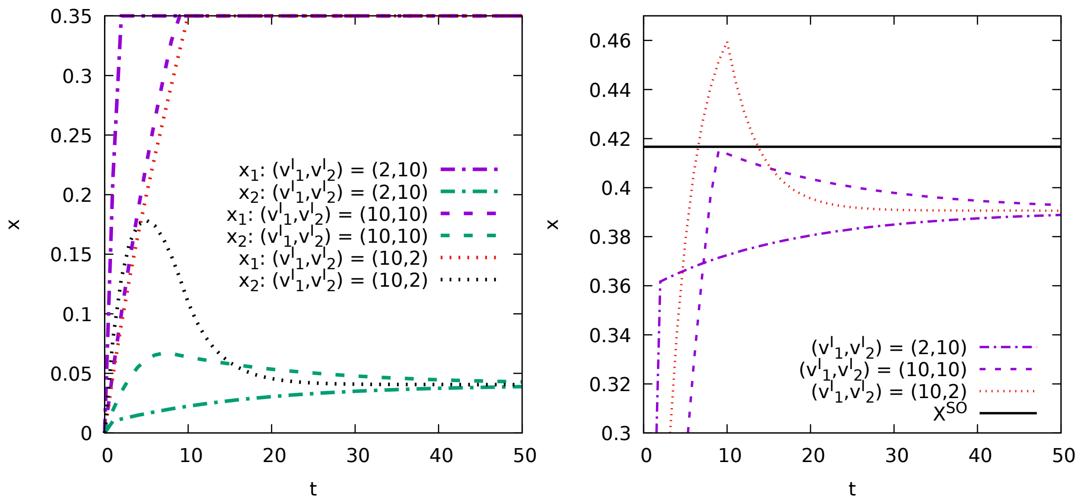


Figure 12: Individual (left) and total (right) contributions; case [bII.1] with excess then shortage; varying $v_1^{\mathrm{I}}$

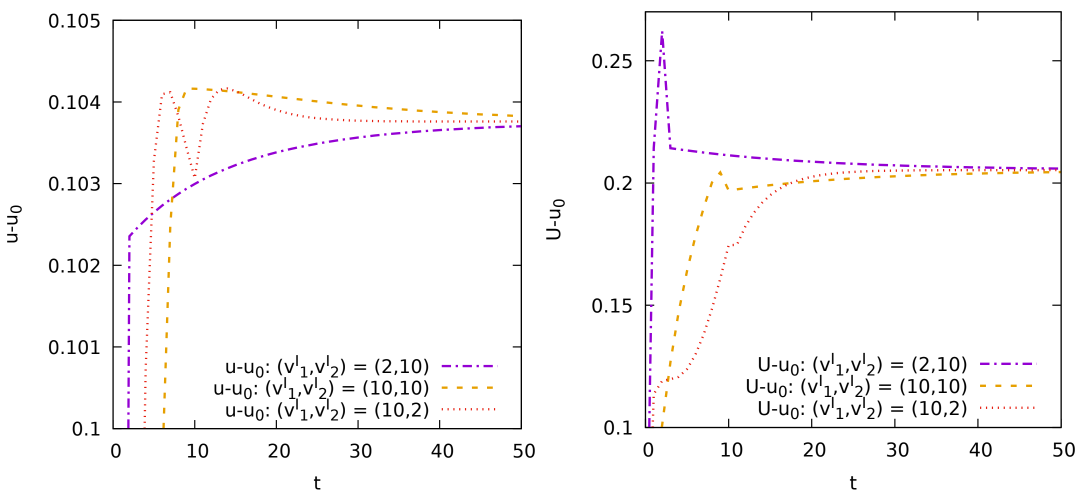


Figure 13: Total intrinsic utility $u$ (left) and total utility $U$ (right), relative to utility without consumption ($u_0 = w_1 + w_2$); case [bII.1] with excess then shortage; varying $v_1^{\mathrm{I}}$

to the large inertia parameter $v_1^{\mathrm{I}}$ and relatively small value of $\varepsilon\alpha_1$ (specifically, $v_1^{\mathrm{I}}/(v_1^{\mathrm{I}}+\varepsilon\alpha_1) = 75/79$, see Appendix F.2). For the specified parameters values, total contribution increases steadily towards its long-run value that surpasses the social optimum $X^{\mathrm{N}} = 0.75 > X^{\mathrm{SO}} = 0.625$. Correspondingly, the global intrinsic utility grows until the moment when the total contribution reaches $X^{\mathrm{SO}}$ and decays to its long-run value thereafter.

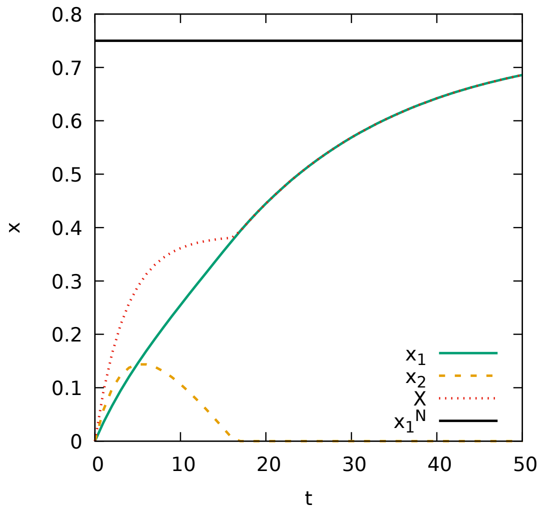


Figure 14: Overshooting for a static equilibrium; case [bII.3]

## 7 Conclusions

Conspicuous consumption of a private good is typically associated with too high consumer spending to maintain social status and hence inefficiency in terms of social welfare. However, when the positional good is the private provision of a public good then, positional concerns can induce positive contributions by selfish agents. This is analyzed by a two-player game where each agent gets joy from absolute consumption (intrinsic utility) but also receives a positional payoff from contributing more than her opponent. Players can be asymmetric in their positional concerns and their public good valuation. In consequence, they typically wish for different amount of public good.

The strategic interaction between the two agents is analyzed first as a one-shot game. Players positively contribute if their positional concerns are strong enough. In that case different equilibria are possible, depending on the players' positional concerns, on endowments and on public good valuation. The player who wishes the most always provides a positive amount of public good. Conversely, the player who wishes the least only contributes if she finds that some amount is still missing once the contribution by the other player is accounted for.

Intrinsic utility increases with public good consumption at a decreasing rate. Too large positional concerns can induce too much contribution placing the intrinsic utility above its social optimum. We characterize the conditions for shortage or excess contribution. Moreover, we also show that exceedingly large positional concerns can push contribution too high so that intrinsic utility falls below the no contribution case.

If players differ in their positional concerns (but equally value the public good), then the region in the parameters' space where positive contributions increase intrinsic utility is wider than the region where it decreases, unless people satiate fast and give a small value to public good consumption. If public good valuation is also asymmetric, then a rise in the intrinsic utility with positional effects is more likely if the player with higher positional concern also values the public good the most and *vice versa*.

Social welfare is defined as the addition of the total intrinsic utility plus the positional payoffs. For this global measure the quest for status when the positional good is the contribution to a public good very likely enhances social welfare. This is true in the symmetric case as long as positional concerns are not too large and too even at the same time. In the asymmetric case, if the player with the greatest positional concern also values public good consumption the most, then positionality can worsen social welfare if the two players give similar importance to status, even when this is small.

The second part of the paper analyzes the strategic interaction in a dynamic setting, assuming that players are reluctant to changes, i.e. show inertia from previous actions. Moreover, the status concern is defined with respect to the opponent's previous contribution. Although players are concerned on previous actions, they act myopically. We first characterize the optimal trajectory of the dynamical system both in the interior as well as in the boundary. It is proved that the contributions converge to one of the static Nash equilibria. Then we present some numerical simulation for each type of Nash equilibrium, assuming for simplicity that players are symmetric in their valuation of public good consumption. Starting from the initial situations of zero contribution (which constitutes the Nash equilibrium without positional effects), the player with the greatest positional concern monotonously raises her contribution. The contribution of the player with the smallest positional concern can also rise monotonously or, if she shows relatively less inertia that her opponent, then her contribution can show overshooting. If strong enough, overshooting by this player can induce overshooting in total contribution.

When overshooting happens for the total contribution, its effect on the trajectory for the intrinsic utility depends on whether the total contribution in the long-run (Nash equilibrium) is greater or lower than its value at the social optimum. When overshooting occurs, and if contributions are in shortage in the long-run, then total intrinsic utility along the transition is typically above its long-run value, contrary to the case without overshooting, characterized by convergence from below. The opposite is true if contributions are in excess in the long run.

