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STRONG CONSISTENCY OF THE LOCAL LINEAR RELATIVE REGRESSION ESTIMATOR FOR CENSORED DATA

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Abstract. In this paper, we combine the local linear approach to the relative error regression estimation method to build a new estimator of the regression operator when the response variable is subject to random right censoring. We establish the uniform almost sure consistency with rate over a compact set of the proposed estimator. Numerical studies, firstly on simulated data, then on a real data set concerning the death times of kidney transplant patients, were conducted. These practical studies clearly show the superiority of the new estimator compared to competitive estimators.

Keywords: censored data, local linear approach, relative error, regression function, uniform almost sure convergence.

Mathematics Subject Classification: 62N01, 62N02, 62G08, 62G35, 62P10.

1. INTRODUCTION

To explore the relationship between two random variables (rv), regression models have appeared as a common and flexible tool in various disciplines, such as biology, medicine, economics, insurance and others. Consider a random vector (X, T) taking values in $\mathbb{R} \times \mathbb{R}_\star^+$ where T is the interest rv with unknown distribution function (d.f.) F and X is an explanatory variable with a density function $f(\cdot)$. Hence, let us consider the following nonparametric regression model:

$$T = m(X) + \varepsilon,$$

where $m(\cdot)$ is an unknown regression operator and ε is the unobservable error term which is a rv with $\mathbb{E}[\varepsilon] = 0$, a finite second moment and independent of X. The regression function appears as a quantity that contains all the information about the dependence structure. Usually, to estimate the regression function, we minimize the following loss function:

$$m_{CR}(x) := \arg\min_{m\in\mathbb{R}} \mathbb{E}\left[(T - m(x))^2 \,|X = x\right], \tag{1.1}$$

for a fixed x. However, it is well known that the last loss function is inefficient in the presence of outliers in data, which is a common case in practical situations.

The aim of the present paper is to propose a new approach which reduces these drawbacks. Relative error regression (RER) estimation has been recently used in regression analysis as an alternative to the restrictions imposed by the classical regression approach. So, we consider the following minimizing problem of the mean squared relative error loss function, that is, for $T > 0$

$$m_{RER}(x) := \arg\min_{m\in\mathbb{R}} \mathbb{E}\left[\left(\frac{(T - m(x))}{T}\right)^2 \Big|X = x\right]. \tag{1.2}$$

This criterion has been widely studied for parametric models, without pretending to exhaustivity, we refer to [6] for a discussion about the previous works and [18] for a real example on the electricity consumption. When the first two conditional inverse moments of T given X are finite, [26] showed that the solution of (1.2) can be written by the following ratio:

$$m_{RER}(x) = \frac{\mathbb{E}[T^{-1}|X = x]}{\mathbb{E}[T^{-2}|X = x]} =: \frac{m_1(x)}{m_2(x)} =: \frac{r_1(x)}{r_2(x)}. \tag{1.3}$$

Here $m_\ell(x) = r_\ell(x)/f(x)$ and $r_\ell(x) = \displaystyle\int t^{-\ell} f_{X,T}(x,t)dt$, for $\ell = 1, 2$, with $f_{X,T}(\cdot,\cdot)$ and $f(\cdot)$ are the joint and marginal density of the couple (X, T) and X respectively. Among the first contributors to the RER method in the nonparametric framework, we refer to [27] and [28] with applications in finance and image analysis respectively. In the recent literature, the RER method has received an increasing interest, we can quote [5] where it was considered the regression function estimation for a functional explanatory variable while [1] have looked into the case where the data are from a strictly stationary spacial process. [30] constructed an estimator based on a deconvolution problem. [19] established the consistency and the asymptotic normality of the regression function based on a least product relative error.

It is well-known that the local linear (LL) method has several advantages over the classical kernel smoothing. Especially, one of its known advantages is the reduction of side effects. More details on the importance of the LL approach can be found in [12, 13] and [14]. For recent works on the estimation of the regression function using the LL method, we refer to [20] for independent complete data, [10, 11] for dependent censored data and [3] for independent right censored data.

