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Abstract

We give the explicit solution of the optimal control problem which consists in minimizing the epidemic peak in the SIR model when the control is an attenuation factor of the infectious rate, subject to a L^1 constraint on the control which represents a budget constraint. The optimal strategy is given as a feedback control which consists in a singular arc maintaining the infected population at a constant level until the immunity threshold is reached, and no intervention outside the singular arc. We discuss and compare this strategy with the one that minimizes the peak when fixing the duration of a single intervention, as already proposed in the literature. Numerical simulations illustrate the benefits of the proposed control.

Key words: Optimal control; maximum cost; feedback control; epidemiology; SIR model.

1 Introduction

Since the pioneering work of Kermack and McKendrick [16], the SIR model has been very popular in epidemiology, as the basic model for infectious diseases with direct transmission (see for instance [22,18] as introductions on the subject). It retakes great importance nowadays due to the recent coronavirus pandemic. In face of a new pathogen, non-pharmaceutical interventions (such as reducing physical distance in the population) are often the first available means to reduce the propagation of the disease, but this has economic and social costs. In [20,19], the authors underline the need of control strategies for epidemic mitigation by "flattering the epidemic curve", rather than eradication of the disease that might be too costly. Several works have applied the optimal control theory considering interventions as a control variable that reduces the effective transmission rate of the SIR model, and studied optimal strategies with criteria based on running and terminal cost over fixed finite interval or infinite horizon [4,7,8,15,21,5,9,12,17,6]. However, the highest peak of the epidemic appears to be the highly relevant criterion to be minimized (especially when there is an hospital pressure to save individuals with severe forms of the infection). In [20], the authors studied the minimization of the peak of the infected population under the constraint that interventions occur on a single time interval of given duration. They obtained that the optimal control consists in four phases: no intervention, apply interventions to maintain the prevalence constant, apply a full intervention that stops the disease transmission and finally release any intervention. This control presents thus three switches and relies on a full break of the transmission. In the present work, we consider the same criterion, but under a budget constraint on the control, as an integral cost. We believe such a constraint to be more relevant as it takes into account the strength of the interventions and does not impose an *a priori* single time interval of given length for the interventions to take place. We have been able to prove that the optimal solution consists indeed in having interventions on a single time interval but with a control strategy that differs from the one obtained in [20]. We shall see that it consists in applying interventions to maintain the prevalence at a constant precise value, so that the interventions are reduced progressively to the point where they are no longer necessary. Therefore, this strategy does not require to apply a full intervention as in [20], which in practice is much less demanding. Let us also mention a more recent work [1] that considers a kind of

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"dual" problem, which consists in minimizing an integral cost of the control under the constraint that the epidemic stays below a prescribed value and an additional constraint on the state at a fixed time. The structure of the optimal strategy given by the authors in [1] is similar to the one we obtained without having to fix a time horizon and a terminal constraint. All the cited works rely on numerical methods to provide the effective control. Here, we give an explicit analytical expression of the optimal control.

Let us stress that optimal control problems with maximum cost are not in the usual Mayer, Lagrange or Bolza forms of the optimal control theory [10], for which the necessary optimality conditions of Pontryagin's Principle apply, but fall into the class of optimal control with L^∞ criterion, for which characterizations have been proposed in the literature mainly in terms of the value function (see for instance [3]). Although necessary optimality conditions and numerical procedures have been derived from theses characterizations (see for instance [2,11]), these approaches remain quite difficult and numerically heavy to be applied on concrete problems. On the other hand, for minimal time problems with planar dynamics linear with respect to the control variable, comparison tools based on the application of the Green's Theorem have shown that it is possible to dispense with the use of necessary conditions to prove the optimality of a candidate solution [14]. Although our criterion is of different nature, we show in the present work that it is also possible to implement this approach for our problem.

The paper is organized as follows. In the next section, we posit the problem of peak minimization to be studied. In Section 3, we define a class of feedback strategies that we called "NSN", and give some preliminary properties. Section 4 proves the existence of an NSN strategy which is optimal for our problem, and makes it explicit. Finally, Section 5 illustrates the optimal solutions on numerical simulations and discusses about the optimal strategy.