The trajectory for the social welfare, which encompasses intrinsic utility plus the positional payoffs, is strongly determined by these latter. Without overshooting, and regardless of whether the long-run equilibrium is in shortage or in excess, a gap in the players' contributions is initially opened in favor of the player who values contribution the most, raising positional payoffs and social welfare. As this gap narrows, social welfare decreases towards its long-run value. Overshooting is generated by the player with the smallest positional concern who initially contributes but reduces her contribution from a given time on, widening the contribution gap in favor of the player with the greatest positional concern. In consequence, from this moment on, positional payoffs for society rise and so does social welfare, that converges to its long-run value from below.

The simulations carried out for the dynamic model show that it is possible that contributions become temporary intrinsic welfare reducing within the transition, although in the long-run they improve intrinsic welfare. Examples have been found when the long-run equilibrium is characterized by excess contribution, but not in the more realistic case of shortage contribution. A better understanding of the dynamic model could shed light on whether this is actually possible.

The analysis presented in this paper opens the doors to several extensions. A first extension would be the assumption of multiple agents. With more than two players, the positional concern of a given player could be defined as the result from the comparison of her contribution against either the average or the maximum of all other players' contributions. We believe that the first alternative is probably suited for an analytical solution, at least in the quadratic framework of the present paper. The second alternative looks more challenging.

Another direction for extensions is towards the introduction of other kinds of subjective behavior. When taking into account the influence of social context, we have focused on the "snob effect", or the desired of the people for exclusivity, or to distinguish themselves from the "common herd". However, one could also focus on the desire of some consumers to be "in style", known in the literature as the "bandwagon effect". Likewise, one could think of the existence of individuals with more pro-social behavior.

In the dynamic model, we have started by considering myopic agents. An interesting extension would consider farsighted agents, who care not only on other players' current or past contributions, but also on the accumulated amount.

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# A Equilibria of the static game

**Proof of Proposition 1.** Let $(x_1^{\mathrm{N}}, x_2^{\mathrm{N}})$ be a Nash equilibrium. Not considering the constraints on contributions $x_i$, the first-order conditions are:

$$0 = \frac{\partial U_i}{\partial x_i} = -1 + b_i'(x_1 + x_2) + v_i^{\mathrm{P}}, \quad i = 1, 2. \tag{25}$$

Since $b_i'(\cdot)$ is strictly decreasing ($b_i$ is strictly concave), there is at most one solution $X \geq 0$ to the equation $b_i'(X) = 1 - v_i^{\mathrm{P}}$. Assume first that there exists, for each $i = 1, 2$, such a contribution $A_i \geq 0$ such that $b_i'(A_i) = 1 - v_i^{\mathrm{P}}$. Then if the Nash equilibrium solves (25), this implies $X^{\mathrm{N}} = x_1^{\mathrm{N}} + x_2^{\mathrm{N}} = A_i$ for $i = 1, 2$. If $A_i = A_j \geq 0$, every couple $(x_1, x_2)$ that satisfies this equality and the constraints on contributions, is a Nash equilibrium. This is statement $bI$).

On the other hand, if $A_i \neq A_j$, there are no solutions to this equation and hence no interior solution. We then have to compute the best response functions. The best response of Player $i$ to Player $j$'s play $x_j$ is:

$$x_i^b(x_j) = \begin{cases} 0 & \text{if } A_i \leq x_j \\ A_i - x_j & \text{if } A_i - w_i \leq x_j \leq A_i \\ w_i & \text{if } x_j \leq A_i - w_i. \end{cases} \tag{26}$$

A general diagram of the superposition of both Players' best responses is as in Figure 15.

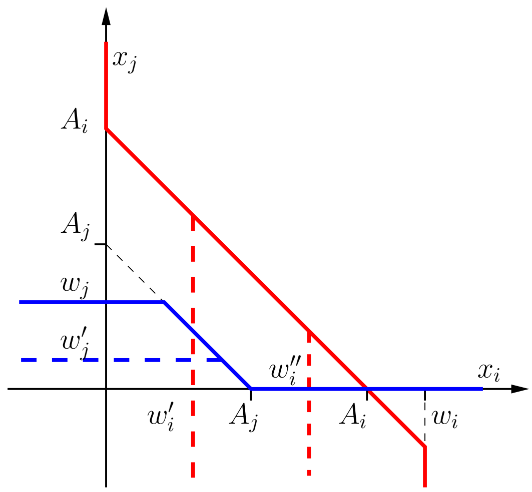


Figure 15: Superposition of best responses

From this figure, we derive the following cases. Without loss of generality, $A_i > A_j$: the reverse case can be obtained by exchanging the roles of $i$ and $j$.

- If $\min\{A_i, A_j\} \geq w_1 + w_2$, there is a unique intersection at: $x_i = w_i$, $i = 1, 2$. This configuration is marked with $w_i'$ and $w_j'$ in Figure 15. This is statement $a$);
- If $A_i \leq 0$ and $A_j \leq 0$, there is a unique intersection at $x_i = 0$, $i = 1, 2$. This is statement $c$).

- If $w_i \leq A_j < A_i$ but $w_1 + w_2 > A_j$, there is a unique intersection at:

$$x_i = w_i, \quad x_j = A_j - w_i.$$

This is the configuration marked with $w_i'$ and $w_j$ in Figure 15. It corresponds to statement [bII.1].

- If $A_j \leq w_i \leq A_i$ and $A_j < A_i$, there is a unique Nash equilibrium at:

$$x_i = w_i, \quad x_j = 0.$$

This is the configuration marked with $w_i''$ in the figure. It corresponds statement [bII.2].

- If $w_i \geq A_i > A_j$, there is a unique Nash equilibrium at:

$$x_i = A_i, \quad x_j = 0.$$

This is the configuration marked with $w_i$ and $w_j$ in the figure. It corresponds to statement [bII.3].

There remains to consider the situation where no positive contribution $A_i$ solves the equation $0 = -1 + v_i^{\mathrm{P}} + b_i'(A_i)$, at least for one $i \in \{1, 2\}$. If $b_i'(X) < -1 + v_i^{\mathrm{P}}$ for all $X \geq 0$, the best response $x_i^b(x_j)$ is always 0. The preceding reasoning applies with $A_i = 0$. On the other hand, if $b_i'(X) > -1 + v_i^{\mathrm{P}}$ for all $X$, the best response of Player $i$ is always $w_i$. The preceding reasoning applies then with $A_i = w_i$. This concludes the proof of Proposition 1. ■

As a complement, this parametric discussion is synthesized again in Figure 16 with $A_i$ and $A_j$ as variables instead of $w_i$ and $w_j$. The situation illustrated is such that $w_j \leq w_i$. The reverse situation can be obtained by exchanging Player $i$ and $j$.

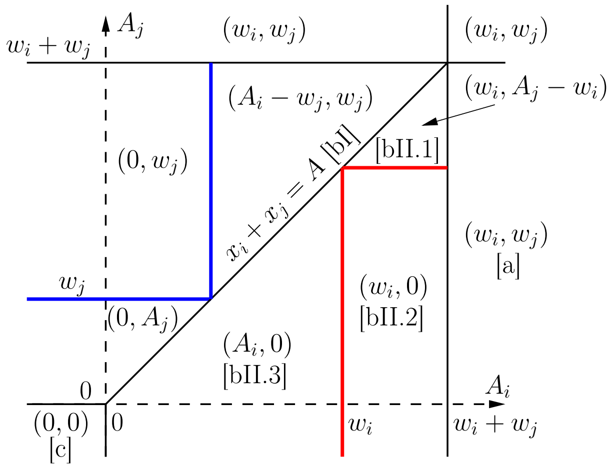


Figure 16: Nash equilibria as a function of $(A_i, A_j)$

# B Proof of Proposition 2

**Proof.** In the symmetric case the assumption $A_i > A_j$ is equivalent to $v_i^{\mathrm{P}} > v_j^{\mathrm{P}}$. Under this assumption the three equilibria in [bII] and the conditions for IW-I or IW-R can be summarized as:
with $\hat{\Theta} = 1 - \alpha + 2\alpha\varepsilon w = \Theta + \alpha\varepsilon w > \Theta$.

| | [bII.1] | [bII.2] | [bII.3] |
|---|---|---|---|
| Condition | $\Theta < v_j^{\mathrm{P}} < \hat{\Theta}$ | $v_j^{\mathrm{P}} \le \Theta < v_i^{\mathrm{P}}$ | $v_j^{\mathrm{P}} < v_i^{\mathrm{P}} < \Theta$ |
| IW-I | $v_j^{\mathrm{P}} < \alpha$ | $\varepsilon < \hat{\varepsilon}$ | $v_i^{\mathrm{P}} < \alpha$ |
| IW-R | $v_j^{\mathrm{P}} > \alpha$ | $\varepsilon > \hat{\varepsilon}$ | $v_i^{\mathrm{P}} > \alpha$ |