All these works concern the complete data except the last two papers. However, in many situations, the data can not be observed completely. Important examples are the survival time of patients or the unemployment time and many others in different fields. A frequent problem in survival analysis is right-censoring, which may be due to different causes: the loss of some subjects under study, and the end of the follow-up period. Examples of situations where this kind of data occur can be found in [23].

Inspired by all the articles above, our work in this paper aims to contribute to the research on nonparametric models by combining the two methods RER and LL when the data are censored. We extend the work of [20] to the censoring framework by stating a strong result. We point out that in the last paper, only a pointwise of the bias and variance terms have been investigated. In our paper, we build the local linear relative error regression (LLRER) estimator and establish its uniform almost sure consistency with rate over a compact set under appropriate conditions. Simulation experiments emphasize that the LLRER, is highly competitive to the existing estimators for regression function. To the best of our knowledge, this problem is open up to now and there is no analogous result.

This paper is organized as follows. The general idea of the LL fit of the mean squared relative error regression function in the censoring framework is described in Section 2. Assumptions and theoretical results are given in Section 3 and some simulation results that illustrate the performance of the proposed procedure are given in Section 4. Finally, Section 5 is devoted to auxiliary results and technical details.

2. THE MODEL

In this section, our interest is on the estimation of the relative regression operator $m_{RER}(\cdot)$ given by (1.3). According to the right-censoring mechanism, instead of observing T we only observe (Y, δ) where $Y = \min(T, C)$ and $\delta = \mathbb{1}_{\{T \leq C\}}$, here $\mathbb{1}(\cdot)$ is the indicator function. The rv C represent the censoring time, which is independent of T and with d.f. G. The observed data becomes (X, Y, δ). From now on, we will always make the following assumption:

$$(X, T) \text{ and } C \text{ are independent.} \tag{2.1}$$

This assumption is required to make the estimation of the censoring distribution easier. However, it is reasonable only when the censoring is not associated with the characteristic of the individuals under study. Let $\{(X_i, Y_i, \delta_i),\ i = 1, \ldots, n\}$ be n independent and identically distributed (iid) vectors as (X, Y, δ). Our main aim is to estimate the RER function defined in (1.3) using the LL fit. The extension of nonparametric LL procedures to the censored framework requires to replace the unavailable data by a suitable construction of the observed data given by

$$\widetilde{T}_i^{-\ell} = \frac{\delta_i Y_i^{-\ell}}{\overline{G}(Y_i)}, \quad \text{for } \ell = 1, 2 \text{ and } 1 \leq i \leq n, \tag{2.2}$$

where $\overline{G}(\cdot) = 1 - G(\cdot)$ denotes the survival function of the rv C. The new variables defined in (2.2) are called "synthetic data" and permit to consider the effect of censoring in the construction of our estimator, for more details, we refer to [4] and [24].

In this spirit, based on this construction of the data, using the conditional expectation property and under the assumption (2.1), for $\ell = 1, 2$, we have

$$\begin{aligned}
\mathbb{E}\left[\widetilde{T}_1^{-\ell}|X_1 = x\right] &= \mathbb{E}\left[\frac{\delta_1 Y_1^{-\ell}}{\overline{G}(Y_1)}\Big|X_1 = x\right] \\
&= \mathbb{E}\left[\frac{T_1^{-\ell}}{\overline{G}(T_1)}\mathbb{E}\left[\mathbb{1}_{\{T_1 \le C_1\}}|T_1\right]\Big|X_1 = x\right] \\
&= \mathbb{E}\left[T_1^{-\ell}|X_1 = x\right] = m_\ell(x).
\end{aligned}$$

