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Optimal lockdown and vaccination policies to contain the spread of a mutating infectious disease

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Abstract

We develop a piecewise deterministic control model to study optimal lockdown and vaccination policies to manage a pandemic. Lockdown is modeled as an impulse control that allows the system to switch from one restriction regime of restrictions to another. Vaccination policy is a continuous control. Decisions are taken under the risk of mutations of the disease, with repercussions on the transmission rate. The decision maker follows a cost minimization objective. We first characterize the optimality conditions for impulse control and show how the prospect of a mutation affects the decision maker's choice by inducing her to anticipate the relative benefit of a regime change after a mutation has occurred. Under some parametric conditions, our problem admits infinitely many value functions. We show the existence of a minimum value function that is a natural candidate to the solution given the nature of the problem. Focusing on this specific value function, we finally study the features of the optimal policy, especially the timing of impulse control. We prove that uncertainty surrounding future "bad" vs. "good" mutation of the disease expedites vs. delays the adoption of lockdown measures.

Keywords: pandemic, lockdown, vaccination, mutation, impulse control, uncertainty

JEL classification: C61, D81, I18

1 Introduction

The onset of the COVID-19 pandemic marked the renewed interest of economists for the analysis of the impacts of epidemics and the design of public policies to cope with them. This is best illustrated by the impressive number of papers that have been published on these topics over the course of the last two years.

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Most contributions throughout the first year of the pandemic have been devoted to the analysis of lockdown, or quarantine or more generally restriction measures, as the main instrument to control the evolution of the epidemic. This was of course primarily motivated by the absence of alternative intervention (before the discovery of vaccines, and still in the absence of cure). But even after the widespread use of vaccination, lockdown measures remain a credible policy tool in the eyes of policy-makers, and are considered as such in the current paper.

Specifically, our aim is to develop an original piecewise deterministic optimal control model to study optimal lockdown and vaccination policies to manage an infectious disease. Our approach is original is that it combines the following three main ingredients. First, we consider two alternative options to control the spread of the pandemic: vaccination and lockdown. Second, we take lockdown measures as impulse controls to echo the evidence that policy makers are subject to a wide set of (administrative, political, economic) constraints that prevent them from changing the level of restrictions on an everyday basis. Last but not least, again motivated by COVID-19, we account for the uncertainty that surrounds the severity of the disease and results from the frequent mutations of the virus. As perfectly outlined by Boucekkine et al. (2021), this together with policy interventions are the fundamental drivers of the rate of transmission of the disease, and its resulting economic and health impacts. In this setting, our primary concern has to do with the study of the interplay between lockdown restrictions, as impulse controls, and mutations modeled as random and discrete shifts in the virus contagiousness. We especially seek to determine whether lockdown should be used as a prevention policy in the prospect of handling better a future (potentially more severe) mutation, or as an adaptation policy to such event, once and if realized. In addition, we are also interested in the effect of possible mutation on the timing of the lockdown policy: does the prospect of a mutation delay, or on the contrary expedite the lockdown policy? The second issue we want to address is about the interplay between the vaccination and lockdown policies: Are they substitute or complement tools in the hands of the policy makers?

It goes without saying that any attempt to provide a comprehensive review of the literature on epidemics and COVID-19 would be a vain exercise given how fast it grows. Rather, we more modestly give a brief overview of both the "classics," that is the pre-COVID reference papers merging epidemiological and economic models, and the most recent and relevant post-COVID papers dealing with topics similar to ours.

Gersovitz and Hammer (2004) are the first to investigate the connection between the spread of an infectious disease and economic outcomes. By comparing a representative agent problem with the optimal solution, they discuss how standard economic instruments work to internalize the externalities surrounding the epidemic propagation. In a series of excellent papers, Goenka and Liu (2012, 2020) and Goenka, Liu and

Nguyen (2014) develop full-fledged analyses of the impact of an epidemic on the macro-economy. For that purpose, they consider several versions of a model merging SIS dynamics and neoclassical growth. Goenka and Liu (2012) focus on the impact of the pandemic on endogenous labor supply, show how the pandemic dynamics can generate chaos and cycles, and discuss the type of policy intervention capable of stabilizing endogenous fluctuations. Goenka, Liu and Nguyen (2014) go a step further by taking account of the two-way interaction between the economy and the pandemic. On the one hand, the disease negatively affects the (exogenous here) labor supply and production, while health expenditure are meant to slow down the virus transmission. Both papers adopt a social planner perspective by characterizing either the optimal growth path, or the optimal policy. Finally, in a similar framework, Goenka and Liu (2020) analyze the decentralized equilibrium when private agents also invest in human capital. They emphasize the existence of multiple balanced growth paths, with very distinct features in terms of economic performance and disease prevalence.

As to the post-COVID-19 literature, our attention was drawn to the contributions that consider lockdown measures as the single instrument to control the evolution and severity of the epidemic situation. Alvarez et al. (2021) is an excellent representative of this line of research. They authors study the optimal control of a pandemic thanks to lockdown restrictions. They consider lockdown as a continuous control, the decision maker choosing the intensity of the lockdown, or the share of population subject to a lockdown, at every instant. They capture the basic trade-off between the economic cost and health benefit of lockdown (reduction of the transmission rate). In addition, they model the pandemic dynamics with the SIR model, and adopt a short run perspective by assuming that the planning horizon is finite but unknown as it is determined by the arrival of a vaccine. They investigate the features—timing, duration and intensity—of the optimal policy both analytically and by means of a calibrated model. Other related papers differ from this approach along several ways.[1] The most relevant to our approach are those *i/* that depart from the modeling of lockdown as a continuous control, and *ii/* that introduce uncertainty in the analysis. Aspri et al. (2021) observe that in the real world, policies such as lockdown cannot be revised instantaneously and further, cannot be revised before a certain amount time elapses. Based on this observation, they study the optimal design of

[1] Goenka, Liu and Nguyen (2021) study optimal lockdown by accounting for the waning immunity (thereby using a SIRS model). They keep working with a neoclassical growth model and focus on long run outcomes. Loerstcher and Miur (2021) do not take an optimal policy perspective. Rather they assume that the decision maker's objective is to make sure that the health system capacity cannot be overwhelmed because of the increase in the number of infected people requiring special care. The limited capacity constraint of the health system is also present in Jones et al. (2021), where the authors compare the representative agent equilibrium with the optimal solution. They use an aggregate transmission rate defined as a function of consumption and working decisions and analyze the reactions, in terms of social distancing and remote working, following the announcement of an outbreak of the pandemic. See Eichenbaum et al. (2021) for a similar approach but with different perspective. Caulkins et al. (2021) also consider the optimal design of lockdown policy by incorporating several novelties such as dealing with the level of economic activity and lockdown "fatigue" as additional states of the system. The analysis of the quite complex dynamics generated by their model notably show the existence of multiple Skiba points. It is also worth mentioning the contribution of Acemoglu et al. (2021) who develop a multi-group SIR model where the population is divided into three age classes (youth, middle age, old) to assess the performance of targeted lockdown policy. Gollier (2020a) investigates the very same topic. He does not analyze the optimal solution, though. Rather he considers two types of lockdown interventions, strong vs. softer.

lockdown policies, when available policies remain constant along some time interval, within a SEIRAD model (that is they also take care of the asymptomatic health status). In a more standard model, Caulkins et al. (2020) also choose to deal with lockdown policies as impulse rather than continuous controls. Adopting this perspective, they are able to address the issue of the optimal timing, onset and exit, of a one-shot lockdown regime.

As mentioned above, uncertainty seems like an essential feature not only for the management of pandemic situations, but also and more generally for the understanding of both individuals behaviors and policy performance in many economic problems (ranging from finance, to investment and resource management problems). Yet there are very few studies that have incorporated this dimension into the analysis of optimal policy to fight against COVID-19. Exceptions are Bandyopadhyay et al. (2021), Gollier (2020b), and Federico and Ferrari (2021). Gollier (2020b) studies the impact of uncertainty surrounding the transmission rate when lockdown restrictions are lifted and the role of learning about it on the optimal lockdown policy within a two-stage decision problem. He shows that introducing uncertainty tends to reduce the optimal rate of lockdown by lowering the expected cost of a less strict lockdown. Bandyopadhyay et al. (2021) conduct the same kind of analysis in terms of uncertainty and learning in a three-period problem. They consider that imposing a lockdown prevents the decision maker from learning about the actual contagiousness of the disease. Moreover, they introduce an additional cost of delaying lockdown, namely the lost opportunity of habit formation.[2] On the other hand, Federico and Ferrari (2021) develops a stochastic optimal control problem where the decision maker has to choose the lockdown policy while being subject to an uncertainty not about the level but the evolution of the transmission rate. For that purpose, they model the dynamics of the transmission rate thanks to a diffusion (Wiener) process and take lockdown as a means to reduce the trend (deterministic part) of the process. In this setting, they conduct numerical experiments that help highlight the features of the lockdown policy.

In sum, there is no study that combines lockdown as an impulse control, uncertainty surrounding the evolution of the transmission rate, and consider both lockdown and vaccination policies. This exactly where the contribution of our paper lies. Precisely, we first develop a general optimal control model to study optimal lockdown and vaccination policies in pandemic times. Lockdown is modeled as an impulse control that allows the system to switch from one regime of restrictions to another (stricter or softer). Vaccination policy, on the other hand, is a continuous control. Decisions are taken under the risk of mutations of the disease, with repercussions on the transmission rate. The decision maker follows a cost minimization objective. Moving to a simplified model where the virus can mutate only once and there exist only two

[2] Their argument being that habit formation should normally take place during a lockdown and give people incentives to behave more cautiously once it ends.

lockdown regimes, with the possibility to go back and forth between them, allows us to draw a series of interesting results. We first characterize the optimality conditions for impulse control and show how the prospect of a mutation affects the decision maker's choice. In fact, it induces her to anticipate the relative benefit of a regime change after a mutation has occurred, which may or may not increase the incentive to set a lockdown. Under some parametric conditions, our problem admits infinitely many value functions. We then show the existence of a minimum value function that is a natural candidate to the solution given the nature of the optimization program. Focusing on this specific value function, we finally study the features of the optimal policy, especially the timing of impulse control. We prove that uncertainty about future possible "bad" vs. "good" mutation (in terms of contagiousness) of the disease tends to expedite vs. delay the adoption of lockdown measures. This conclusion closely parallels those of the two strands of literature on decision making under uncertainty. The first one analyzes the impact of the occurence of costly events (Crepin and Naedval, 2020, for a review), whereas the second one emphasizes the role of uncertainty and learning (Dixit and Pyndick, 1994).

The paper is organized as follows. Section 2 is devoted to our modeling strategy. Section 3 deals with the optimality conditions for impulse controls and emphasize the difference of deciding before or after a mutation. Section 4 conducts a dynamic analysis and show how the dynamic behavior is intricately linked to the features of the value function. Section 5 addresses the issue of the timing of impulse control, reviews the different cases possible and illustrates them numerically. Section 6 concludes.

2 Model

2.1 The general problem

Following the vast majority of the literature (Goenka and Liu, 2012, Alvarez et al., 2021, Federico and Ferrari, 2021, among others), we adopt the fully centralized perspective of a decision maker (DM) who has to manage a pandemic. It means that we do not address externality problems associated with the disease transmission. It sounds like a natural starting point for the study of the optimal management of epidemics.

2.1.1 Pandemic dynamics and control variables

As discussed in the Introduction, most analyses that conducts an economic analysis of the optimal control of an epidemic make use of SIS and SIR models, and some variations of these. The main pre-COVID-19 contributions, that merge epidemiological and economic models, are based on the SIS model (Goenka and Liu, 2012, Goenka, Liu and Nguyen, 2014). In the SIS model, the (constant) population is split into two groups, the "susceptible" and the "infected" and two state equations described changes in the health status. Most recent post-COVID-19 papers rather rely on the SIR model, whereby upon infection people get full or partial immunity and enter the third category of "recovery," or die (Acemoglu et al., 2021, Jones et al., 2021).

In what follows, we model the epidemic evolution along the line of a SIS, instead of a SIR, model. The reason for this choice is threefold. Firstly, based on the latest evidence on COVID-19, it is now clear that immunity acquired through an infection is not permanent, and actually lasts for a short period. This makes the standard SIR model inadequate to reproduce the dynamics of pandemic such as COVID-19. One may then opt for the more general SIRS model which allows recovered people to return to the susceptible health status after a while (Caulkins et al., 2021, Goenka, Liu and Nguyen, 2021). But the SIS model then looks like an acceptable simplification of the problem because it keeps the dynamical system one-dimensional. Secondly, the other advantage of the SIR model, over the SIS, is that it takes account of disease induced mortality. But again, since we do not account for demographic aspects and the actual average death rate by COVID-19 is very low, the SIS alternative may seem more attractive based on the same tractability argument. Last but not least, we want to incorporate vaccination decision, as an alternative to lockdown restrictions. Now, we know that COVID-19 vaccines do not obtain long lasting immunity either, especially in the event of a mutation of the virus. This means that we cannot use standard epidemiological models that deal with vaccinated people as an additional state of the system (see for instance, Choi and Shim, 2020), and have to find an alternative.

Precisely, as in Goenka and Liu (2012) and Goenka, Liu and Nguyen (2014), we abstract from disease related mortality. We assume a constant population, whose size is normalized to one. Then we adapt the SIS model in a way that is consistent with the logistic approach. Following the logistic approach boils down to working with only one state variable, the number of infected people. It produces similar dynamics as the SIS model, once reduced to a one dimension problem too (Burger et al., 2021). In addition, it has the advantage of being flexible. By flexibility, we mean that it allows us to model the vaccination control pretty simply. Let $s(t) \in [0,1]$ be the share of infected population. Its dynamics are given by:

$$\dot{s}(t) = \theta s\left(t\right)\left(a\left(r(t), K\right) - s\left(t\right)\right), \text{ with } a\left(r(t), K\right) = 1 - r(t) - \frac{K}{\theta}, \tag{1}$$

with θ the infection rate, and K the quality of the health system. The quality of the health system encompasses both the qualitative and quantitative dimensions of hospital infrastructure and medical staff that of course partly determine the ease with which a country can cope with the pandemic.[3] In the coming analysis

[3]The role of the quality and/or capacity of the health system has been analyzed in a series of contributions by Goenka, Liu

it is taken as given because we choose to focus on policies and shocks affecting the infection rate.[4]

Beside θ whose crucial role is explained just below, we first define the rate of vaccination, $r(t) \geq 0$. We assume that the DM chooses this rate, this decision being modeled as a continuous control as the DM can progressively make people get vaccinated by means of a wide set of more or less coercitive measures and incentives. Then, we can interpret $a(.)$ as the population at risk of getting infected and of transmitting the virus. This population is negatively affected by r, the vaccination rate, or the share of the population that is not sick and receives the vaccine.[5]

The important variable in state equation (1) is the rate of diffusion of the pandemic, θ. This rate changes across time thanks to the second DM's control variable and random shifts: $\theta = \theta(L_k, \omega)$. This approach is very much in line with Jones et al. (2021)'s observation that "Infected people transmit the virus to susceptible people at a rate that depends on the nature of the virus and the frequency of social interactions." As to the second control variable, we assume that the DM can decide on the lockdown regime L_k,[6] with k taken within a finite set of integers, imposed upon the economy. By convention, we say that lockdown regime L_n is stricter than L_m if and only if $L_m < L_n$ and $\theta'_{L_k} < 0$ for all k. This is line with evidence supporting that imposing a stricter lockdown allows health authorities to reduce the infection rate and slow down the virus diffusion. More importantly, we choose to model this second decision as an impulse control. Accordingly, we denote as $t_k \in [0, \infty)$, for all k, the date of moving to lockdown regime L_k. This is the actual control variable. The reason for this modeling option is twice. Most of the literature on lockdown defines this policy as a continuous control (Acemoglu et al., 2021, Alvarez et al., 2021, Goenka, Liu and Nguyen, 2021). This means that the decision maker chooses, at every instant, the intensity of the lockdown, or the share of the population subject to it.[7] However, it is clear that policy makers cannot adjust such policy decisions on a daily basis. This is perfectly acknowledged by Aspri et al. (2021) who argue that policies are not adjusted instantaneously, and further that there exists some inertia. Considering lockdown measures as impulse controls follows this observation. This is also the avenue taken by Caulkins et al. (2020). Moreover, the impulse control nature of this decision remarkably echoes the other type of discrete change that the system can experience, which we introduce now.