1. If $\varepsilon < \hat{\varepsilon}(\Theta < \alpha)$, then

$$\begin{array}{lll} [bII.1] & \Theta < \min\{v_j^{\mathrm{P}}, \alpha\} & \Rightarrow \; IW - I(v_j^{\mathrm{P}} < \alpha) \text{ or } IW - R(v_j^{\mathrm{P}} > \alpha) \\ [bII.2] & w > X^{E0} & \Rightarrow \; IW - I \\ [bII.3] & v_i^{\mathrm{P}} < \Theta < \alpha & \Rightarrow \; IW - I \end{array}$$

2. If $\varepsilon > \hat{\varepsilon}(\alpha < \Theta)$, then

$$\begin{array}{lll} [bII.1] & \alpha < \Theta < v_j^{\mathrm{P}} & \Rightarrow \; IW - R \\ [bII.2] & w < X^{E0} & \Rightarrow \; IW - R \\ [bII.3] & \Theta > \max\{v_i^{\mathrm{P}}, \alpha\} & \Rightarrow \; IW - R(v_i^{\mathrm{P}} > \alpha) \text{ or } IW - I(v_i^{\mathrm{P}} < \alpha). \end{array}$$

The areas in the $(v_i^{\mathrm{P}}, v_j^{\mathrm{P}})$ plane of the IW-I and IW-R regions in cases [a] and [b], where some public good is provided can be compared as follows:

1. If $\varepsilon < \hat{\varepsilon} \iff \Theta < \alpha$,

$$IW - I = \frac{1}{2} - (1-\alpha)^2 \quad IW - R = \frac{(1-\alpha)^2}{2}.$$

   Region IW-I is larger or smaller than IW-R if and only if $(1-\alpha)^2 \lesseqgtr 1/3$. Since $(1-\alpha)^2 \le 1/4$ for $\alpha \in (1/2, 1)$, one gets $IW - I > IW - R$ for all feasible $\alpha \in (1/2, 1)$.

2. If $\varepsilon > \hat{\varepsilon} \iff \alpha < \Theta$,

$$IW - I = \frac{2\alpha - 1}{2}, \quad IW - R = \frac{1-\alpha^2}{2}.$$

   Region $IW - I$ is larger or smaller than $IW - R$ if and only if $\alpha^2 + 2\alpha - 2 \gtreqless 0$. This convex parabola has a negative root, $-1-\sqrt{3}$, and a positive root, $-1+\sqrt{3}$. Because $\alpha > 1/2$, one gets:

$$\begin{array}{ll} \alpha \in \left(\frac{1}{2}, \sqrt{3}-1\right) & IW - I < IW - R, \\ \alpha \in \left(\sqrt{3}-1, 1\right) & IW - I > IW - R. \end{array}$$

■

# C Proof of Proposition 3

**Proof.** The slope of the isoclines for the social welfare, in the symmetric case can be computed from (12) as:

$$(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) \equiv -\frac{\frac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}}}{\frac{dUv}{dv_i^{\mathrm{P}}}}, = \begin{cases} 1 - \alpha \dfrac{v_i^{\mathrm{P}} + v_j^{\mathrm{P}} - 1}{v_j^{\mathrm{P}} - \hat{\Theta}} & [bII.1], \\ 1 & [bII.2], \\ \dfrac{v_i^{\mathrm{P}} - (1-\alpha)}{\alpha - v_j^{\mathrm{P}}} & [bII.3]. \end{cases} \tag{27}$$

In cases [bII.1] and [bII.2] $\dfrac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}} > 0$ and hence $\dfrac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}} + \dfrac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}} \gtreqless 0$ is equivalent to $(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) \lesseqgtr 1$. Case [bII.1] is characterized by $v_j^{\mathrm{P}} < \hat{\Theta}$ and then $(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) \lesseqgtr 1 \Leftrightarrow v_i^{\mathrm{P}} + v_j^{\mathrm{P}} \lesseqgtr 1$. In case [bII.2] $(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}})$ is always equal to one.

In case [bII.3] $\dfrac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}} < 0$ and

$$\operatorname{sign}(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) = \operatorname{sign}\left(\frac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}}\right) = \operatorname{sign}\left(\alpha - v_j^{\mathrm{P}}\right).$$

Thus, for $v_j^{\mathrm{P}} < \alpha$, $\dfrac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}} + \dfrac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}} \gtreqless 0$ is again equivalent to $(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) \lesseqgtr 1$ i.e. $v_i^{\mathrm{P}} + v_j^{\mathrm{P}} \lesseqgtr 1$. However, for $v_j^{\mathrm{P}} > \alpha$ (the case with $(v_i^{\mathrm{P}})'(v_j^{\mathrm{P}}) < 0$), $\operatorname{sign}\left(\frac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}}\right) = \operatorname{sign}\left(\frac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}}\right) < 0$ and in consequence $\dfrac{dU^{\mathrm{N}}}{dv_i^{\mathrm{P}}} + \dfrac{dU^{\mathrm{N}}}{dv_j^{\mathrm{P}}} < 0$. Note that in this case, because we are assuming $v_i^{\mathrm{P}} > 1 - \alpha$, it immediately follows that $v_i^{\mathrm{P}} + v_j^{\mathrm{P}} > 1$. ■

# D Proof of statement [R0] in Proposition 4

**Proof.** The best reaction function of the static case is (see (26))

$$r_i^S(x_j) = \begin{cases} w_i & \text{if } r_i^b(x_j) \geq w_i \\ r_i^b(x_{jt}) & \text{if } 0 \leq r_i^b(x_j) \leq w_i \\ 0 & \text{if } r_i^b(x_j) \leq 0, \end{cases} \quad \text{where} \quad r_i^b(x_j) = A_i - x_j \tag{28}$$

and the Nash static equilibrium verifies $x_i = r_i^S(x_j)$ for $i, j \in \{1,2\}$, $i \neq j$.
If the dynamic process has a steady state, at the steady state the reaction function reads $r_i^{D\infty}(x_i, x_j) = r_i^D(x_j; x_i)$, with:

$$r_i^{D\infty}(x_i, x_j) = \begin{cases} w_i & \text{if } r_i^\infty(x_i, x_j) \geq w_i \\ r_i^\infty(x_i, x_j) & \text{if } 0 \leq r_i^\infty(x_i, x_j) \leq w_i \\ 0 & \text{if } r_i^\infty(x_i, x_j) \leq 0, \end{cases} \text{ where } r_i^\infty(x_i, x_j) = \frac{\alpha_i \varepsilon (A_i - x_j) + v_i^{\mathrm{I}} x_i}{v_i^{\mathrm{I}} + \alpha_i \varepsilon}, \tag{29}$$

and the steady state satisfies $x_i^\infty = r_i^{D\infty}(x_i^\infty, x_j^\infty)$ for $i \neq j$.

We are going to see that

$$x_i^\infty = r_i^{D\infty}(x_i^\infty, x_j^\infty) \quad \Longleftrightarrow \quad x_i^\infty = r_i^S(x_j^\infty).$$

In fact

$$0 < r_i^\infty(x_i^\infty, x_j^\infty) < w_i \quad \Longleftrightarrow 0 < x_i^\infty = \frac{\alpha_i \varepsilon (A_i - x_j^\infty) + v_i^{\mathrm{I}} x_i^\infty}{v_i^{\mathrm{I}} + \alpha_i \varepsilon} < w_i,$$

$$\Longleftrightarrow \quad w_i(v_i^{\mathrm{I}} + \alpha_i \varepsilon) > \frac{\Phi_i - \alpha_i \varepsilon x_j^\infty}{\alpha_i \varepsilon} > 0 \quad \Longleftrightarrow \quad w_i > r_i^b(x_j^\infty) > 0,$$

$$\Longleftrightarrow \quad 0 < x_i^\infty = \frac{\Phi_i - \alpha_i \varepsilon x_j}{\alpha_i \varepsilon} < w_i \quad \Longleftrightarrow \quad x_i^\infty = r_i^S(x_j^\infty).$$

The same reasoning applies if $r_i(x_j^\infty) \leq 0$ or $r_i(x_j^\infty) \geq w_i$. ■

# E Analysis of the free dynamics

## E.1 General principles

Consider a linear recurrence of the form:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M \begin{pmatrix} x_{it-1} \\ x_{jt-1} \end{pmatrix} + \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \tag{30}$$

which is the case of (21). The general solution to recurrence (30) is:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \left(M^{t-1} + M^{t-2} + \dots + I\right) \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix}. \tag{31}$$

The general analysis of powers of matrices concludes that, in the case where matrix $M$ has two distinct eigenvalues, $\lambda_1$ and $\lambda_2$, or one unique eigenvalue but with an eigenspace of dimension 2, there exist two rank-1 matrices $M_1$ and $M_2$ such that for any integer $n$,

$$M^n = M_1 \lambda_1^n + M_2 \lambda_2^n \ .$$

This is known as a spectral decomposition. By convention, $|\lambda_2| \le |\lambda_1|$.

