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Abstract

In anaerobic digestion plants, it is often crucial to measure state variables such as the substrate or biomass densities for monitoring the process. But, this task can be very difficult to achieve because the sensors are sometimes unavailable, unreliable or very expensive. For these reasons software sensors can be sought instead, using indirect measurements. In this paper, we design a new observer for the chemostat model with the single measurement of biogas flow rate. This observer is positive and we show its adjustable convergence. Through simulations we also show its robustness under noise measurements. The key point is to construct a non-linear observer using hidden symmetries in the framework of the theory of symmetry preserving nonlinear systems. The effectiveness of the proposed methodology is shown through simulations.

Keywords:
Nonlinear system, invariant system, invariant observer, chemostat model.

1. Introduction

The state reconstruction of anaerobic digestion models by software sensors or observers has received great attention in the literature. For an overview, one can see [15, 5] and the references therein. Most of the time, some of the

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state variables of the model are assumed to be measured, such as the biomass or substrate densities. Unfortunately the sensors measuring these quantities, if available, are too expensive or unreliable [29, 4, 32, 22, 10, 11]. However, in the context of anaerobic digestion, measuring the biogas production is an easy and cheap alternative [18, 30, 31, 17]. In this work, we address the problem of reconstructing the state variables of the chemostat model, which is often considered as a good representation of the anaerobic digestion process, with the single measurement of biogas flow rate.

The well-known (nonlinear) model of the chemostat obtained by mass balance is given by the differential equations

$$\begin{aligned} \dot{s} &= D(t)(s_{in} - s) - k\mu(s, K)x, \\ \dot{x} &= [\mu(s, K) - D(t)]\, x, \end{aligned} \tag{1}$$

where the vector $(s, x) \in X := \mathbb{R}_+^2$ represents the substrate and biomass concentrations respectively. $D(t)$ is the dilution rate, k the conversion factor, s_{in} the input substrate concentration and $\mu(\cdot, K)$ the specific growth rate per unit of biomass with K a parameters vector. Most of the time, $\mu(.)$ is a function of the variable s only, and is parameterized by one or several constants that can be gathered in a vector denoted K. We emphasize here the dependence of μ on K in order to apply below the theory of symmetry preserving systems. In the present work, we consider the output variable

$$y(t) = \mu(s(t), K)x(t) \in Y := \mathbb{R}_+, \tag{2}$$

which represents the biogas flow rate, up to a stoichiometric parameter (typically, the biogas is produced at a rate proportional to the growth rate). In bioprocess and automatic control literature, one mainly find state estimations with sensors measuring reactant concentrations, and only some recent works have tackled the state estimation problem with biogas measurements. In [7], an asymptotic observer has been proposed for anaerobic digestion, which is robust with respect to the knowledge of the function $\mu(\cdot, K)$, but its convergence speed cannot be assigned and depends strongly on the control function $D(\cdot)$. In particular if $D(\cdot)$ takes small values on certain time interval the speed of convergence will be very slow. Moreover, it has been chosen to be sensitive to the knowledge of the stoichiometric coefficients [14]. Several techniques based on linearization have been also applied, such as bundles of local observers [9], extended Kalman filter [23], or unscented Kalman Filter [21]. However, as underlined in [15], these approaches are not satisfactory for operating conditions too far from nominal ones and can reserve some surprises with certain time-varying control functions $D(\cdot)$, differently to the asymptotic observer. More recently, a new kind of observer has been proposed, which consists in considering the biogas flow rate as an additional state variable with an unmodeled part of the dynamics, and an nonlinear observer that is robust with respect to the unmodeled part [24]. This observer has been shown to behave better that more local ones, although its convergence analysis could not have been achieved exhaustively. Another kind of observer based on sliding mode technique is proposed in [28]. It uses the measurement of the biogas and the substrate to estimate the unknown inputs and biomasses for an anaerobic digestion model with four states variables. Here, we aim at providing an exact an adjustable observer with the measurement of the biogas only and proving its convergence.

Let us underline that y is not a state variable, as it is often the case in observer design, and that it is not easy to find a smooth invertible change of coordinates with y as a state variable, to write the system in the canonical nonlinear normal form [19].

Recently, a new methodology based on the so-called "invariant observers" exploiting the symmetries of the dynamics has been proposed [1, 8]. The aim of the present work is to investigate how it can be applied to the present observation problem. It consists in two steps:

1. find invariants to propose a form of a observer whose estimations remain positive,
2. find the conditions on the gains to prove the convergence of the observer.

As we shall see, these tasks are far to be straightforward, but the final observer possesses several advantages compared to the other approaches mentioned previously. In particular, the proposed observer is positive as the original system is a positive system. This avoids for instance some estimation peaking that could provide negative values during the transient. The question of designing positive observers for positive systems has been addressed in the literature mainly for linear systems [12, 20, 6, 3], but not for nonlinear dynamics up to our knowledge.

In this paper, we will consider the invariant domain

$$\mathcal{D} := \{(s, x) \in X,\ (s, x) > 0\}. \tag{3}$$

Note that the estimation in batch i.e. with D identically null has already been tackled in [16, 26] where a new exact observer has been proposed, although the trajectories of the system converge to a subset of the state state that is not observable. Here we consider the "true" chemostat for which we shall assume persistently exciting inputs D.

Assumption 1. There exists numbers D_{min}, D_{max} such that

$$0 < D_{min} < D(t) < D_{max}, \quad \forall t \geq 0. \tag{4}$$

Observe also that the subset

$$\mathcal{I} := \{(s, x) \in \mathcal{D},\ kx + s = s_{in}\} \tag{5}$$

is invariant by the dynamics (1) and that the dynamics is reduced to a scalar dynamics on this subset

$$\dot{s} = (s_{in} - s)(D(t) - \mu(s, K)). \tag{6}$$

We shall focus here on the true planar dynamics, assuming that the (unknown) initial condition does not belong to the particular subset $\mathcal{I}$ (which cannot be reached in finite time). It is indeed very unlikely in practice that the initial condition belongs exactly to this set. The observability analysis on $\mathcal{D} \setminus \mathcal{I}$ is given in Appendix B, under an additional technical assumption.