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and Nguyen (2014), Caulkins et al. (2021), Jones et al. (2021) and Loertscher and Miur (2021), among others.

[4]So we do not consider any other type of control capable of changing K, like investment in the health system. But this is a potential interesting extension of the analysis.

[5]Taking r as a continuous control certainly is a strong simplification. As explained later, we introduce a cost of vaccination $c_r = c(r)$. So a mathematically equivalent formulation would consist in defining c as vaccination expenditure, a continuous control, and re-interpret $r = c^{-1}(c_r)$ as the return on vaccination. In any event, we do not account for the possibility to build a protection capital, or shield, for the population thanks to vaccination.

[6]Hereafter we use the terms regime or mode interchangeably.

[7]This introduces an additional quadratic term in the dynamical system, which is crucial to understand the effect of such a policy on the epidemic and the economy.

2.1.2 Stochastic events

As recent experience has shown, there are many uncertainties surrounding COVID-19 pandemic, more than two years after its outbreak. However most contributions on the optimal design of lockdown policies get rid of uncertainty (or at best conduct sensitivity analyses of calibrated models). The noticeable exceptions are Gollier (2020b), Bandyopadhyay et al. (2021), and Federico and Ferrari (2021). In different settings, the first two contributions account for the uncertainty in the transmission rate and analyze how uncertainty and the possibility of learning about the exact severity of the disease shape the decision maker decision. The last paper is the closest to ours because the authors model the transmission rate as a random variable whose evolution is driven by a diffusion process (and where lockdown measures affect the trend part of it). Dealing with the random evolution of the disease is seems to be very important, again in light of the COVID-situation. We however believe that virus mutations are better described by jump processes, such as the Poisson process, as they do not occur continuously but rather at certain points of time. This is where our approach differs from the contributions above.

Indeed, we consider that the DM, when designing her policy, may be subject to two kinds of uncertainty. Both affect the severity and evolution of the pandemic. Specifically, we take account of the possibility of the virus mutates, while spreading across the population, with repercussions on its infectious power. As a result, there is uncertainty surrounding not only the time when a mutation occurs, but also the nature of this mutation (contagiousness of mutated virus). Let us define as ω the virus regime, with $\omega \in \Omega$, the finite set of possible mutations, and $t_\omega \in [0, \infty)$ the date of occurrence of mutation ω. Mutations can positively or negatively affects the rate of diffusion of the pandemic, θ. Again by convention, the pandemic regime is supposed to get worse if and only if for any two successive realizations of the random variable $\{\omega, \omega'\}$, we have $\omega < \omega'$ and $\theta'_\omega > 0$ for all ω.

In addition, as seen from instant t_ω when the regime with mutation ω starts, the available information about a possible new shift from mutation ω to ω' at $t \geq t_\omega$ is summarized by a probability distribution function (PDF): $F(t; \omega) = \Pr(t_{\omega'} < t | t \geq t_\omega; \omega)$, defined over the support $[t_\omega, \infty)$, and density $f(t; \omega)$.[8] Then considering any two successive realizations of the random variable $\{\omega, \omega'\}$, the second source of uncertainty is depicted by another PDF, $G(. \mid \omega)$. It represents the conditional distribution of the next regime ω' defined over Ω, which only depends on the preceding regime ω.[9]

[8]We could endogenize the PDF by making it dependent of the state of the system s.

[9]This second source of uncertainty, that has to do with the intensity of the mutation, has been introduced by Sakamoto (2014), within a different context.

2.1.3 Objective function

Here our aim is to capture the basic cost-benefit analysis underlying the choice of the vaccination and lockdown policies. For that purpose, we adopt a cost minimization (of the pandemic) perspective, like in for instance Aspri et al. (2021) and Gollier (2020b). The DM wants to minimize the total cost of the pandemic, which is regime dependent. In the economy a regime is defined by the pair $\{\omega, L_k\}$. Then, the cost has two components. First, there is an instantaneous cost that depends on both vaccination policy, lockdown policy and number of infected people. For simplicity, we shall work with a separable cost function whose arguments are r, L_k and s:

$$C(r, L_k, s) = c(r) + h_k + d(s), \tag{2}$$

with $c(.), d(.)$ strictly increasing and convex for $r, s > 0$. The trade-off associated with the vaccination policy is as follows: Vaccination policy is costly because it requires to purchase vaccine doses, and develop temporary vaccination infrastructure. But it allows the DM to reduce the social cost of sickness by reducing the spread of the pandemic.[10] The benefit of the lockdown policy is of the same nature, while the term $h_k \geq 0$ captures the economic costs inherent in operating lockdown regime L_k. Depending on the level of restrictions, the DM faces additional expenditure to take care of people working remotely or no longer working, the level of economic activity shrinks etc. In general, we may consider that the cost of a lockdown measure is increasing over time because the longer it lasts the costlier it is all the involved dimensions. But as a starting point, we can take h_k constant for given L_k.

Beside the instantaneous cost of lockdown measures, we also introduce a second lump-sum cost incurred when changing the lockdown regime from L_m to L_n, and denote it by $\Gamma_n \geq 0$. This cost is in essence of political or social nature. This is in the line with the "lockdown fatigue" approach developed by Caulkins et al. (2021), or with the argument of "political backslash" put forward by Bandyopadhyay et al. (2021). People get angry when the level of freedom is impaired. It is meant to account for the asymmetry that exists between imposing a lockdown measure and removing the very same measure.

Within any regime $\{\omega, L_m\}$, the general optimization problem can be written as:

$$V(s;\omega) = \min_{r(\cdot), 0 \leqq t_k \leqq \infty} \mathbb{E}\left\{ \int_{t_m}^{t_{\omega'}} C(r, L_m, s) e^{-\rho(t-t_m)} dt + e^{-\rho(t_{\omega'}-t_m)} \mathbb{E}\left[V(s(t_{\omega'}); \omega') | \omega)\right] \right\} + \sum_{k=1}^{n} e^{-\rho(t_k - t_m)} \Gamma_k$$

for all $t_k \leq t_{\omega'} \leqq \infty$ and where $V(.)$ is the value function, the first expectation operator refers to the random date of an epidemic change and the second one to the random realization of the next mutation:

[10] Infected people incur both economic (reduced income) and psychological costs. They may also impose a cost to society, especially where the health care system is publicly funded.

$\mathbb{E}\left[V(s(t_{\omega'});\omega')|\omega)\right] = \int_\Omega V(s,\omega')dG(\omega' \mid \omega)$. The optimization is subject to (1) with $\theta_m(\omega) = \theta(L_m,\omega)$ and $s(t_k) = s_k > 0$, given.

Now, we know that it is possible to re-express this optimization problem as a deterministic problem within any mutation regime. This gives:

$$\min_{r(\cdot),0 \leqq t_k \leqq \infty} \int_{t_m}^{\infty} \left(C(r(t),L_m,s(t))(1-F(t;\omega)) + f(t;\omega)\mathbb{E}\left[V(s(t);\omega')|\omega)\right]\right) e^{-\rho(t-t_m)}dt + \sum_{k=1}^{n} e^{-\rho(t_k-t_m)}\Gamma_k$$

subject to the same set of constraints. Or,

$$\min_{r(\cdot),0 \leqq t_k \leqq \infty} \int_{t_m}^{\infty} (1-F(t;\omega)) \Big(C(r(t),L_m,s(t))e^{-\rho(t-t_m)} + \lambda(t;\omega)e^{-\rho(t-t_m)}\mathbb{E}\left[V(s(t);\omega')|\omega)\right]\Big)\, dt + \sum_{k=1}^{n} e^{-\rho(t_k-t_m)}\Gamma_k$$

with $\lambda(t;\omega)$ the hazard rate generically defined as: $\lambda(t;\omega) = \frac{f(t;\omega)}{1-F(t;\omega)}$.

At this stage, one can already notice that our approach is original in that it combines both impulse controls, that is controlled regime shifts, and random shifts. The key contribution of the paper is actually to study the interplay between these two relevant characteristics of the optimal control of pandemics. Given this objective, and in order to have a chance to get analytical insights, we now present a simplified model that will serve as a vehicle for the coming analysis.

2.2 A simplified problem

Hereafter we make a series of simplifications in order to identify the fundamentals drivers of the optimal management of the pandemic.

As a first step toward tractability, we shall work with an exponential PDF, that yields a constant hazard, λ ($F(t) = 1 - e^{-\lambda t}$). Next, we restrict the analysis to the case where there are two pandemic regimes and two lockdown regimes. As to the pandemic, it is assumed that there are two states of the virus only, and at most one mutation from one to the other: $\Omega = \{\underline{\omega}, \overline{\omega}\}$ with $\overline{\omega} > \underline{\omega}$. So, pandemic under $\overline{\omega}$ is supposed to be more severe than under $\underline{\omega}$. Those two states are known. This means that we switch-off the second source of uncertainty. In addition, in order to handle a mutation in the virus, if any, the DM can choose between two lockdown regimes, $\{L_m, L_n\}$, one stricter than the other. For instance, we may have $L_m < L_n$. For now, we do not need to impose a particular ranking. Finally, we allow the DM to go back and forth between these two regimes.[11] In terms of model specification, we choose a quadratic specification of the cost function (as in Federico and Ferrari, 2021):

$$C(r,L_k,s) = \frac{r^2}{2} + h_k + \beta\frac{s^2}{2}.$$

[11] Actually, we can simply oppose two regimes. The one in which the economy is locked-down and the one where it is "open."

In the following sections, our aim is to analyze this pandemic situation characterized by four regimes, denoted by $(L_m, \overline{\omega})$, $(L_n, \overline{\omega})$, $(L_m, \underline{\omega})$ and $(L_n, \underline{\omega})$. We use $\overline{V}_k(s)$ and $\underline{V}_k(s)$ to denote the value function in regime $(L_k, \overline{\omega})$ and $(L_k, \underline{\omega})$, respectively, for $k \in \{m, n\}$. Similarly, the infection rate in regime $(L_k, \underline{\omega})$ is denoted by $\theta(L_k, \underline{\omega}) = \underline{\theta}_k$ and in regime $(L_k, \overline{\omega})$ by $\theta(L_k, \overline{\omega}) = \overline{\theta}_k$ for $k = m, n$.

3 Optimality conditions

We first state the optimality conditions for both continuous and impulse controls. In particular, particular attention is paid to the ones characterizing the latter type of control. All of the proofs are gathered in the Appendix.

3.1 Continuous and impulse controls

In regime$(L_m, \overline{\omega})$, by dynamic programming, $\overline{V}_m(s)$ satisfies the inequality

$$\rho \overline{V}_m(s) \leq \overline{H}^*_m\left(s, \overline{V}'_m(s)\right)$$

where

$$\overline{H}^*_m(s, \mu) = s\mu\left[\overline{\theta}_m(1-s) - K\right] - \frac{\left[\overline{\theta}_m s\mu\right]^2}{2} + h_m + \frac{\beta}{2}s^2 \tag{3}$$

is the current value Hamiltonian, and the minimizer r^* satisfies

$$r^* = \overline{V}'_m(s)\,\overline{\theta}_m s. \tag{4}$$

In order for the solution to be well-behaved, one must have $\overline{V}'_m(s) \geq 0$. Then, how exactly the vaccination rate changes with s depends on the particular shape of the value function. If it is convex, then the larger the infection rate, the larger the vaccination rate. In addition, any impulse control $L_m \to L_n$, satisfying $L_m < L_n$ (case of a stricter lockdown), will induce a drop in the vaccination rate. This is the first evidence of the substitutable nature of the two controls.

The DM does not take the impulse control $L_m \to L_n$ if doing so is not profitable. Hence, mode m continues if

$$\overline{V}_m(s) \leq \overline{V}_n(s) + \Gamma_n.$$

Since at least one of the inequalities must hold true, we can formulate the quasi-variational inequality (QVI):

$$\max\left\{\rho\overline{V}_m(s)-\overline{H}_m^*\left(s,\overline{V}_m'(s)\right),\overline{V}_m(s)-\overline{V}_n(s)-\Gamma_n\right\}=0\qquad\text{for } s\in(0,1). \tag{5}$$

Similarly, in regimes $(L_m,\underline{\omega})$, the value functions $\underline{V}_m(s)$ satisfies

$$(\rho+\lambda)\underline{V}_m(s)\leq\underline{H}_m^*\left(s,\underline{V}_m'(s),\lambda\overline{V}_m(s)\right)$$

with

$$\underline{H}_m^*(s,\mu,W)=s\mu\left[\underline{\theta}_m(1-s)-K\right]-\frac{\left[\underline{\theta}_m s\mu\right]^2}{2}+h_m+\frac{\beta}{2}s^2+W \tag{6}$$

and minimizer r^* given by

$$r^*=\underline{V}_m'(s)\,\underline{\theta}_m s. \tag{7}$$

Also, condition

$$\underline{V}_m(s)\leq\underline{V}_n(s)+\Gamma_n$$

holds for mode m to remain unchanged. Thus, the second QVI:

$$\begin{aligned}\max\{&(\rho+\lambda)\underline{V}_m(s)-\underline{H}_m^*\left(s,\underline{V}_m'(s),\lambda\overline{V}_m(s)\right),\\&\underline{V}_m(s)-\underline{V}_n(s)-\Gamma_n\}=0\qquad\text{for } s\in(0,1).\end{aligned} \tag{8}$$

In particular, at a value s_n^* where the impulse control $L_m\to L_n$ is taken, we have

$$\overline{V}_m(s^*)=\overline{V}_n(s^*)+\Gamma_n \tag{9}$$

after mutation, and

$$\underline{V}_m(s^*)=\underline{V}_n(s^*)+\Gamma_n \tag{10}$$

before mutation.

We now prove the following criteria for the timing of an impulse control.