In the practical situations we face in this paper, it is known that $|\lambda_1| \le 1$ and $|\lambda_2| < 1$. When replacing the spectral decomposition in (31), three cases should be distinguished.

$\lambda_1 = 1$ **and** $|\lambda_2| < 1$**:** then

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M_1 \left[ \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + t \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right] + M_2 \left[ \lambda_2^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{1-\lambda_2^t}{1-\lambda_2} \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right]. \tag{32}$$

$\lambda_1 = -1$ **and** $|\lambda_2| < 1$**:** then

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M_1 \left[ (-1)^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{1-(-1)^t}{2} \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right] + M_2 \left[ \lambda_2^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{1-\lambda_2^t}{1-\lambda_2} \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right]. \tag{33}$$

$|\lambda_1| < 1$ **and** $|\lambda_2| < 1$**:** then

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = M_1 \left[ \lambda_1^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{1-\lambda_1^t}{1-\lambda_1} \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right] + M_2 \left[ \lambda_2^t \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{1-\lambda_2^t}{1-\lambda_2} \begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix} \right]. \tag{34}$$

This last form will not be used in the present paper.

## E.2 Application to the positional dynamics

There are three dynamics that can be constructed with the positional game, depending on how the utility of players at time $t$ depends on present and past contributions. The main dynamics (defined with (13) in Section 5) has the intrinsic utility based on present values and the positional part based on the past value of the opponent; an alternative "based on present values" would have the positional term depend on the present contribution of the opponent; finally, an alternative "based on past values" would have all the utility depend solely on the past value of the opponent's contributions.

As far as the definition of the dynamics is concerned, the main dynamics of Section 5 and the alternative based on present values are equivalent: this is due to the fact that the first-order conditions for both players are the same. On the other hand, the alternative based on past values leads to a distinct dynamics. Observe however that when each player bases their utility on past contributions of opponent, there is no game to be solved anymore: players simply solve an optimal control problem.

In the following analysis, we first derive general formulas that apply to all variants of the dynamics. We then specialize these formulas to the main dynamics of Section 5 (in Section E.2.3), then to the alternative based on past values (in Section E.2.4).

### E.2.1 Features common to all dynamics

The dynamics studied here being linear, we adopt a matrix/vector notation. We will denote with $x'$ the transpose of vector $x$.

The recurrences obtained by solving the first-order equations can be written as: $x_t = Mx_{t-1} + V^0$. By construction, in all cases the matrix $M$ has $\lambda_1 = 1$ as an eigenvalue, and $u = (1, -1)'$ as corresponding eigenvector. This implies (from the spectral decomposition introduced in Section E.1) that there exists a row vector $v'$ and a column vector $\bar{v}$ perpendicular to $v'$, such that $M_1 = u.v'$ and $M_2 = \bar{v}.\bar{u}'$ with $\bar{u}' = (1, 1)$, and $M = M_1 + M_2$. It also holds that $v'.u = \bar{u}'.\bar{v} = 1$.

The solution of the recurrence is then given by (32). We reorganize this expression as follows, using the property that $M_1 + M_2 = I$:

$$\begin{aligned} x_t &= M_1(x_0 + tV^0) + M_2\left(\lambda_2^t x_0 + \frac{1-\lambda_2^t}{1-\lambda_2}V^0\right) \\ &= x_0 + M_1V^0\, t + M_2\left(-x_0 + \lambda_2^t x_0 + \frac{1-\lambda_2^t}{1-\lambda_2}V^0\right) \\ &= x_0 + (v'.V^0)\, t\begin{pmatrix}1\\-1\end{pmatrix} + (1-\lambda_2^t)\left(\frac{\bar{u}'.V^0}{1-\lambda_2} - \bar{u}'.x_0\right)\bar{v}. \end{aligned} \tag{35}$$

The *total contribution* $X_t = x_{it} + x_{jt} = \bar{u}'.x_t$ is deduced from (35) as:

$$\begin{aligned} X_t &= X_0 + (1-\lambda_2^t)\left(\frac{\bar{u}'.V^0}{1-\lambda_2} - X_0\right)\bar{u}'.\bar{v} \\ &= X_0 + (1-\lambda_2^t)\,(X^* - X_0)\ , \end{aligned} \tag{36}$$

where

$$X^* = \frac{\bar{u}'.V^0}{1-\lambda_2}. \tag{37}$$

In the case where $|\lambda_2| < 1$, this sequence converges to $X^*$ which can then be interpreted as a *limit contribution*. Finally, introducing $\delta = v'.V^0$, we can write the solution of the dynamics as:

$$x_t = x_0 + \delta t u + (1-\lambda_2^t)\,(X^* - X_0)\ \bar{v}. \tag{38}$$

The geometric interpretation is that points of the trajectory are obtained from the initial position with a displacement proportional to $t$ in the direction $u = (1, -1)'$, plus a displacement in the direction $\bar{v}$.

### E.2.2 Overshooting

Overshooting occurs when the contribution of a player exceeds that of the "ideal" or "equilibrium" situation at some point of the dynamics, given that the contribution is initially lower.

**Overshooting for the total contribution.** In order to analyze the possibility of overshooting for a part of the trajectory that lies inside the domain of constraints for contributions, consider first the total contribution $X_t$, given by (36), or equivalently by:

$$X^* - X_t \;=\; \lambda_2^t \,(X^* - X_0)\,.$$

Since the initial contribution is assumed to be lower than the "ideal" one, represented here by $X^*$, we have: $X^* - X_0 > 0$. In both dynamic systems, it turns out that $-1 \le \lambda_2 < 1$. If $\lambda_2 \in [0,1)$, the sequence $X_t$ is then monotone and no overshooting occurs. On the other hand, if $\lambda_2 \in [-1,0)$, the sign of $X^* - X_t$ alternates between positive and negative, and overshooting necessarily occurs for at least one of the players.

When $\lambda_2 \in [0,1)$, the sequence of total contributions is increasing monotonously, as long as it stays inside the domain of constraints. However, when one of the constraints is met, the dynamics change, and it is possible that this sequence of total contributions decrease. Overshooting can therefore occur because of this phenomenon.

**Overshooting for individual contributions.** We continue the discussion under the hypothesis that $\lambda_2 \in [0,1)$: there is still the possibility that overshooting occurs for one of the players, in the part of the trajectory that lies inside the domain. The explicit solution for the dynamics is given by (35) or (38). Taking the difference between values $x_t$ and $x_{t-1}$, we have:

$$x_t - x_{t-1} \;=\; \delta u + \lambda_2^{t-1}(1-\lambda_2)(X^* - X_0)\bar{v}\ .$$

As it turns out, the components of vector $\bar{v}$ are both nonnegative, and have seen that $u = (1,-1)'$. Since $u_i = 1$, the component of vector $x_t - x_{t-1}$ corresponding to player $i$ is always positive. This means that contributions of this player are monotonously increasing and no overshooting occurs for her. For the component corresponding to player $j$, we have the equivalences:

$$\begin{aligned} (x_t - x_{t-1})_j &\lesseqgtr 0 \\ -\delta + \lambda_2^{t-1}(1-\lambda_2)(X^* - X_0)\bar{v}_j &\lesseqgtr 0 \\ \lambda_2^{t-1} &\lesseqgtr \frac{\delta}{(1-\lambda_2)(X^* - X_0)\bar{v}_j}\ . \end{aligned} \tag{39}$$

Overshooting occurs if, and only if, the inequality varies from ">" to "<" as $t$ increases. This in turn is equivalent to right-hand side of (39) being in the interval $(0,1)$. In that case indeed, the sequence $\lambda_2^{t-1}$ decreases from 1 to 0 and the direction of the inequality changes. The necessary and sufficient condition for overshooting is therefore:

$$0 < \delta < (1-\lambda_2)(X^* - X_0)\bar{v}_j. \tag{40}$$

Observe that if $A_i = A_j$, that is, if $\delta = 0$, no overshooting can occur: the sequence of contribution of both players is monotonously increasing.