2 Definitions and problem statement

We consider the SIR model

$$\begin{cases} \dot{S} = -\beta S I(1-u) \\ \dot{I} = \beta S I(1-u) - \gamma I \\ \dot{R} = \gamma I \end{cases} \tag{1}$$

where S, I and R denotes respectively the proportion of susceptible, infected and recovered individuals in a population of constant size. The parameters β and γ are the transmission and recovery rates of the disease. The control u, which belongs to $U := [0,1]$, represents the efforts of interventions by reducing the effective transmission rate. For simplicity, we shall drop in the following the R dynamics. Throughout the paper, we shall assume that the *basic reproduction number* $\mathcal{R}_0$ is larger than one, so that an epidemic outbreak may occur.

Assumption 1

$$\mathcal{R}_0 := \frac{\beta}{\gamma} > 1.$$

For a positive initial condition $(S(0), I(0)) = (S_0, I_0)$ with $S_0 + I_0 \leq 1$, we consider the optimal control problem which consists in minimizing the epidemic peak under a budget constraint

$$\inf_{u(\cdot)\in\mathcal{U}} \max_{t\geq 0} I(t), \tag{2}$$

where $\mathcal{U}$ denotes the set of measurable functions $u(\cdot)$ that take values in U and satisfy the L^1 constraint

$$\int_0^{+\infty} u(t)dt \leq Q.$$

In the epidemiological context, the integral of the control measures the cumulative efforts in reducing the incidence rate, and Q is a given value not be to exceeded.

Remark 1 *From equations* (1)*, one can easily check that the solution* $I(t)$ *tends to zero when* t *tends to* $+\infty$ *whatever is the control* $u(\cdot)$*, so that the supreme of* $I(\cdot)$ *over* $[0,+\infty)$ *in* (2) *is reached.*

Equivalently, one can consider the extended dynamics.

$$\begin{cases} \dot{S} = -\beta S I(1-u), \\ \dot{I} = \beta S I(1-u) - \gamma I, \\ \dot{C} = -u, \end{cases} \tag{3}$$

with the initial condition $(S(0), I(0), C(0)) = (S_0, I_0, Q)$ and the state constraint

$$C(t) \geq 0, \quad t \geq 0. \tag{4}$$

A solution of (3) is *admissible* if the control $u(\cdot)$ takes its values in U and the condition (4) is fulfilled.

3 The NSN feedback

Let us denote the *immunity threshold*

$$S_h := \mathcal{R}_0^{-1} = \frac{\gamma}{\beta} < 1.$$

Note that $S(\cdot)$ is a non increasing function and that one has $\dot{I} \leq 0$ when $S \leq S_h$, whatever is the control. If

$S_0 \leq S_h$, the maximum of $I(\cdot)$ is thus equal to I_0 for any control $u(\cdot)$, which solves the optimal control problem. We shall now consider that the non-trivial case.

Assumption 2

$$S_0 > S_h.$$

Under this assumption, we thus know that for any admissible solution, the maximum of $I(\cdot)$ is reached for $S \geq S_h$. Consider the dynamics with the control $u(t) = 0$ for all $t \geq 0$. One can check from equations (1) that the quantity

$$t \mapsto S(t) + I(t) - S_h \log(S(t)) \tag{5}$$

is constant. Then, one has

$$S(t) + I(t) - S_h \log(S(t)) = S_0 + I_0 - S_h \log(S_0) \tag{6}$$

for any $t > 0$, and the maximum of $I(\cdot)$ reached for $S = S_h$ is given by the value

$$I_h := I_0 + S_0 - S_h - S_h \log\left(\frac{S_0}{S_h}\right).$$

We define the "NSN" (for null-singular-null) strategy as follows.

Definition 1 *For $\bar{I} \in [I_0, I_h]$, consider the feedback control*

$$\psi_{\bar{I}}(I, S) := \begin{cases} 1 - \frac{S_h}{S}, & \text{if } I = \bar{I} \text{ and } S > S_h, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

We denote the L^1 norm associated to the NSN control

$$\mathcal{L}(\bar{I}) := \int_0^{+\infty} u^{\psi_{\bar{I}}}(t)dt, \quad \bar{I} \in [I_0, I_h],$$

where $u^{\psi_{\bar{I}}}(\cdot)$ is the control generated by the feedback (7).