Modeling by the LL method, we assume that the first derivative of $m(x)$ at the point x exists and is continuous, so that $m(X)$ can be approximated by a linear function, that is, $m(X) \approx m(x) + m'(x)(X - x) =: \beta_1 + \beta_2(X - x)$. Then, (1.2) is the solution of the following optimization problem:

$$\arg \min_{(\beta_1,\beta_2)\in\mathbb{R}^2} \left\{\sum_{i=1}^n \widetilde{T}_i^{-2} \left(Y_i - \beta_1 - \beta_2(X_i - x)\right)^2 K_h(X_i - x)\right\} \tag{2.3}$$

where $K_h(\cdot) := K\left(\frac{\cdot}{h}\right)$ is a kernel density function and $h := h_n$ denotes a smoothing parameter converging to 0 with an increasing sample size. By elementary calculus, the solution of the least relative squares problem (2.3) yields

$$\widetilde{\beta}_1 =: \widetilde{m}_{LLRER}(x) =: \frac{\widetilde{r}_1(x)}{\widetilde{r}_2(x)},$$

with

$$\widetilde{r}_\ell(x) = \frac{1}{(nh)^2}\sum_{i,j=1}^n \widetilde{w}_{i,j}^\ell(x), \tag{2.4}$$

where

$$\widetilde{w}_{i,j}^\ell(x) = (X_i - x)\left((X_i - x) - (X_j - x)\right) K_h(X_i - x)K_h(X_j - x)\widetilde{T}_i^{-2}\widetilde{T}_j^{-\ell}, \tag{2.5}$$

for $\ell = 1, 2$. Of course in data analysis, the survival function $\overline{G}(\cdot)$ is unknown and needs to be estimated. This can be done via [21] (KM) as an estimator of $\overline{G}(\cdot)$ defined by

$$\overline{G}_n(t) = \begin{cases} \prod_{i=1}^n \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathbb{1}\left\{Y_{(i)} \le t\right\}} & \text{if} \quad t < Y_{(n)}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.6}$$

where $Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}$ are the order statistics of the Y_i and $\delta_{(i)}$ is the corresponding uncensored indicator. The properties of $\overline{G}_n(t)$ have been studied by many authors. So, (2.2) becomes

$$\widehat{T}_i^{-\ell} = \frac{\delta_i Y_i^{-\ell}}{\overline{G}_n(Y_i)}, \quad \text{for } \ell = 1, 2 \text{ and } 1 \le i \le n. \tag{2.7}$$

Replacing (2.7) in (2.5) gives us a feasible LLRER estimate expressed as

$$\widehat{m}_{LLRER}(x) =: \frac{\widehat{r}_1(x)}{\widehat{r}_2(x)}, \tag{2.8}$$

with

$$\widehat{r}_\ell(x) = \frac{1}{(nh)^2} \sum_{i,j=1}^{n} \widehat{w}_{i,j}^{\ell}(x) \tag{2.9}$$

where

$$\widehat{w}_{i,j}^{\ell}(x) = (X_i - x)\left((X_i - x) - (X_j - x)\right) K_h(X_i - x) K_h(X_j - x) \widehat{T}_i^{-2} \widehat{T}_j^{-\ell}, \tag{2.10}$$

for $\ell = 1, 2$. For technical reasons, we will give a second form to the LLRER estimator which will be used in the proofs. So, (2.9) becomes for $\ell = 1, 2$:

$$\widehat{r}_\ell(x) = \widehat{S}_{2,2}(x)\widehat{S}_{\ell,0}(x) - \widehat{S}_{2,1}(x)\widehat{S}_{\ell,1}(x) \tag{2.11}$$

where for $\gamma = 0, \ell$

$$\widehat{S}_{\ell,\gamma}(x) = \frac{1}{nh} \sum_{i=1}^{n} \widehat{T}_i^{-\ell} (X_i - x)^{\gamma} K_h(X_i - x). \tag{2.12}$$

Then, replacing (2.11) in (2.8) gives us another way to write the LLRER estimator. We want to mention that the pseudo estimator can also be written in this second form, just by replacing (2.12) by

$$\widetilde{S}_{\ell,\gamma}(x) = \frac{1}{nh} \sum_{i=1}^{n} \widetilde{T}_i^{-\ell} (X_i - x)^{\gamma} K_h(X_i - x). \tag{2.13}$$

In what follows, we will adopt the convention $0/0 = 0$ in such a case that if, for example, $\widehat{r}_1(\cdot) = 0$ and $\widehat{r}_2(\cdot) = 0$, the ratio $\widehat{r}_1(\cdot)/\widehat{r}_2(\cdot)$ in (2.8) will be interpreted as zero.