### E.2.3 Application to the main dynamics

Consider the dynamics specifically defined by (21). In this situation, the spectral decomposition of matrix $M$ takes the form:

$$M = \underbrace{\frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_j v_i^{\mathrm{I}} & -\alpha_i v_j^{\mathrm{I}} \\ -\alpha_j v_i^{\mathrm{I}} & \alpha_i v_j^{\mathrm{I}} \end{pmatrix}}_{M_1} + \underbrace{\frac{v_i^{\mathrm{I}} v_j^{\mathrm{I}}}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})}}_{\lambda_2} \underbrace{\frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_i v_j^{\mathrm{I}} & \alpha_i v_j^{\mathrm{I}} \\ \alpha_j v_i^{\mathrm{I}} & \alpha_j v_i^{\mathrm{I}} \end{pmatrix}}_{M_2}. \tag{41}$$

In the notation of the general solution (35), we have:

$$\lambda_2 = \frac{v_i^{\mathrm{I}} v_j^{\mathrm{I}}}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})}, \quad v' = \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_j v_i^{\mathrm{I}} & -\alpha_i v_j^{\mathrm{I}} \end{pmatrix}, \quad \bar{v} = \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \begin{pmatrix} \alpha_i v_j^{\mathrm{I}} \\ \alpha_j v_i^{\mathrm{I}} \end{pmatrix}. \tag{42}$$

The vector $V^0$ of the dynamics (21) is here:

$$V^0 = \frac{\varepsilon}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \begin{pmatrix} v_j^{\mathrm{I}} \alpha_i A_i + \varepsilon \alpha_i \alpha_j (A_i - A_j) \\ v_i^{\mathrm{I}} \alpha_j A_j + \varepsilon \alpha_i \alpha_j (A_j - A_i) \end{pmatrix}.$$

Still referring to the explicit solution of the recurrence given by (35), we evaluate the leading coefficient $v.V^0$. It is simplified as:

$$\begin{aligned}
\delta := v'.V^0 &= \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \frac{\varepsilon}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \begin{pmatrix} \alpha_j v_i^{\mathrm{I}} & -\alpha_i v_j^{\mathrm{I}} \end{pmatrix} \begin{pmatrix} v_j^{\mathrm{I}} \alpha_i A_i + \varepsilon \alpha_i \alpha_j (A_i - A_j) \\ v_i^{\mathrm{I}} \alpha_j A_j + \varepsilon \alpha_i \alpha_j (A_j - A_i) \end{pmatrix} \\
&= \frac{\varepsilon}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \frac{\alpha_j v_i^{\mathrm{I}}(v_j^{\mathrm{I}} \alpha_i A_i + \varepsilon \alpha_i \alpha_j (A_i - A_j)) - \alpha_i v_j^{\mathrm{I}}(v_i^{\mathrm{I}} \alpha_j A_j + \varepsilon \alpha_i \alpha_j (A_j - A_i))}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \\
&= \frac{\varepsilon}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} \frac{\alpha_i \alpha_j v_i^{\mathrm{I}} v_j^{\mathrm{I}} (A_i - A_j) + \varepsilon \alpha_i \alpha_j (A_i - A_j)(\alpha_j v_i^{\mathrm{I}} + \alpha_i v_j^{\mathrm{I}})}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \\
&= \frac{\varepsilon \alpha_i \alpha_j (A_i - A_j)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}}.
\end{aligned} \tag{43}$$

Next, the limit contribution $X^*$ defined in (37) is:

$$\begin{aligned}
X^* &= \frac{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})}{\varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \frac{\varepsilon}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} v_j^{\mathrm{I}} \alpha_i A_i + \varepsilon \alpha_i \alpha_j (A_i - A_j) \\ v_i^{\mathrm{I}} \alpha_j A_j + \varepsilon \alpha_i \alpha_j (A_j - A_i) \end{pmatrix} \\
&= \frac{v_j^{\mathrm{I}} \alpha_i A_i + v_i^{\mathrm{I}} \alpha_j A_j}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}}.
\end{aligned} \tag{44}$$

This is a convex combination of the wished amounts $A_i$ and $A_j$, so the quantity $X^*$ lies between these values.

**Overshooting.** Applying (40) to this case, we find that the necessary and sufficient condition for overshooting (of Player $j$'s contribution) can be written as:

$$0 < \frac{\varepsilon \alpha_i \alpha_j (A_i - A_j)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} < \frac{\varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})}{v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})} (X^* - X_0) \frac{\alpha_j v_i^{\mathrm{I}}}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}}$$

$$0 < \alpha_i (A_i - A_j)(v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})) < (\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}) \left( \frac{v_j^{\mathrm{I}} \alpha_i A_i + v_i^{\mathrm{I}} \alpha_j A_j}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}} - X_0 \right) v_i^{\mathrm{I}}$$

$$0 < \alpha_i (A_i - A_j)(v_i^{\mathrm{I}} v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})) < v_i^{\mathrm{I}}(\alpha_i v_j^{\mathrm{I}}(A_i - X_0) + \alpha_j v_i^{\mathrm{I}}(A_j - X_0)) \; .$$

So, in addition to the necessary condition $A_i \geq A_i$, we must have:

$$\begin{aligned}
\alpha_i(A_i - A_j)(v_i^{\mathrm{I}}v_j^{\mathrm{I}} + \varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})) &< v_i^{\mathrm{I}}(\alpha_i v_j^{\mathrm{I}}(A_i - A_j) + (\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})(A_j - X_0)) \\
\alpha_i(A_i - A_j)\varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}})) &< v_i^{\mathrm{I}}(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}}))(A_j - X_0) \\
A_i - A_j &< \frac{v_i^{\mathrm{I}}}{\varepsilon\alpha_i}(A_j - X_0) \ . \qquad (45)
\end{aligned}$$

### E.2.4 Application to the dynamics based on past values

Alternative dynamics can be constructed if the assumptions in (13) are different. Each agent could use the last observation of the opponent's action to estimate the total contribution to the PG. Conversely, each agent could use the current play of the opponent in the subjective part.

In the latter case, the only difference in the utility functions is the term "$v_i^{\mathrm{P}}(x_{it} - x_{jt-1})$" which becomes "$v_i^{\mathrm{P}}(x_{it} - x_{jt})$". However, this does not modify the first-order conditions of the player's optimization problem. The solution is then exactly the one studied in sections E.2.3.

In the former case, the problem is effectively modified. Consider the utility for Player $i$:

$$U_i(\bullet) = w_i - x_{it} + \alpha_i\left[x_{it} + x_{jt-1} - \frac{\varepsilon}{2}(x_{it} + x_{jt-1})^2\right] - \frac{v_i^{\mathrm{I}}}{2}(x_{it} - x_{it-1})^2 + v_i^{\mathrm{P}}(x_{it} - x_{jt-1}). \qquad (46)$$

Solving for the first-order equations, we obtain a recurrence of the form $x_t = Mx_{t-1} + V^0$ with elements:

$$M = \begin{pmatrix} \frac{v_i^{\mathrm{I}}}{\varepsilon\alpha_i + v_i^{\mathrm{I}}} & -\frac{\varepsilon\alpha_i}{\varepsilon\alpha_i + v_i^{\mathrm{I}}} \\ -\frac{\varepsilon\alpha_j}{\varepsilon\alpha_j + v_j^{\mathrm{I}}} & \frac{v_j^{\mathrm{I}}}{\varepsilon\alpha_j + v_j^{\mathrm{I}}} \end{pmatrix} \qquad V^0 = \begin{pmatrix} \frac{\varepsilon\alpha_i A_i}{\varepsilon\alpha_i + v_i^{\mathrm{I}}} \\ \frac{\varepsilon\alpha_j A_j}{\varepsilon\alpha_j + v_j^{\mathrm{I}}} \end{pmatrix} .$$

In this situation, the spectral decomposition of matrix $M$ takes the form:

$$\begin{aligned}
M = &\underbrace{\frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) & -\alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \end{pmatrix}}_{M_1} \\
&+ \underbrace{\frac{v_i^{\mathrm{I}}v_j^{\mathrm{I}} - \varepsilon^2\alpha_1\alpha_2}{(v_i^{\mathrm{I}} + \varepsilon\alpha_i)(v_j^{\mathrm{I}} + \varepsilon\alpha_j)}}_{\lambda_2}\underbrace{\frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}\begin{pmatrix} \alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \\ \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}}_{M_2} . \qquad (47)
\end{aligned}$$