This control strategy consists in three phases:

(1) no intervention until the prevalence I reaches $\bar{I}$ (null control),
(2) maintain the prevalence I equal to $\bar{I}$ by adjusting the interventions until S reaches S_h or the budget is entirely consumed (singular control),
(3) no longer intervention when $S < S_h$ (null control).

Remark 2 *There is no switch of the control between phases 2 and 3, because $u(t)$ tends to zero when $S(t)$ tends to S_h, according to expression (7).*

We shall discuss the practicability of this control in Section 5.

One can check straightforwardly the following properties are fulfilled.

Lemma 1 *For any $\bar{I} \in [I_0, I_h]$, the maximal value of the control $u^{\psi_{\bar{I}}}(\cdot)$ is given by*

$$u_{max}(\bar{I}) := 1 - \frac{S_h}{\bar{S}} < 1,$$

where $\bar{S}$ is solution of

$$\bar{S} - S_h \log \bar{S} = S_0 + I_0 - S_h \log S_0 - \bar{I}.$$

Moreover, any solution given by the NSN strategy verifies

$$\max_{t \geq 0} I(t) = \bar{I}.$$

4 Optimal strategy

We first show that the function $\mathcal{L}$ can be made explicit.

Proposition 1 *One has*

$$\mathcal{L}(\bar{I}) = \frac{I_h - \bar{I}}{\beta S_h \bar{I}}, \quad \bar{I} \in [I_0, I_h]. \tag{8}$$

PROOF. Note first that whatever is $\bar{I}$, $S(\cdot)$ is decreasing with the control (7). One can then equivalently parameterize the solution $I(\cdot)$, $C(\cdot)$ by

$$\sigma(t) := S_0 - S(t),$$

instead of t. Let us put $\sigma_h := \sigma(t_h) = S_0 - S_h$.

As long as $I < \bar{I}$, one has $u = 0$ which gives

$$\begin{cases} \dfrac{dI}{d\sigma} = f(\sigma) := 1 - \frac{S_h}{S_0 - \sigma} > 0, \\ \dfrac{dC}{d\sigma} = 0. \end{cases}$$

Remind, from the definition of I_h, that the solution $I(\cdot)$ with $u = 0$ reaches I_h in finite time. Therefore, one can define the number

$$\bar{\sigma} := \inf\{\sigma \geq 0,\ I(\sigma) = \bar{I}\} \leq \sigma_h,$$

which verifies

$$\int_0^{\bar{\sigma}} f(\sigma)\, d\sigma = \bar{I} - I_0. \tag{9}$$

For $\sigma \in [\bar{\sigma}, \sigma_h]$, one has $u = 1 - S_h/S$, that is

$$\begin{cases} \dfrac{dI}{d\sigma} = 0, \\ \dfrac{dC}{d\sigma} = -\frac{1}{\beta S_h \bar{I}} \left(1 - \frac{S_h}{S_0 - \sigma}\right) = -\frac{f(\sigma)}{\beta S_h \bar{I}} < 0. \end{cases}$$

One then obtains

$$\mathcal{L}(\bar{I}) = C(0) - C(\sigma_h) = \frac{1}{\beta S_h \bar{I}} \int_{\bar{\sigma}(\bar{I})}^{\sigma_h} f(\sigma)\, d\sigma,$$

and with (9) one can write

$$\mathcal{L}(\bar{I}) = \frac{1}{\beta S_h \bar{I}} \left(\int_0^{\sigma_h} f(\sigma)\, d\sigma + I_0 - \bar{I} \right).$$

On the other hand, one has

$$\int_0^{\sigma_h} f(\sigma)\, d\sigma = \sigma_h + S_h \log\left(\frac{S_h}{S_0}\right) = I_h - I_0,$$

which finally gives the expression (8).

Then, the best admissible NSN control can be given as follows.

Corollary 1 *When $Q \leq \frac{I_h - I_0}{\beta S_h I_0}$, the smallest $\bar{I} \in [I_0, I_h]$ for which the solution with the NSN strategy is admissible, is given by the value*

$$\bar{I}^\star(Q) := \frac{I_h}{Q\beta S_h + 1} \tag{10}$$

and one has

$$\mathcal{L}(\bar{I}^\star(Q)) = Q. \tag{11}$$

We give now our main result that shows that the NSN strategy is optimal.