The paper is organized as follows. In Section 2, we show step by step the construction of an "invariant observer", leading to a "pre-observer" form. Then, in Section 3 we give our main result which guarantees the convergence of the proposed observer for a right choice of its gains. Finally Section 4 shows numerical simulations and discuss about this new observer.

2. Design of the observer

Let us consider the vector

$$u = (s_{in}, k, K) \in U := \mathbb{R}_+^3 \tag{7}$$

as the "virtual" control, and deploy the methodology of invariant observers introduced in [8] to characterize symmetries in the system, as it has been done for bioprocess systems [13]. Note that the true input of the system is s_{in}. Since k and K are constant parameters, we will keep them at constant values as part of the virtual control. Here, K is chosen as a single scalar parameter because we shall consider the following assumption on the growth function. Let us underline that the use of an extended vector of controls including the true control and some parameters of the system is new and has not been introduced before.

Assumption 2. The growth function $\mu(\cdot, K)$ is homogeneous i.e. $\mu(s, K) = \mu_*\left(\frac{s}{K}\right)$, where μ_* is an analytical function in $\mathbb{R}_+^\star$.

We consider transformations $G := \{G_{\lambda_1,\lambda_2}\}_{\lambda_1,\lambda_2}$ parameterized by $\lambda_1 > 0$ and $\lambda_2 > 0$

$$G_{\lambda_1,\lambda_2} : X \times U \times Y \mapsto X \times U \times Y \tag{8}$$

of the form

$$G_{\lambda_1,\lambda_2}((s, x), u, y) = (\varphi_{\lambda_1,\lambda_2}(s, x), \psi_{\lambda_1,\lambda_2}(s_{in}, k, K), \rho_{\lambda_1,\lambda_2}(y)) \tag{9}$$

such that

$$\begin{array}{l} \varphi_{\lambda_1,\lambda_2}(s, x) = (\lambda_1 s, \lambda_2 x)\,, \\ \psi_{\lambda_1,\lambda_2}(s_{in}, k, K) = \left(\lambda_1 s_{in}, \frac{\lambda_1}{\lambda_2} k, \lambda_1 K\right), \\ \rho_{\lambda_1,\lambda_2}(y) = \lambda_2 y. \end{array} \tag{10}$$

We shall show that the dynamics (1) is invariant by such G-actions, that is for any $\lambda_1 > 0$, $\lambda_2 > 0$. Here, the word invariance refers to the symmetry preserving property under the action of a group of transformations, that was introduced and developed by P. Rouchon et al. (we invite a reader not familiar with this concept to consult the reference [8]). Posit

$$f(t, (s, x), (s_{in}, k, K)) := \left[\begin{array}{c} D(t)(s_{in} - s) - k\mu(s, K)x \\ \left[\mu(s, K) - D(t)\right] x \end{array} \right] \tag{11}$$

as the right hand side of the differential equation (1). On one hand, one has

$$\begin{array}{l} f\left(t, (\lambda_1 s, \lambda_2 x), \left(\lambda_1 s_{in}, \dfrac{\lambda_1}{\lambda_2} k, \lambda_1 K\right)\right) = \\ \qquad \left[\begin{array}{c} D(t)(\lambda_1 s_{in} - \lambda_1 s) - \frac{\lambda_1}{\lambda_2} k\mu(\lambda_1 s, \lambda_1 K)\lambda_2 x \\ \left[\mu(\lambda_1 s, \lambda_1 K) - D(t)\right] \lambda_2 x \end{array} \right] \end{array} \tag{12}$$

and on the other hand

$$D_{\varphi_{\lambda_1,\lambda_2}}(s, x).f = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right] f \tag{13}$$

where $D_{\varphi_{\lambda_1,\lambda_2}}$ denotes the differential of $\varphi_{\lambda_1,\lambda_2}$, which proves the invariance.

2.1. Invariant scalar functions and vector fields

We consider now functions $J : X \times U \times Y \mapsto \mathbb{R}$ that are invariant by the G-action, i.e. that are such that

$$J\left(\left(\lambda_1 s, \lambda_2 x\right), \left(\lambda_1 s_{in}, \frac{\lambda_1}{\lambda_2} k, \lambda_1 K\right), \lambda_2 y\right) = J\left(\left(s, x\right), \left(s_{in}, k, K\right), y\right). \quad (14)$$

If J is C^1, we can differentiate J with respect to λ_1, for instance at $\lambda_1 = \lambda_2 = 1$, and obtain a first p.d.e. (partial differential equation)

$$s\frac{\partial J}{\partial s} + s_{in}\frac{\partial J}{\partial s_{in}} + k\frac{\partial J}{\partial k} + K\frac{\partial J}{\partial K} = 0. \quad (15)$$

Symmetrically, we differentiate J with respect to λ_2 at $\lambda_1 = \lambda_2 = 1$, and we obtain a second p.d.e.

$$x\frac{\partial J}{\partial x} - k\frac{\partial J}{\partial k} + y\frac{\partial J}{\partial y} = 0. \quad (16)$$

Using the well-known method of characteristics for p.d.e. (see for instance [27] Chap 5, p. 220) to solve (15) and (16), we get solutions of the form

$$J\left((s, x), (s_{in}, k, K), y\right) = L\left(\frac{s_{in}}{s}, \frac{kx}{s}, \frac{K}{s}, \frac{y}{x}\right) \quad (17)$$

where L is any smooth function defined on an invariant domain of the dynamics. In the following, we will restrict, for simplicity, the class of functions L to "one variable" invariant functions (but other choices are of course possible)

$$J\left((s, x), (s_{in}, k, K), y\right) = L\left(\frac{y}{x}\right). \quad (18)$$

We shall characterize L more precisely latter on. Following the "methodology" (exposed in [1]) to build invariant pre-observers, let $\omega = (\omega_1, \omega_2)$ be an invariant vector field. We have then

$$\begin{aligned} \omega_1(\lambda_1 s, \lambda_2 x) = &\ \lambda_1 \omega_1(s, x), \\ \omega_2(\lambda_1 s, \lambda_2 x) = &\ \lambda_2 \omega_2(s, x), \end{aligned} \quad (19)$$

for any (s, x), and we proceed in the same way as for the scalar invariant functions to get as candidate solutions

$$\omega_1(s, x) = as, \qquad \omega_2(s, x) = bx, \quad (20)$$

where a and b are two constants which will play the role of "gains" of the observer.

