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NOTE ON THE APPROXIMATION OF THE CONDITIONAL INTENSITY OF NON-STATIONARY CLUSTER POINT PROCESSES

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ABSTRACT

In this note we consider non-stationary cluster point processes and we derive their local intensity, i.e. the intensity of the process given the locations of one or more events of the process. We then provide some approximations of this local intensity.

Keywords: conditional intensity, Neyman-Scott process, point process.

INTRODUCTION

The problem of conditioning is an old problem in the point process theory. The conditional distribution of the spatial point process Φ given a realisation of some events of Φ was introduced by Palm (1943) for stationary point processes on the real line and recently summed up in Cœurjolly *et al.* (2017) in the general case. Usually, authors call conditional intensity $\lambda(x_o|\Phi)$ the intensity knowing that the point process is observed everywhere else but x_o. Here we consider locally finite point processes Φ defined in a compact set $S \subset \mathbb{R}^2$ and we focus on a different conditioning: $\lambda(x_o|\Phi_W)$, which is the intensity given the point process is observed in $W \subset S$, but not in $S \backslash W$. Gabriel *et al.* (2017; 2022) refer to it as the spatial *local intensity* of Φ, that they define by the limit of $\mathbb{E}[\Phi(\mathrm{d}x_o)|\Phi_W]/|\mathrm{d}x_o|$ as $|\mathrm{d}x_o| \to 0$, where $|\mathrm{d}x_o|$ denotes the area of $\mathrm{d}x_o$. The local intensity is tractable for very few processes. Gabriel *et al.* (2017) and Gabriel *et al.* (2022) define a "model-free" predictor of $\lambda(x_o|\Phi_W)$ for stationary and non-stationary processes, in sense that it is only related to the first and second-order moments of the point process. In this note, we derive the local intensity of non-stationary cluster point processes and provide some approximations for practical applications.

LOCAL INTENSITY OF NON-STATIONARY CLUSTER POINT PROCESSES

Cluster point processes, developed by Neyman *et al.* (1958), are formed by a simple procedure, with a homogeneous Poisson process Ψ with intensity κ generating *parent* points at a first step and a random pattern of *offspring* points around each parent point at a second step. The number of offspring points has a Poisson distribution with mean μ and the offspring points are independently and identically distributed with a bounded support kernel k depending on the distance from offspring to parent. The cluster point process $\tilde{\Phi}$ is the set of offspring points, regardless their parentage. This process is stationary with intensity $\tilde{\lambda} = \kappa\mu$.

We focus on the $p(x)-$thinned process of $\tilde{\Phi}$, where $p(x)$ is a deterministic function on $\mathbb{R}^2$ with $0 \leq p(x) \leq 1$. If the point x belongs to $\tilde{\Phi}$, it is deleted with probability $1 - p(x)$ and again its deletion is independent of locations and possible deletions of any other points. Let Φ be the $p(x)-$thinned process. The process Φ is second-order intensity reweighted stationary with $\lambda(x) = \kappa\mu p(x)$ (see Chiu *et al.* (2013)).

Here we want to know the local intensity of $\Phi_{S\backslash W}$ given Φ_W. We denote ∂W the border of the observation window W, with width defined by the range of the dispersion kernel k, say r. In other words, $\partial W = W_{\oplus r} \backslash W$. Fig. 1 illustrates the different steps of the generating procedure.

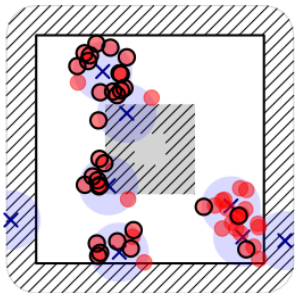


Fig. 1: Generating procedure of the $p(x)$-thinned cluster process. Parent points (Ψ =blue crosses) are generated in the union of the observation window W (white area) and ∂W (hatched area). Offspring ($\tilde{\Phi}$ =red dots) are generated within the circle around the parent points (blue area). The final process (Φ =black dots) is obtained by $p(x)$-thinning. The window of interest S is delineated in black. The prediction window is represented by the grey shaded central square.