Remark 2.1.

(1) If $\beta_2 = 0$ in (2.3), we come back to the RER function estimator of (1.3) defined in [22].

(2) In the case of complete data, i.e. we replace $\widehat{T}^{-\ell}$ for $\ell = 1, 2$ by the variable of interest T in (2.10), the resulting estimator (2.8) of (1.3) has been defined in [20].

(3) A crucial point in censored regression is to extend the identifiability assumption on the independence of T and C defined in (2.1) to the case where the explanatory variables are present. In this spirit, one may impose that T and C are independent conditionally to X. Then, the synthetic data given in (2.7) becomes

$$\widehat{T}_i = \frac{\delta_i Y_i^{-\ell}}{\overline{G}_n(Y_i|X_i)}, \qquad \text{for} \qquad \ell = 1, 2 \quad \text{and} \quad 1 \leq i \leq n \tag{2.14}$$

where $\overline{G}_n(Y_i|X_i)$ is Beran's estimator of the conditional survival function of the rv C given X, for more details see [2]. The property of this estimator has been studied by [7, 8]. Replacing (2.14) in (2.10) we obtain a feasible estimate (2.8) of (1.3).

Throughout this paper, we denote by

$$\tau_F := \sup\left\{x : \overline{F}(x) > 0\right\} \quad \text{and} \quad \tau_G := \sup\left\{x : \overline{G}(x) > 0\right\}$$

be the right support endpoints of $\overline{F}$ and $\overline{G}$, respectively. Let τ such that $0 < \tau < \tau_F < \infty$, $\overline{G}(\tau) > 0$ which implies $\tau < \tau_F \leq \tau_G$, which were also assumed in [17].

3. ASSUMPTIONS AND MAIN RESULTS

Let $\mathcal{C}$ referring to a compact set of $\mathcal{C}_0$, where $\mathcal{C}_0 = \{x \in \mathbb{R} : f(x) > 0\}$ is an open set. Let C be any generic positive constant whose value is allowed to change. In addition, as T is a lifetime, it can be supposed to be bounded. In the following, we present all required technical conditions for the asymptotic result.

(H1) The bandwidth h satisfies $\lim\limits_{n\to\infty} \dfrac{nh^2}{\log n} = +\infty$ and $\lim\limits_{n\to\infty} \dfrac{nh^6}{\log n} = 0$.

(H2) The kernel $K(\cdot)$ is symmetric and non-negative function. Furthermore, for $\gamma = 2, 3$,
  (i) $\displaystyle\int |t|^\gamma K(t)dt < \infty$,
  (ii) $\displaystyle\int |t|^\gamma K^2(t)dt < \infty$.

(H3) The density function $f(\cdot)$ is continuously differentiable.

(H4) The functions $\phi_\ell(\cdot) = \displaystyle\int \frac{t^{-\ell}}{\overline{G}(t)} f_{X,T}(\cdot, t)dt$, for $\ell = 2, 3, 4$, are continuously differentiable.

(H5) The functions $r_\ell(\cdot)$, for $\ell = 1, 2$, are continuously differentiable.

(H6) The functions $m_\ell(\cdot)$, for $\ell = 1, 2$, are twice continuously differentiable.

Comments on the assumptions: The assumption (H1) concerns the bandwidth and is very common in nonparametric estimation. The assumption (H2) regards the Kernel K and are needed to obtain the rate of convergence of the bias (of order two) and the variance. Analogous assumptions on the kernel has been also made by [12]. The assumption (H3) deals with the density function $f(\cdot)$. The assumptions (H4), (H5) and (H6) are regularity conditions for the functions $\phi_\ell(\cdot)$, $r_\ell(\cdot)$ and $m_\ell(\cdot)$, for $\ell = 1, 2$, respectively.