2.2. The pre-observer

For the class of functions L chosen above, we get the invariant "pre-observer" form

$$\begin{aligned} \dot{\hat{s}} &= D(t)(s_{in} - \hat{s}) - k\mu(\hat{s}, K)\hat{x} + a\hat{s}\left[L\left(\frac{y(t)}{\hat{x}}\right) - L\left(\frac{\hat{y}}{\hat{x}}\right)\right] \\ \dot{\hat{x}} &= \left[\mu(\hat{s}, K) - D(t)\right]\hat{x} + b\hat{x}\left[L\left(\frac{y(t)}{\hat{x}}\right) - L\left(\frac{\hat{y}}{\hat{x}}\right)\right] \end{aligned} \tag{21}$$

where $\hat{y} = \mu(\hat{s}, K)\hat{x}$. To prove its convergence of the estimation, we seek for an error expression that is invariant and which tends to 0 when t tends to $+\infty$.

For the following choice of the function L

$$L\left(\frac{y}{x}\right) = \ln\left(\frac{y}{x}\right), \tag{22}$$

the observer is made explicit from the pre-observer form (21), and we naturally consider the error variables

$$e_1(t) = \ln\left(\frac{s(t)}{\hat{s}(t)}\right), \quad e_2(t) = \ln\left(\frac{x(t)}{\hat{x}(t)}\right). \tag{23}$$

Note that these error variables are well defined due to the positivity of the variables on $\mathcal{D}$, and we shall study the convergence of e towards 0. One can easily check that the positive orthant is invariant by this observer whatever are the gains a, b, differently to classical observers, such as Luenberger observers. This is why this observer is called a "positive observer".

Let us also assume some properties of the growth function.

Assumption 3. The function $\mu_*(.)$ is C^2, concave, increasing and bounded on $\mathbb{R}_+$ with $\mu_*(0) = 0$. Moreover, we assume that its first and second derivatives are bounded.

A typical instance of a function μ that fulfills Assumptions 2 and 3 is given by the well-known Monod function

$$\mu(s, K) = \frac{\mu_{max} s}{K + s} \tag{24}$$

with

$$\mu_*(s) = \frac{\mu_{max} s}{1 + s} \tag{25}$$

where μ_{max} is the maximum growth rate and K the half saturation or substrate affinity constant.

3. Convergence of the observer

We give now our main result about the asymptotic property of the state estimation provided by the observer of the form (21) for the choice (22) of the function L.

Theorem 1. *Let Assumptions 1, 2 and 3 be fulfilled. For the system (1) with (2) as output, the following dynamics*

$$\begin{aligned} \dot{\hat{s}} &= D(t)(s_{in} - \hat{s}) - k\mu(\hat{s}, K)\hat{x} + a\,\hat{s}\ln\left(\frac{y(t)}{\hat{y}}\right), \\ \dot{\hat{x}} &= [\mu(\hat{s}, K) - D(t)]\,\hat{x} + b\,\hat{x}\ln\left(\frac{y(t)}{\hat{y}}\right), \end{aligned} \tag{26}$$

with $\hat{y} = \mu(\hat{s}, K)\hat{x}$, is an observer on $\mathcal{D}\setminus\mathcal{I}$ for any $D(\cdot)$ that fulfills Assumption 1. $(0, 0)$ *is a uniformly asymptotically stable equilibrium of the dynamics of the error variables (23), for non positive a and positive b, provided that $|a|$ is not too large and b large enough.*

Proof. To simplify the writing, we shall drop the time dependency of the variables s, x, y, $\hat{s}$, $\hat{x}$, $\hat{y}$ and e_1, e_2.

One can straightforwardly check that the time derivatives of the error variables defined in (23) satisfy

$$\begin{aligned} \dot{e}_1 &= s_{in}D(t)\left(\frac{1}{s} - \frac{1}{\hat{s}}\right) - k\left[\mu(s, K)\frac{x}{s} - \mu(\hat{s}, K)\frac{\hat{x}}{\hat{s}}\right] - a\ln\left(\frac{y(t)}{\hat{y}}\right), \\ \dot{e}_2 &= [\mu(s, K) - \mu(\hat{s}, K)] - b\ln\left(\frac{y(t)}{\hat{y}}\right). \end{aligned} \tag{27}$$

From the expressions of the errors, we get

$$\hat{s} = se^{-e_1} \qquad \text{and} \qquad \hat{x} = xe^{-e_2}\,.$$

Using Assumption 2, one has

$$\frac{y}{\hat{y}} = \frac{\mu_*\left(\frac{s}{K}\right)}{\mu_*\left(\frac{\hat{s}}{K}\right)}\frac{x}{\hat{x}} \tag{28}$$

and

$$\ln\left(\frac{y}{\hat{y}}\right) = e_2 - \ln\left(\frac{\mu_*\left(\frac{se^{-e_1}}{K}\right)}{\mu_*\left(\frac{s}{K}\right)}\right). \tag{29}$$

Then the error dynamics (27) can be written as follows.

$$
\begin{aligned}
\dot{e}_1 &= G_1(t,e_1,e_2) := \frac{s_{in}D(t)}{s}\left(1-e^{e_1}\right) \\
&\quad -\frac{kx}{s}\left[\mu_*\left(\frac{s}{K}\right)-\mu_*\left(\frac{se^{-e_1}}{K}\right)e^{e_1-e_2}\right]-a\ln\left(\frac{y(t)}{\hat{y}}\right), \\
\dot{e}_2 &= G_2(t,e_1,e_2) := \left[\mu_*\left(\frac{s}{K}\right)-\mu_*\left(\frac{se^{-e_1}}{K}\right)\right]-b\ln\left(\frac{y(t)}{\hat{y}}\right).
\end{aligned}
\tag{30}
$$

One can write also

$$
\begin{aligned}
G_1(t,e_1,e_2) &= \frac{s_{in}D(t)}{s}\left(1-e^{e_1}\right)-\frac{kx}{s}\left[\mu_*\left(\frac{s}{K}\right)-\mu_*\left(\frac{se^{-e_1}}{K}\right)e^{e_1-e_2}\right] \\
&\qquad -ae_2+a\ln\left[\frac{\mu_*\left(\frac{se^{-e_1}}{K}\right)}{\mu_*\left(\frac{s}{K}\right)}\right], \\
G_2(t,e_1,e_2) &= \left[\mu_*\left(\frac{s}{K}\right)-\mu_*\left(\frac{se^{-e_1}}{K}\right)\right]-be_2+b\ln\left[\frac{\mu_*\left(\frac{se^{-e_1}}{K}\right)}{\mu_*\left(\frac{s}{K}\right)}\right].
\end{aligned}
\tag{31}
$$