Proposition 1. *For $x_o \in S \backslash W$, the local intensity of a $p(x)$-thinned cluster process is:*

$$\lambda(x_o|\Phi_W) = \int_{W\cup\partial W} \mu p(x_o) k(y-x_o) \rho(y|\Phi_W) dy + \mu\kappa \int_{b(x_o,r)\backslash(W\cup\partial W)} p(x_o) k(y-x_o) dy, \quad (1)$$

where $b(x_o,r)$ denotes the disc of centre x_o and radius r and $\rho(y|\Phi_W)$ is the intensity of parent points in $W\cup\partial W$ given the offspring points in W.

The proof is rather straightforward as (i) for a cluster process we know the intensity given the realisation of parent points, (ii) the parent process is Poisson, (iii) knowing the offspring points in W is not informative on parent points in $b(x_o,r)\backslash(W\cup\partial W)$ and (iv) finally we get

$$\lambda(x_o|\Phi_W) = \int \left[\sum_{y\in\Psi_{W\cup\partial W}} \mu p(x_o) k(y-x_o) + \mu\kappa \int_{b(x_o,r)\backslash(W\cup\partial W)} p(x_o) k(y-x_o) \mathrm{d}y \right] \times \mathrm{d}\mathbb{P}[\Psi_{W\cup\partial W}|\Phi_W],$$

that leads to (1) by Campbell's theorem (Chiu *et al.*, 2013).

Baudin (1983) derived the following formula for $\rho(y|\Phi_W)$ for Neyman-Scott processes:

$$\rho(y|\Phi_W) = \kappa G(1-F(W_y)) + \sum_{j=1}^{2^n-1} \sum_{b\in\mathscr{B}} b(a_j) \prod_{i=1}^{2^n-1} S(\Phi,W,a_i)^{b(a_i)} \times \left[\frac{\kappa G^{|a_j|}(1-F(W_y)) \prod_{\ell=1}^n k(x_\ell - y)^{a_{j\ell}}}{S(\Phi,W,a_j)} \right]^{b(a_j)} \times \left[\sum_{b\in\mathscr{B}} \prod_{i=1}^{2^n-1} S(\Phi,W,a_i)^{b(a_i)} \right]^{-1}, \quad (2)$$

where

- G is the probability generating function of the number of points in a cluster,
- $F(\mathrm{d}x)$ is the probability distribution of offspring points, with density k,
- $\{x_1,\ldots,x_n\} = \Phi_W$,
- $W_y = -y + W$,
- $a_1 = (0,\ldots,0,1)$, $a_2 = (0,\ldots,0,1,0)$, $\ldots$, $a_{2^n-1} = (1,\ldots,1)$: vectors of length n,
- $\mathscr{B}$ is the set of all functions $b : \{a_1,\ldots,a_{2^n-1}\} \to \{0,1\}$ such that $\sum_{i=1}^{2^n-1} b(a_i)a_i = (1,\ldots,1)$, $a_i = (a_{i1},\ldots,a_{in})$, $|a_i| = a_{i1}+\cdots+a_{in}$,
- $S(\Phi,W,a_i) = \kappa \int G^{|a_i|}(1-F(W_y)) \times \prod_{\ell=1}^n k(x_l - y)^{a_{i\ell}} \mathrm{d}y$.

However, this intensity is based on combinations and is just too complicated in practice. van Lieshout *et al.* (1943) interpreted the problem of identifying parent points as a statistical estimation problem with a Bayesian inference based on MCMC methods; see also (lawson, 2002) for similar approaches.

