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► **To cite this version:**

Emilio Molina, Alain Rapaport, Hector Ramirez Cabrera. Equivalent formulations of the minimum peak problem and applications. French German Portuguese Conference on Optimization - FGP22, May 2022, Porto, Portugal. hal-03694722

**HAL Id: hal-03694722**

**<https://hal.inrae.fr/hal-03694722v1>**

Submitted on 25 Jul 2022

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# Equivalent formulations of the minimum peak problem and applications

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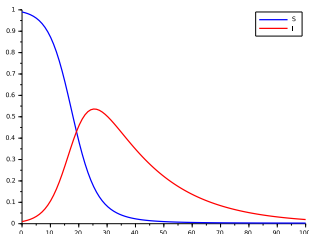
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FGP 2022, Porto

# Motivation of the work

$$\begin{cases} \dot{S} = -\beta SI \\ \dot{I} = \beta SI - \gamma I \\ \dot{R} = \gamma I \end{cases} \quad \blacktriangleright$$



$$\begin{cases} \dot{S} = -\beta(1-u)SI \\ \dot{I} = \beta(1-u)SI - \gamma I \\ \dot{R} = \gamma I \end{cases} \quad \text{with} \quad \int_0^{\infty} u(\tau) d\tau \leq Q$$

How to reduce at best  $\max_i I(t)$  ?

# Optimal control with $L^\infty$ criterion

$$\dot{x} = f(x, u) \quad \rightarrow \quad \inf_{u(\cdot)} \max_{t \in [0, T]} h(x(t))$$

is not in the classical form of a Mayer or Bolza problem...

- ▶ Characterization of the value function (Barron, Ishii, Jensen, Liu, Gonzalez, Aragone...) in terms of a variational inequality :

$$\min \left( \partial_t V + \inf_u \partial_x V \cdot f(x, u) , V - h(x) \right) = 0 .$$

- ▶ Approximation of the  $L^\infty$  norm by the  $L^p$  norm.

## Optimal control with $L^\infty$ criterion

$$(\mathcal{D}) : \quad \begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \end{cases} \quad \text{with} \quad \begin{cases} x \in \mathbb{R}^n \\ y \in \mathbb{R} \\ u \in U \subset \mathbb{R}^p \end{cases}$$

*Assumptions.*  $U$  is compact;  $f$  and  $g$  are  $C^1$  with linear growth.

$$\mathcal{P} : \quad \inf_{u(\cdot)} \max_{t \in [0, T]} y(t)$$

**Question.** Is it possible to write an equivalent problem in Mayer or Bolza form (and use numerical tools such as Bocop)?

$$\text{Remark} : \quad \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x} = f(x, u) \\ \dot{y} = g(x, u) := \nabla h(x)^T \cdot f(x, u) \end{cases}$$

## A first formulation with state constraint

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = 0 \end{cases}$$

The problem

$$\mathcal{P}_0 : \inf_{u(\cdot)} z(T)$$

under the state constraint

$$z(t) \geq y(t), \quad t \in [0, T]$$

and **the two boundaries condition**

$$z(T) = z(0)$$

is equivalent to  $\mathcal{P}$ .

*Remark.* Here  $z(0)$  is free.

## Another formulation as a Cauchy problem

$$(\mathcal{D}_a) : \begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \end{cases} \quad v \in [0, 1]$$

**Proposition.** With  $z_0 = y_0$ , the problem

$$\mathcal{P}_1 : \inf_{u(\cdot), v(\cdot)} z(T)$$

under the state constraint

$$\mathcal{C} : z(t) \geq y(t), \quad t \in [0, T]$$

admits an optimal solution, and is equivalent to problem  $\mathcal{P}$ .

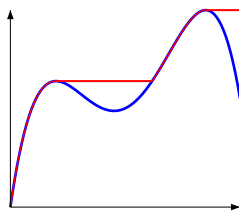
## Sketch of proof

$$I := \{t \in (0, T); y(t') > y(t) \text{ for some } t' < t\}.$$

Sun Rising Lemma :  $I \neq \emptyset \Rightarrow \text{int } I = \bigcup_n (a_n, b_n)$  with

- i)  $y(a_n) = y(b_n)$  if  $b_n \neq T$ ,
- ii) if  $b_n = T$ , then  $y(a_n) \geq y(b_n)$ .