Proposition 2 *Let Assumptions 1 and 2 be fulfilled. Then, the NSN feedback is optimal with*

$$\bar{I} = \begin{cases} \bar{I}^\star(Q), & Q < \frac{I_h - I_0}{\beta S_h I_0}, \\ I_0, & Q \geq \frac{I_h - I_0}{\beta S_h I_0}, \end{cases}$$

where $\bar{I}^\star(Q)$ is defined in (10)*, and $\bar{I}$ is the optimal value of problem* (2)*.*

PROOF. When $Q \geq \frac{I_h - I_0}{\beta S_h I_0}$, the NSN strategy is admissible and the corresponding solution verifies

$$\max_{t \geq 0} I(t) = I_0,$$

which is thus optimal.

Consider now $Q < \frac{I_h - I_0}{\beta S_h I_0}$. Let $(S^\star(\cdot), I^\star(\cdot), C^\star(\cdot))$ be the solution generated by the NSN strategy with $\bar{I} = \bar{I}^\star(Q)$, and denote $u^\star(\cdot)$ the corresponding control. Let

$$\bar{S} := S^\star(\bar{t}) \text{ where } \bar{t} = \inf\{t > 0,\ I^\star(t) = \bar{I}\},$$

and

$$t_h^\star := \inf\{t > \bar{t},\ S^\star(t) = S_h\}.$$

We consider in the (S, I) plane the curve

$$\mathcal{C}^\star := \{(S^\star(t), I^\star(t));\ t \in [0, t_h^\star]\}.$$

For $S \geq \bar{S}$, the control (7) is null and a upward normal to $\mathcal{C}^\star$ is given by the expression

$$\vec{n}(S, I) = \begin{bmatrix} \beta S I - \gamma I \\ \beta S I \end{bmatrix},\ (S, I) \in \mathcal{C}^\star \text{ with } S \in [\bar{S}, S_0].$$

On the other hand, the vector field in the (S, I) plane of any admissible solution is

$$\vec{v}(S, I, u) = \begin{bmatrix} -\beta S I (1 - u) \\ \beta S I (1 - u) - \gamma I \end{bmatrix}.$$

Then, one has

$$\vec{n}(S, I).\vec{v}(S, I, u) = -\beta\gamma S I^2 u \leq 0,$$

for any $(S, I) \in \mathcal{C}^\star$ with $S \in [\bar{S}, S_0]$, which shows that any admissible solution is below the curve $\mathcal{C}^\star$ in the (S, I) plane for $S \in [\bar{S}, S_0]$. For $S \in [S_h, \bar{S}]$, the curve $\mathcal{C}^\star$ is an horizontal line with $I = \bar{I}$. Therefore, if there exists an admissible solution $(S(\cdot), I(\cdot), C(\cdot))$ with $\max_t I(t) < \bar{I}$, its trajectory in the (S, I) plane has to be below the curve $\mathcal{C}^\star$ for any $S \in [S_h, S_0]$. Let

$$t_h := \inf\{t > 0,\ S(t) = S_h\}.$$

One has thus $I(t_h) < \bar{I}$. Define

$$T := t_h^\star + \frac{1}{\gamma} \log\left(\frac{\bar{I}}{I(t_h)}\right) > t_h^\star,$$

and consider the non-admissible solution $(\tilde{S}(\cdot), \tilde{I}(\cdot), \tilde{C}(\cdot))$ of (3) on $[0, T]$ defined by the control

$$\tilde{u}(t) = \begin{cases} u^\star(t), & t \in [0, t_h^\star), \\ 1, & t \in [t_h^\star, T]. \end{cases}$$

One can straightforwardly check with equations (3) that the solution $(\tilde{S}(t), \tilde{I}(t), \tilde{C}(t))$ is

$$\begin{cases} (S^\star(t), I^\star(t), C^\star(t)), & t \in [0, t_h^\star), \\ (S_h, \bar{I}\exp(-\gamma(t - t_h^\star)), C^\star(t_h^\star) + t_h^\star - t), & t \in [t_h^\star, T]. \end{cases}$$