The Taylor's formula of order 1 with Lagrange remainder writes

$$
\begin{aligned}
G_1(t,e_1,e_2) &= G_1(t,0,0)+e_1\tfrac{\partial G_1}{\partial e_1}(t,0,0)+e_2\tfrac{\partial G_1}{\partial e_1}(t,0,0)+R_1(t,e_1,e_2), \\
G_2(t,e_1,e_2) &= G_2(t,0,0)+e_1\tfrac{\partial G_2}{\partial e_1}(t,0,0)+e_2\tfrac{\partial G_2}{\partial e_2}(t,0,0)+R_2(t,e_1,e_2),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
R_1(t,e_1,e_2) &= \tfrac{1}{2}\Big[e_1^2\tfrac{\partial^2 G_1}{\partial e_1^2}(t,\theta_1(t)e_1,\theta_1(t)e_2) \\
&\qquad +2e_1e_2\tfrac{\partial^2 G_1}{\partial e_1\partial e_2}(t,\theta_1(t)e_1,\theta_1(t)e_2)+e_2^2\tfrac{\partial^2 G_1}{\partial e_2^2}(t,\theta_1(t)e_1,\theta_1(t)e_2)\Big], \\
R_2(t,e_1,e_2) &= \tfrac{1}{2}\Big[e_1^2\tfrac{\partial^2 G_2}{\partial e_1^2}(t,\theta_2(t)e_1,\theta_2(t)e_2) \\
&\qquad +2e_1e_2\tfrac{\partial^2 G_2}{\partial e_1\partial e_2}(t,\theta_2(t)e_1,\theta_2(t)e_2)+e_2^2\tfrac{\partial^2 G_2}{\partial e_2^2}(t,\theta_2(t)e_1,\theta_2(t)e_2)\Big],
\end{aligned}
\tag{33}
$$

with $\theta_1(t)\in(0,1)$ and $\theta_2(t)\in(0,1)$. Note that R_1 and R_2 can be written as

$$
\begin{aligned}
R_1(t,e_1,e_2) &= \tfrac{1}{2}e^T H_{G_1}(t,\theta_1(t)e)e, \\
R_2(t,e_1,e_2) &= \tfrac{1}{2}e^T H_{G_2}(t,\theta_2(t)e)e,
\end{aligned}
\tag{34}
$$

where

$$H_{G_1}(t,\varepsilon) = \begin{bmatrix} \frac{\partial^2 G_1}{\partial e_1^2}(t,\varepsilon) & \frac{\partial^2 G_1}{\partial e_1 \partial e_2}(t,\varepsilon) \\ \frac{\partial^2 G_1}{\partial e_1 \partial e_2}(t,\varepsilon) & \frac{\partial^2 G_1}{\partial e_2^2}(t,\varepsilon) \end{bmatrix}, \tag{35}$$

$$H_{G_2}(t,\varepsilon) = \begin{bmatrix} \frac{\partial^2 G_2}{\partial e_1^2}(t,\varepsilon) & \frac{\partial^2 G_2}{\partial e_1 \partial e_2}(t,\varepsilon) \\ \frac{\partial^2 G_2}{\partial e_1 \partial e_2}(t,\varepsilon) & \frac{\partial^2 G_2}{\partial e_2^2}(t,\varepsilon) \end{bmatrix}. \tag{36}$$

The error dynamics is non-autonomous and $(0,0)$ is an equilibrium point. In order to show that this equilibrium is locally uniformly asymptotically stable, we use a result given in [27] (see Theorem 2 in Appendix A). To prove that the map

$$R(t,e) = \begin{bmatrix} R_1(t,e_1,e_2) \\ R_2(t,e_1,e_2) \end{bmatrix} \tag{37}$$

verifies the condition of Theorem 2, that is

$$\forall \epsilon > 0,\ \exists \delta_\epsilon > 0,\ \|e\| \le \delta_\epsilon \Rightarrow \|R(t,e)\| \le \epsilon(\|e\|),\ \forall t \ge T,$$

any norm $\|.\|$ on $\mathbb{R}^2$ can be used. Here we choose the L_1 norm. Under Assumption 3, one has

$$\|R(t,e)\| = |R_1(t,e_1,e_2)| + |R_2(t,e_1,e_2)| \le h(\delta_\epsilon)(|e_1| + |e_2|) \tag{38}$$

with

$$\begin{aligned} h(\delta_\epsilon) \quad = \quad & \frac{1}{2}\Bigg\{ \frac{s_{in} D_{max}}{\underline{s}} + \frac{k\overline{x}}{\underline{s}} \left[4\mu_{max} + 3\frac{s_{in}}{K}\mu'_{max} e^{\delta_\epsilon} + \frac{s_{in}^2}{K^2}\mu''_{max} \right] \\ & + (|a| + |b|)(1 + e^{\delta_\epsilon}) + \frac{s_{in}}{K}\mu'_{max} + \frac{s_{in}^2}{K^2}\mu''_{max} e^{\delta_\epsilon} \Bigg\} \delta_\epsilon, \end{aligned} \tag{39}$$

where μ_{max}, μ'_{max} and μ''_{max} denotes upper bounds of μ, μ' and $-\mu''$, respectively,

$$\bar{x} = \max(s_{in}, x(0)), \quad \underline{s} = \min(s(0), \tilde{s}) \tag{40}$$

with

$$\tilde{s} = \inf\left\{ s > 0 : D_{min}(s_{in} - s) - k\mu(s)\bar{x} > 0 \right\}. \tag{41}$$

Note that the function $h(\cdot)$ is continuous with $h(0) = 0$. So, for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $h(\delta_\epsilon) < \epsilon$.