Then, the control  $v(t) = \begin{cases} 1, & t \in \text{int } I \\ 0, & t \notin \text{int } I \end{cases}$  is optimal





## A formulation with mixed constraint

$$(\mathcal{D}_a) : \begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \end{cases} \quad v \in [0, 1]$$

**Proposition.** For  $z_0 = y_0$ , the problem

$$\mathcal{P}_2 : \inf_{u(\cdot), v(\cdot)} z(T)$$

under the mixed constraint

$$\mathcal{C}_m : \max(y(t) - z(t), 0)(1 - v(t)) + z(t) - y(t) \geq 0, \quad t \in [0, T]$$

is equivalent to  $\mathcal{P}$ .

## Formulation without constraint

Consider  $Z = (x, y, z)^\top$ ,  $\dot{Z} \in F(Z) := \bigcup_{(u,v) \in U \times [0,1]} \begin{bmatrix} f(x, y, u) \\ g(x, y, u) \\ h(x, y, z, u, v) \end{bmatrix}$

with  $h(x, y, u, v) = \max(g(x, y, u), 0)(1 - v\mathbb{1}_{\mathbb{R}^+}(z - y))$

Let  $\mathcal{S} := \{Z(\cdot) \text{ a.c. ; } \dot{Z}(t) \in F(Z(t)) \text{ a.e. and } Z(0) = (x_0, y_0, y_0)\}$

*Remark.*  $F$  is only upper semi-continuous.

**Proposition.** Assume  $G(x, y) := \bigcup_{u \in U} \begin{bmatrix} f(x, y, u) \\ g(x, y, u) \end{bmatrix}$  is convex. Then

$$\mathcal{P}_3 : \quad \inf_{Z(\cdot) \in \mathcal{S}} z(T)$$

admits a solution  $Z^*(\cdot)$  and  $(x^*(\cdot), y^*(\cdot))$  is optimal for problem  $\mathcal{P}$ .

# Sketch of proof

1. Consider the augmented dynamics :

$$\dot{Z} \in F^\dagger(Z) := \bigcup_{(u,v,\alpha) \in U \times [0,1]^2} \begin{bmatrix} f(x, y, u) \\ g(x, y, u) \\ h^\dagger(x, y, z, u, v, \alpha) \end{bmatrix}$$

$$h^\dagger(x, y, z, u, v, \alpha) = (1 - \alpha)h(x, y, z, u, v) + \alpha \max_{w \in U} h(x, y, z, w, 0)$$

$\Rightarrow \mathcal{S}^\dagger$  compact  $\Rightarrow \exists (x^*(\cdot), y^*(\cdot), z^*(\cdot))$  optimal.

2. Any  $Z(\cdot)$  sol. of  $\mathcal{D}_a$  with  $\mathcal{C}_m$  belongs to  $\mathcal{S} \subset \mathcal{S}^\dagger$

$\Rightarrow z^*(T) \leq \inf\{z(T); (x(\cdot), y(\cdot), z(\cdot)) \text{ sol. of } \mathcal{D}_a \text{ with } \mathcal{C}_m\}$ .

## Sketch of proof

3. If  $E := \{t \in (0, T); z(t) < y(t)\} \neq \emptyset$ , then one has  $\dot{z}(t) - \dot{y}(t) \geq 0$  for a.e.  $t \in E \Rightarrow z(0) - y(0) < 0$  : contradiction.

$$\Rightarrow z(T) \geq \max_{t \in [0, T]} y(t)$$

4. Filipov's Lemma **applied to  $G$**  :

$$\Rightarrow z^*(T) \geq \max_{t \in [0, T]} y^*(t) \geq \inf_{u \in \mathcal{U}} \left\{ \max_{t \in [0, T]} y(t); (x(\cdot), y(\cdot)) \text{ sol. of } \mathcal{D} \right\}$$

where  $(x^*(\cdot), y^*(\cdot))$  is solution of  $\mathcal{D}$  for a certain  $u^*(\cdot)$ .

## Approximation from below

$$(\mathcal{D}_a^\theta) : \begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = h_\theta(x, y, z, u, v) \end{cases}$$

with  $h_\theta(x, y, z, u, v) = \max(g(x, y, u), 0) (1 - v e^{-\theta \max(y-z, 0)})$

**Proposition.** For any increasing sequence  $\theta_n \rightarrow +\infty$ , the problem

$$\mathcal{P}_3^{\theta_n} : \inf_{Z(\cdot) \in \mathcal{S}^{\theta_n}} z(T)$$

admits an optimal solution  $(x_n(\cdot), y_n(\cdot), z_n(\cdot))$ .