Remind, from Corollary 1, that one has $C^\star(t_h^\star) = 0$ by equation (11)). Clearly, one has $(\tilde{S}(T), \tilde{I}(T)) = (S_h, I(t_h))$ and $\tilde{C}(T) < 0$. We consider now in the (S, I) plane the simple closed curve Γ which is the concatenation of the trajectory $(\tilde{S}(\cdot), \tilde{I}(\cdot))$ on forward time with the trajectory $(S(\cdot), I(\cdot))$ in backward time:

$$\Gamma := \{(\tilde{S}(\tau), \tilde{I}(\tau)),\ \tau \in [0, T]\} \cup \{(S(T + t_h - t), I(T + t_h - t)),\ \tau \in [T, T + t_h]\},$$

that is anticlockwise oriented by $\tau \in [0, T + t_h]$. Then one has

$$\tilde{C}(T) - C(t_h) = \oint_\Gamma dC.$$

From equations (3), one gets

$$dC = -\frac{dS}{\beta SI} - dt = -\frac{dS}{\beta SI} + \frac{dS+dI}{\gamma I} = \left(1 - \frac{S_h}{S}\right)\frac{dS}{\gamma I} + \frac{dI}{\gamma I}$$

and thus

$$\tilde{C}(T) - C(t_h) = \oint_\Gamma P(S, I)dS + Q(S, I)dI,$$

with

$$P(S, I) = \left(1 - \frac{S_h}{S}\right)\frac{1}{\gamma I}, \quad Q(S, I) = \frac{1}{\gamma I}.$$

By the Green's Theorem, one obtains

$$\begin{aligned}\tilde{C}(T) - C(t_h) &= \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial S}(S, I) - \frac{\partial P}{\partial I}(S, I)\right) dSdI \\ &= \iint_{\mathcal{D}} \left(1 - \frac{S_h}{S}\right)\frac{1}{\gamma I^2}\, dSdI > 0,\end{aligned}$$

where $\mathcal{D}$ is the domain bounded by Γ (see Figure 1 as an illustration). This implies $C(t_h) < \tilde{C}(T) < 0$ and thus a contradiction with the admissibility condition (4) of the solution $(S(\cdot), I(\cdot), C(\cdot))$. We conclude that $(S^\star(\cdot), I^\star(\cdot), C^\star(\cdot))$ is optimal.

5 Numerical illustrations and discussion of practical considerations

We illustrate the behavior of the optimal trajectories with numerical simulations for parameters and initial condition borrowed from [20] , which studied the initial stage of COVID-19 disease. (see Table 1). For these values, one computes

$$\mathcal{R}_0 = 3, \quad S_h = \frac{1}{3}, \quad I_h \simeq 0.3.$$

Figure 2 presents a simulation of the optimal NSN strategy for the budget $Q = 28$, as an example (the minimum peak is reached for $\bar{I} \simeq 0.1015$). One can see that for maintaining the size I at the constant level I_h, the control u is decreasing with respect to time until it reaches the value 0 (no intervention), exactly when the size of susceptible S reaches the immunity value S_h. A particular feature of the optimal strategy is to do not have discontinuity of the control when the optimal trajectory leaves the singular arc. In practice, one might not have a precise measurement of the size of the susceptible population S to apply the feedback law (7). However, assuming that the proportion of the infected population I is measured with a relatively good accuracy, say daily, one could adjust the level of restrictions u to keep I as close as possible to I_h. Let us underline that one does not necessarily need to measure S to know when to stop the interventions: they have simply to be applied until the

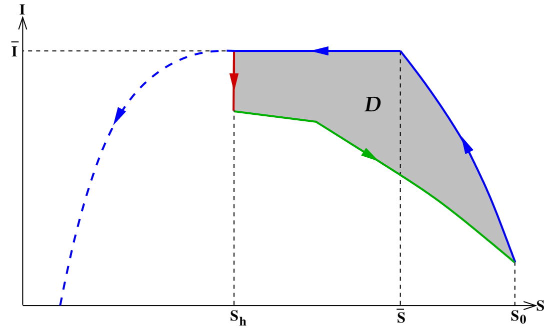

Fig. 1. The closed curve Γ is composed of the trajectory $(S^\star(\cdot), I^\star(\cdot))$ in blue up to to the point $(S_h, \bar{I})$, the additional part $(\tilde{S}(\cdot), \tilde{I}(\cdot))$ in red and the hypothetical better trajectory $(S(\cdot), I(\cdot))$ in backward time in green.