Theorem 1 *Suppose the DM takes the impulse control $L_m\to L_n$ after mutation at an interior point $\overline{s}_n^*\in(0,1)$. If value functions $\overline{V}_m(s)$ and $\overline{V}_n(s)$ are both differentiable in $(0,1)$, then $\overline{s}_n^*$ satisfies the equation*

$$\rho\left[\overline{V}_n(\overline{s}_n^*)+\Gamma_n\right]=\overline{H}_m^*\left(\overline{s}_n^*,\overline{V}_n'(\overline{s}_n^*)\right). \tag{11}$$

Similarly, suppose the impulse control is taken before mutation at an interior point $\underline{s}_n^* \in (0,1)$*. If* $\underline{V}_m(s)$ *and* $\underline{V}_n(s)$ *are differentiable in* $(0,1)$*, then* $\underline{s}_n^*$ *is defined by*

$$(\rho+\lambda)\left[\underline{V}_n(\underline{s}^*)+\Gamma_n\right]=\underline{H}_m^*\left(\underline{s}_n^*,\underline{V}_n'(\underline{s}_n^*),\lambda\overline{V}_m(\underline{s}_n^*)\right). \tag{12}$$

Conditions (11) and (12) are essentially transversality conditions that govern the optimal transition between different regimes. In (deterministic) optimal control literature, they are usually stated in terms of the maximized Hamiltonians, holding before and after the regime change (see for instance Boucekkine et al. 2013). In the absence of lump-sum cost, these optimality conditions impose the continuity of the Hamiltonians at the date of regime switching, when the impulse control is taken, and the continuity of the co-state variable(s) as long as the level of the state variable(s) at which the decision is taken is free. In dynamic programming, these correspond to the well-known value matching and smooth pasting conditions, expressed in terms of value functions and their first order derivative. Here, we merge both approaches because of the specificity of our problem. Indeed, the comparison between conditions (11) and (12) reveals how the prospect of a mutation affects the impulse control. Indeed, we observe that both the hazard rate and the marginal value following a mutation in lockdown regime L_m, $\overline{V}_m(.)$, show up in the optimality condition before any mutation.

For further discussion, let us express these conditions in the specified model. Because $\overline{V}_n$ and $\underline{V}_n$ are differentiable at $\overline{s}_n^*$ and $\underline{s}_n^*$ respectively, they satisfy the HJB equations

$$\begin{aligned}\rho\overline{V}_n(\overline{s}_n^*) &= \overline{H}_n^*\left(\overline{s}_n^*,\overline{V}_n'(\overline{s}_n^*)\right) \qquad \text{and}\\ (\rho+\lambda)\underline{V}_n(\underline{s}_n^*) &= \underline{H}_n^*\left(\underline{s}_n^*,\underline{V}_n'(\underline{s}_n^*),\lambda\overline{V}_n(s)\right).\end{aligned}$$

By subtracting the corresponding sides in (11) and (12), we obtain

$$\overline{H}_m^*\left(\overline{s}_n^*,\overline{V}_n'(\overline{s}_n^*)\right)=\overline{H}_n^*\left(\overline{s}_n^*,\overline{V}_n'(\overline{s}_n^*)\right)+\rho\Gamma_n,$$

and

$$\underline{H}_m^*\left(\underline{s}_n^*,\underline{V}_n'(\underline{s}_n^*),\overline{V}_m(\underline{s}_n^*)\right)=\underline{H}_n^*\left(\underline{s}_n^*,\underline{V}_n'(\underline{s}_n^*),\lambda\overline{V}_n(\underline{s}_n^*)\right)+(\rho+\lambda)\Gamma_n.$$

Using the specific forms of $\overline{H}_m^*$ and $\underline{H}_m^*$ in (3) and (6), respectively, the equations become

$$\begin{aligned}&\overline{s}_n^*\overline{V}_n'(\overline{s}_n^*)\left(\overline{\theta}_n-\overline{\theta}_m\right)(1-\overline{s}_n^*)-\frac{1}{2}\left[\overline{s}_n^*\overline{V}_n'(\overline{s}_n^*)\right]^2\left(\overline{\theta}_n^2-\overline{\theta}_m^2\right)\\ &+h_n-h_m+\rho\Gamma_n=0,\end{aligned} \tag{13}$$

and

$$\underline{s}_n^* \underline{V}_n' \left(\underline{s}_n^*\right) \left(\underline{\theta}_n - \underline{\theta}_m\right) \left(1 - \underline{s}_n^*\right) - \frac{1}{2} \left[\underline{s}_n^* \underline{V}_n' \left(\underline{s}_n^*\right)\right]^2 \left(\underline{\theta}_n^2 - \underline{\theta}_m^2\right) \\ + h_n - h_m + \lambda \left(\overline{V}_n \left(\underline{s}_n^*\right) - \overline{V}_m \left(\underline{s}_n^*\right)\right) + (\rho + \lambda) \Gamma_n = 0, \tag{14}$$

respectively.

We immediately notice that the optimality condition before a mutation is modified in two ways, compared to the condition after. First, we get that being subject to uncertainty induces the DM to use an augmented discount rate, that is the sum of the pure rate of time preference and the hazard rate. This is a well-known effect of considering the occurence of stochastic (exogenous) events in optimal control problems. To stay close to the topic under scrutiny, this is actually similar to what is obtained in the "short-term analyses" of optimal lockdown policy that assume that the planning horizon is finite but uncertain (Alvarez et al. 2021, Jones et al., 2021 etc.), except that the source of uncertainty is different.[12] Second, there is an extra term that involves the difference of the value functions obtained in the two lockdown regimes, after the mutation. This term represents the net benefit of operating under lockdown regime L_n, instead of L_m (the former being the stricter regime iff $L_n > L_m$). It means that when contemplating the opportunity to take a lockdown measure before a mutation, the DM has to take into account the net gain to be in a different lockdown regime if and once a mutation occurs. This feature is very much in line with what Long et al. (2017) have shown in different context.

The influence of stochastic mutation on impulse control raises a series of questions: how does the DM adapt her policy to the risk of virus mutation? Is there an incentive to use lockdown as a prevention to a potential mutation, or as an adaptation to an actual one? Does the prospect of a mutation hasten, or on the contrary, delay lockdown measures? The next sections address these issues. To start with, we provide necessary conditions for a lockdown measures and discuss them, then we also show sufficient conditions for impulse controls.

3.2 Necessary conditions for impulse control

We investigate under which conditions the DM may want to take an impulse control $L_m \to L_n$ in mode m, whatever the pandemic state. That is, we provide necessary conditions for an impulse control.

Theorem 2 *If the impulse control $L_m \to L_n$ is taken for some $\overline{s}_n^* \in (0,1)$ after mutation, then*

$$\frac{\overline{\theta}_m - \overline{\theta}_n}{2(\overline{\theta}_m + \overline{\theta}_n)} \geq h_n - h_m + \rho\Gamma_n. \tag{15}$$

[12]These papers typically assume that the horizon is given by the arrival of a vaccine, modeled as a Poisson process, which boils down to working with a deterministic infinite horizon optimal control problem with an increased discount rate (equal to the sum of the rate of time preferences and the constant hazard rate) and salvage value function.

If the impulse control $L_m \to L_n$ is taken for some $\underline{s}_n^* \in (0,1)$ before mutation, then

$$\frac{\underline{\theta}_m - \underline{\theta}_n}{2(\underline{\theta}_m + \underline{\theta}_n)} \geq h_n - h_m + (\rho + \lambda)\,\Gamma_n + \lambda \left(\overline{V}_n \left(\underline{s}_n^* \right) - \overline{V}_m \left(\underline{s}_n^* \right) \right) \tag{16}$$

After a mutation, the necessary condition involves the net total cost, which is the sum of the lump-sum cost and of the difference between operation costs $(h_n - h_m)$, and the health net gain $(\overline{\theta}_m - \overline{\theta}_n)$ of switching lockdown regime. Then ,we observe that using the impulse control after a mutation has occurred can be worth only when the health gain is sufficiently large and/or the cost imposed to society is low enough.

In the second scenario in which adopting lockdown measure is considered as a policy option before the mutation is realized, the necessary condition is changed. Suppose first that the value functions after mutation are the same. The eventuality of a lockdown regime switching in that scenario requires the health gain be much larger. With different value functions, what happens after the mutation also matters. If the DM expects that the overall cost of the pandemic is going to be larger $(\overline{V}_n\,(.) - \overline{V}_m(.) > 0)$, once the mutation occurs, in the new lockdown regime L_n, then the necessary condition becomes even harder to meet. If on the contrary, the DM considers that the economy will be better prepared to handle the pandemic with the new regime $(\overline{V}_n\,(.) - \overline{V}_m(.) < 0)$, then the necessary condition for impulse control is less demanding.

For the sake of completeness, we can also provide sufficient conditions for an impulse control in any regime. For that purpose, we need to introduce a couple of additional concepts. Let $\overline{U}_k\,(s)$ and $\underline{U}_k\,(s)$, for $k = m, n$, be the solutions to the following HJB equations:

$$\rho \overline{U}_k\,(s) = \overline{H}_k^* \left(s, \overline{U}_k'\,(s) \right), \qquad \text{for } s \in (0,1)\,, \tag{17}$$

and

$$(\rho + \lambda)\,\underline{U}_k\,(s) = \underline{H}_k^* \left(s, \underline{U}_k'\,(s)\,, \lambda \overline{V}_k\,(s) \right) \qquad \text{for } s \in (0,1)\,, \tag{18}$$

where $\overline{H}_k^*$ and $\underline{H}_k^*$ are defined in (3) and (6), respectively, and $\overline{V}_k$ is the value function in mode k after mutation. We remark that $\overline{U}_k\,(s)$ and $\underline{U}_k\,(s)$ need not be the value functions. These are optimal values without possibility of impulse control. Hence, by optimality of the value functions, it is necessary that

$$\overline{V}_k\,(s) \leq \overline{U}_k\,(s)\,, \qquad \underline{V}_k\,(s) \leq \underline{U}_k\,(s) \qquad \text{for } s \in (0,1)\,, \quad k = m, n.$$

Then, we can establish that:[13]

[13]Note that the results are presented in the most general way, i.e., hold for any pandemic regime

Theorem 3 *Suppose $\Gamma_m + \Gamma_n > 0$. Let U_k be either $\overline{U}_k$ after mutation, or $\underline{U}_k$ before mutation, for $k = m, n$. The following claims are true.*

1. *If*

$$h_n - h_m + \rho\Gamma_n < 0 \tag{19}$$

and

$$U_m(1) < U_n(1) + \Gamma_n. \tag{20}$$

Then impulse control $L_m \to L_n$ must occur at some $s^* \in (0, 1)$.

2. *If*

$$h_n - h_m + \rho\Gamma_n > 0 \tag{21}$$

and

$$U_m(1) > U_n(1) + \Gamma_n \tag{22}$$

Then $L_m \to L_n$ must occur at some $s_n^* \in (0, 1)$.

These sufficient conditions all sound pretty natural. For an interpretation, it is enough to focus on part 2. of Theorem 3. Consider that the impulse control $L_m \to L_n$ corresponds a tightening of the lockdown policy. Then, logically the total net cost of the measure should be positive ($h_n - h_m + \rho\Gamma_n > 0$). Given this, condition (22) simply states that the DM has to find it worth to place the economy under the most restrictive lockdown regime in the worst case scenario where a hundred percent of the population gets infected.

4 Dynamical system

Hereafter, we summarize the basic features of the dynamical system in any regime. This is a prerequisite for the coming analysis of the potential outcomes for our problem.

4.1 General insights

The system dynamics in regime k before or after mutation is governed by the differential equation

$$\dot{s} = s\left[\theta_k(1 - s) - K - \theta_k^2 s V_k'(s)\right] \tag{23}$$

where θ_k and V_k are either $\overline{\theta}_k$ and $\overline{V}_k$, or $\underline{\theta}_k$ and $\underline{V}_k$, $k = m, n$. Since $V_k' \geq 0$, from (4) and (7), $s(t)$ is decreasing over time if

$$\theta_k (1 - s) \leq K. \tag{24}$$

Hence, if $\theta_k \leq K$ then $s(t)$ monotonically decrease to 0 as $t \to \infty$.

If on the contrary, $\theta_k > K$, the dynamic and the steady state analyses are more complicated as they depend on the shape of the value function, vaccination policy, and of course are regime-dependent. Moreover, as there is no boundary condition to the HJB equation, there may be infinitely many solutions, which makes the study even more complicated. We come back to this issue in the next section. In the meantime, we simply analyze the features of the dynamics within a particular regime and for a particular value function. This will be enough to discuss the possible timing of impulse controls.

From equation (23), $s = 0$ is always one unstable steady state when $\theta_k > K$. The other possible steady states solve

$$\theta_k(1 - s) = K + \theta_k^2 s\, V_k'(s).$$

Denote the left hand side as $f(s) = \theta_k(1 - s)$ and right hand side as $g(s) = K + \theta_k^2 s\, V_k'(s)$. At $s = 0$, we have $g(0) = K < f(0) = \theta_k$, while at $s = 1$, $g(1) = K + \theta_k^2 V_k'(1) > K > f(1) = 0$. So there exists at least one non trivial ($s \neq 0$) steady state as well. Furthermore $f'(s) = -\theta_k < 0$, and

$$g'(s) = \theta_k^2 V_k'(s)\,(1 - \epsilon(s)) \geq 0 \Leftrightarrow 1 \geq \epsilon(s)$$

where $\epsilon(s) = -\frac{sV''_k(s)}{V_k'(s)} (> 0)$ is the elasticity of marginal value function with respect to s. We observe that if this elasticity is sufficiently low for all $s \in [0, 1]$, which means that the DM is not too sensitive to a change in the number of infected, then $g(s)$ is monotonically increasing and thus the steady state is unique, $\widehat{s_k} \in (0, 1)$, and is at least locally asymptotically stable.

In general however, there is an odd number of non trivial steady states. The infimum and the supremum of the set of steady states, $\hat{s}_k$, $\tilde{s}_k$, are such that $0 < \hat{s}_k \leq \tilde{s}_k < 1$ and

$$\begin{aligned} \lim_{t\to\infty} s(t) &= \hat{s}_k \qquad \text{if } 0 < s(0) < \hat{s}_k, \\ \lim_{t\to\infty} s(t) &= \tilde{s}_k \qquad \text{if } \tilde{s}_k < s(0) \leq 1. \end{aligned}$$

Furthermore, since $\tilde{s}_k$ is a positive root of function on the right-hand side of (23), it follows that

$$\theta_k (1 - \tilde{s}_k) = K + \theta_k^2 \tilde{s}_k V_k' (\tilde{s}_k) \geq K.$$

Hence,

$$\tilde{s}_k \leq 1 - \frac{K}{\theta_k} \equiv s_k.$$

As apparent from the discussion above, the ranking between the θ_ks and parameter K (efficiency of the health system) is quite important when it comes to the pandemic dynamics. When $\theta_k < K$, the share of infected people decreases even in the absence of vaccination policy, r. This implies that this policy is only useful to control the speed of decrease of the contagion. This may not sound like the most interesting case to study. However, even in this case we cannot conclude that the public policy is of lesser importance because it may well be that θ_k falls below K as a result of the lockdown measure. This would be the case if for instance, after a mutation, $\overline{\theta}_m > K > \overline{\theta}_n$. Anyway, for the sake of completeness of the analysis, we consider all of the possible cases.