$(x_n(\cdot), y_n(\cdot))$  converges (up to a sub-sequence) uniformly to an optimal solution  $(x^*(\cdot), y^*(\cdot))$  of problem  $\mathcal{P}$ , and  $(\dot{x}^*(\cdot), \dot{y}^*(\cdot))$  converges weakly to  $(\dot{x}^*(\cdot), \dot{y}^*(\cdot))$  in  $L^2$ .

Moreover,  $z_n(T)$  is an increasing sequence that converges to  $\max_{t \in [0, T]} y^*(t)$ .

## A particular class of dynamics

$$\begin{cases} \dot{x} = f(x) \\ \dot{y} = g(x, u) \end{cases} \quad u \in U$$

**Proposition** A feedback control  $x \mapsto \phi^*(x)$  such that

$$g(x, \phi^*(x)) = \min_{u \in U} g(x, u), \quad x \in X$$

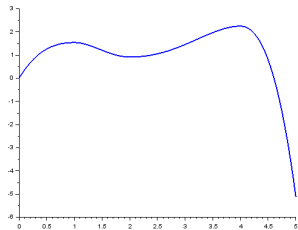
is optimal for problem  $\mathcal{P}$ .

## An example

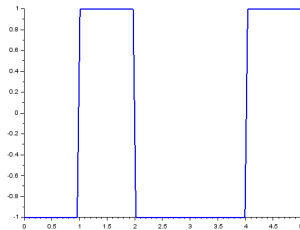
$$\begin{cases} \dot{x} = 1, & x(0) = 0 \\ \dot{y} = (1-x)(2-x)(4-x)(1+u/2), & y(0) = 0 \end{cases} \quad u \in [-1, 1]$$

$\Rightarrow \phi^*(x) = -\text{sign}((1-x)(2-x)(4-x))$  is optimal

*Optimal solution for  $T = 5$  :*

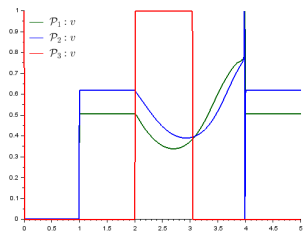
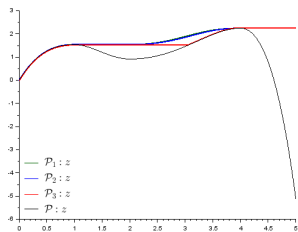
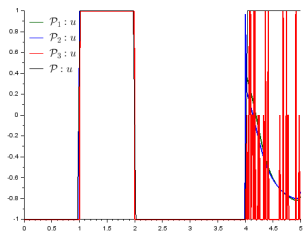
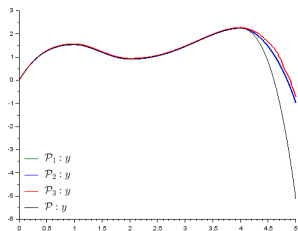


$y(\cdot)$



$u(\cdot)$

# Numerical resolutions with Bocop





# Comparison of the methods

Problems  $\mathcal{P}_1$  and  $\mathcal{P}_2$  : Bocop (direct method) with 500 time steps

Problems  $\mathcal{P}_3$  : BocopHJB with 500 time steps on a  $200 \times 200$  grid

problem	$\max_{t \in [0, T]} y(t)$	error	computation time
$\mathcal{P}$	2.24985	0	—
$\mathcal{P}_1$	2.24989	0.001%	1.5 s
$\mathcal{P}_2$	2.24986	0.0004%	4.8 s
$\mathcal{P}_3$	2.26778	0.8%	248 s

# The SIR problem

$$\begin{cases} \dot{S} = -(1-u)\beta SI \\ \dot{I} = (1-u)\beta SI - \gamma I \end{cases} \quad u \in [0, \bar{u}] \text{ with } \bar{u} \leq 1$$

→  $\inf_{u(\cdot)} \max_{t \in [0, T]} I(t)$  under the budget constraint  $\int_0^T u(t) dt \leq Q$

Consider the augmented dynamics

$$\begin{cases} \dot{S} = -(1-u)\beta SI \\ \dot{I} = (1-u)\beta SI - \gamma I \\ \dot{q} = u, \quad q(0) = 0 \end{cases} \quad \text{with the target } q(T) = Q$$

# The SIR problem

Posit  $\mathcal{R}_0 = \beta/\gamma$  and  $S_h = \mathcal{R}_0^{-1}$ .