β	γ	$S(0)$	$I(0)$
0.21	0.07	$1 - 10^{-6}$	10^{-6}

Table 1
Chosen SIR parameters and initial condition, as in [20].

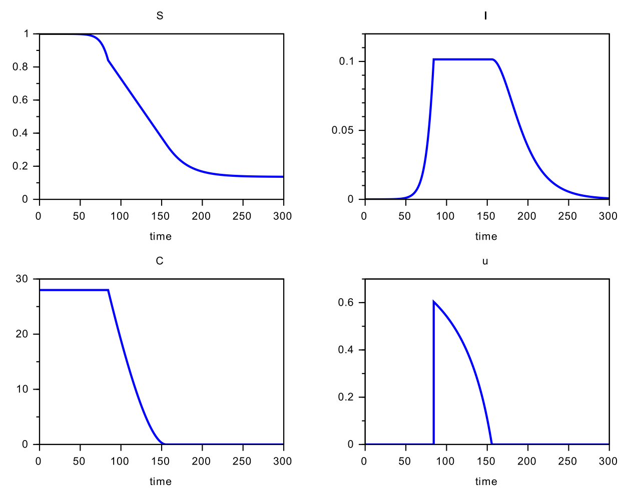

Fig. 2. State variables and control over time for the optimal strategy with the budget $Q = 28$.

budget Q is consumed. Equivalently, as the corresponding effort u decreases progressively until vanishing, the interventions phase ceases also when the control u takes the value 0.

As a comparison, we have computed the optimal strategy which minimizes the epidemic peak for a fixed time duration of interventions without consideration of any budget, obtained by Morris et al. in [20] (see Figure 3). It consists in four phases: no intervention, maintain I constant, apply the maximal control (i.e. $u = 1$) and stop the intervention. This control presents thus three switches and relies on a full break of the transmission, differently to the NSN strategy which presents only one switch (see Remark 2) and does not require a full break (see the maximal value of the control given in Lemma 1). Applying an NSN strategy appears thus less restrictive to be applied in practice. The strategy proposed by Morris et al. induces also a second peak: after the third phase, the prevalence I increases again up to a peak which has to be equal to the level maintained during the second phase if it is optimally chosen. But this second peak turns out to be non robust under a mischoice (or mistiming) of the second phase (see [20] for more details). Comparatively, the NSN is naturally robust with respect to a bad choice of $\bar{I}$: the maximum value of I is always guaranteed to be equal to $\bar{I}$. However, a mischoice of $\bar{I}$ has an impact on the budget of the NSN strategy, given by expression (8) and illustrated in Table 2 (for model parameters given in Table 1 and $Q = 28$).

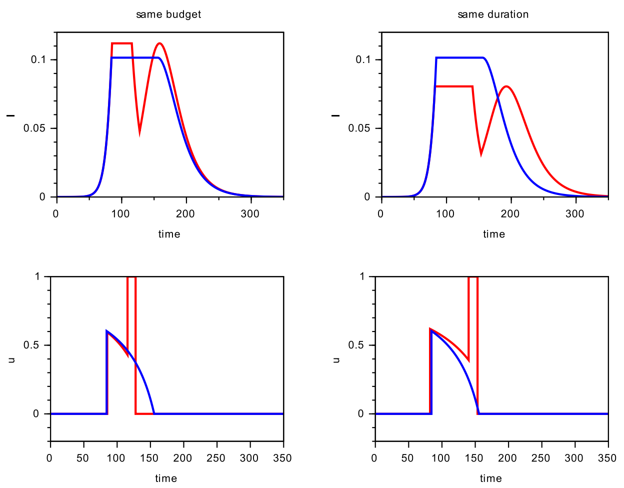

Fig. 3. Comparison of the time evolution of the infected population I and the control u between the optimal NSN strategy (in blue) and the optimal one of Morris et al. (in red)) [20].

