Equivalent formulations of the minimum peak problem and applications

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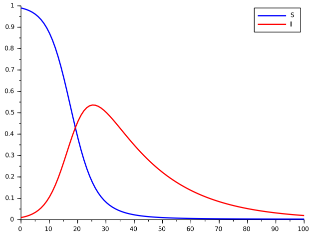
Motivation of the work

$$\begin{cases} \dot S = -\beta S I \\ \dot I = \beta S I - \gamma I \\ \dot R = \gamma I \end{cases}$$



$$\begin{cases} \dot S = -\beta(1-u) S I \\ \dot I = \beta(1-u) S I - \gamma I \\ \dot R = \gamma I \end{cases} \quad \text{with } \int_0^{\infty} u(\tau)\, d\tau \leq Q$$

How to reduce at best $\max_{i} I(t)$?

Optimal control with L^∞ criterion

$$\dot{x} = f(x,u) \quad \rightarrow \quad \inf_{u(\cdot)} \max_{t\in[0,T]} h(x(t))$$

is not in the classical form of a Mayer or Bolza problem...

- Characterization of the value function (Barron, Ishii, Jensen, Liu, Gonzalez, Aragone...) in terms of a variational inequality :

$$\min\left(\partial_t V + \inf_u \partial_x V.f(x,u)\ ,\ V - h(x)\right) = 0\ .$$

- Approximation of the L^∞ norm by the L^p norm.

Optimal control with L^∞ criterion

$$(\mathcal{D}) : \quad \left\{ \begin{array}{l} \dot{x} = f(x,y,u) \\ \dot{y} = g(x,y,u) \end{array} \right. \quad \text{with} \quad \left| \begin{array}{l} x \in \mathbb{R}^n \\ y \in \mathbb{R} \\ u \in U \subset \mathbb{R}^p \end{array} \right.$$

Assumptions. U is compact ; f and g are C^1 with linear growth.

$$\mathcal{P} : \quad \inf_{u(\cdot)} \max_{t \in [0,T]} y(t)$$

Question. Is it possible to write an equivalent problem in Mayer or Bolza form (and use numerical tools such as Bocop) ?

Remark : $\left\{ \begin{array}{l} \dot{x} = f(x,u) \\ y = h(x) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \dot{x} = f(x,u) \\ \dot{y} = g(x,u) := \nabla h(x)^T \cdot f(x,u) \end{array} \right.$

A first formulation with state constraint

$$\begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = 0 \end{cases}$$

The problem

$$\mathcal{P}_0 : \quad \inf_{u(\cdot)} z(T)$$

under the state constraint

$$z(t) \geq y(t), \ t \in [0, T]$$

and the two boundaries condition

$$z(T) = z(0)$$

is equivalent to $\mathcal{P}$.

Remark. Here $z(0)$ is free.

Another formulation as a Cauchy problem

$$(\mathcal{D}_a): \quad \begin{cases} \dot{x} = f(x, y, u) \\ \dot{y} = g(x, y, u) \\ \dot{z} = \max(g(x, y, u), 0)(1 - v) \end{cases} \qquad v \in [0, 1]$$

Proposition. With $z_0 = y_0$, the problem

$$\mathcal{P}_1: \quad \inf_{u(\cdot), v(\cdot)} z(T)$$

under the state constraint

$$\mathcal{C}: \quad z(t) \geq y(t), \; t \in [0, T]$$

admits an optimal solution, and is equivalent to problem $\mathcal{P}$.

Sketch of proof

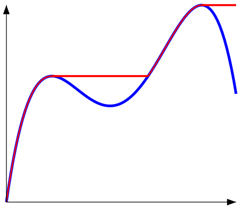
$$I := \left\{t \in (0,T);\ y(t') > y(t) \text{ for some } t' < t\right\}.$$

Sun Rising Lemma : $I \neq \emptyset \Rightarrow \operatorname{int} I = \bigcup_n (a_n, b_n)$ with

i) $y(a_n) = y(b_n)$ if $b_n \neq T$,

ii) if $b_n = T$, then $y(a_n) \geq y(b_n)$.

Then, the control $v(t) = \begin{cases} 1, & t \in \operatorname{int} I \\ 0, & t \notin \operatorname{int} I \end{cases}$ is optimal



A formulation with mixed constraint

$$(\mathcal{D}_a): \quad \begin{cases} \dot{x} = f(x,y,u) \\ \dot{y} = g(x,y,u) \\ \dot{z} = \max(g(x,y,u),0)(1-v) \end{cases} \qquad v \in [0,1]$$

Proposition. For $z_0 = y_0$, the problem

$$\mathcal{P}_2: \quad \inf_{u(\cdot),v(\cdot)} z(T)$$

under the mixed constraint

$$\mathcal{C}_m: \quad \max(y(t)-z(t),0)(1-v(t))+z(t)-y(t) \geq 0, \quad t \in [0,T]$$

is equivalent to $\mathcal{P}$.