Theorem 3.1. *Under assumptions* (H1)–(H6)*, for n large enough, we have*

$$\sup_{x\in\mathcal{C}} |\widehat{m}_{LLRER}(x) - m_{RER}(x)| = O\left(h^2\right) + O_{a.s.}\left(\sqrt{\frac{\log n}{nh^2}}\right).$$

The proof of Theorem 3.1 is made up on the following decomposition:

$$\begin{aligned}\widehat{m}_{LLRER}(x) - m_{RER}(x) = \frac{1}{\widehat{r}_2(x)}\Big\{&\widehat{r}_1(x) - \widetilde{r}_1(x) + \widetilde{r}_1(x) - \mathbb{E}[\widetilde{r}_1(x)] + \mathbb{E}[\widetilde{r}_1(x)]\\ &- m_1(x)m_2(x) + m_{RER}(x)\{m_2^2(x) - \mathbb{E}[\widetilde{r}_2(x)]\\ &+ \mathbb{E}[\widetilde{r}_2(x)] - \widetilde{r}_2(x) + \widetilde{r}_2(x) - \widehat{r}_2(x)\}\Big\}.\end{aligned}$$

Remark that by assumption (H6), there exists $\eta > 0$ such that $\inf_{x\in\mathcal{C}} |m_2(x)| = \eta$. Then, by a triangle inequality, we have

$$\begin{aligned}&\sup_{x\in\mathcal{C}} \left|\widehat{m}_{LLRER}(x) - m_{RER}(x)\right|\\ &\leq \frac{1}{\eta^2 - \sup_{x\in\mathcal{C}} \left|\widehat{r}_2(x) - m_2^2(x)\right|} \left\{\sup_{x\in\mathcal{C}} \left|\widehat{r}_1(x) - \widetilde{r}_1(x)\right|\right.\\ &\quad + \sup_{x\in\mathcal{C}} \left|\widetilde{r}_1(x) - \mathbb{E}[\widetilde{r}_1(x)]\right| + \sup_{x\in\mathcal{C}} \left|\mathbb{E}[\widetilde{r}_1(x)] - m_1(x)m_2(x)\right|\\ &\quad + \sup_{x\in\mathcal{C}} \left|m_{RER}(x)\right| \left\{\sup_{x\in\mathcal{C}} \left|\mathbb{E}[\widetilde{r}_2(x)] - m_2^2(x)\right|\right.\\ &\quad \left.\left. + \sup_{x\in\mathcal{C}} \left|\widetilde{r}_2(x) - \mathbb{E}[\widetilde{r}_2(x)]\right| + \sup_{x\in\mathcal{C}} \left|\widehat{r}_2(x) - \widetilde{r}_2(x)\right|\right\}\right\}.\end{aligned}$$

The proof will be achieved with the following propositions.

Proposition 3.2. *Under assumptions* (H1), (H2), (H4) *and* (H5), *for* $\ell = 1, 2$ *and* n *large enough, we have*

$$\sup_{x\in\mathcal{C}} |\widehat{r}_\ell(x) - \widetilde{r}_\ell(x)| = O_{a.s.}\left(\sqrt{\frac{\log\log n}{n}}\right).$$

Proposition 3.3. *Under assumptions* (H1), (H2), (H4) *and* (H5), *for* $\ell = 1, 2$ *and* n *large enough, we have*

$$\sup_{x\in\mathcal{C}} \left|\widetilde{r}_\ell(x) - \mathbb{E}[\widetilde{r}_\ell(x)]\right| = O_{a.s.}\left(\sqrt{\frac{\log n}{nh^2}}\right).$$