Table 3 gives the asymptotic size $S_\infty < S_h$ of the susceptible population for the two strategies under budget or duration constraint, corresponding to Figure 3. Note that for both strategies one has $I = \bar{I}$ (where $\bar{I}$ is the peak value of I) and $S = S_h$ when entering the last phase with $u = 0$. Then, S_∞ can be determined with the invariant property (5), as the solution of the equation

$$f(S_\infty) := S_\infty - S_h \log(S_\infty) = S_h + \bar{I} - S_h \log(S_h).$$

where f is decreasing for $S_\infty < S_h$. This shows that for both constraints a lower peak implies a larger value of the final size of S (i.e. fewer susceptible individuals that have contracted the disease).

$\bar{I} - \bar{I}^\star$	-10%	-5%	-1%	$+5\%$	$+10\%$
$\mathcal{L}(\bar{I}) - Q$	$+17\%$	$+8\%$	$+1.5\%$	-7%	-14%

Table 2
Variation of the control budget of the NSN strategy under a mischoice of $\bar{I}$.

	same budget	same duration
NSN	$S_\infty \simeq 0.135$	
Morris et al.	$S_\infty \simeq 0.129$	$S_\infty \simeq 0.152$

Table 3
Comparison of the final size of S for the two strategies, corresponding to Figure 3.

In case of a new epidemic among a large population, one can consider that the initial number of infected individuals is very low, while all the remaining population is susceptible. Therefore, one has $S_0 + I_0 = 1$ with I_0 very small, and the optimal value of $\bar{I}$ can be well approximated by its limiting expression for $I_0 = 0$, that is

$$\bar{I}_\ell := \frac{1 - S_h + S_h \log(S_h)}{Q\beta S_h + 1}. \quad (12)$$

From property (6), one also gets an approximation of the value $\bar{S}_\ell$ of S when I reaches $\bar{I}_\ell$ with $u = 0$, as the solution of the equation

$$\bar{S}_\ell + \bar{I}_\ell - S_h \log(\bar{S}_\ell) = 1,$$

and then an approximation of the duration of the intervention is given by

$$d_\ell := \frac{S_h - \bar{S}_\ell}{\gamma \bar{I}_\ell}$$

(one can easily check that along the singular arc $I = \bar{I}$, one has $\dot{S} = -\gamma\bar{I}$). For the parameters of Table 1, one obtains the limiting values given in Table 4. Provided that parameters β and γ of the disease are known (or estimated), and a budget Q is given, one can thus determine the minimal value of the peak and the optimal strategy to apply, without the knowledge of the initial size of the infected population.

In practice, one might not know precisely the initial condition. However, assuming that the proportion of the

$\bar{I}_\ell$	$\bar{S}_\ell$	d_ℓ
0.1015	0.8406	71.39

Table 4
The limiting optimal values for arbitrarily small I_0 (with $Q = 28$)

infected population is measured with a relatively good accuracy when it is not too small, one could apply the approximate feedback which consists in waiting the proportion I to be about the value $\bar{I}_l$ adjusting daily the effort u so that I remains close to $\bar{I}_l$ during the time interval d_l.

The question of parameters estimation in the SIR model from data is out of the scope of the present work. However, while reaching $I = \bar{I}_\ell$ without intervention, one may expect refinement of the estimates of β and γ and thus an adjustment of the value of $\bar{I}_\ell$.

Note that if it is rather the height of the peak $\bar{I}$ that is imposed, the corresponding effort can be determined with expression (12), that is

$$Q = \frac{1}{\beta S_h}\left(\frac{1-S_h}{\bar{I}} - 1\right),$$

as well as the duration of the intervention.