Formulation without constraint

Consider $Z=(x,y,z)^\top$, $\quad \dot{Z}\in F(Z):=\displaystyle\bigcup_{(u,v)\in U\times[0,1]}\begin{bmatrix} f(x,y,u)\\ g(x,y,u)\\ h(x,y,z,u,v)\end{bmatrix}$

$$\text{with } h(x,y,u,v)=\max(g(x,y,u),0)(1-v\mathbb{1}_{\mathbb{R}^+}(z-y))$$

Let $\mathcal{S}:=\{Z(\cdot)\text{ a.c. };\ \dot{Z}(t)\in F(Z(t))\text{ a.e. and } Z(0)=(x_0,y_0,y_0)\}$

Remark. F is only upper semi-continuous.

Proposition. Assume $G(x,y):=\displaystyle\bigcup_{u\in U}\begin{bmatrix} f(x,y,u)\\ g(x,y,u)\end{bmatrix}$ is convex. Then

$$\mathcal{P}_3:\quad \inf_{Z(\cdot)\in\mathcal{S}} z(T)$$

admits a solution $Z^\star(\cdot)$ and $(x^\star(\cdot),y^\star(\cdot))$ is optimal for problem $\mathcal{P}$.

Sketch of proof

1. Consider the augmented dynamics :

$$\dot{Z} \in F^\dagger(Z) := \bigcup_{(u,v,\alpha)\in U\times[0,1]^2} \begin{bmatrix} f(x,y,u) \\ g(x,y,u) \\ h^\dagger(x,y,z,u,v,\alpha) \end{bmatrix}$$

$$h^\dagger(x,y,z,u,v,\alpha) = (1-\alpha)h(x,y,z,u,v) + \alpha \max_{w\in U} h(x,y,z,w,0)$$

$\Rightarrow \mathcal{S}^\dagger$ compact $\Rightarrow \exists (x^\star(\cdot), x^\star(\cdot), z^\star(\cdot))$ optimal.

2. Any $Z(\cdot)$ sol. of $\mathcal{D}_a$ with $\mathcal{C}_m$ belongs to $\mathcal{S} \subset \mathcal{S}^\dagger$

$$\Rightarrow z^\star(T) \leq \inf\{z(T);\ (x(\cdot), y(\cdot), z(\cdot)) \text{ sol. of } \mathcal{D}_a \text{ with } \mathcal{C}_m\}.$$

Sketch of proof

3. If $E := \{t \in (0,T);\; z(t) < y(t)\} \neq \emptyset$, then one has $\dot{z}(t) - \dot{y}(t) \geq 0$ for a.e. $t \in E \Rightarrow z(0) - y(0) < 0$: contradiction.

$$\Rightarrow z(T) \geq \max_{t \in [0,T]} y(t)$$

4. Filipov's Lemma applied to G :

$$\Rightarrow z^\star(T) \geq \max_{t \in [0,T]} y^\star(t) \geq \inf_{u \in \mathcal{U}} \left\{ \max_{t \in [0,T]} y(t);\; (x(\cdot), y(\cdot)) \text{ sol. of } \mathcal{D} \right\}$$

where $(x^\star(\cdot), y^\star(\cdot))$ is solution of $\mathcal{D}$ for a certain $u^\star(\cdot)$.

Approximation from below

$$(\mathcal{D}_a^\theta) : \quad \begin{cases} \dot{x} = f(x,y,u) \\ \dot{y} = g(x,y,u) \\ \dot{z} = h_\theta(x,y,z,u,v) \end{cases}$$

with $h_\theta(x,y,z,u,v) = \max(g(x,y,u),0)\left(1 - v\, e^{-\theta \max(y-z,0)}\right)$

Proposition. For any increasing sequence $\theta_n \to +\infty$, the problem

$$\mathcal{P}_3^{\theta_n} : \quad \inf_{Z(\cdot)\in\mathcal{S}^{\theta_n}} z(T)$$

admits an optimal solution $(x_n(\cdot), y_n(\cdot), z_n)(\cdot))$.