To show that $(0,0)$ is locally uniformly asymptotically stable for the non-linear system (27), it remains to show that $(0,0)$ is uniformly asymptotically stable for the linear part of (27) *i.e.* for $\dot{e}(t) = A(t)e(t)$ where

$$A(t) = \begin{bmatrix} \frac{\partial G_1}{\partial e_1}(t,0,0) & \frac{\partial G_1}{\partial e_2}(t,0,0) \\ \\ \frac{\partial G_2}{\partial e_1}(t,0,0) & \frac{\partial G_2}{\partial e_2}(t,0,0) \end{bmatrix} \tag{42}$$

with

$$\begin{aligned} \frac{\partial G_1}{\partial e_1}(t,0,0) &= -\frac{s_{in}}{s}D(t) + \frac{kx}{s}\left[\mu_*\left(\frac{s}{K}\right) - \frac{s}{K}\mu_*'\left(\frac{s}{K}\right)\right] - a\frac{s}{K}\frac{\mu_*'\left(\frac{s}{K}\right)}{\mu_*\left(\frac{s}{K}\right)}, \\ \frac{\partial G_1}{\partial e_2}(t,0,0) &= -\frac{kx}{s}\mu_*\left(\frac{s}{K}\right) - a, \\ \frac{\partial G_2}{\partial e_1}(t,0,0) &= \frac{s}{K}\mu_*'\left(\frac{s}{K}\right) - b\frac{s}{K}\frac{\mu_*'\left(\frac{s}{K}\right)}{\mu_*\left(\frac{s}{K}\right)}, \\ \frac{\partial G_2}{\partial e_2}(t,0,0) &= -b. \end{aligned} \tag{43}$$

For this, we consider the Losinskii measure $\mathcal{L}(.)$ (see [25] for more details) associated to the L_1 norm for which this measure is negative (that is $\|x\| = \sum_{j=1}^{n}|x_j|$ for a vector $x = (x_1, x_2, ..., x_n)^T$ and $\|M\| = \max_j\left\{\sum_{i=1}^{n}|m_{ij}|\right\}$ for a matrix $M = [m_{ij}] \in \mathcal{M}_{n\times n}$), which is given by the following expression

$$\mathcal{L}(M) = \max_j\left\{m_{jj} + \sum_{i\neq j}|m_{ij}|\right\}, \tag{44}$$

In dimension two, one gets

$$\mathcal{L}(M) = \max\left\{m_{11} + |m_{12}|\,,\, m_{22} + |m_{21}|\right\}. \tag{45}$$

Here, we have

$$\mathcal{L}(A(t)) = \max\left\{-av + u_1(t) + |a + u_2|\,,\, -b + v\left|b - \mu_*\left(\frac{s}{K}\right)\right|\right\} \tag{46}$$

with

$$v = \frac{s}{K} \frac{\mu_*'\left(\frac{s}{K}\right)}{\mu_*\left(\frac{s}{K}\right)} > 0, \tag{47}$$

$$u_1(t) = -\frac{s_{in}}{s} D(t) + \frac{kx}{s} \left[\mu_* \left(\frac{s}{K} \right) - \frac{s}{K} \mu_*' \left(\frac{s}{K} \right) \right], \tag{48}$$

$$u_2 = \frac{kx}{s} \mu_* \left(\frac{s}{K} \right) > 0. \tag{49}$$

As the function μ_* is concave, one has the inequality

$$\mu_*(0) = 0 \leq \mu_*(s) - s\mu_*'(s), \tag{50}$$

which implies $v \leq 1$. Note that one has also $u_1(t) + u_2 < 0$ for any $t \geq 0$. Then, for $a \geq -u_2$, one gets

$$-av + u_1(t) + |a + u_2| = a(1 - v) + u_1(t) + u_2 \tag{51}$$

which is negative for any $t \geq 0$, provided that a is non positive and $|a|$ not too large to ensure $a \geq -u_2$ along the trajectory of the system. For $b > \mu_{max}$, one gets

$$-b + v \left| b - \mu_* \left(\frac{s}{K} \right) \right| = b(v - 1) - v\mu_* \left(\frac{s}{K} \right) < 0. \tag{52}$$

Finally, one obtains that $\mathcal{L}(A(t))$ is negative for any $t \geq 0$, which ends the proof.

4. Numerical illustrations

We have performed numerical simulations with the Monod growth function (24), which verifies Assumptions 2 and 3. The simulations were carried out using the parameter values and initial conditions given in Tables 1 and 2, over a period of 5 days for a variable dilution rate depicted on Fig. 1. The corresponding time measurement of biogas flow rate is given on Fig. 2.

We have compared the performances of three observers for the reconstruction of the state variables s and x.

1. The proposed invariant observer (26) with the gains $a = -40$ and $b = 50$.

k	6.6
μ_{max}	$1.2\,h^{-1}$
K	$4.95\,mg.l^{-1}$
s_{in}	$9\,mg.l^{-1}$

Table 1: Parameter values

model	observer
$s(0) = 3\,mg.l^{-1}$ $x(0) = 0.5\,mg.l^{-1}$	$\hat{s}(0) = 2\,mg.l^{-1}$ $\hat{x}(0) = 0.8\,mg.l^{-1}$

Table 2: Initial conditions

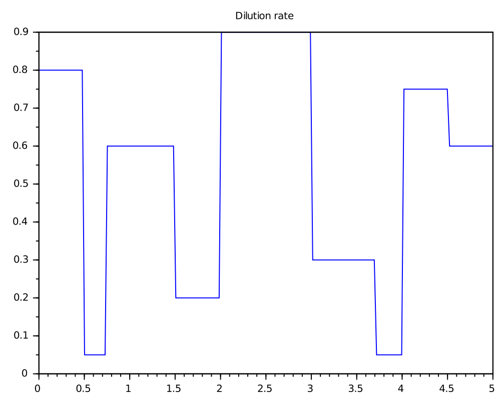

Figure 1: Dilution rate

2. The asymptotic observer

$$\begin{aligned}\dot{\hat{s}} &= D(t)(s_{in} - \hat{s}) - ky(t),\\ \dot{\hat{x}} &= y(t) - D(t)\hat{x},\end{aligned}$$

which is not adjustable.