Proposition 3.4. *Under assumptions* (H2)(i), (H3) *and* (H6), *for* $\ell = 1, 2$ *and* n *large enough, we have*

$$\sup_{x\in\mathcal{C}} |\mathbb{E}[\widetilde{r}_\ell(x)] - m_\ell(x)m_2(x)| = O\left(h^2\right).$$

Remark 3.5. In the simulation part, we will compare the new approach to the estimator of the classical regression (CR) function given by (1.1) using the LL approach (LLR for "local linear regression") defined in [3] by

$$\widehat{m}_{LLR}(x) := \frac{\sum_{i,j=1}^{n} v_{i,j}(x)\widehat{T}_j}{\sum_{i,j=1}^{n} v_{i,j}(x)}, \tag{3.1}$$

where

$$v_{i,j}(x) = (X_i - x)\left((X_i - x) - (X_j - x)\right) K_h\left(X_i - x\right) K_h\left(X_j - x\right),$$

and $\widehat{T} = \widehat{T}^{-\ell}$ for $\ell = -1$.

The CR function estimator using the kernel method has been defined in [17] by

$$\widehat{m}_{CR}(x) = \frac{\widehat{S}_{-1,0}(x)}{\widehat{f}(x)}, \tag{3.2}$$

where for $\gamma = 0$ and $\ell = -1$, $\widehat{S}_{\ell,\gamma}(\cdot)$ is given in (2.12) and $\widehat{f}(\cdot)$ is the marginal density function estimator defined by [29]. In the case of complete data, (3.2) has been defined by [25] and [31]. In addition, we can derive the RER function estimator using the kernel method defined in [22] by

$$\widehat{m}_{RER}(x) = \frac{\widehat{S}_{1,0}(x)}{\widehat{S}_{2,0}(x)}, \tag{3.3}$$

where for $\gamma = 0$ and $\ell = 1, 2$, $\widehat{S}_{\ell,\gamma}(\cdot)$ is given in (2.12).

The optimal bandwidth: The main idea of the bandwidth choice is the cross-validation method, which chooses h by minimizing:

$$\begin{aligned}
CV_{CR}(x) &= \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \widehat{m}_{CR}^{-i}(x)\right)^2, \\
CV_{LLR}(x) &= \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \widehat{m}_{LLR}^{-i}(x)\right)^2, \\
CV_{RER}(x) &= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{Y_i - \widehat{m}_{RER}^{-i}(x)}{Y_i}\right)^2, \\
CV_{LLRER}(x) &= \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{Y_i - \widehat{m}_{LLRER}^{-i}(x)}{Y_i}\right)^2,
\end{aligned} \tag{3.4}$$

where $\widehat{m}_{CR}^{-i}(\cdot)$, $\widehat{m}_{LLR}^{-i}(\cdot)$, $\widehat{m}_{RER}^{-i}(\cdot)$ and $\widehat{m}_{LLRER}^{-i}(\cdot)$ are CR, LLR, RER and LLRER estimators respectively defined in (3.2), (3.1), (3.3) and (2.8) respectively without the i-th observation (X_i, Y_i, δ_i).

4. NUMERICAL STUDY

To evaluate the quality of this method, we perform several simulations of the proposed estimator (2.8) with different levels of censoring and outliers in data. For that, we generate the data as follows:

Inputs: Generate n iid $\{X_i \rightsquigarrow \mathcal{W}(1,1),\ C_i \rightsquigarrow \exp(\lambda)$ and $\varepsilon_i \rightsquigarrow \mathcal{N}(0,0.2)\}$, for $1 \le i \le n$, where λ is a constant that adjusts the censoring percentage (C.P.).

Step 1. Calculate the interest variable according to the two following models:

$$\begin{aligned}\textbf{Model 0 (M0):}\quad & T_i = 2X_i + 1 + \varepsilon_i,\\ \textbf{Model 1 (M1):}\quad & T_i = 3 + \sin(X_i) + \varepsilon_i.\end{aligned}$$

Step 2. Determinate the observed variable $Y_i = \min(T_i, C_i)$ and the corresponding indicator δ_i.