4.2 Minimum value function and corresponding dynamical system

Hereafter we show the existence of a minimum value function when $\theta_m > K$. This value function is particularly appealing given the nature of the optimization program. Selecting this one, that displays interesting dynamic properties, will prove to be very useful for the analysis conducted in the next Section, and devoted to the features of the optimal policy.

As mentioned above, in the case $\theta_m > K$, the value function U_m is not unique. For example, in a regime after mutation, at a point s such that

$$\bar{U}_m(s) > q_m(s) \equiv \frac{h_m}{\rho} + \frac{\beta}{2\rho}s^2$$

the HJB equation

$$\rho \bar{U}_m = s\bar{\theta}_m \bar{U}_m'(s)(s_m - s) - \frac{\left[\bar{\theta}_m s \bar{U}_m'(s)\right]^2}{2} + \rho q_m(s)$$

leads to two possible vaccination rates,

$$\bar{r}_m(s) = \bar{\theta}_m s \bar{U}_m'(s) = s_m - s \pm \sqrt{\left[(s_m - s)^2 + 2\rho\left[q_m(s) - \bar{U}_m(s)\right]\right]},$$

which are both positive. Even if we choose $\bar{r}_m(s)$ to be the smaller nonnegative root for any s, there are still infinitely many solutions to the HJB equation.

Nonetheless, it can be shown that there exists a minimum solution among all possible solutions to the HJB equation. This particular value function has the advantage of generating much simpler dynamics, thereby allowing us to provide a clear characterization of outcomes in the long run. Lemma 1, in the Appendix D, constructs this minimum value function in the following way:

Suppose $\theta_m > K$ and either the regime is after mutation or before mutation with $\lambda = 0$. Let

$$s'_m = \frac{2s_m}{1+\sqrt{1+\frac{2\beta}{\rho}\theta_m s_m}} \, (< 1) \, .$$

Denote $U_m(s) = U_m(s; s'_m)$ the solution of HJB (a nonlinear differential equation)

$$\rho U_m(s) = s\theta_m U'_m(s)(s_m - s) - \frac{[\theta_m s U'_m(s)]^2}{2} + \rho q_m(s) \tag{25}$$

with boundary condition

$$U_m(s'_m; s'_m) = q_m(s'_m) \, . \tag{26}$$

Then, Lemma 1 establishes the following result: for $0 < s < 1$, the function $U_m(s; s'_m)$, that satisfies

$$U'_m(s; s'_m) = \frac{1}{\theta_m s}\left[s_m - s - \sqrt{(s_m - s)^2 + 2\rho\,[q_m(s) - U_m(s; s'_m)]}\right], \tag{27}$$

is the minimum value function among all solutions to the HJB equation, and is associated with the minimum (nonnegative) vaccination rate $r_m(s) = \theta_m s U_m(s; s'_m)$ at all s.

Equipped with this minimum value function, we can conduct the analysis of the dynamical system to obtain (Appendix E):

Proposition 1 *Suppose $\theta_m > K$, and either the regime is after mutation or before mutation with $\lambda = 0$. For the value function $U_m(s; s'_m)$, that is the solution of (25), we have*

$$\frac{ds}{dt}\begin{cases} > 0 & \text{if} \quad s < s'_m, \\ < 0 & \text{if} \quad s > s'_m. \end{cases}$$

In addition, $s(t)$ reaches s'_m in finite time from any point $s > 0$.

Obviously, under the assumption of the above proposition, s'_m is the globally asymptotically stable long-run steady state, which checks $s'_m \in (0, 1)$. This result will be extensively used in the next Section, where we especially want to understand if being subject to a risk of mutation induces the DM to delay, or on the contrary expedite the lockdown decision.

5 Features of the optimal policy

From now one, for the ease of exposition, we pay attention to the following specific scenario (and replace general indexes m, n with $0, 1$):

$$h_1 > h_0, \qquad \Gamma_1 > 0 = \Gamma_0, \quad \overline{\theta}_0 > \overline{\theta}_1, \quad \underline{\theta}_0 > \underline{\theta}_1. \tag{28}$$

All in all, this means that L_0 is the reference situation with no lockdown constraints imposed upon society, while L_1 refers to the situation in which the economy is locked-down. Accordingly, the move $L_0 \to L_1$ corresponds to a lockdown of the economy (introduction of some constraints), whereas $L_1 \to L_0$ captures a reopening (removal of these constraints). The ranking between cost and infection rate parameters follow directly from this characterization.

In what follows, we first state some results—from general to more specific—about the features of the solution, especially the timing for impulse controls. Then, we present the potential outcomes, and how they relate to the evolution of the number of infected across time. Finally, we provide a numerical illustration to highlight the main characteristics of the solution.

5.1 Timing for impulse control

Based on Theorem 2 and conditions (15)-(16), we can define critical levels for lockdown in every possible regime:

$$\begin{aligned}
\overline{\delta}_1 &= 2\frac{\overline{\theta}_0 + \overline{\theta}_1}{\overline{\theta}_0 - \overline{\theta}_1}\left[h_1 - h_0 + \rho\Gamma_1\right], \\
\underline{\delta}_1 &= 2\frac{\underline{\theta}_0 + \underline{\theta}_1}{\underline{\theta}_0 - \underline{\theta}_1}\left[h_1 - h_0 + \rho\Gamma_1 + \lambda \max_{\{0\le s\le 1\}}\left(\overline{V}_1(s) + \Gamma_1 - \overline{V}_0(s)\right)\right], \\
\overline{\delta}_0 &= 2\frac{\overline{\theta}_0 + \overline{\theta}_1}{\overline{\theta}_0 - \overline{\theta}_1}\left[h_1 - h_0\right], \\
\underline{\delta}_0 &= 2\frac{\underline{\theta}_0 + \underline{\theta}_1}{\underline{\theta}_0 - \underline{\theta}_1}\left[h_1 - h_0 + \lambda \max_{\{0\le s\le 1\}}\left(\overline{V}_1(s) - \overline{V}_0(s)\right)\right].
\end{aligned}$$

Then, we obtain some general results connecting lockdown with reopening type of impulse controls' occurrence (Appendix F):

Proposition 2 *Suppose (28) holds. Let s_0^*, δ_0, V_0 and U_0 be either $\overline{s}_0^*$, $\overline{\delta}_0$, $\overline{V}_0$, and $\overline{U}_0$ after mutation, respectively, or $\underline{s}_0^*$, $\underline{\delta}_0$, $\underline{V}_0$, and $\underline{U}_0$, respectively. The following are true.*

1. *If lockdown occurs at a point $s_1^* \in (0,1)$, then $\delta_1 < 1$. In this case, s_1^* that satisfies*

$$s_1^* \le 1 - \sqrt{\delta_1}. \tag{29}$$

Conversely, if lockdown does not occur at an interior point, then $V_0(s) = U_0(s)$ *for all* $s \in [0,1]$*. In this case, either* $V_1(s) = V_0(s)$ *for all* $s \in [0,1]$*, or there is* $s_0^* \in (0,1)$ *such that*

$$\begin{aligned} &V_1(s) = V_0(s) && \text{for } s \leq s_0^*, \\ &V_1(s) < V_0(s) \leq V_1(s) + \Gamma_1 && \text{for } s_0^* < s \leq 1. \end{aligned} \tag{30}$$

In the latter case reopening occurs at $s = s_0^*$*.*

2. *If reopening occurs at a point* $s_0^* \in (0,1)$*, then* $\delta_0 < 1$*. In this case,* s_0^* *that satisfies*

$$s_0^* \leq 1 - \sqrt{\delta_0}. \tag{31}$$

Conversely, if reopening does not occur at an interior point, then lockdown also does not happen at an interior point. In this case

$$V_1(s) = V_0(s) = U_0(s) \qquad \text{for } 0 \leq s \leq 1. \tag{32}$$

Proposition (2) has two parts, divided into two claims. The first claim in each part (conditions for lockdown and reopening at an interior point) follows directly from Theorem 2. Let us then discuss the second one. If a lockdown does not happen at an interior point where some individuals are infected, then the DM does not impose lockdown at the current regime. In this case, by definition the value function $U_0(s)$ yields the lowest social cost ($V_0(s) = U_0(s)$), and as lockdown is never beneficial. Then, there are two possibilities. Either the social cost of lockdown is higher than that without lockdown, or the social cost of lockdown is lower but adding the lumpsum start up cost the total cost is too high. In the former case naturally the DM will never impose lockdown in the current regime, and if the regime is after mutation and the state is locked down before mutation, once mutation occurs, the DM will immediately reopen. The latter case can only happen if the share of the infected population is already high ($s_0^* < s \leq 1$). In the case where lockdown is imposed before mutation when the infected population is low, i.e., at some $s \leq s_0^*$, the DM would immediately reopen when the mutation occurs. If the mutation occurs with high infected population, i.e., at some $s > s_0^*$, since the social cost with lockdown is lower, the DM will keep the state locked down until the share of the infected people drops to s_0^*, and then reopen at this moment.

Finally, if reopening does not occur at an interior point, the only possibility is that $s_0^* = 1$. This means even if the entire population is infected, lockdown is not a choice. So, certainly lockdown does not happen for any size of infected population.

For an impulse control $L_0 \to L_1$ to occur at an interior point, it is necessary that $\overline{\delta}_1 < 1$ or $\underline{\delta}_1 < 1$. Since by (5)

$$\overline{V}_1(s) \leq \overline{V}_0(s) \leq \overline{V}_1(s) + \Gamma_1$$

for all $s \in [0,1]$, it follows that $\underline{\delta}_0$ is nonincreasing in λ whereas $\underline{\delta}_1$ is nondecreasing in λ. Hence, reopening at an interior point before mutation can be impossible without mutation, but becomes possible with mutation, and lockdown can be possible without mutation, but becomes impossible with mutation. To be more precise, based on the above results, we see that if reopening at an interior point is not possible after mutation, then

$$\underline{\delta}_0 = 2\frac{\underline{\theta}_0 + \underline{\theta}_1}{\underline{\theta}_0 - \underline{\theta}_1}(h_1 - h_0)$$

is independent of λ, but

$$\underline{\delta}_1 = 2\frac{\underline{\theta}_0 + \underline{\theta}_1}{\underline{\theta}_0 - \underline{\theta}_1}[h_1 - h_0 + (\rho + \lambda)\Gamma_1]$$

is strictly increased by a multiple of λ. This means lockdown before mutation can become less possible.

In addition, under specific conditions, we can show that the prospect of mutation delays both types of impulse control before mutation (Appendix G).

Proposition 3

1. *Suppose lockdown occurs before mutation at an interior point $\underline{s}_1^*(0) \in (0,1)$ with $\lambda = 0$ which is less than the least positive steady state. If after mutation lockdown does not occur at some $\overline{s}_1^* \leq \underline{s}_1^*(0)$, then there is $\varepsilon > 0$ such that lockdown occurs at some $\underline{s}_1^*(\lambda) > \underline{s}_1^*(0)$ for $0 < \lambda < \varepsilon$. If after mutation lockdown occurs at some $\overline{s}_1^* < \underline{s}_1^*(0)$, then there is $\varepsilon > 0$ such that lockdown occurs at some $\underline{s}_1^*(\lambda) \leq \underline{s}_1^*(0)$ for $0 < \lambda < \varepsilon$.*

2. *Suppose $\underline{\theta}_1 < K$ and that reopening occurs before mutation at an interior point $\underline{s}_0^*(0) \in (0,1)$ with $\lambda = 0$. If after mutation reopening occurs at some $\overline{s}_0^* < \underline{s}_0^*(0)$, then there is $\varepsilon > 0$ such that reopening occurs at some $\underline{s}_0^*(\lambda) < \underline{s}_0^*(0)$ for $0 < \lambda < \varepsilon$. If after mutation reopening occurs at some $\overline{s}_0^* > \underline{s}_0^*(0)$, then there is $\varepsilon > 0$ such that reopening occurs at some $\underline{s}_0^*(\lambda) \geq \underline{s}_0^*(0)$ for $0 < \lambda < \varepsilon$.*

The proof is built on the comparison between threshold levels for impulse control, in the absence of uncertainty, in the pandemic conditions of before vs. after a mutation. Based on this comparison, we can then determine whether being subject to a risk of mutation makes one delay or on the contrary expedite impulse control. We further consider scenarios in which in the absence of policy intervention, the share of infected would increase in every pandemic regime. The first part of Proposition 3 deals with the impact of uncertainty about the virus on the lockdown decision. We obtain that if in the absence of uncertainty, it would be optimal for the DM to take the lockdown decision "sooner" in the worst case scenario (in terms of infectivity) than in the best case one, then uncertainty expedites lockdown. Put differently, the prospect of a "bad mutation" at some uncertain future date induces the DM to act more cautiously. In this particular context, this boils down to imposing a lockdown sooner (than in the absence of risk of mutation) in order to be better prepared to the future likely event of a mutation. On the contrary, the risk of experiencing a good mutation delays lockdown measures. In this case, given the costs of this policy (that are known for sure) and its uncertain, yet likely low, benefit, the DM prefers to wait and see the evolution of the pandemic situation before taking this kind of decision. The second part of the Proposition provides symmetric results for reopening decision: a bad mutation delays reopening before it happens and then decision is surrounded by uncertainty, and a good one does the opposite.

We can draw a parallel between our conclusions and the two main strands of the literature on decision making under uncertainty. First and foremost, there is a long tradition of papers studying the impact of the occurrence of random events on optimal decision making dating back to Dasgupta and Heal (1974) and Cropper (1976). Many papers precisely ask how being subject to costly (sometimes catastrophic) events shapes decisions, with many applications in environmental and resources economics (see Crepin and Naedval, 2020, for a recent overview of the literature). One of the main messages is that the optimal response to a risk of costly event is to behave more cautiously (in terms of resource extraction or polluting emissions for instance). Our result in the bad mutation case clearly echoes those obtained in the literature, and extends them to the class of impulse controls. Second, these results also have a broader connection with the real option value literature (Dixit and Pindyck, 1994) that emphasizes the role of uncertainty and learning in forming decisions (see Bandyophadyay et al., 2021, Gollier, 2020b, for contributions on the control of epidemics). In our setting, because the DM can take the lockdown and reopening decisions whenever she wants, there exists (at least in the good mutation case) an incentive to wait and possibly experience the mutation before acting, as upon a mutation, the information about the disease contagiousness is revealed.

5.2 Possible outcomes

Assuming (28), then by (5) and (8),

$$V_1(s) \leq V_0(s) \leq V_1(s) + \Gamma_0 \qquad \text{for } s \in [0,1]$$

for both before and after mutation, where V_k is either $\overline{V}_k$ or $\underline{V}_k$ for $k = 0, 1$. Furthermore, since $h_1 > h_0$, and

$$U_k(0) = \frac{h_k}{\rho} \qquad \text{for } k = 0, 1,$$

it follows that

$$V_1(0) \le V_0(0) \le \frac{h_0}{\rho} < U_1(0).$$

Hence, $V_1(s) \ne U_1(s)$ for small s. This implies that

$$V_1(s) = V_0(s) \tag{33}$$

for s sufficiently small. Either (33) holds for all $s \in [0, 1]$ or there is $s_0^* \in (0, 1)$ such that (33) holds only for $0 \le s \le s_0^*$.