In the notation of the general solution (35), we have:

$$\lambda_2 = \frac{v_i^{\mathrm{I}}v_j^{\mathrm{I}} - \varepsilon^2\alpha_1\alpha_2}{(v_i^{\mathrm{I}} + \varepsilon\alpha_i)(v_j^{\mathrm{I}} + \varepsilon\alpha_j)}, \qquad \bar{v} = \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}\begin{pmatrix} \alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \\ \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) \end{pmatrix} , \qquad (48)$$

and $v'$ the row vector perpendicular to $\bar{v}$ with the same norm. The explicit solution of the recurrence is given by (35). The coefficient $\delta$ of the leading term is:

$$\begin{aligned}
\delta = v'.V^0 &= \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j} \begin{pmatrix} \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) & -\alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \end{pmatrix} \begin{pmatrix} \dfrac{\varepsilon\alpha_i A_i}{v_i^{\mathrm{I}} + \varepsilon\alpha_i} \\ \dfrac{\varepsilon\alpha_j A_j}{v_j^{\mathrm{I}} + \varepsilon\alpha_j} \end{pmatrix} \\
&= \frac{1}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j} \left(\varepsilon\alpha_i\alpha_j A_i - \varepsilon\alpha_i\alpha_j A_j\right) \\
&= \frac{\varepsilon\alpha_i\alpha_j(A_i - A_j)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}.
\end{aligned} \tag{49}$$

Next, particular total contribution $X^*$ defined in (37) is:

$$\begin{aligned}
X^* &= \frac{(v_i^{\mathrm{I}} + \varepsilon\alpha_i)(v_j^{\mathrm{I}} + \varepsilon\alpha_j)}{\varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j)} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\varepsilon\alpha_i A_i}{v_i^{\mathrm{I}} + \varepsilon\alpha_i} \\ \dfrac{\varepsilon\alpha_j A_j}{v_j^{\mathrm{I}} + \varepsilon\alpha_j} \end{pmatrix} \\
&= \frac{(v_i^{\mathrm{I}} + \varepsilon\alpha_i)(v_j^{\mathrm{I}} + \varepsilon\alpha_j)}{\varepsilon(\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j)} \frac{\varepsilon\alpha_i A_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) + \varepsilon\alpha_j A_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i)}{(v_i^{\mathrm{I}} + \varepsilon\alpha_i)(v_j^{\mathrm{I}} + \varepsilon\alpha_j)} \\
&= \frac{\alpha_i A_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) + \alpha_j A_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}.
\end{aligned} \tag{50}$$

This is again a convex combination of the wished amounts of each player.

In summary, the solution of the recurrence is then:

$$\begin{pmatrix} x_{it} \\ x_{jt} \end{pmatrix} = \delta t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + (1 - \lambda_2^t) \frac{X^* - X_0}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j} \begin{pmatrix} \alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \\ \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) \end{pmatrix}, \tag{51}$$

where now, according to (48), (49) and (50),

$$\delta = \frac{\varepsilon\alpha_i\alpha_j(A_i - A_j)}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j} \quad \lambda_2 = \frac{v_i^{\mathrm{I}} v_j^{\mathrm{I}} - \varepsilon^2\alpha_i\alpha_j}{(\varepsilon\alpha_i + v_i^{\mathrm{I}})(\varepsilon\alpha_j + v_j^{\mathrm{I}})} \quad X^* = \frac{\alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j)A_i + \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i)A_j}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j}.$$

This second eigenvalue is such that $|\lambda_2| \leq 1$ and there is equality when $v_i^{\mathrm{I}} = v_j^{\mathrm{I}} = 0$ (no inertia), in which case $\lambda_2 = -1$. Observe that, in contrast with the case studied in Section E.2.3, this eigenvalue $\lambda_2$ can be negative. When there is inertia, the situation is formally equivalent to the one studied in Section E.2.3, and we have globally the same conclusions. Note however that since $\lambda_2$ is not necessarily positive, oscillations are possible in the trajectories. These oscillations converge when there is inertia (because $|\lambda_2| < 1$) but not when $\lambda_2 = -1$. We proceed with the details in each case.

**Players with inertia – dynamics based on the past.** When players have the same wished amount $A_i = A_j = A$, the leading term $\delta t$ in (51) vanishes and we observe a convergence to the limit point

$$\begin{pmatrix} x_i^* \\ x_j^* \end{pmatrix} = \begin{pmatrix} x_{i0} \\ x_{j0} \end{pmatrix} + \frac{X^* - X_0}{\alpha_i v_j^{\mathrm{I}} + \alpha_j v_i^{\mathrm{I}} + 2\varepsilon\alpha_i\alpha_j} \begin{pmatrix} \alpha_i(v_j^{\mathrm{I}} + \varepsilon\alpha_j) \\ \alpha_j(v_i^{\mathrm{I}} + \varepsilon\alpha_i) \end{pmatrix}.$$

This limit point depends on the initial contribution, and its geometric interpretation is the same as in Section 5.3.2.

When players have different wished amounts, we have the same asymptotic behavior as described in Section 5.3.2. Without the constraints on budget and positivity of contributions, the sequence of contribution vectors would go to infinity, with an asymptotic transfer of $\delta$ units from the player who wishes the largest amount to the other player.

Because of the constraints, the trajectory will hit the boundary of feasible contributions in finite time. The behavior after it does should be again similar to the situation in Section 5.3.2 when $0 \leq \lambda_2 < 1$. It may however be more complex if $\lambda_2 < 1$. The details are not addressed in this document.

**Players without inertia – dynamics based on the past.** When there is no inertia, the recurrence (51) actually reduces to

$$x_{it} = A_i - x_{jt-1}$$
$$x_{jt} = A_j - x_{it-1}.$$

This implies that $x_{it} = A_i - A_j + x_{it-2}$ with the obvious solution $x_{it} = t(A_i - A_j)/2 + x_{i0}$ when $t$ is even, or $x_{it} = t(A_i - A_j)/2 + (A_i + A_j)/2 - x_{j0}$ when it is odd. The total contribution oscillates between $X_{2m} = X_0$ and $X_{2m+1} = X^* - X_0 = (A_i + A_j)/2 - X_0$.

If $A_i \neq A_j$, the situation is similar to that of Section 5.3.2, except that contribution vectors keep oscillating as they diverge in the direction $(1, -1)'$, until they hit the boundary of feasible contributions.[16] If $A_i = A_j = A$, the contributions alternate between $(x_{i0}, x_{j0})$ and $(A - x_{j0}, A - x_{i0})$ (provided this second vector of contributions satisfies the budget constraints). The point in the middle is the symmetric Nash equilibrium $(A, A)$. It does not depend on the initial position.

# F Dynamics on the boundary

In this section, we discuss the dynamics on the boundary of the domain of constraints, for the dynamics described in Section 5. The corresponding analysis for the alternate dynamics presented in Section E.2.4 is left for future research.

Without loss of generality, we restrict the discussion to the case $A_i \geq A_j$. Of particular interest, in view of our simulations, are the cases where either $x_i = w_i$, or $x_j = 0$. The case where $x_j = w_j$ is also relevant.

With the purpose of simplifying some expressions, we introduce the new notation $\theta_i := v_i^{\mathrm{I}}/(\alpha_i \varepsilon)$ for $i = 1, 2$.

## F.1 Dynamics on $x_i = w_i$

When a contribution vector is $x_{t-1} = (w_i, x_{jt-1})$, it is not always the case that the next contribution will still be on the boundary $x_i = w_i$. This will be the case however if the contribution $x_{t-1}$ is close enough to the equilibrium and the equilibrium lies on the same boundary. We analyze this situation now.

[16] The dynamics after this occurs is not studied in this document.

Assume that the state of the dynamical system is $x_{t-1} = (w_i, y)$ for some $0 \le y \le w_j$. Assume also that the equilibrium is of type [a], [bII.1] or [bII.2], referring to the typology of Nash equilibria identified in Proposition 1. These are the cases where $x_i^N = w_i$. Observe that in these three cases, $A_i > 0$, but that $A_j$ may be zero in case [bII.2]. According to the dynamic best response of players specified in (18), we find that, whenever $A_j > 0$:

$$r_i^D(x_j) = \begin{cases} w_i & \text{if } x_j \le A_i - w_i \\ \dfrac{A_i - x_j + \theta_i w_i}{1 + \theta_i} & \text{if } A_i - w_i \le x_j \le A_i + \theta_i w_i \\ 0 & \text{if } x_j \ge A_i + \theta_i w_i \end{cases} \tag{52}$$

$$r_j^D(x_i) = \begin{cases} w_j & \text{if } x_i \le A_j - w_j + \theta_j(y - w_j) \\ \dfrac{A_j - x_i + \theta_j y}{1 + \theta_j} & \text{if } A_j - w_j + \theta_j(y - w_j) \le x_i \le A_j + \theta_j y \\ 0 & \text{if } x_i \ge A_j + \theta_j y. \end{cases}$$

We are interested in characterizing the situations where $x_{t-1} = (w_i, y)$ implies $x_t = (w_i, y')$. The latter condition is equivalent to

$$w_i = r_i^D(y') \qquad \text{and} \qquad y' = r_j^D(w_i).$$

which is itself equivalent to:

$$y' \le A_i - w_i \qquad \text{and} \qquad y' = r_j^D(w_i). \tag{53}$$

The principal results are summarized in the following proposition. It essentially states that when Player $i$ arrives from the interior to the frontier $x_i = w_i$, her contribution stays forever at $w_i$, and that of Player $j$ converges monotonously to the static equilibrium.