$(x_n(\cdot), y_n(\cdot))$ converges (up to a sub-sequence) uniformly to an optimal solution $(x^\star(\cdot), y^\star(\cdot))$ of problem $\mathcal{P}$, and $(\dot{x}^\star(\cdot), \dot{y}^\star(\cdot))$ converges weakly to $(\dot{x}^\star(\cdot), \dot{y}^\star(\cdot))$ in L^2.

Moreover, $z_n(T)$ is an increasing sequence that converges to $\max_{t\in[0,T]} y^\star(t)$.

A particular class of dynamics

$$\begin{cases} \dot{x} = f(x) \\ \dot{y} = g(x,u) \end{cases} \qquad u \in U$$

Proposition A feedback control $x \mapsto \phi^\star(x)$ such that

$$g(x,\phi^\star(x)) = \min_{u \in U} g(x,u), \quad x \in X$$

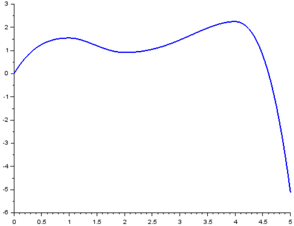
is optimal for problem $\mathcal{P}$.

An example

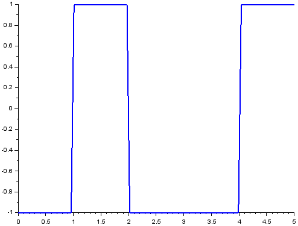
$$\begin{cases} \dot{x} = 1, \; x(0) = 0 \\ \dot{y} = (1-x)(2-x)(4-x)(1+u/2), \; y(0) = 0 \end{cases} \qquad u \in [-1, 1]$$

$$\Rightarrow \phi^\star(x) = -\operatorname{sign}\Big((1-x)(2-x)(4-x)\Big) \text{ is optimal}$$

Optimal solution for $T = 5$ *:*

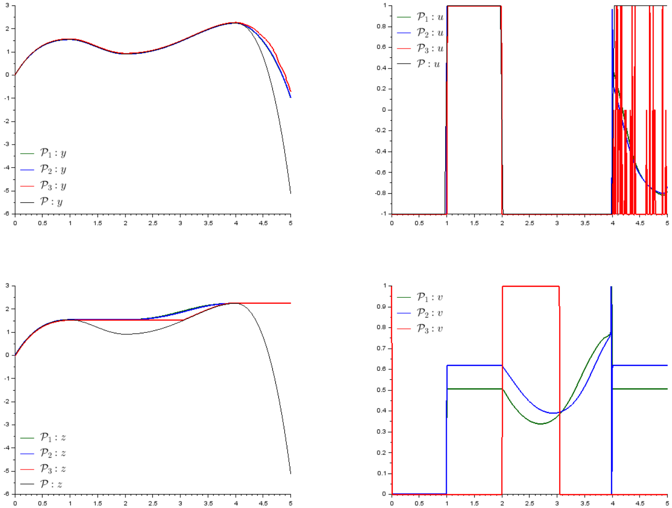


$y(\cdot)$



$u(\cdot)$

Numerical resolutions with Bocop



Comparison of the methods

Problems $\mathcal{P}_1$ and $\mathcal{P}_2$: Bocop (direct method) with 500 time steps
Problems $\mathcal{P}_3$: BocopHJB with 500 time steps on a 200×200 grid

problem	$\max\limits_{t\in[0,T]} y(t)$	error	computation time
$\mathcal{P}$	2.24985	0	–
$\mathcal{P}_1$	2.24989	0.001%	1.5 s
$\mathcal{P}_2$	2.24986	0.0004%	4.8 s
$\mathcal{P}_3$	2.26778	0.8%	248 s

The SIR problem

$$\begin{cases} \dot{S} = -(1-u)\beta SI \\ \dot{I} = (1-u)\beta SI - \gamma I \end{cases} \qquad u \in [0, \bar{u}] \text{ with } \bar{u} \leq 1$$

$$\rightarrow \inf_{u(\cdot)} \max_{t \in [0,T]} I(t) \text{ under the budget constraint } \int_0^T u(t)\, dt \leq Q$$

Consider the augmented dynamics

$$\begin{cases} \dot{S} = -(1-u)\beta SI \\ \dot{I} = (1-u)\beta SI - \gamma I \quad \text{with the target } q(T) = Q \\ \dot{q} = u, \;\; q(0) = 0 \end{cases}$$

The SIR problem

Posit $\mathcal{R}_0 = \beta/\gamma$ and $S_h = \mathcal{R}_0^{-1}$.