In case s_0^* exists,

$$V_1(s) < V_0(s) < V_1(s) + \Gamma_1 \tag{34}$$

holds for $s > s_0^*$ and is near s_0^*. Either (34) holds for all $s \in (s_0^*, 1]$ or there is $s_1^* < 1$ such that (34) only holds for $s_0^* < s < s_1^*$. For $s > s_1^*$ the relation

$$V_0(s) = V_1(s) + \Gamma_1$$

must hold.

With this in mind, we can discuss the features of the solution, in terms of impulse control and induced evolution of the pandemic. This discussion is conducted whatever the pandemic regime. In order to have a full picture of the optimal, one will then have to combine the properties of the solution before mutation, given that a mutation will occur eventually (i.e. before the steady state is achieved), with the ones of the solution after, when there is no more room for a mutation of the virus. There are three possible scenarios. In all scenarios, we provide figures representing the ranking between value functions and how it evolves across time.

5.2.1 Case 1: Neither s_0^* nor s_1^* exists

Since neither s_0^* nor s_1^* exists, $V_0(s) = V_1(s)$ for all $s \in [0, 1]$.

If at $t = 0$ the economy is locked down, then it is optimal for the DM to reopen immediately. In the un-locked down regime, lockdown will never happen. If $\theta_0 \le K$, the $s(t)$ decreases to zero as $t \to \infty$. If

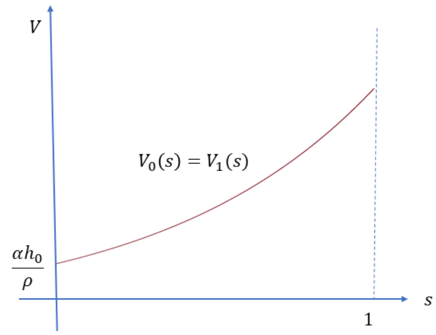

Figure 1: Neither s_0^* nor s_1^* exists

$\theta_0 > K$, then

$$\lim_{t\to\infty} s(t) = \tilde{s}_0 \quad \text{if } s(0) > \tilde{s}_0,$$
$$\lim_{t\to\infty} s(t) = \hat{s}_0 \quad \text{if } s(0) < \hat{s}_0.$$

This is the situation in which the impact of the pandemic is not so severe and/or the economic and social cost of a lockdown is too large for the DM to find it optimal to place the economy under a lockdown for a non-degenerated period of time (see the reverse conditions of those imposed in Theorem 2). There is also no room for reopening at an interior point, $t \in (0, \infty)$.

5.2.2 Case 2: Only s_0^* exists

In this case, for $s \leq s_0^*$, $V_1(s) = V_0(s)$ and for $s_0^* < s \leq 1$,

$$V_1(s) < V_0(s) < V_1(s) + \Gamma_1. \tag{35}$$

Then depending on the initial state of the pandemic, the impulse control decision can be the following.

If at $t = 0$ the state is locked down and $s(0) \leq s_0^*$, reopening immediately occurs. The state of the pandemic is so low that it is optimal to remove the lockdown as soon as possible. Then $s(t) \to \hat{s}_0$ if $s(0) < \hat{s}_0$ and $s(t) \to \tilde{s}_0$ if $s(0) > \tilde{s}_0$. On the other hand, if $s(0) > s_0^*$ while the state is locked down, $s(t)$ converges to a steady state or decreases to s_0^*. In the latter case reopening occurs as $s(t)$ reaches s_0^*, and then the state remains open and $s(t)$ converges to a steady state. So, when the initial share of infected people is sufficiently high, it is optimal for the DM to keep the lockdown in operation until the pandemic situation is under control again. A permanent lockdown is possible otherwise.

As in Case 1, if at $t = 0$ the state is not locked-down, lockdown will not occur, and $s(t)$ will converge to a steady state. In the case $\theta_0 \leq K$, $s(t)$ decreases to zero. Otherwise, $s(t)$ converges to a nonzero steady state. The value functions are now depicted by Figure 2.

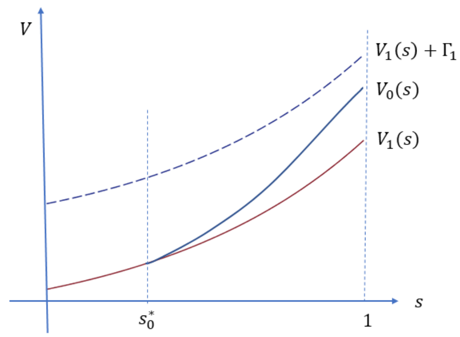

Figure 2: Only s_0^* exists

5.2.3 Case 3: Both s_0^* and s_1^* exist

This is the richest case that provides the most diverse scenarios, as depicted by Figure 3.

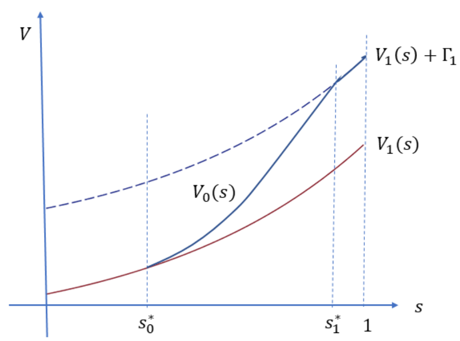

Figure 3: Both s_0^* and s_1^* exist

As shown in the graph, for $s \leq s_0^*$, $V_1(s) = V_0(s)$. For $s_0^* < s < s_1^*$, (35) holds, and for $s_1^* \leq s \leq 1$, $V_0(s) = V_1(s) + \Gamma_1$.

If at $t = 0$ the state is locked down and $s(0) \leq s_0^*$, reopening immediately occurs. After that, either $s(t)$ approaches to a steady state and the state remains un-locked down, or $s(t)$ approaches to s_1^* triggering lockdown. In the latter case, either $s(t)$ approached a steady state while the state remains locked down, or it approaches s_0^* triggering reopening. In the latter case $s(t)$ will approach to s_1^* and trigger lockdown again. The pattern repeats until the virus mutate if one considers the situation before a mutation. So in the case, we obtain a quite sophisticated whereby the DM adapts to the evolution of the pandemic by switching-on and off the lockdown button. As expected, beside the vaccination policy, the DM uses the impulse control to manage the spread of the virus across the population. When the situation gets worse, she take the lockdown decision that is later removed when it improves.

Before we move to a numerical example, let us conclude this discussion with an additional result regarding the asymptotic behavior of the system (Appendix H).

Proposition 4 *Suppose (28) holds. If $\theta_0 > K$, then $s(t)$ does not converge to zero. If $\theta_0, \theta_1 \leq K$, then the state is un-locked down for large t and $s(t)$ converges to zero.*

Going back to a previous remark, the ranking between the θ_ks, and especially θ_0 the infection rate in the absence of lockdown, and K is crucial to characterize the asymptotic behavior of the optimal solution. When $\theta_0 > K$, the share of infected people will converge to a positive value whatever the case. The public policy proves itself worth for controlling the pandemic, but it never allows the system to erase it. By contrast, in the best case scenario where the health system is very efficient, $\theta_0, \theta_1 \leq K$, the pandemic will necessary vanish eventually, making the vaccination and lockdown policy less essential.

5.3 Numerical example

For the illustration, we consider two examples. In line with the discussion above, we compare the situation where imposing a lockdown allows the economy to bring back the infection rate below K whatever the pandemic regime with the situation where it does not.

5.3.1 Example 1

Here we use the following set of parameter values:

$$\begin{aligned} K &= 0.4, \quad \rho = 0.2, \quad \beta = 0.5, \\ \lambda &= 0.3, \quad \Gamma_1 = 0.2, \quad h_0 = 0, \quad h_1 = 0.04. \end{aligned}$$

We first consider the case in which the disease transmission rates are

$$\bar{\theta}_0 = 2, \quad \bar{\theta}_1 = 0.2, \quad \underline{\theta}_0 = 1.75, \quad \underline{\theta}_1 = 0.1.$$

Thus, after mutation, the transmission rates with and without lockdown are both increased.

The graphs of the value functions $\overline{V}_0$ and $\overline{V}_1$ after mutation are shown below (see Figure 4). The value functions before mutation have similar graphs.

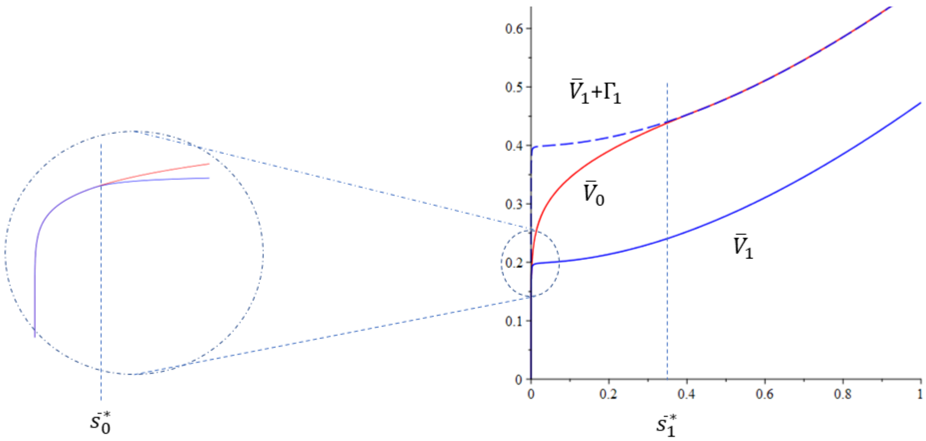

Figure 4: Value functions in example 1

Dynamics: The point of intersection between $\bar{V}_0$ and $\bar{V}_1 + \Gamma_1$, at which the lockdown comes into force, is $\bar{s}_1^* \approx 0.345$, whereas the point for reopening is $\bar{s}_0^* \approx 0.00048$. Computation shows that

$$\frac{ds}{dt} = \begin{cases} \bar{\theta}_0 (1-s) - K - \bar{\theta}_0 s \bar{V}_0' (s) > 0 & \text{for } s < \bar{s}_1^*, \text{ and} \\ \bar{\theta}_1 (1-s) - K - \bar{\theta}_1 s \bar{V}_1' (s) < 0 & \text{for } s \in [0,1]. \end{cases} \tag{36}$$

Therefore, before lockdown, $s(t)$ increases to $\bar{s}_1^*$, triggering lockdown, and after lockdown, $s(t)$ decreases to $\bar{s}_0^*$, triggering reopening.

A similar pattern is observed before a mutation, however with different switching points. The lockdown point is $\underline{s}_1^* \approx 0.4$ and the reopening point is $\underline{s}_0^* \approx 0.0095$. The dynamics are still given by (36), when replacing upper bars by lower bars. So $s(t)$ increases to $\underline{s}_1^*$ triggering lockdown, and after lockdown, $s(t)$ decreases to $\underline{s}_0^*$ triggering reopening.

Vaccination rates: The numerical illustration is interesting because it allows us to investigate further the interplay between lockdown and vaccination policies. For both before and after mutation, computations show that the vaccination rates, $r_k(s) \equiv \theta_k s V_k'(s)$ are increasing in s, and $r_0(s)$ is much greater than $r_1(s)$. This perfectly illustrate what we already noticed: lockdown and vaccination policies are substitutes in the eyes of the DM. One indirect benefit of imposing a lockdown is that the need for vaccination becomes urgent. See the illustration below, for the regime after a mutation.

Effect of mutation on impulse controls before mutation: Finally, we highlight the impact of the risk of mutation on the impulse control by comparing lockdown and reopening points with and without the

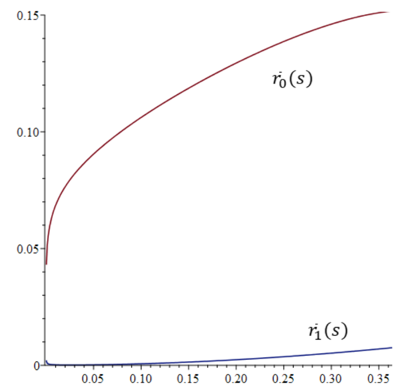

Figure 5: Comparison between vaccination rates with and without a lockdown

risky mutation. If there is no possibility of mutation, the point of lockdown changes to $\underline{s}_1^*(0) \approx 0.425$, and the point of reopening changes to $\underline{s}_0^*(0) \approx 0.001$. So both thresholds are smaller with $\lambda = 0.3$ than with $\lambda = 0$ (no mutation). This means the prospect of mutation expedites lockdown and delays reopening before mutation. These results clearly confirms Part 1 of Proposition 3 regarding the case where $\bar{s}_1^* < \underline{s}_1^*(0)$ (note that $\bar{s}_1^* \approx 0.345 < 0.425 \approx \underline{s}_1^*(0)$) and Part 2 of Proposition 3 (note that $\bar{s}_0^* \approx 0.00048 < 0.001 \approx \underline{s}_0^*(0)$).

For comparison, let us consider a case of "good" mutation with the disease transmission rates

$$\bar{\theta}_0 = 1.8, \quad \bar{\theta}_1 = 0.2, \quad \underline{\theta}_0 = 1.75, \quad \underline{\theta}_1 = 0.3.$$

So, after mutation the transmission rate without lockdown is somewhat increased, but that with lockdown is decreased. Computations show that in this case the lockdown point after mutation is $\bar{s}_1^* \approx 0.39$. On the other hand, the lockdown point before mutation when there is no risk of mutation is $\underline{s}_1^*(0) \approx 0.385$, and with risk of mutation when $\lambda = 0.3$ is $\underline{s}_1^*(0.3) \approx 0.43$. Hence, the prospect of mutation delays the lockdown before mutation. This confirms Part 1 of Proposition 3 regarding the case where $\bar{s}_1^* > \underline{s}_1^*(0)$. It is interesting to note that the steady state before mutation is approximately 0.408, which is between $\underline{s}_1^*(0)$ and $\underline{s}_1^*(0.3)$. Hence, without prospect of mutation the DM will lockdown before the number of infected people reaches the steady state, but with prospect of mutation the DM would delay lockdown so much as to letting the number of infected people reaches the steady state. After mutation, the steady state is approximately 0.406, which exceeds the lockdown point (0.39). Hence, the steady state is never reached.

The reopening points also confirms the result in Proposition 3, with

$$\bar{s}_0^* \approx 0.0011 < 0.0018 \approx \underline{s}_0^*(0) \approx \underline{s}_0^*(0.3).$$

5.3.2 Example 2

We now present an example such that after mutation, $\overline{\theta}_0 > \overline{\theta}_1 > K$. Using the same parameter values for K, ρ, and β, and further supposing

$$\overline{\theta}_0 = 0.8, \quad \overline{\theta}_1 = 0.6, \quad \Gamma_1 = 0.03, \quad h_0 = 0, \quad h_1 = 0.02.$$

By computation, lockdown occurs at an interior point $\bar{s}_1^* \approx 0.13$ and the graph of the value functions $\overline{V}_0$ and $\bar{V}_1$ are depicted in Figure 6. Without possibility of lockdown, there exists a unique steady state is

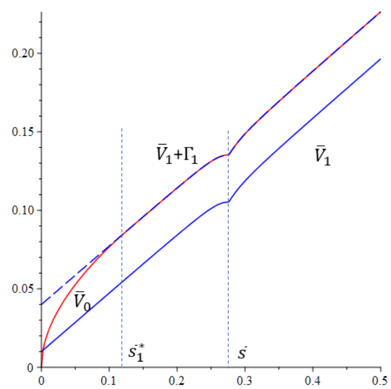

Figure 6: Value functions, example 2

$\bar{s} \approx 0.37$. After lockdown, the unique steady state is changed to $\bar{s} \approx 0.28$. Then, there are three possible outcomes:

Case 1. Mutation occurs when the state is not locked down and with $s_0(t_\omega) < \bar{s}_1^*$, where t_ω is the time when mutation occurs. In this case $s(t)$ first increases to $\bar{s}_1^*$, which triggers lockdown. Then $s(t)$ continue to increase to the steady state $\bar{s}$.