To have a better insight of the impacts of the available budget Q on the course of the epidemic, we have considered four characteristics numbers:

- t_i: the starting date of the intervention,
- d: the duration of the intervention,
- $\bar{I}$: the height of the peak,
- u_{max}: the maximal value of the control,

of the optimal solution, depicted on Figure 4 as a function of Q for $I_0 = 10^{-6}$ and $S_0 + I_0 = 1$. Let us note that the maximal budget Q under which it is not possible to immediately slow down the progress of the epidemic is given, according to Proposition 2, by

$$Q_{max} := \frac{I_h - I_0}{\beta S_h I_0} \simeq 4.3\,10^6,$$

which is quite high. Moreover, the maximal value of the control is bounded by the value

$$u_{max}(\bar{I}) \leq 1 - S_h = \frac{2}{3},$$

far from the value 1 (that would consists in a total lockdown of the population). On Figure 4, one can see that the peak $\bar{I}$ can be drastically reduced under a reasonable budget, and that taking larger budgets slows down

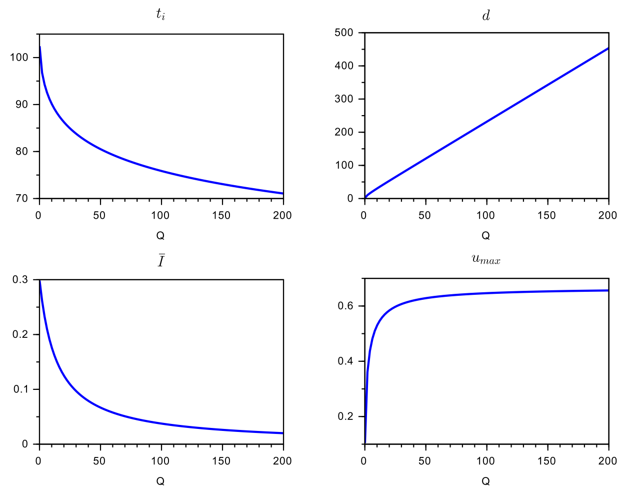

Fig. 4. Characteristics numbers as functions of Q.

the decrease of the peak, while the duration of the intervention carries on increasing, almost linearly. Indeed, one has $\dot{S} = -\gamma\bar{I}$ on the singular arc and one thus gets $d = (\bar{S} - S_h)/(\gamma\bar{I})$. For an optimal value of $\bar{I}$, one has $Q = (I_h - \bar{I})/(\gamma\bar{I})$ from (10) and then one obtains

$$d = \frac{\bar{S} - S_h}{I_h - \bar{I}}\,Q.$$

For large values of Q, $\bar{I}$ is small and $\bar{S}$ closed to one, which gives an approximation of d as the linear function of Q

$$d \simeq \frac{1 - S_h}{I_h}\,Q \simeq 2.194\,Q.$$

This implies that for a long duration, fixing the budget Q or the duration d tends to be equivalent. Therefore, for the same large duration, the optimal peak gets near from the optimal one of the strategy of Morris et al. which constraints the duration only, but the difference of the budgets of these two strategies gets increasing with always a lower one for the NSN strategy, as one can see on Figure 5.

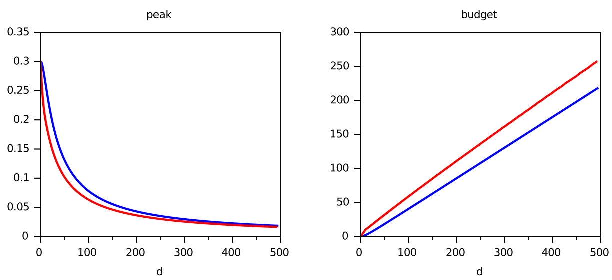

Fig. 5. Comparison of the performances of the optimal strategies with same duration (NSN in blue and Morris et al. [20] in red).

Finally, this analysis highlights (as already mentioned in [20,19]) the importance to do not intervene too early (unless one has a very large budget) and to choose the "right" time to launch interventions. We believe that curves as in Figure 4 might be of some help for decision makers.

6 Conclusion

In this note, we have shown how to use the Green's Theorem as a geometric tool to prove the optimality of the "null-singular-null" control strategy for the minimization of the epidemic peak under a budget constraint. This strategy turns out to be different than the one proposed by Morris et al. [20] which minimizes the peak fixing a single interval length for interventions, and is simpler to apply. Although the objective of the present work was not to optimize the asymptotic size of the susceptible population (as for instance in [6]), the study of a compromise between the epidemic peak and the final size could be the matter of future investigations.

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