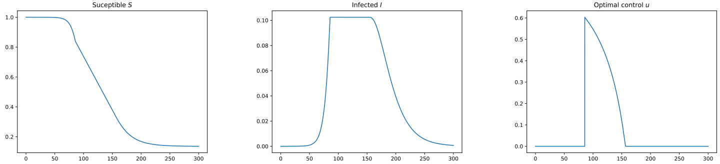
Proposition. [1] The *"0-singular-0"* feedback control

$$\psi(I,S) := \begin{cases} 1 - \frac{S_h}{S} & \text{if } I = \bar{I} \text{ and } \; S > S_h \\ 0 & \text{otherwise} \end{cases}$$

with

$$\bar{I} := \frac{I_0 + S_0 - S_h - S_h \log\left(\frac{S_0}{S_h}\right)}{Q\beta S_h + 1}$$

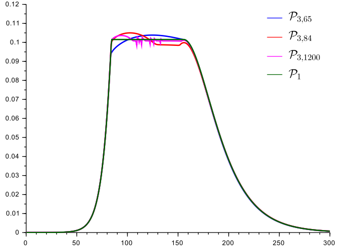
is optimal.



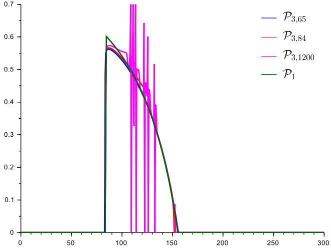
1. Molina, E. and Rapaport, A., *An optimal feedback control that minimizes the epidemic peak in the SIR model under a budget constraint.* Preprint hal-03604958.

Numerical illustrations with Bocop

$$\beta = 0.21, \ \gamma = 0.07, \ Q = 28, \ I(0) = 10^6$$



$I(\cdot)$



$u(\cdot)$

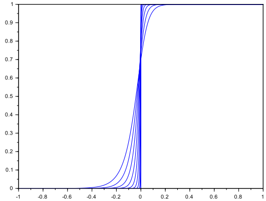
Numerical resolutions with Bocop

	$\mathcal{P}_1$	$\mathcal{P}_3^{65}$	$\mathcal{P}_3^{84}$	$\mathcal{P}_3^{233}$	$\mathcal{P}_3^{460}$	$\mathcal{P}_3^{1200}$
I_{max}	0.101508	0.0684	0.0823	0.09542	0.0993	0.101028

- direct method with 500 time steps
- $\max(\dot{y},0) \simeq \log\left(exp(\lambda_1 \dot{y}) + 1\right)/\lambda_1$
- $\exp(-\theta \max(y-z,0)) \simeq \exp\left(-\theta \log(\exp(\lambda_2(y-z)) + 1)/\lambda_2\right)$

$$\lambda_2 = \frac{\log\left(\varepsilon^{\frac{\log(2)}{\log(1-\epsilon)}} - 1\right)}{\varepsilon^2}$$

$$\theta = -\frac{\log(1-\varepsilon)}{\log(2)}\lambda_2$$



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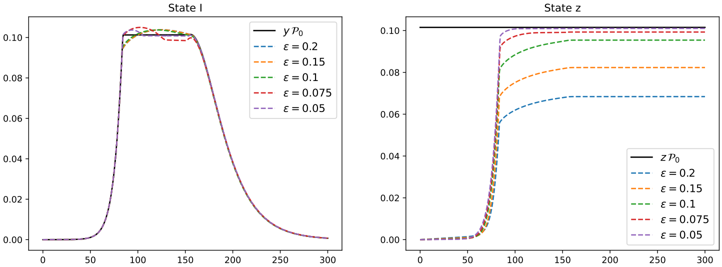
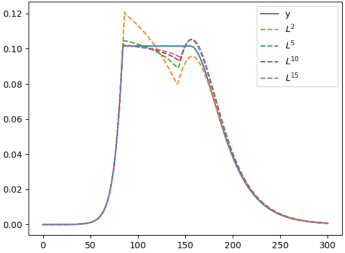
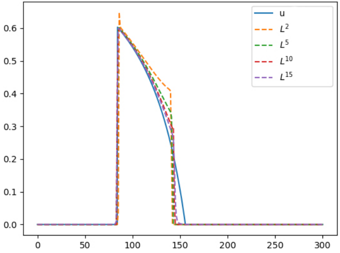


$I(\cdot)$ $\qquad$ $z(\cdot)$

Comparison with L^p approximation



$I(\cdot)$



$u(\cdot)$

Conclusions

We have proposed equivalent formulations that allow to use existing numerical methods, but...

- with state constraint : not qualified...
- without state constraint : discontinuous differential inclusion...

- one cannot use (available) Maximum Principles...

Open problem : How to derive necessary optimality conditions ?