Case 2. Mutation occurs when the state is not locked down and at $s_0(t_\omega) \geq \bar{s}_1^*$. In this case lockdown immediately happens, and $s(t)$ approaches to the steady state $\bar{s}$.

Case 3. Mutation occurs when the state is locked down. Then the state remains locked down while $s(t)$ converges to $\bar{s}$.

In this second example, we take the worst case scenario (infection rate always larger than K, inducing a worsening of the pandemic), and observe that the system is always locked down eventually.

6 Conclusion

This is the first paper that combines lockdown as an impulse control, vaccination and uncertainty surrounding the evolution of the transmission rate to analyze the optimal control of a pandemic. The aim of the paper is to analyze the interplay between lockdown decisions and random mutations. Lockdown is modeled as an impulse control that allows the system to switch from one regime of restrictions to another (stricter or softer). This can be a valuable option, together with vaccination, to control the spread of the disease. More fundamentally, in our setting, lockdown can serve as a way to anticipate a mutation or to respond to it. Indeed, decisions are taken under the risk of mutations of the disease, with repercussions on the transmission rate. The decision maker follows a cost minimization objective. In a simplified model where the virus can mutate only once and there exist only two lockdown regimes, we first characterize the optimality conditions for impulse control and show how the prospect of a mutation affects the decision maker's choice. In fact, it induces her to anticipate the relative benefit of a regime change after a mutation has occurred, which may or may not increase the incentive to set a lockdown. Under some parametric conditions, our problem admits infinitely many value functions. We then show the existence of a minimum value function that is a natural candidate to the solution given the nature of the optimization program. We finally study the features of the optimal policy and notably prove that uncertainty surrounding future mutation of the disease expedites lockdown intervention whenever mutation increases contagiousness. This conclusion strikingly echoes those of the literature dealing with the impact of the occurrence of (random) costly events on decision making, and extends them to the classe of impulse control.

A Proof of Theorem 1

Suppose DM takes the impulse control $L_m \to L_n$ after mutation at $s^* \in (0,1)$. Then the value function $\overline{V}_m(s)$ satisfies the HJB equation

$$\rho \overline{V}_m(s) = \overline{H}_m^*\left(s, \overline{V}_m'(s)\right) \tag{37}$$

where $H_m^*(s,\mu)$ is given by (3). The differential equation is supplemented by the boundary condition (9). The HJB equation is quadratic in $\overline{V}_m'(s)$. Let $\overline{Q}_m(s,\nu)$ be the minimum positive root, μ, of the equation

$$\rho\nu = H_m^*(s,\mu)$$

if solution exists, we can write the equation in the form

$$\overline{V}_m'(s) = \overline{Q}_m\left(s, \overline{V}_m(s)\right)$$

Thus $\overline{V}_m(s)$ satisfies the integral equation

$$\overline{V}_m(s) = \overline{V}_n(s^*) + \Gamma_n + \int_{s^*}^{s} \overline{Q}_m\left(u, \overline{V}_m(u)\right) du$$

for s on the side of s^* on which the HJB equation holds equal. Since the DM chooses s^* to maximize $\overline{V}_m(s)$, the derivative of the right-hand side with respect to s^* vanishes. Hence

$$\overline{V}_n'(s^*) - \overline{Q}_m\left(s^*, \overline{V}_m(s^*)\right) = 0.$$

By the terminal condition (9), we also have

$$\overline{V}_n'(s^*) - \overline{Q}_m\left(s^*, \overline{V}_n(s^*) + \Gamma_n\right) = 0$$

which is equivalent to (11). This proves the first part of the proposition.

Suppose the impulse control $L_m \to L_n$ is taken before mutation at $s^* \in (0,1)$. Then for s on the side of s^* the value function $\underline{V}_m(s)$ satisfies the HJB equation

$$(\rho + \lambda)\underline{V}_m(s) = \underline{H}_m^*\left(s, \underline{V}_m'(s), \overline{V}_m(s)\right) \tag{38}$$

for s on the side of s^* before the impulse control is taken, where $\underline{H}_m^*(s,\mu,W)$ is given by (6). In addition, $\underline{V}_m$ also satisfies the boundary condition (10). Let $\underline{Q}_m(s,\nu,W)$ be the minimum positive root, μ, of the equation

$$\left(\rho+\lambda\right)\nu=\underline{H}_m^*\left(s,\mu,W\right),$$

if the root exists. Then

$$\underline{V}_m\left(s\right)=\underline{V}_n\left(s^*\right)+\Gamma_n+\int_{s^*}^{s}\underline{Q}_m\left(u,\underline{V}_m\left(u\right),\overline{V}_m\left(u\right)\right)du.$$

Differentiate the right-hand side with respect to s^*. It follows from the terminal condition that

$$\underline{V}_n'\left(s^*\right)-\underline{Q}_m\left(s^*,\underline{V}_n\left(s^*\right)+\Gamma_n,\overline{V}_m\left(s^*\right)\right)=0.$$

This leads to (12). The proof is complete.

B Proof of Theorem 2

To prove (15), we note that equation (13) is quadratic in $\overline{s}_n^*\overline{V}_n'\left(\overline{s}_n^*\right)$. For this equation to be solvable, it is necessary that

$$\left(1-\overline{s}_n^*\right)^2+2\frac{\overline{\theta}_m+\overline{\theta}_n}{\overline{\theta}_n-\overline{\theta}_m}\left[h_n-h_m+\rho\Gamma_n\right]\geq 0.$$

As $\overline{s}_n^*\in(0,1)$, we further need to impose that the RHS is not greater than 1, which leads to (15).

Similarly, equation (14) is quadratic in $\underline{s}^*\underline{V}_n'\left(\underline{s}^*\right)$. To have a real root, it is necessary that

$$\left(1-\underline{s}_n^*\right)^2+2\frac{\underline{\theta}_m+\underline{\theta}_n}{\underline{\theta}_n-\underline{\theta}_m}\left[h_n-h_m+\left(\rho+\lambda\right)\Gamma_n+\lambda\left(\overline{V}_n\left(\underline{s}_n^*\right)-\overline{V}_m\left(\underline{s}_n^*\right)\right)\right]\geq 0.$$

Again, because $\underline{s}_n^*\in(0,1)$, the RHS must be no greater than 1, so we get (16).

C Proof of Theorem 3

Part 1. Suppose (19) and (20) both hold but $L_m\to L_n$ does not occur at an interior point $s^*\in(0,1)$. We first show that $V_m\left(0\right)\neq U_m\left(0\right)$. From HJB equations (17) and (18) we find

$$\overline{U}_m\left(0\right)=\frac{h_m}{\rho},\quad \underline{U}_m\left(0\right)=\frac{h_m+\lambda\overline{V}_m\left(0\right)}{\rho+\lambda}\quad\text{for } m=0,1. \tag{39}$$

In the case after mutation, by (19) $\overline{V}_m(0) \neq \overline{U}_m(0)$ since otherwise

$$\overline{V}_m(0) = \frac{h_m}{\rho} > \frac{h_n}{\rho} + \Gamma_n \geq \overline{V}_n(0) + \Gamma_n,$$

violating (5) at $s = 0$. In the case before mutation, since

$$\overline{V}_m(0) = \frac{h_n}{\rho} + \Gamma_n, \quad \overline{V}_n(0) = \frac{h_n}{\rho},$$

by (39)

$$\underline{U}_m(0) = \frac{h_m + \overline{V}_m(0)}{\rho + \lambda} > \frac{h_n}{\rho} + \Gamma_n, \quad \underline{U}_n(0) = \frac{h_n + \lambda h_n/\rho}{\rho + \lambda} = \frac{h_n}{\rho}.$$

Hence, we again have $\underline{V}_m(0) \neq \underline{U}_m(0)$.

Since the regime change does not occur, we must have $V_m(s) = V_n(s) + \Gamma_n$ for $s \in [0,1]$. This equivalent to

$$V_n(s) = V_m(s) - \Gamma_n < V_m(s) + \Gamma_m.$$

By (5) or (8) with n and m interchanged, we have satisfies

$$\rho\overline{V}_n(s) = \overline{H}_n^*(s, V_n'(s)) \quad \text{or } \rho\underline{V}_n(s) = \underline{H}_n\left(s, \underline{V}_n'(s), \lambda\overline{V}_n(s)\right) \tag{40}$$

for $s \in (0,1)$. Therefore, $V_n(s) = U_n(s)$ for $s \in (0,1)$. However, by (20),

$$V_m(1) \leq U_m(1) < U_n(1) + \Gamma_n = V_n(1) + \Gamma_n.$$

This is a contradiction.

Part 2. Suppose (21) and (22) both hold but $L_m \to L_n$ does not occur at an interior point. We show that $V_m(0) \neq V_n(0) + \Gamma_n$. If $V_m(0) = V_n(0) + \Gamma_n$ holds, then

$$V_n(0) = V_m(0) - \Gamma_n < V_m(0) + \Gamma_m.$$

In the case after mutation, $\overline{V}_n(s)$ satisfies the first equation in (40). Therefore $\overline{V}_n(0) = h_n/\rho$. However, by (21)

$$\overline{V}_m(0) \leq \overline{U}_m(0) = \frac{h_m}{\rho} < \frac{h_n}{\rho} + \Gamma_n = \overline{V}_n(0) + \Gamma_n.$$

This is a contradiction. Hence $\overline{V}_m(0) < \overline{V}_n(0) + \Gamma_n$. In the case before mutation, Since $\overline{V}_m(0) = h_m/\rho$

and $\overline{V}_n(0) = h_n/\rho$, by (39)

$$\underline{U}_m(0) = \frac{h_m + \lambda\overline{V}_m(0)}{\rho+\lambda} = \frac{h_m}{\rho}, \quad \underline{U}_n(0) = \frac{h_n + \lambda\overline{V}_n(0)}{\rho+\lambda} = \frac{h_n}{\rho}.$$

Since $\underline{V}_n(s)$ satisfies the second equation in (40), it follows that $\underline{V}_n(0) = \underline{U}_n(0) = h_n/\rho$. This again leads to

$$\underline{V}_m(0) \leq \underline{U}_m(0) = \frac{h_m}{\rho} < \frac{h_n}{\rho} + \Gamma_n = \underline{V}_n(0) + \Gamma_n,$$

contradicting the assumption. Hence, in any case $V_m(0) \neq V_n(0) + \Gamma_n$.

By (5) and (8), $V_m(s)$ satisfies either

$$\rho\overline{V}_m(s) = \overline{H}^*_m\left(s, \overline{V}'_m(s)\right) \text{ or } (\rho+\lambda)\underline{V}_m(s) = \underline{H}^*_m\left(s, \underline{V}'_m(s), \lambda\overline{V}_m(s)\right) \tag{41}$$

for s near 0. Since $L_m \to L_n$ does not occur, it follows that $V_m(s) \leq V_n(s) + \Gamma_n$. In particular, $V_m(1) \leq V_n(1) + \Gamma_n$. Furthermore, $V_m(s)$ satisfies (41) for all $s \in (0,1)$. Thus $V_m(s) = U_m(s)$ for all $s \in (0,1)$. Therefore, by (22),

$$V_m(1) = U_m(1) > U_n(1) + \Gamma_n \geq V_n(1) + \Gamma_n.$$

This is a contradiction. Thus $L_m \to L_n$ must occur at an interior point.

The proof is complete.

D Minimum value function

Lemma 1 *Suppose $\theta_m > K$ and either the regime is after mutation or before mutation with $\lambda = 0$. Let*

$$s'_m = \frac{2(\theta_m - K)}{\theta_m\left[1 + \sqrt{1 + \frac{2\beta}{\rho}(\theta_m - K)}\right]}.$$

Then, there is $s''_m > s'_m$ such that for any $s_0 \in (s'_m, s''_m)$ the HJB equation

$$\rho U_m(s) = sU'_m(s)\left[\theta_m(1-s) - K\right] - \frac{\left[\theta_m sU'_m(s)\right]^2}{2} + \rho q_m(s) \tag{42}$$

has a solution $U_m(s; s_0)$ that satisfies

$$U_m(s_0; s_0) = q_m(s_0) \tag{43}$$

and

$$U_m'(s;s_0) = \frac{1}{\theta_m s}\left[s_m - s - \sqrt{(s_m - s)^2 + 2\rho\left[q_m(s) - U_m(s;s_0)\right]}\right] \qquad \textit{for } 0 < s < s_0. \tag{44}$$

Furthermore, $U_m(s;s_0)$ *is increasing in* s_0.

To prove this, let

$$M = \min_{0\le s\le 1}\left\{\frac{(s_m - s)^2}{2\rho} + q_m(s)\right\}$$

with $s_m = 1 - K/\theta_m$, and let s_m'' satisfies

$$q_m(s_m'') = M.$$

It can be shown that $s_m' < s_m'' < s_m$. Let $U_m(s;s_0)$ be the solution to the HJB equation (42) with the initial condition (43), where s_0 satisfies $s_m' \le s_0 \le s_m''$.

We first show that $U_m(s;s_0)$ exists and is positive for all $s \in (0, s_0)$. To see that $U_m(s;s_0)$ exists for all $s \in (0, s_0)$, it suffices to show that the right-hand side of (44) is real for such s. If not, then there is some $s_1 \in (0, s_0)$ such that the quantity in the square root is positive for $s_1 < s < s_0$ and it becomes zero at $s = s_1$. This implies that

$$U_m(s_1;s_0) = \frac{(s_m - s_1)^2}{2\rho} + q_m(s_1) < U_m(s_0;s_0) = q_m(s_0).$$

However, since $q_m(s)$ is increasing and $s_0 \le s_m''$, it follows that

$$q_m(s_0) \le q_m(s_m'') = \min_{0\le s\le 1}\left\{\frac{(s_m - s)^2}{2\rho} + q_m(s)\right\} \le \frac{(s_m - s_1)^2}{2\rho} + q_m(s_1).$$

This is impossible. Therefore, the right-hand side of (44) is real for all $0 < s \le s_0$.

We next show that $U_m(s;s_0) > q_m(s)$ for $0 < s < s_0$. If this is not true, then there is $s_2 \in (0, s_0)$ such that $U_m(s;s_0) > q_m(s)$ for $s_2 < s < s_0$ and $U_m(s_2;s_0) = q_m(s_2)$. Therefore,

$$\begin{aligned} U_m'(s_2;s_0) &= \lim_{h\to 0}\frac{U_m(s_2+h;s_0) - U_m(s_2;s_0)}{h} \\ &\ge \lim_{h\to 0}\frac{q_m(s_2+h) - q_m(s_0)}{h} = q_m'(s_2) = \frac{\beta}{\rho}s_2 > 0. \end{aligned}$$

However, by (44), $U_m'(s_2;s_0) = 0$, contradicting the above inequalities.