3. The classical Luenberger observer

$$\begin{aligned}\dot{\hat{s}} &= D(t)(s_{in} - \hat{s}) - k\mu(\hat{s}, K)\hat{x} + g_1(y - \hat{y}),\\ \dot{\hat{x}} &= [\mu(\hat{s}, K) - D(t)]\,\hat{x} + g_2(y - \hat{y}),\end{aligned}$$

with the gains $g_1 = 100$, $g_2 = 100$.

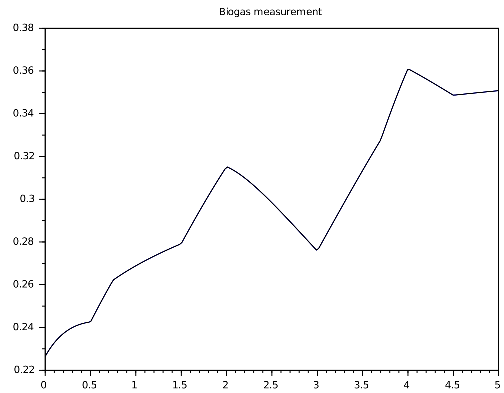

Figure 2: Biogas measurement

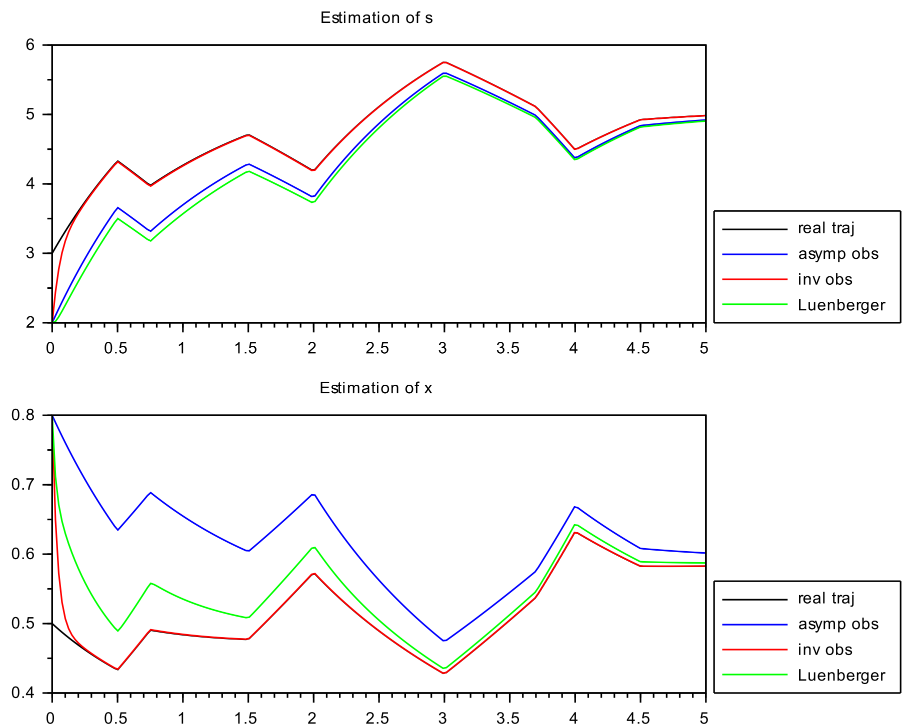

Figure 3: Estimations of the substrate and biomass concentrations

The comparison of the three estimators given on Fig. 3 shows that the invariant observer provides a very fast convergence, faster than the other observers. For the Luenberger observer, we did not obtained faster convergence, even with very large gains g_1, g_2. The asymptotic observer does converge but its convergence speed is low, especially for the estimation of the variable x. Despite the relatively large values of the gains a and b, one can observe that the invariant observer does not suffer from any peaking in the transients, due to its structure that preserves positivity of the variables.

To test the robustness of the observers, we have considered uncertainty on the knowledge of the parameter μ_{max}, with observers using the value of μ_{max} 20% higher than the real value. Fig. 4 shows that the invariant observer, with the same gains as before, does not behave well...

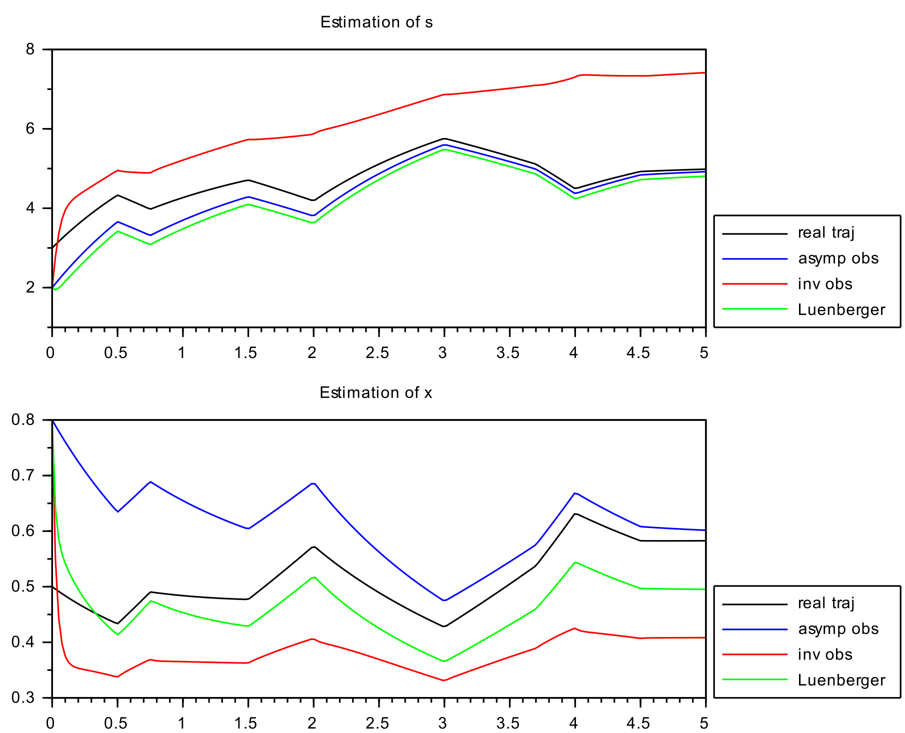

Figure 4: Estimations of the substrate and biomass concentrations with a bad knowledge of the parameter μ_{max}

If one reduces the absolute value of the gain a, accordingly to the statement of Theorem 1, taking $a = -5$ instead of $a = -40$, Fig. 5 shows that the estimation of the variable s is satisfactorily reconstructed, in a better way than for the two other observers. A compromise with a lower speed of convergence has therefore to be chosen. However, the reconstruction of the variable x presents a bias, as for the Luenberger one (indeed the product $\mu_{max}x(t)$ is well estimated, which provides an estimation of $x(t)$ 20% lower). On the contrary, the asymptotic observer does not present an asymptotic bias because its dynamics does not rely on the knowledge of the function μ, but its estimation error remains very large during a long period of time because of its slow convergence.