**Proposition 7** *Assume the vector of contributions at time $t-1$ is such that $x_{it-1} = w_i$. Then,*

- *a) in case [a], then $x_t = (x_{it}, x_{jt}) = (w_i, r_j^D(w_i))$; moreover, the sequence $x_{jt}$ converges monotonically thereon to the Nash equilibrium;*
- *b) in cases [bII.1] or [bII.2], then $x_t = (x_{it}, x_{jt}) = (w_i, r_j^D(w_i))$ if the condition*

$$\theta_j x_{jt-1} \le A_i - A_j + \theta_j(A_i - w_i) \tag{54}$$

  *is satisfied; in that case, the sequence $x_{jt}$ converges monotonically thereon to the Nash equilibrium.*
- *c) if $x_{it-1}$ results itself from one step of the dynamics, then Condition* (54) *is satisfied.*

Note that, according to Statement [R1a], any trajectory in cases [bII.1] or [bII.2] will eventually hit the boundary $x_i = w_i$. Then because of Proposition 7 c), there must exist a $t$ such that $x_{jt-1}$ verifies (54).

We prove the proposition for each type of static Nash equilibrium.

**Type [a].** The conditions for this case are: $A_i, A_j \geq w_i + w_j$. Since $A_i - w_i \geq w_j$ and $x_j \leq w_j$, the first case in the definition of $r_i^D$ always holds and $r_i^D(x_j) = w_i$ whatever $x_j$ may be. We now discuss the position of $x_{jt}$ with respect to $x_{jt-1}$.

Since $A_j \geq w_i$, the last case in the definition of $r_j^D(w_i)$ never holds. Then, $r_j^D(w_i)$ is either $w_j$, in which case it is larger than $x_{jt-1}$; or it is given by

$$x_{jt} = \frac{A_j - w_i + \theta_j x_{jt-1}}{1 + \theta_j}. \tag{55}$$

Since this is the convex combination of $A_j - w_i \geq w_j$ and $x_{jt-1} \leq w_j$, this value is therefore also larger than $x_{jt-1}$. In both cases $x_{jt} \geq x_{jt-1}$. This proves statement a) of Proposition 7.

**Type [bII.1].** The conditions for this case are $w_i < A_j < w_i + w_j$ and $A_j \leq A_i$. As in the previous case, since $A_j > w_i$, the third case in the definition of $r_j^D(w_i)$ never holds. But since $w_i > A_j - w_j$ and $\theta_j(y - w_j) \leq 0$, the first case in this definition never occurs either. The only remaining case is that $y'$ is given by (55). Then, the conditions (53) are equivalent to $y' \leq A_i - w_i$, that is:

$$\begin{aligned} A_j - w_i + \theta_j y &\leq (1 + \theta_j)(A_i - w_i) \\ \theta_j y &\leq A_i - A_j + \theta_j(A_i - w_i), \end{aligned}$$

which is (54).

We now turn to the monotonicity of the sequence $x_{jt}$. Since $x_{jt}$ is given by (55), it is a convex combination of the Nash equilibrium $A_j - w_i$ and the previous location $x_{jt-1}$: it is therefore closer to the Nash equilibrium $A_j - w_i$. If $x_{jt-1} \geq A_j - w_i$, then $x_{jt} \leq x_{jt-1}$ so that $x_{jt}$ still satisfies (54). If $x_{jt-1} < A_j - w_i$, then $x_{jt}$ is still smaller than $A_j - w_i$ and it also satisfies (54). Therefore the new contributions $x_t$ still satisfy the conditions of Proposition 7 and the trajectory stays on the boundary forever, while getting closer to the Nash equilibrium. This proves statement b) of Proposition 7 for equilibria of type [bII.1].

**Type [bII.2].** The conditions for this case are $A_j \leq w_i < A_i$. We develop the argument for the case where $A_j > 0$ and $r_j^D(w_i)$ is given by the formula above. When $A_j = 0$, the argument still holds with $A_j$ replaced with $\Phi_j/(\alpha_j \varepsilon) \leq 0$. The inequality $A_j \leq w_i$ implies that the first case in the definition of $r_j^D(w_i)$ never holds. Then, the conditions (53) are equivalent to:

$$y' \leq A_i - w_i \quad \text{and} \quad y' = \begin{cases} \dfrac{A_j - w_i + \theta_j y}{1 + \theta_j} & \text{if } A_j - w_j + \theta_j(y - w_j) \leq w_i \leq A_j + \theta_j y \\ 0 & \text{if } w_i \geq A_j + \theta_j y. \end{cases}$$

A first situation occurs when

$$w_i - A_j \leq \theta_j y \leq w_i + w_j - A_j + \theta_j w_j. \tag{56}$$

Then $y'$ is given by (55) and the conditions (53) reduce to (54) as in the case of type [bII.1] equilibria. A second situation occurs when

$$\theta_j y \leq w_i - A_j. \tag{57}$$

Then, since $y' = 0 \leq A_i - w_i$, the conditions (53) are satisfied. In summary, if either (57) holds, or both (54) and (56) hold, then $(w_i, y')$ is a Nash equilibrium. In all other cases, it is not.

Since (57) implies (54), we conclude that when Condition (54) is satisfied, the next-step Nash equilibrium is $(w_i, y')$.

Concerning monotonicity and convergence: when $x_t = (w_i, y')$, it is checked that $x_{jt} \le x_{jt-1}$: either $x_{jt}$ is 0, or it is given by (55) which is a convex combination of some negative quantity and $x_{jt-1}$. In both cases, $x_{jt} \le x_{jt-1}$ and $x_{jt}$ also satisfies (54). This proves statement b) of Proposition 7 for equilibria of type [bII.2]. The proof of this statement is now complete.

**Joining the boundary from the interior.** There remains to prove statement c) of Proposition 7. Assume indeed that $x_{t-1} = (w_i, x_{jt-1})$ is a Nash equilibrium of Problem (14) for some $(x_{it-2}, x_{jt-2})$. Then $x_{it} = w_i$ is the best response of Player $i$, so that, from (18) and (16) (remember that, since $A_i \ge 0$, $\alpha_i - 1 + v_i^{\mathrm{P}} = \alpha_i \varepsilon A_i$) we must have

$$\begin{aligned} r_i(x_{jt-1}; x_{it-2}) &\ge w_i \\ A_i - x_{jt-1} + \theta_i x_{it-2} &\ge w_i(1+\theta_i) \\ x_{jt-1} &\le A_i - w_i - \theta_i(w_i - x_{it-2}) \;\le\; A_i - w_i. \end{aligned}$$

This inequality implies (54) holds. This proves statement c) of Proposition 7.

## F.2 Dynamics on $x_j = 0$

When a contribution vector is $x_{t-1} = (x_{it-1}, 0)$, it is not always the case that the next contribution will still be on the boundary $x_j = 0$. This will be the case however if the contribution $x_{t-1}$ is close enough to the equilibrium and this equilibrium lies on the same boundary. We analyze this situation now.

Assume that the state of the dynamical system is $x_{t-1} = (z, 0)$ for some $0 \le z \le w_i$. Assume also that the static Nash equilibrium is of type [bII.2], [bII.3] or [c], that is, such that $x_j^N = 0$. According to the dynamic best response of players specified in (18), we find that, whenever $A_i > 0$ and $A_j > 0$:

$$r_i^D(x_j) = \begin{cases} w_i & \text{if } x_j \le A_i - w_i - \theta_i(w_i - z) \\ \dfrac{A_i - x_j + \theta_i z}{1+\theta_i} & \text{if } A_i - w_i - \theta_i(w_i - z) \le x_j \le A_i + \theta_i z \\ 0 & \text{if } x_j \ge A_i + \theta_i z \end{cases}$$

$$r_j^D(x_i) = \begin{cases} w_j & \text{if } x_i \le A_j - w_j - \theta_j w_j \\ \dfrac{A_j - x_i}{1+\theta_j} & \text{if } A_i - w_j - \theta_j w_j \le x_i \le A_j \\ 0 & \text{if } x_i \ge A_j. \end{cases} \tag{58}$$

If $A_j = 0$, these formulas hold with $A_j$ replaced with $\Phi_j/(\alpha_j \varepsilon)$, possibly a negative quantity. Similarly for $A_i = 0$.