Here we propose to approximate it as follows,

$$\widehat{\rho}(y|\Phi_W) = \frac{c(y)}{\mu p(y)} \sum_{x\in\Phi_W} k(x-y) + \kappa \exp\left(-\mu \int_W p(z) k(y-z) \mathrm{d}z \right), \quad (3)$$

where $c(y)$ ensures that $\mathbb{E}[\rho(y|\Phi_W)] = \kappa$. The heuristic behind this approximation is that it depends on both the offspring observed in W (first term) and the unobserved offspring due to thinning or their proximity to the boundary of W (second term).

VALIDATION PROCEDURE

In this section, we aim at comparing parent points with intensity $\widehat{\rho}(y|\Phi_W)$ and those of the original thinned Matérn cluster process, hereafter we call the latter observed parent points. We thus use complementary statistics to test the interactions (i) between parent points, (ii) between parent and offspring points and (iii) between parent points and the inner (or outer) boundary of W. Let $\Phi(A)$ (resp. $\Psi(B)$) be the number of points of Φ in a Borel set A (resp. of Ψ in B). We denote Ψ_S the observed parent points in S and $\nu(B)$ the Lebesgue measure of B. Then,

(i) Interaction statistic between parent points

Let $\widehat{H}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and $H(d)$ the theoretical one:

$$\begin{aligned}\widehat{H}(d) &= \frac{1}{\Psi(S)}\sum_{y_1,y_2\in\Psi_S}^{\neq}\mathbb{I}_{\{\|y_1-y_2\|\leq d\}},\\ H(d) &= \frac{1}{\int_S\rho(y|\Phi_W)\mathrm{d}y}\int_S\int_{b(z,d)\cap S}\rho(y|\Phi_W)\\ &\quad\times\rho(z|\Phi_W)\mathrm{d}y\mathrm{d}z.\end{aligned}$$

(ii) Interaction statistic between parent and offspring points

Let $\widehat{E}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and observed offspring points Φ_W and $E(d)$ the theoretical one:

$$\begin{aligned}\widehat{E}(d) &= \frac{1}{\Phi(W)}\sum_{x\in\Phi_W}\sum_{y\in\Psi_S}\mathbb{I}_{\{\|x-y\|\leq d\}},\\ E(d) &= \frac{1}{\Phi(W)}\sum_{x\in\Phi_W}\int_{b(z,d)\cap S}\rho(z|\Phi_W)\mathrm{d}z.\end{aligned}$$

(iii) Interaction statistic between parent points and the boundary, denoted b_W, of W

Let $\widehat{B}(d)$ be the empirical cumulative distribution function between observed parent points Ψ_S and the inner boundary ($b_W = W\backslash W_{\ominus r} = B^{inner}$) or outer boundary ($b_W = W_{\oplus r}\backslash W = B^{outer}$) of W, and $B(d)$ the theoretical one:

$$\begin{aligned}\widehat{B}(d) &= \frac{1}{\nu(b_W)}\sum_{y\in\Psi_S}\int_{b_W}\mathbb{I}_{\{\|\ell-y\|\leq d\}}\mathrm{d}\ell\\ &= \frac{1}{\nu(b_W)}\sum_{y\in\Psi_S}\nu(b_W\cap b(y,d)),\\ B(d) &= \frac{1}{\nu(b_W)}\int_{b_W}\int_{b(\ell,d)\cap S}\rho(z|\Phi_W)\mathrm{d}z\mathrm{d}\ell.\end{aligned}$$

We illustrate the results for a thinned Matérn cluster process Φ. For this process, k is the uniform distribution on the disc of radius r and the local intensity is

$$\begin{aligned}\lambda(x_o|\Phi_W) = &\frac{\mu p(x_o)}{\pi r^2}\int_{b(x_o,r)\cap(W\cup\partial W)}\rho(y|\Phi_W)\mathrm{d}y\\ &+\frac{\kappa\mu p(x_o)}{\pi r^2}\nu\left(b(x_o,r)\backslash(W\cup\partial W)\right),\end{aligned}$$

with

$$\begin{aligned}\rho(y|\Phi_W) = &\frac{1}{\mu p(y)\pi r^2}\sum_{x\in\Phi_W}\mathbb{I}_{b(x,r)}(y)\\ &+\kappa\exp\left(-\frac{\mu}{\pi r^2}\int_{b(y,r)\cap W}p(z)\mathrm{d}z\right).\end{aligned}$$