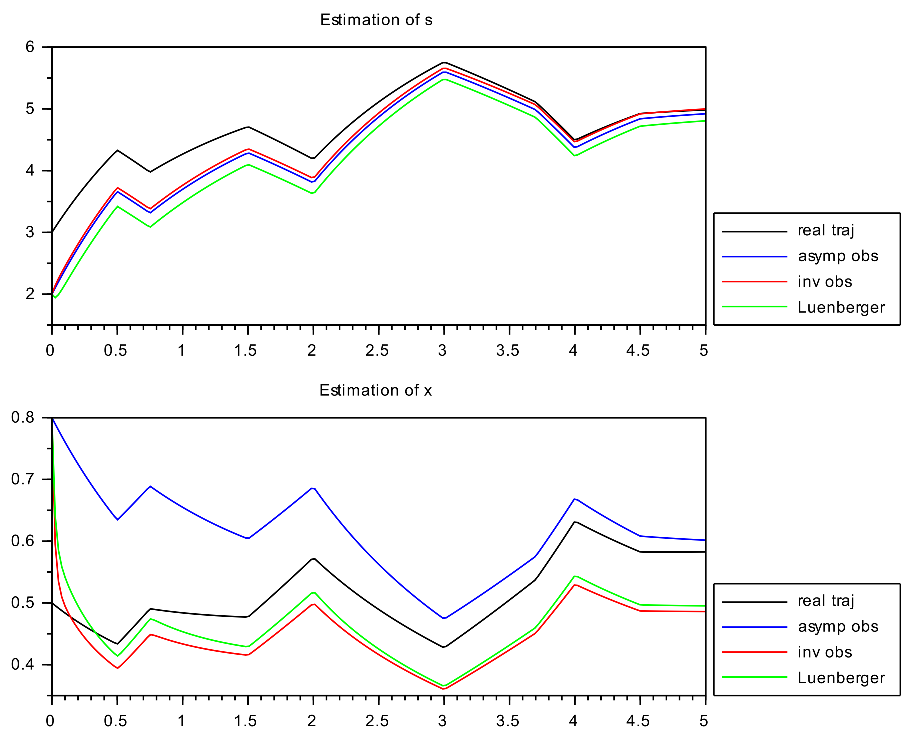

Figure 5: Estimations of the substrate and biomass concentrations under a bad knowledge of the parameter μ_{max} with the gain $a = -5$ for the invariant observer

Finally, we have considered in the first simulations an additive corrupted noise on the measurements (with a Gaussian noise of variance 10^{-3}), depicted on Fig. 6. The simulations of the three observers show that the invariant observer still behaves well (better than the Luenberger one), while the asymptotic observer is not much affected by the noise (this is due to the fact that this estimator filters the measurement signal without any amplification gain). Despite this property of filtering of the asymptotic observer, the invariant observer is better from the point of view of convergence to the true signal.

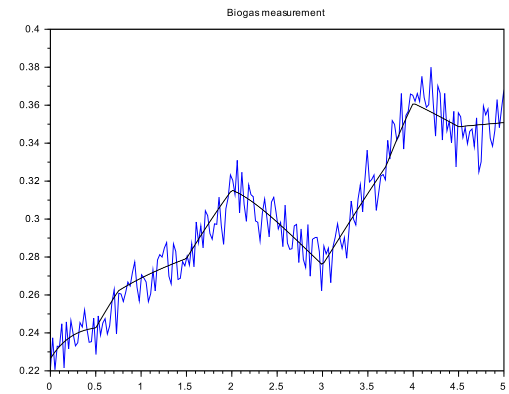

Figure 6: Biogas measurement with noise

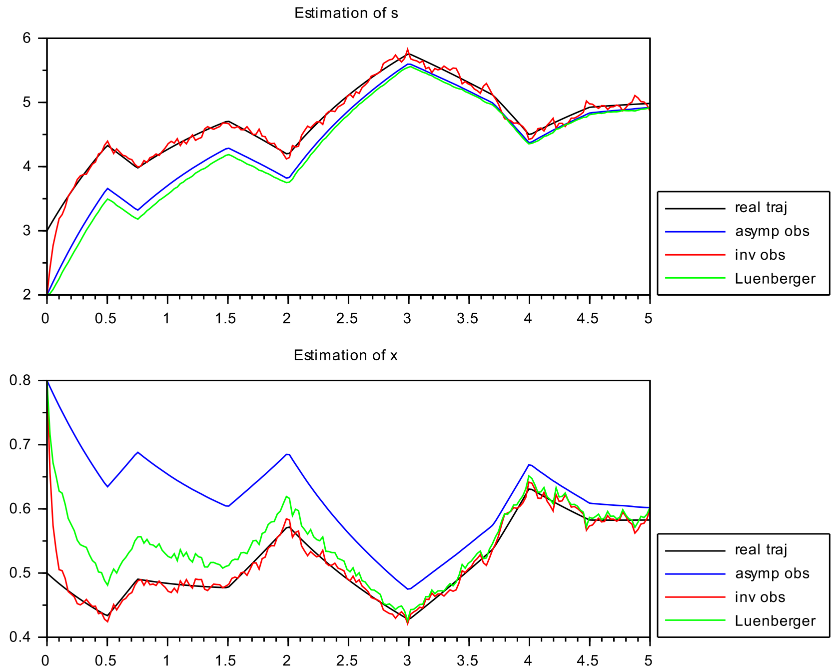

Figure 7: Estimations of the substrate and biomass concentrations with measurement noise

5. Conclusion

In this work, we have designed a nonlinear observer using hidden symmetries for the chemostat model with the single measurement of biogas flow rate. The novelty is the use of an extended vector of controls including the true control and some parameters. The main advantage of this observer is its positivity and its convergence without having to choose high gains as it often occurs with other nonlinear techniques. Important features of this study is that any growth rate function μ can be considered under some mild assumptions, and the convergence of the observer and the local observability have been theoretically proven. A future work will concern its extension to more general classes of growth functions to weaken the assumptions on the growth function, and allow non-monotonic ones such as the Haldane function [2].