We are interested in situations where $x_{t-1} = (z, 0)$ implies $x_t = (z', 0)$. The latter condition is equivalent to $r_i^D(0) = z'$ and $r_j^D(z') = 0$. This in turn is equivalent to

$$r_i^D(0) = z' \qquad \text{and} \qquad z' \ge A_j. \tag{59}$$

The principal results are summarized in the following proposition. Similar to Proposition 7, it states that when Player $j$ arrives from the interior to the frontier $x_j = 0$, her contribution stays forever at 0, and that of Player $i$ converges monotonously to the static equilibrium.

**Proposition 8** *Assume the vector of contributions at time* $t-1$ *is such that* $x_{jt-1} = 0$. *Then,*

a) *in case [c], then* $x_t = (x_{it}, x_{jt}) = (r_i^D(0), 0)$*; moreover, the sequence* $x_{jt}$ *converges monotonically thereon to the Nash equilibrium;*

b) *in cases [bII.2] or [bII.3], then* $x_t = (x_{it}, x_{jt}) = (r_i^D(0), 0)$ *if the condition*

$$\theta_i(A_j - x_{it-1}) \leq A_i - A_j \tag{60}$$

*is satisfied; in that case, the sequence* $x_{it}$ *converges thereon to the Nash equilibrium;*

c) *if* $x_{jt-1}$ *results itself from one step of the dynamics, then Condition* (60) *is satisfied.*

We prove the proposition for each type of static Nash equilibrium.

**Type [c].** The conditions for this case are $A_i = A_j = 0$. Then it holds that $r_j^D(x_j) = 0$ whatever $x_j$ may be. Since $A_i = 0$, the first case in the definition of $r_i^D(0)$ never holds. Then $x_{it} = r_i^D(0)$ is either 0, or given by:

$$x_{it} = \frac{\Phi_i + v_i^{\mathrm{I}} x_{it-1}}{\alpha_i \varepsilon + v_i^{\mathrm{I}}} \leq \frac{v_i^{\mathrm{I}} x_{it-1}}{\alpha_i \varepsilon + v_i^{\mathrm{I}}} \leq x_{it-1}. \tag{61}$$

In both cases, $x_{it} \leq x_{it-1}$. The vector of contributions stays on the boundary and the contributions of player $i$ are monotonously decreasing. This proves statement a) from Proposition 8.

**Type [bII.3].** The conditions for this case are $A_j \leq A_i \leq w_i$ with $A_i > 0$. Since $A_i + \theta_i z > 0$, the condition for $r_i^D(0)$ to be 0 is never satisfied. Since $A_i - w_i \leq 0$ and $-\theta_i(w_i - z) \leq 0$, the condition for $r_i^D(x_j)$ to be $w_i$ is also never satisfied. Then the conditions (59) are equivalent to:

$$z' \geq A_j \quad \text{and} \quad z' = r_i^D(0) = \frac{A_i + \theta_i z}{1 + \theta_i}.$$

We have then the equivalence:

$$\begin{aligned} r_i^D(0) &\geq A_j \\ A_i + \theta_i z &\geq A_j(1 + \theta_i) \\ \theta_i(A_j - z) &\leq A_i - A_j \ , \end{aligned}$$

the last condition being exactly (60).

If $z$ satisfies Condition (60) (which happens when $z \geq A_j$ or $z \leq A_j$ but close enough to $A_j$), then $x_{it}$ is given by (61). This new contribution is a convex combination of the previous one with $A_i$ (remember that $\Phi_i = \varepsilon \alpha_i A_i$), which is the equilibrium value. Because $A_i \geq A_j$, it then still satisfies Condition (60), so that the trajectory stays on the boundary. We have proven that, once Condition (60) is satisfied, the dynamics converges monotonously, that is, statement b) of Proposition 8 for static Nash equilibria of type [bII.3].

**Type [bII.2].** The conditions for this case are: $A_j \le w_i \le A_i$ and $w_i + w_j \ge A_j$. As in the previous case, the condition for $r_i^D(0)$ to be 0 never holds. Two situations remain where the equilibrium $x_t$ is located on the boundary $x_j = 0$.

A first situation occurs when

$$0 \le A_i - w_i - \theta_i(w_i - z). \tag{62}$$

In that case, $r_i^D(0) = w_i$ and the conditions (59) are equivalent to: $r_i^D(0) = w_i \ge A_j$, which is true by assumption.

The second situation occurs if

$$0 \ge A_i - w_i - \theta_i(w_i - z). \tag{63}$$

Then, $r_i^D(0)$ is given by (61) and, as for type [bII.3] equilibria, the conditions (59) are equivalent to (60). In summary, if either (62) holds, or both (63) and (60) hold at the same time, then the next-step Nash equilibrium is $(z', 0)$. It is easily seen that condition (62) implies (60), because $A_j \le w_i$. As a consequence, whenever (60) holds, conditions (59) hold also, whether (62) is true or (63) is true.

Then the next point of the dynamics is either $x_{it} = w_i$, or given by (61). In the first case, since $A_i \ge A_j$, the value $z = w_i$ itself satisfies Condition (60), so that this vector of contribution, which is the Nash equilibrium, is indeed stable under the dynamics. This fact was known because of Proposition 4 [R0]. In the second case, this is a convex combination of the previous contribution $x_{it-1}$ and $A_i \ge w_i$. In both cases, it is therefore larger than $x_{it-1}$ and, a fortiori, still satisfies (60). We have proven that, once Condition (60) is satisfied, the dynamics converges monotonously, that is, statement b) of Proposition 8 for static Nash equilibria of type [bII.2]. This completes the proof of this statement.

**Joining the boundary from the interior.** We now argue that if the current vector of contributions $x_{t-1} = (x_{it-1}, 0)$ results from a step of the dynamics that comes from the interior of the domain, then it satisfies Condition (60). Assume indeed that $(x_{it-1}, 0)$ is a Nash equilibrium of Problem (14) for some $(x_{it-2}, x_{jt-2})$. Then $x_{jt-1} = 0$ is the best response of Player $j$, so that, from (18) and (16), we must have

$$\begin{aligned} 0 &\ge \alpha_j \varepsilon A_j + v_j^{\mathrm{I}} x_{jt-1} - \alpha_j \varepsilon x_{it-1} \\ x_{it-1} &\ge A_j + \theta_j x_{jt-1} \ \ge A_j. \end{aligned}$$

This implies that the left-hand side of (60) is negative, and then that (60) holds. This proves statement c) of Proposition 8.

## F.3 Dynamics on $x_j = w_j$

The purpose of this section is to prove the following result.

**Proposition 9** *Assume $A_i > A_j$. Assume the vectors of contributions at time $t-1$ and time $t$ are such that $x_{jt-1} = x_{jt} = w_j$. Then $x_{it-1} \ge x_{it}$.*

**Proof.** Using (18), we find that the condition $x_{jt} = w_j$ is equivalent to $r_j^D(x_{it}; w_j) = w_j$, itself equivalent to $A_j - w_j \geq x_{it}$. On the other hand, $x_{it} = r_i^D(w_j; x_{it-1})$ is given by:

$$x_{it} = \begin{cases} w_i & \text{if } A_i + \theta_i x_{it-1} - w_j \geq (1+\theta_i) w_i \\ \dfrac{A_i + \theta_i x_{it-1} - w_j}{1+\theta_i} & \text{if } 0 \leq A_i + \theta_i x_{it-1} - w_j \leq (1+\theta_i) w_i \\ 0 & \text{if } A_i + \theta_i x_{it-1} - w_j \leq 0. \end{cases}$$

If $x_{it} = w_i$ then obviously $x_{it} \geq x_{it-1}$. If $x_{it}$ is given by the interior case, then the condition $x_{it} \leq A_j - w_j$ is equivalent to:

$$\begin{aligned} x_{it} = \frac{A_i + \theta_i x_{it-1} - w_j}{1+\theta_i} &\leq A_j - w_j \\ A_i + \theta_i x_{it-1} - w_j &\leq A_j + \theta_i A_j - w_j - w_j \theta_i \\ x_{it-1} + w_j &\leq A_j + \frac{A_j - A_i}{\theta_i}. \end{aligned}$$

Then it follows that

$$\begin{aligned} x_{it} - x_{it-1} &= \frac{A_i - x_{it-1} - w_j}{1+\theta_i} \\ &\geq \frac{1}{1+\theta_i}\left(A_i - A_j - \frac{A_j - A_i}{\theta_i}\right) = \frac{A_i - A_j}{\theta_i} \geq 0. \end{aligned}$$

Finally, the case $x_{it} = 0$ would require $A_i + \theta_i x_{it-1} - w_j \leq 0$, which implies $A_i \leq w_j$. But we also need $A_j - w_j \geq 0$, which leads to $A_i \leq A_j$, a contradiction. In all possible cases, we have shown that $x_{it} \geq x_{it-1}$, hence the proposition. ■

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