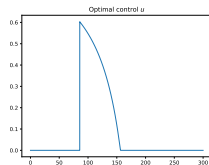
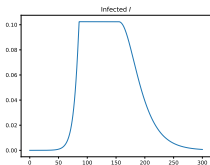
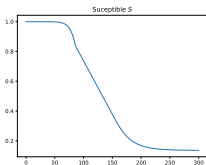
**Proposition.**<sup>1</sup> The "0-singular-0" feedback control

$$\psi(I, S) := \begin{cases} 1 - \frac{S_h}{S} & \text{if } I = \bar{I} \text{ and } S > S_h \\ 0 & \text{otherwise} \end{cases}$$

with

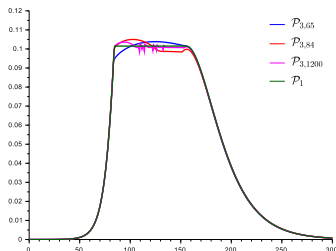
$$\bar{I} := \frac{I_0 + S_0 - S_h - S_h \log\left(\frac{S_0}{S_h}\right)}{Q\beta S_h + 1}$$

is optimal.

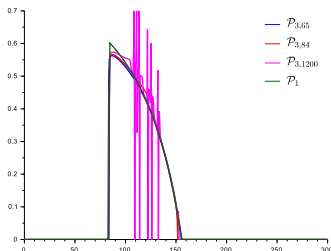


# Numerical illustrations with Bocop

$$\beta = 0.21, \gamma = 0.07, Q = 28, I(0) = 10^6$$



$I(\cdot)$



$u(\cdot)$

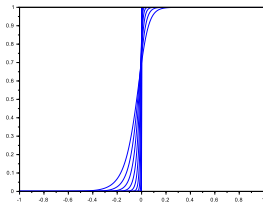
# Numerical resolutions with Bocop

	$\mathcal{P}_1$	$\mathcal{P}_3^{65}$	$\mathcal{P}_3^{84}$	$\mathcal{P}_3^{233}$	$\mathcal{P}_3^{460}$	$\mathcal{P}_3^{1200}$
$I_{max}$	0.101508	0.0684	0.0823	0.09542	0.0993	0.101028

- ▶ direct method with 500 time steps
- ▶  $\max(\dot{y}, 0) \simeq \log(\exp(\lambda_1 \dot{y}) + 1) / \lambda_1$
- ▶  $\exp(-\theta \max(y - z, 0)) \simeq \exp(-\theta \log(\exp(\lambda_2(y - z)) + 1) / \lambda_2)$

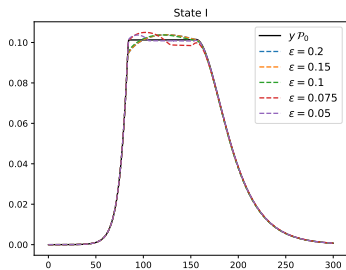
$$\lambda_2 = \frac{\log\left(\varepsilon \frac{\log(2)}{\log(1-\varepsilon)} - 1\right)}{\varepsilon^2}$$

$$\theta = -\frac{\log(1-\varepsilon)}{\log(2)} \lambda_2$$

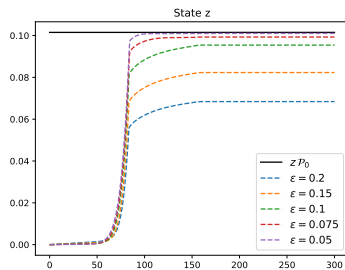


# Numerical illustrations with Bocop

$$\beta = 0.21, \gamma = 0.07, Q = 28, I(0) = 10^6$$

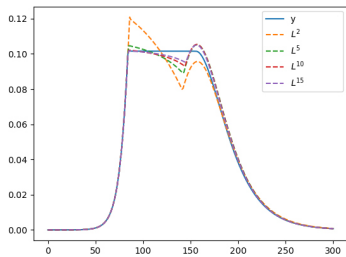


$I(\cdot)$

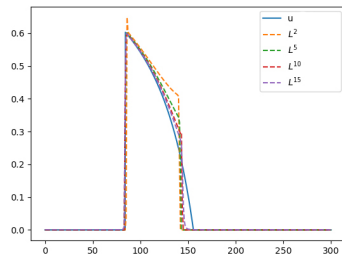


$z(\cdot)$

# Comparison with $L^p$ approximation



$I(\cdot)$



$u(\cdot)$

# Conclusions

We have proposed equivalent formulations that allow to use existing numerical methods, but...

- ▶ with state constraint : not qualified...
  - ▶ without state constraint : discontinuous differential inclusion...
- ▶ one cannot use (available) Maximum Principles...

**Open problem** : How to derive necessary optimality conditions ?