This proves the existence and positivity of $U_m(s;s_0)$ for $0 < s < s_0$.

At $s = s_0$, the right-hand side of (44) vanishes, while $q_m'(s_0) > 0$. Hence, if U_m continue to satisfy (44), one would have $U_m(s;s_0) < q_m(s)$ for $s > s_0$ and is near s_0. However, this would lead to $U_m'(s;s_0) < 0$.

Hence it is necessary that

$$U_m'(s;s_0) = \frac{1}{\theta_m s}\left[s_m - s + \sqrt{(s_m - s)^2 + 2\rho\,[q_m(s) - U_m(s;s_0)]}\right] \qquad \text{for } s > s_0. \tag{45}$$

It follows that

$$\lim_{s\to s_0^+} U_m'(s;s_0) = \frac{2(s_m - s_0)}{\theta_m s_0}.$$

To ensure $U_m(s;s_0) \le q_m(s)$ for $s > s_0$, it is necessary that the above slope is less than that of q_m at s_0. I.e.,

$$\frac{2(s_m - s_0)}{\theta_m s_0} \le q_m'(s_0) = \frac{\beta}{\rho}s_0.$$

Hence

$$\beta\theta_m s_0^2 \ge 2\rho(s_m - s_0).$$

This inequality leads to

$$s_0 \ge \frac{2(\theta_m - K)}{\theta_m\left[1 + \sqrt{1 + \frac{2\beta}{\rho}(\theta_m - K)}\right]} = s_m'.$$

We next show the solution of (45) with initial condition (43) exists for all $s_0 \le s \le 1$. We first show that $U_m(s;s_0) < q_m(s)$ for $s > s_0$. If it is not true, then there is $s_3 > s_0$ such that $U_m(s;s_0) < q_m(s)$ for $s_0 < s < s_3$ and $U_m(s_3;s_0) = q_m(s_3)$. Hence

$$\begin{aligned} U_m'(s_3;s_0) &= \lim_{h\to 0}\frac{q_m(s_3) - U_m(s_3 - h;s_0)}{h} \\ &\ge \lim_{h\to 0}\frac{q_m(s_3) - q_m(s_3 - h)}{h} = q_m'(s_3) = \frac{\beta}{\rho}s_3. \end{aligned}$$

On the other hand, by (45)

$$U_m'(s_3;s_0) = \frac{2(s_m - s_3)}{\theta_m s_3} \text{ if } s_3 < s_m \text{ or } U_m'(s_3;s_0) = 0 \text{ if } s_3 \ge s_m.$$

The latter case is obviously impossible. The former case leads to

$$\theta_m\beta s_3^2 \le 2\rho(s_m - s_3)$$

and so $s_3 \le s_m' \le s_0$. It is also impossible. So, no such s_3 exists.

Since $U_m(s;s_0) < q_m(s)$ for all $s > s_0$, the right-hand side of (45) exists and is positive for such s. This proves that $U_m(s;s_0)$ exists and is increasing for $s > s_0$.

It remains to show that $U_m(s;s_0)$ is increasing in s_0. Suppose $s'_m \le s'_0 < s''_0 \le s''_m$. By definition,

$$U_m(s'_0;s'_0) = q_m(s'_0) < U_m(s'_0;s''_0).$$

Suppose there is an $s_4 \in (0,1)$ such that $U_m(s_4;s'_0) = U_m(s_4;s''_0)$. If $s_4 < s'_0$, then both $U_m(s;s'_0)$ and $U_m(s_4;s''_0)$ are solutions to the initial value problem

$$\begin{aligned} Y'(s) &= \frac{1}{\theta_m s}\left[s_m - s - \sqrt{(s_m - s)^2 + 2\rho\left[q_m(s) - Y(s)\right]}\right] \qquad \text{for } s < s_4, \\ Y(s_4) &= U_m(s_4;s'_0) = U_m(s_4;s''_0). \end{aligned}$$

This contradicts the uniqueness of solution. (Note that the right-hand side of the differential equation satisfies the Lipschitz condition.) If $s > s''_0$, then both $U_m(s;s'_0)$ and $U_m(s_4;s''_0)$ are solutions to the initial value problem

$$\begin{aligned} Z'(s) &= \frac{1}{\theta_m s}\left[s_m - s + \sqrt{(s_m - s)^2 + 2\rho\left[q_m(s) - Z(s)\right]}\right] \qquad \text{for } s < s_4, \\ Z(s_4) &= U_m(s_4;s'_0) = U_m(s_4;s''_0), \end{aligned}$$

again violating the uniqueness of solution. Finally, for any $s'_0 < s < s''_0$ we have

$$U_m(s;s'_0) < q_m(s) < U_m(s;s''_0).$$

So no such s_4 exists. This proves the monotonicity of $U_m(s;s_0)$ with respect to s_0.

The proof of the lemma is complete.

Based on the above lemma, $U_m(s;s'_m)$ is the minimum value function among all solutions to the HJB equation with least nonnegative vaccination rate $r_m(s)$ at all s.

E Proof of Proposition 1

We first show that $ds/dt < 0$ for $s > s'_m$. By (45) with $s_0 = s'_m$,

$$\theta_m s U'_m(s;s'_m) = s_m - s + \sqrt{(s_m - s)^2 + 2\rho\left[q_m(s) - U_m(s;s'_m)\right]}.$$

Hence,

$$\theta_m s U_m'(s; s_m') \begin{cases} > 2(s_m - s) & \text{if } s_m' < s \le s_m \\ > 0 & \text{if } s > s_m. \end{cases}$$

In view of (23),

$$\frac{ds}{dt} = \theta_m(1-s) - K - \theta_m^2 s U_m'(s) < -\theta_m[s_m - s] \le 0$$

if $s_m' < s \le s_m$, and

$$\frac{ds}{dt} \le s_m - s < 0$$

if $s > s_m$. Furthermore,

$$\lim_{s \to s_m'^+} \theta_m s U_m'(s) = 2(s_m - s_m').$$

Hence,

$$\begin{aligned} \lim_{s \to s_m'^+} \frac{ds}{dt} &= \theta_m(1-s_m') - K - 2\theta_m(s_m - s_m') \\ &= -(\theta_m - K) + \frac{2(\theta_m - K)}{1 + \sqrt{1 + \frac{2\beta}{\rho}(\theta_m - K)}} < 0. \end{aligned}$$

Hence, ds/dt has negative upper bound for $s > s_m'$. Consequently, $s(t)$ decreases to s_m' in finite time from any initial value $s(t_0) > s_m'$.

We next show that $ds/dt > 0$ if $s < s_m'$. For such s,

$$\begin{aligned} \theta_m s U_m'(s) &= s_m - s - \sqrt{(s_m - s)^2 + 2\rho[q_m(s) - U_m(s)]} \\ &< s_m - s \end{aligned}$$

if the regime is after mutation or before mutation with $\lambda = 0$. Hence,

$$\frac{ds}{dt} = \theta_m(1-s) - K - \theta_m^2 s U_m'(s) > 0 \qquad \text{for } s < s_m'.$$

In addition, as $s \to s_m'$ from left, $\theta_m s U_m'(s) \to 0$. Hence

$$\frac{ds}{dt} \to \theta_m(s_m - s_m') = \theta_m - K - \frac{2(\theta_m - K)}{1 + \sqrt{1 + \frac{2\beta}{\rho}(\theta_m - K)}} > 0.$$

Hence, ds/dt has a positive lower bound for $s < s_m'$. Therefore, $s(t)$ increases to s_m' in finite time. This completes the proof of the proposition.

F Proof of proposition 2

Relations (29) and (31) directly come from the definitions of theδs and conditions stated in Theorem 2.

Suppose $L_0 \to L_1$ does not occur at a finite time. Then $V_0(s)$ satisfies the HJB equation (37) after mutation or (38) before mutation for all $s \in (0,1)$. Hence, $V_0(s) = U_0(s)$ for $s \in [0,1]$. Furthermore, by (5) or (8)

$$V_1(s) \le V_0(s) \le V_1(s) + \Gamma_1 \qquad \text{for all } s \in [0,1].$$

From HJB equations (17) and (18) we find

$$\overline{U}_m(0) = \frac{h_m}{\rho}, \quad \underline{U}_m(0) = \frac{h_m + \lambda \overline{V}_m(0)}{\rho + \lambda} \quad \text{for } m = 0, 1.$$

Since $h_1 > h_0$, it follows that from $V_1 \le V_0$ that

$$\overline{V}_1(0) = \overline{V}_0(0) = \frac{h_0}{\rho} < \overline{U}_1(0).$$

This leads to

$$\underline{U}_0(0) = \frac{h_0 + \lambda h_0/\rho}{\rho + \lambda} = \frac{h_0}{\rho}, \quad \underline{U}_1(0) = \frac{h_0 + \lambda h_1/\rho}{\rho + \lambda} > \frac{h_0}{\rho}.$$

Hence,

$$\underline{V}_1(0) \le \underline{V}_0(0) = \frac{h_0}{\rho} < \underline{U}_1(0).$$

Therefore, in any case, $V_1(s) \ne U_1(s)$ for small s. Hence $V_1(s) = V_0(s)$ for small s. If there is $s \in [0,1]$ such that $V_1(s) < V_0(s)$, then the infimum of such s is the transition point between locked down and unlockdown. I.e., it is the point of reopening, s_0^*. So, (30) holds. If there is no such point $s \in [0,1]$, then $V_1(s) = V_0(s)$ for all $s \in [0,1]$. This proves Part 1.

Suppose reopening occurs at a finite time. Furthermore, by (15),

$$\delta_0 \equiv 2\frac{\theta_0 + \theta_1}{\theta_0 - \theta_1}[h_1 - h_0] \le (1 - s_0^*)^2 < 1.$$

The above inequality also implies (31).

Suppose $L_1 \to L_0$ does not occur at a finite time. Since $h_1 > h_0$, $V_1(s)$ does not satisfy the HJB equation. Hence $V_1(s) = V_0(s)$ for small s. It is not possible that $V_1(s) < V_0(s)$ for some $s \in (0,1)$, because, otherwise, at the infimum of such s is $L_1 \to L_0$ takes place. Hence, $V_1(s) = V_0(s)$ for all $s \in [0,1]$. This implies that $V_0(s) < V_1(s) + \Gamma_1$ for all $s \in (0,1)$. Hence, lockdown does not happen at an interior point. Hence, by (5), $V_0(s)$ satisfies the HJB equation for all s. Therefore $V_0(s) = U_0(s)$ for $s \in [0,1]$. This proves Part 2.

G Proof of Proposition 3

G.1 Part 1

We define a function F_1 by

$$F_1(s) = s\underline{V}_1'(s;\lambda)(1-s)(\underline{\theta}_1 - \underline{\theta}_0) - \frac{1}{2}\left[s\underline{V}_1'(s;\lambda)\right]^2\left(\underline{\theta}_1^2 - \underline{\theta}_0^2\right) + \alpha(h_1 - h_0) + \rho\Gamma_1$$

where $\underline{V}_1(s;\lambda)$ satisfies

$$(\rho+\lambda)\underline{V}_1(s;\lambda) = \underline{H}_1^*\left(s, \underline{V}_1'(s;\lambda), \lambda\bar{V}(s)\right). \tag{46}$$

Then (14) with $m=0$ and $n=1$ can be written as

$$F_1(\underline{s}_1^*(\lambda)) + \lambda\left[\bar{V}_1(\underline{s}_1^*(\lambda);\lambda) - \bar{V}_0(\underline{s}_1^*(\lambda);\lambda) + \Gamma_1\right] = 0. \tag{47}$$

Suppose after mutation lockdown does not occur at $\underline{s}_1^*(0)$, by (5),

$$\bar{V}_1(s) - \bar{V}_0(s) + \Gamma_1 > 0 \qquad \text{for } s \text{ greater than and is near } \underline{s}_1^*(0).$$

It follows that the second term on the left-hand side of (47) is positive for λ positive and small. Hence

$$F_1(\underline{s}_1^*(\lambda)) < 0 \tag{48}$$

for such λ. We show that $F_1(s) > 0$ for $s < \underline{s}_1^*(0)$ and is near $\underline{s}_1^*(0)$. Once proven, it would imply $\underline{s}_1^*(\lambda) > \underline{s}_1^*(0)$ for λ positive and small.

In terms of $\underline{H}_1^*$ and $\underline{H}_0^*$ defined in (6), (47) with $\lambda = 0$ is equivalent to

$$\begin{aligned}
\underline{H}_1^*\left(\underline{s}_1^*(0), \underline{V}_1'(\underline{s}_1^*(0);0), 0\right) &= \rho\underline{V}_1(\underline{s}_1^*(0);0) \\
\underline{H}_0^*\left(\underline{s}_1^*(0), \underline{V}_1'(\underline{s}_1^*(0);0), 0\right) &= \rho\left[\underline{V}_1(\underline{s}_1^*(0);0) + \Gamma_1\right].
\end{aligned}$$

The equations are quadratic in $\underline{s}_1^*(0)\underline{V}_1'(\underline{s}_1^*(0);0)$. We write the second equation as

$$\underline{V}_1'(\underline{s}_1^*(0);0) = \underline{Q}_0(\underline{s}_1^*(0), \underline{V}_1(\underline{s}_1^*(0);0) + \Gamma_1, 0).$$

Since $\underline{s}_1^*(0)$ is before any steady state, by Proposition 1, $\underline{s}_1^*(0) < \underline{s}_0'$ which is the intersection of $\underline{V}_0(s)$ and

$$\underline{q}_0(s) \equiv \frac{\alpha h_0}{\rho} + \frac{\beta}{2\rho} s^2.$$

Hence, at

$$\begin{aligned} & \underline{Q}_0\left(\underline{s}_1^*(0), \underline{V}_1\left(\underline{s}_1^*(0); 0\right) + \Gamma_1, 0\right) \\ = & \frac{1}{\theta \underline{s}_1^*(0)} \left[\underline{s}_0 - s - \sqrt{\left(\underline{s}_0 - s\right)^2 + 2\alpha h_0 + \beta s^2 - 2\rho\left[\underline{V}_1\left(\underline{s}_1^*(0); 0\right) + \Gamma_1\right]}\right]. \end{aligned}$$

Since lockdown with $\lambda = 0$ occurs at $\underline{s}_1^*(0)$, it follows that

$$\underline{Q}_0\left(s, \underline{V}_1(s; 0) + \Gamma_1, 0\right) > \underline{V}_1'(s; 0) \qquad \text{for } s < \underline{s}_1^*(0).$$

This inequality is equivalent to

$$\underline{H}_0^*\left(s, \underline{V}_1'(s; 0), 0\right) < \rho\left[\underline{V}_1(s; 0) + \Gamma\right] \qquad \text{for } s < \underline{s}_1^*(0).$$

So

$$\underline{H}_1^*\left(s, \underline{V}_1'(s; 0), 0\right) = \rho \underline{V}_1(s; 0) \qquad \text{for all } s \in (0, 1).$$

Hence,

$$\underline{H}_1^*\left(s, \underline{V}_1'(s; 0), 0\right) - \underline{H}_0^*\left(s, \underline{V}_1'(s; 0), 0\right) > -\rho\Gamma_1 \qquad \text{for } s < \underline{s}_1^*(0).$$

By continuity of solutions with respect to parameters, it follows that

$$\underline{H}_1^*\left(s, \underline{V}_1'(s; \lambda), 0\right) - \underline{H}_0^*\left(s, \underline{V}_1'(s; \lambda), 0\right) > -\rho\Gamma_1 \qquad \text{for } s < \underline{s}_1^*(0)$$

if λ is close to 0. This is equivalent to $F_1(s) > 0$ for $s < \underline{s}_1^*(0)$ and is near $\underline{s}_1^*(0)$. Hence, (48) follows.