The non-stationary Matérn cluster process Φ depends on four parameters: the thinning probability $p(x)$, the intensity of parents κ, the mean number of points per parent μ and the radius of dispersion of the offspring around the parent points r. Here we fix $\kappa = 50$ and $\mu = 40$ and we consider:

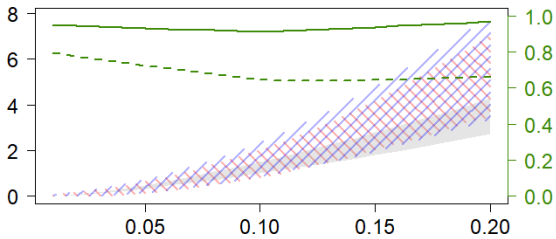
- two thinning probabilities: $p_1(x) = p_1(x_1,x_2) = \alpha_1\mathbb{I}_{\{x_1\leq v\}} + \alpha_2\mathbb{I}_{\{x_1>v\}}$, setting $\alpha_1 = 0.8$, $\alpha_2 = 0.2$ and $v = 0.5$, and $p_2(x) = p_2(x_1,x_2) = 1-x_1$.

- the unit square as study region S. The observation window is $W = S\backslash W_h$, where $W_h = [0.35,0.65]^2$ when using $p_1(x)$ and $W_h = [0.05,0.95]\times[0.36,0.64]$ when using $p_2(x)$.

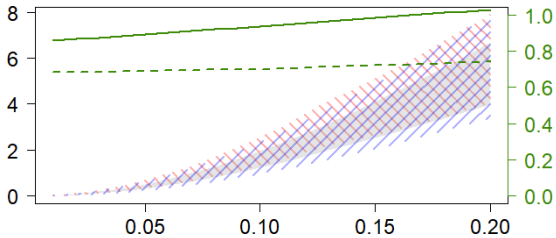
- $r\in\{0.05,0.09,0.13\}$.

For each pair of parameters $(p(x),r)$ we simulate $N = 250$ realisations of the non-stationary Matérn cluster process and compute all the previous interaction statistics, that we denote by $\widehat{H}(d)$, $\widehat{E}(d)$ and $\widehat{B}(d)$. For each of these N realizations, we generate $n = 100$ simulations of parent points from a Poisson process with intensity $\rho(y|\Phi_W)$ and compute the related empirical statistics, that we denote by $\widehat{H}_{sim}(d)$, $\widehat{E}_{sim}(d)$ and $\widehat{B}_{sim}(d)$. Fig. 2 illustrates the 95% envelopes of the empirical statistics computed from the N observed parent

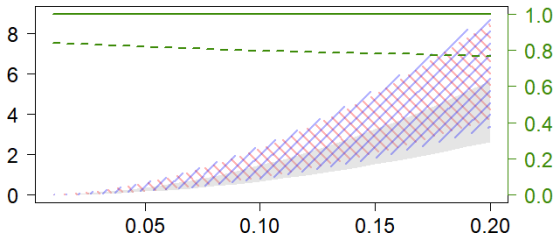
points (red hatching) and from the $N \times n$ simulated parent points (blue hatching). The grey envelopes correspond to the theoretical statistics. In this figure $p(x) = p_1(x)$ and $r = 0.09$.