Appendix A.

We recall the result from [27] (Theorem I.2.1, p. 195):

Theorem 2. *Let $\dot{x} = A(t)x$ be a linear system in $\mathbb{R}^n$, uniformly stable for $t \geq T$. Let $F : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous map such that*

$$\forall \epsilon > 0,\ \exists \eta > 0,\ ||x|| \leq \eta \Rightarrow ||F(t,x)|| \leq \epsilon ||x||,\ \forall t \geq T. \qquad \text{(A.1)}$$

Then, the equilibrium 0 *of the system $\dot{x} = A(t)x + F(t,x)$ is uniformly asymptotically stable for $t \geq T$.*

Appendix B. The observability analysis

For technicalities, we consider here the following class of dilution rate functions $D(\cdot)$, as considered on the example of Section 4.

Assumption 4. $D(\cdot)$ is piecewise constant with at least one discontinuity point on $[0, T]$.

Proposition 3. *Under Assumptions 1, 3 and 4, the system is observable on the domain $\mathcal{D} \setminus \mathcal{I}$.*

Proof. We can prove the observability of the dynamic system (1) for the output $y = \mu(s, K)x$, by expressing the variables s and x from y and $\dot{y}$. We have

$$\dot{y} = \dot{s}\mu'(s,K)x + \dot{x}\mu(s,K) \qquad \text{(B.1)}$$

and then

$$\begin{aligned} \dot{y} = &-k\mu'(s,K)\mu(s,K)x^2 \\ &+ \left[D(t)(s_{in} - s)\mu'(s,K) + (\mu(s,K) - D(t))\mu(s,K)\right] x. \end{aligned} \qquad \text{(B.2)}$$

The Jacobian of the transformation $(s, x) \to (y, \dot{y})$ is

$$Jac(t,s,x) = \begin{bmatrix} \frac{\partial y}{\partial s}(t,s,x) & \frac{\partial y}{\partial x}(t,s,x) \\ \\ \frac{\partial \dot{y}}{\partial s}(t,s,x) & \frac{\partial \dot{y}}{\partial x}(t,s,x) \end{bmatrix} \qquad \text{(B.3)}$$

where

$$\frac{\partial y}{\partial s} = \mu'(s,K)x, \tag{B.4}$$

$$\frac{\partial y}{\partial x} = \mu(s,K), \tag{B.5}$$

$$\begin{aligned}\frac{\partial \dot{y}}{\partial s} &= -k[\mu''(s,K)\mu(s,K) + \mu'(s,K)^2]x^2 \\ &\quad + [2\mu(s,K)\mu'(s,K) - 2D(t)\mu'(s,K) + D(t)(s_{in} - s)\mu''(s,K)]\,, \end{aligned}\tag{B.6}$$

$$\begin{aligned}\frac{\partial \dot{y}}{\partial x} &= -2k\mu'(s,K)\mu(s,K)x \\ &\quad + [D(t)(s_{in} - s)\mu'(s,K) + (\mu(s,K) - D(t))\mu(s,K)]. \end{aligned}\tag{B.7}$$

For a given trajectory in $\mathcal{D} \setminus \mathcal{I}$, let us consider

$$J(t) = det(Jac(t, s(t), x(t))). \tag{B.8}$$

One has

$$\begin{aligned} J(t) \quad = \quad & D(t)\Big[(\mu(s(t),K)\mu''(s(t),K) - \mu'(s(t),K)^2)(s_{in} - s(t)) \\ & -\mu(s(t),K)\mu'(s(t),K)\Big]x(t) + \mu(s(t),K)\Big[kx(\mu'(s(t),K)^2 \\ & -\mu(s(t))\mu''(s(t),K)) + \mu(s(t),K)\mu'(s(t),K)\Big]x(t). \end{aligned}\tag{B.9}$$

To prove that the system is observable on $[0,T]$, it is enough to show that there exists $t \in [0,T]$ such that $J(t) \neq 0$. Suppose on the contrary that J is identically null on the interval $[0,T]$. Note first that D is a positive function under Assumption 1, and that all terms in J are continuous with respect to t except D. Let t^* be a discontinuity point of $D(\cdot)$, and write $\lim\limits_{t \to t_\star^+} J(t) = \lim\limits_{t \to t_\star^-} J(t) = 0$ with $s^* = s(t^*)$ and $x^* = x(t^*)$. This gives the conditions

$$x^* \left[(\mu(s^*,K)\mu''(s^*) - \mu'(s^*,K)^2)(s_{in} - s^*) - \mu(s^*,K)\mu'(s^*,K)\right] = 0, \tag{B.10}$$

$$x^*\mu(s^*,K) \left[kx^*(\mu'(s^*,K)^2 - \mu(s^*,K)\mu''(s^*,K)) + \mu(s^*,K)\mu'(s^*,K)\right] = 0. \tag{B.11}$$

As $x^* \neq 0$ and $\mu(s^*, K) \neq 0$, we get

$$(\mu(s^*, K)\mu''(s^*, K) - \mu'(s^*, K)^2)(s_{in} - s^*) - \mu(s^*, K)\mu'(s^*, K) = 0, \quad \text{(B.12)}$$

$$kx^*(\mu'(s^*, K)^2 - \mu(s^*, K)\mu''(s^*, K)) + \mu(s^*, K)\mu'(s^*, K) = 0. \quad \text{(B.13)}$$

Adding these two last equations raises the condition

$$(\mu(s^*, K)\mu''(s^*, K) - \mu'(s^*, K)^2)(s_{in} - s^* - kx^*) = 0. \quad \text{(B.14)}$$

Outside the set $\mathcal{I}$, one deduces from this last equality that the condition

$$\mu(s^*, K)\mu''(s^*, K) - \mu'(s^*, K)^2 = 0 \quad \text{(B.15)}$$

has to be fulfilled, and thus a contradiction as μ is assumed to be concave and increasing with respect to s. This ends the proof.

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