Suppose after mutation lockdown occurs at some $\bar{s}_1^* < \underline{s}_1^*(0)$. Then, by continuity, $\bar{s}_1^* < \underline{s}_1^*(\lambda)$ for λ near 0. Also by continuity, we have $\underline{s}_1^*(\lambda)$ less than the least steady state before mutation. By (12),

$$(\rho + \lambda)\left[\underline{V}_1\left(\underline{s}_1^*(\lambda); \lambda\right) + \Gamma_1\right] = \underline{H}_0^*\left(\underline{s}_1^*(\lambda), \underline{V}_1'\left(\underline{s}_1^*(\lambda); \lambda\right), \lambda \bar{V}_0\left(\underline{s}_1^*(\lambda)\right)\right) \tag{49}$$

holds for any $\lambda \geq 0$. Since lockdown occurs at $s = \underline{s}_1^*(\lambda)$, it follows that

$$\underline{V}_1(s; \lambda) + \Gamma_1 > \underline{V}_0(s; \lambda), \qquad \underline{V}_0'(s; \lambda) > \underline{V}_1'(s; \lambda) \qquad \text{for } s < \underline{s}_1^*(\lambda).$$

Since $\underline{s}_1^*(\lambda)$ is less than the least steady state before mutation, it follows that

$$\underline{Q}_0\left(s, V, \lambda\bar{V}_0\right) = \frac{1}{\underline{\theta}_0 s}\left[\underline{s}_0 - s - \sqrt{\left(\underline{s}_0 - s\right)^2 + 2\alpha h_0 + \beta s^2 + 2\lambda\bar{V}_0 - 2\left(\rho+\lambda\right)V}\right]$$

which is increasing in V. Hence,

$$\underline{Q}_0\left(s, \underline{V}_1(s;\lambda) + \Gamma_1, \lambda\bar{V}_0(s)\right) > \underline{Q}_0\left(s, \underline{V}_0(s;\lambda), \lambda\bar{V}_0(s)\right) = \underline{V}_0'(s;\lambda) > \underline{V}_1'(s;\lambda)$$

for $s < \underline{s}_1^*(\lambda)$. This inequality is equivalent to

$$\underline{H}_0^*\left(s, \underline{V}_1'(s;\lambda), \lambda\bar{V}_0(s)\right) < (\rho+\lambda)\left[\underline{V}_1(s;\lambda) + \Gamma_1\right].$$

It can be written as

$$\underline{H}_0^*\left(s, \underline{V}_1'(s;\lambda), 0\right) < \rho\left[\underline{V}_1(s;\lambda) + \Gamma_1\right] - \lambda\left[\bar{V}_0(s) - \underline{V}_1(s;\lambda) - \Gamma_1\right] \qquad \text{for } s < \underline{s}_1^*(\lambda). \tag{50}$$

Furthermore, $\underline{V}_1(s;\lambda)$ satisfies the HJB equation

$$(\rho+\lambda)\underline{V}_1(s;\lambda) = \underline{H}_1^*\left(s, \underline{V}_1(s;\lambda), \lambda\bar{V}_1(s)\right) \qquad \text{for any } s \in (0,1)$$

which is equivalent to

$$\underline{H}_1^*\left(s, \underline{V}_1(s;\lambda), 0\right) = \rho\underline{V}_1(s;\lambda) - \lambda\left[\bar{V}_1(s) - \underline{V}_1(s;\lambda)\right] \tag{51}$$

Note that by (5),

$$\bar{V}_0(s) = \bar{V}_1(s) + \Gamma_1 \qquad \text{for } s \geq \bar{s}_1^*.$$

Subtracting the corresponding sides of (50) and (51) yields

$$\underline{H}_1^*\left(s, \underline{V}_1'(s;\lambda), 0\right) - \underline{H}_0^*\left(s, \underline{V}_1'(s;\lambda), 0\right) > -\rho\Gamma_1$$

for any s that satisfies $\bar{s}_1^* < s < \underline{s}_1^*(\lambda)$ and for any λ near 0. However, $\underline{s}_1^*(0)$ satisfies the equations (49) and (51) with $\lambda = 0$ and $s = \underline{s}_1^*(0)$. It follows that

$$\underline{H}_1^*\left(\underline{s}_1^*(0), \underline{V}_1'\left(\underline{s}_1^*(0);0\right)\right) - \underline{H}_0^*\left(\underline{s}_1^*(0), \underline{V}_1'\left(\underline{s}_1^*(0);0\right), 0\right) = -\rho\Gamma_1.$$

Therefore, it is necessary that $\underline{s}_1^*(0) \geq \underline{s}_1^*(\lambda)$.

G.2 Part 2

We define a function F_0 by

$$F_0(s) = s\underline{V}_0'(s;\lambda)(1-s)(\underline{\theta}_0 - \underline{\theta}_1) - \frac{1}{2}\left[s\underline{V}_0'(s;\lambda)\right]^2(\underline{\theta}_0^2 - \underline{\theta}_1^2) + \alpha(h_0 - h_1)$$

where $\underline{V}_0(s;\lambda)$ satisfies

$$(\rho+\lambda)\underline{V}_0(s;\lambda) = \underline{H}_0^*\left(s, \underline{V}_0'(s;\lambda), \lambda\bar{V}_0(s)\right).$$

Then (14) with $m=1$ and $n=0$ can be written as

$$F_0(\underline{s}_0^*(\lambda)) + \lambda\left[\bar{V}_0(\underline{s}_0^*(\lambda);\lambda) - \bar{V}_1(\underline{s}_0^*(\lambda);\lambda)\right] = 0. \tag{52}$$

In the case where after mutation reopening occurs either immediately or at some $\bar{s}_0^* \geq \underline{s}_0^*(0)$, then

$$\bar{V}_1(s) = \bar{V}_0(s) \qquad \text{for } s \leq \underline{s}_0^*(0).$$

In the case where after mutation reopening occurs at $\bar{s}_0^* < \underline{s}_0^*(0)$, by (5),

$$\bar{V}_0(s) - \bar{V}_1(s) > 0 \qquad \text{for } s \text{ near } \underline{s}_0^*(0),$$

it follows that the second term on the left-hand side of (52) is positive. Hence

$$F_0(\underline{s}_0^*(\lambda)) < 0. \tag{53}$$

We show that $F_0(s) > 0$ for $s > \underline{s}_0^*(0)$. Once proven, it would imply $\underline{s}_0^*(\lambda) < \underline{s}_0^*(0)$ for λ positive and small.

In terms of $\underline{H}_1^*$ and $\underline{H}_0^*$ defined in (6), (47) with $\lambda = 0$ is equivalent to

$$\begin{aligned}
\underline{H}_0^*\left(\underline{s}_0^*(0), \underline{V}_0'(\underline{s}_0^*(0)), 0\right) &= \rho\underline{V}_0(\underline{s}_0^*(0);0) \\
\underline{H}_1^*\left(\underline{s}_0^*(0), \underline{V}_0'(\underline{s}_1^*(0)), 0\right) &= \rho\underline{V}_0(\underline{s}_0^*(0);0).
\end{aligned}$$

The equations are quadratic in $\underline{s}_0^*(0)\underline{V}_0'(\underline{s}_0^*(0);0)$. We write the second equation as

$$\begin{aligned}
\underline{V}_0'(\underline{s}_0^*(0);0) &= \underline{Q}_1(\underline{s}_0^*(0), \underline{V}_0(\underline{s}_1^*(0);0), 0) \\
&\equiv \frac{1}{\underline{\theta}_1\underline{s}_0^*(0)}\left[-(s+\underline{\varepsilon}_1) + \sqrt{(s+\underline{\varepsilon}_1) + 2\alpha h_1 + \beta s^2 - 2\rho\underline{V}_0(\underline{s}_0^*(0);0)}\right]
\end{aligned}$$

where $\underline{\varepsilon}_1 = K/\theta_1 - 1$. Since reopening occurs at $\underline{s}_0^*(0)$, it follows that

$$\underline{Q}_1\left(s, \underline{V}_0(s;0), 0\right) < \underline{V}_0'(s;0) \qquad \text{for } s > \underline{s}_0^*(0)$$

This inequality is equivalent to

$$\underline{H}_1^*\left(s, \underline{V}_0'(s;0), 0\right) < \rho \underline{V}_0(s;0) \qquad \text{for } s > \underline{s}_0^*(0).$$

Also,

$$\underline{H}_0^*\left(s, \underline{V}_0'(s;0), 0\right) = \rho \underline{V}_0(s;0) \qquad \text{for all } s.$$

Hence

$$\underline{H}_0^*\left(s, \underline{V}_0'(s;0), 0\right) - \underline{H}_1^*\left(s, \underline{V}_0'(s;0), 0\right) > 0$$

for $s > \underline{s}_0^*(0)$. By the continuity of solutions with respect to λ, we also have

$$\underline{H}_0^*\left(s, \underline{V}_0'(s;\lambda), 0\right) - \underline{H}_1^*\left(s, \underline{V}_0'(s;\lambda), 0\right) > 0$$

This is equivalent to

$$F_0(s) > 0 \qquad \text{for } s > \underline{s}_0^*(0).$$

This proves (53).

Suppose after mutation reopening occurs at some point $\bar{s}_0^*$ such that $\underline{s}_0^*(0) < \bar{s}_0^* \leq 1$. Then

$$\bar{V}_1(s) = \bar{V}_0(s) \qquad \text{for } s \leq \bar{s}_0^*. \tag{54}$$

By continuity, we may suppose that λ is so small such that $\underline{s}_0^*(\lambda) < \bar{s}_0^*$. By (12),

$$(\rho + \lambda)\underline{V}_0\left(\underline{s}_0^*(\lambda);\lambda\right) = \underline{H}_1^*\left(\underline{s}_0^*(\lambda), \underline{V}_0'\left(\underline{s}_0^*(\lambda);\lambda\right), \lambda\bar{V}_1\left(\underline{s}_0^*(\lambda)\right)\right) \tag{55}$$

holds for any $\lambda \geq 0$. Since reopening occurs at $s = \underline{s}_0^*(\lambda)$, it follows that

$$\underline{V}_0(s;\lambda) > \underline{V}_1(s;\lambda), \qquad \underline{V}_0'(s;\lambda) > \underline{V}_1'(s;\lambda) \qquad \text{for } s > \underline{s}_0^*(\lambda).$$

Note that $\underline{\theta}_1 < K$, it follows that

$$\begin{aligned} & \underline{Q}_1(s, V, \lambda W) \\ \equiv \quad & \frac{1}{\underline{\theta}_1 s}\left[-(s+\underline{\varepsilon}_1)+\sqrt{(s+\underline{\varepsilon}_1)+2\alpha h_1+\beta s^2+2\lambda W-2(\rho+\lambda)V}\right] \end{aligned}$$

is decreasing in V. Hence,

$$\underline{Q}_1\left(s, \underline{V}_0(s;\lambda), \lambda\bar{V}_1(s)\right) < \underline{Q}_1\left(s, \underline{V}_1(s;\lambda), \lambda\bar{V}_1(s)\right) < \underline{V}_1'(s;\lambda) < \underline{V}_0'(s;\lambda)$$

for $s > \underline{s}_0^*(\lambda)$. This inequality is equivalent to

$$(\rho+\lambda)\underline{V}_0(s;\lambda) > \underline{H}_1^*\left(s, \underline{V}_0'(s;\lambda), \lambda\bar{V}_1(s)\right) \qquad \text{for } s > \underline{s}_0^*(\lambda),$$

which is the same as

$$\rho\underline{V}_0(s;\lambda) > \underline{H}_1^*\left(s, \underline{V}_0'(s;\lambda), 0\right) + \lambda\left[\bar{V}_1(s) - \underline{V}_0(s;\lambda)\right] \qquad \text{for } s > \underline{s}_0^*(\lambda). \tag{56}$$

In addition, for any $s \in (0,1)$, we also have

$$(\rho+\lambda)\underline{V}_0(s;\lambda) = \underline{H}_0^*\left(s, \underline{V}_0'(s;\lambda), \lambda\bar{V}_0(s)\right)$$

which can be written as

$$\rho\underline{V}_0(s;\lambda) = \underline{H}_0^*\left(s, \underline{V}_0'(s;\lambda), 0\right) + \lambda\left[\bar{V}_0(s) - \underline{V}_0(s;\lambda)\right]. \tag{57}$$

Subtracting the respective sides of (56) and (57) and using (54), we find

$$\underline{H}_0^*\left(s; \underline{V}_0'(s;\lambda), 0\right) - \underline{H}_1^*\left(s, \underline{V}_0'(s;\lambda), 0\right) > 0 \qquad \text{if } \underline{s}_0^*(\lambda) < s < \bar{s}_0^*$$

for λ sufficiently close to 0. However, $\underline{s}_0^*(0)$ satisfies (55) and (57) with $\lambda = 0$. That is,

$$\underline{H}_0^*\left(\underline{s}_0^*(0), \underline{V}_0'(\underline{s}_0^*(0);0), 0\right) - \underline{H}_1^*\left(\underline{s}_0^*(0), \underline{V}_0'(\underline{s}_0^*(0);0), 0\right) = 0.$$

This implies that $\underline{s}_0^*(0) \le \underline{s}_0^*(\lambda)$ for λ sufficiently close to 0.

The proof is complete.

H Proof of Proposition 4

Suppose $\theta_0 > K$. At $t = 0$, either the state is locked down or un-locked down. In the former case, if s_0^* does not exist (Case 1), then reopen occurs immediately. After reopening, the state remains open forever. Since $\theta_0 > K$, as discussed above,

$$\liminf_{t\to\infty} s(t) \geq \hat{s}_0 > 0.$$

If s_0^* exists, then either $s(t)$ converges to a steady state while the state remains locked down, or $s(t)$ approaches to s_0^*, triggering reopening. In the former case,

$$\liminf_{t\to\infty} s(t) \geq s_0^* > 0,$$

In the latter case, as discussed above, $s(t)$ does not converge to zero. Thus, in any case $s(t)$ does not converge to zero.

Suppose $\theta_0, \theta_1 \leq K$. Then either locked down or un-locked down, $s(t)$ is decreasing. So, eventually the state will be un-locked down, and $s(t)$ decreases to zero.

Competing Interests:

The authors have no competing interests to declare that are relevant to the content of this article.

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