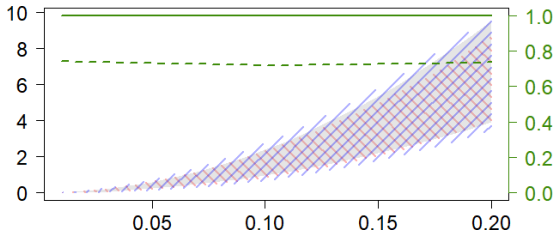
(a) Interactions between parent points.



(b) Interactions between parent and offspring points.



(c) Interactions between parent points and the inner boundary of W.



(d) Interactions between parent points and the outer boundary of W.

Fig. 2: 95% envelopes of the cumulative distribution functions calculated for N realisations of the thinned Matérn cluster process (empirical is in red hatching and theoretical is in grey) and for simulated parent points (blue hatching). The curves represent the cover rates $\tau_1(d)$ (solid) and $\tau_2(d)$ (dashed).

Results for all pairs of parameters are very similar. All overlapping envelopes indicate that the statistics are similar for the observed parent points Ψ_S and for simulated parent points, which further correspond to the theoretical distribution. This is true at any distances and shows that the main characteristics of the approximated distribution of parent points in S given the offspring points in W provided in (3) include those of the original distribution of parent points in S.

For each type of interaction, we computed the global coverage rates between the envelopes obtained from the observed distribution of parent points and the envelopes obtained from the approximated distribution of parent points. E.g., denoting by $\mathscr{E}$ the envelopes, the coverage rates for the interaction statistic between parent points are

$$\tau_1(H) = \nu\left(\mathscr{E}(\widehat{H}) \cap \mathscr{E}(\widehat{H}_{sim})\right) / \nu\left(\mathscr{E}(\widehat{H})\right)$$

and

$$\tau_2(H) = \nu\left(\mathscr{E}(\widehat{H}) \cap \mathscr{E}(\widehat{H}_{sim})\right) / \nu\left(\mathscr{E}(\widehat{H}_{sim})\right).$$

Results for all combination of parameters $(p(x), r)$ are reported in Table 1. The coverage rates are also computed according to the distance and plotted in Fig. 2 ($\tau_1(d)$ in solid line and $\tau_2(d)$ in dashed line, that are one-dimensional versions of $\tau_1(H)$ and $\tau_2(H)$). These results show that for any configuration and interaction range the approximation procedure of the local intensity of parent points in $W \cup \partial W$ given the offspring points in W is conservative.

CONCLUSION

In order to quantify discrepancies between true local intensities and estimated ones (as in Gabriel *et al.* (2022)), we have to both know the local intensity and to get fast computations to browse the space of conditioning realisations. Because existing methods are computationally intensive, not allowing many simulations, instead of simulating the local intensity we proposed to consider an approximating process, whose deviation to the true process can be controlled. We thus propose this approximation if one needs a fast procedure, even if conservative.

Table 1: Coverage rate between the envelopes of the interaction statistics computed from the observed parent points and from the simulated parent points.

r	0.05	0.09	0.13
$p_1(x)$			
$\tau_1(H)$	88.04	93.48	98.73
$\tau_2(H)$	53.66	64.82	69.81
$\tau_1(E)$	97.30	97.40	99.91
$\tau_2(E)$	67.75	71.43	74.68
$\tau_1(B^{inner})$	100.00	99.86	100.00
$\tau_2(B^{inner})$	87.75	78.21	81.69
$\tau_1(B^{outer})$	100.00	100.00	100.00
$\tau_2(B^{outer})$	81.63	72.37	74.30
$p_2(x)$			
$\tau_1(H)$	90.07	96.83	99.37
$\tau_2(H)$	58.31	68.22	77.97
$\tau_1(E)$	97.72	96.24	90.65
$\tau_2(E)$	70.11	69.63	79.14
$\tau_1(B^{inner})$	100.00	100.00	100.00
$\tau_2(B^{inner})$	84.36	88.02	85.92
$\tau_1(B^{outer})$	100.00	100.00	98.57
$\tau_2(B^{outer})$	78.94	70.43	69.06

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