Step 2′: For $1 \le j \le \frac{n}{20}$, we consider $i = 20 \times j$ and multiply Y_i by a multiplicative coefficient (M.C.) that we vary to create the outlier effect.

Step 3. Calculate the KM estimator from (2.6) and compute the synthetic data $\left\{\widehat{T}_i^{-\ell}, 1 \le i \le n\right\}$ for $\ell = 1,2$ from (2.7).

Step 4. Calculate the kernel K as a standard Gaussian function and we select the bandwidth from a sequence of $h \in [0.01, 2]$ by the cross-validation method (see the optimal bandwidth in Section 3).

Ouputs: Compute the LLRER estimator from (2.8) for $x \in [1,4]$ and the optimal bandwidth $h^\star$.

In the following figures, the solid line represent the theoretical curve. Moreover, for each figure, we specify the optimal bandwidth $h^\star$ for all the estimators considered in the study.

4.1. PERFORMANCE OF THE LLRER

Figure 1 represents a test of the effectiveness of the LLRER estimator under the linear model **(M0)**. We plot the true curve together with the LLRER estimator curve. From Figure 1a, we can see that the goodness of fit to the theoretical curve improves as the sample size increases. In Figure 1b, we fix the sample size and vary the C.P. We observe that the LLRER estimator quality is poorly affected by the percentage of censored data. Finally, in order to assess the robustness to outliers of our new estimator, we generate data as in Step 2′. From Figure 1c, we can observe that the quality of fit remains consistent even when the value of M.C. increases. For each curve, we specify the $h^\star$ calculate according to equations (3.4).

a) b) c)

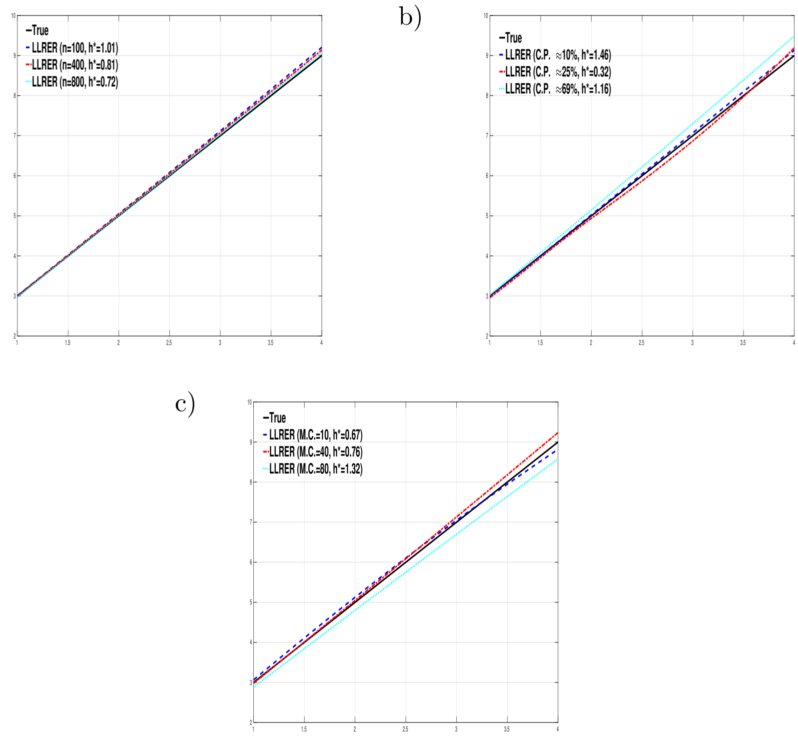

Fig. 1. Performance of the LLRER estimator for **(M0)**: a) Sample size effect for C.P. $\approx$ 25%, b) Censoring effect for $n = 400$, c) Outliers effect for $n = 400$ and C.P. $\approx$ 60%

4.2. NONLINEAR FUNCTIONS

It is important in the nonparametric framework to show that whatever the chosen model, the estimator remains stable. In this subsection, we consider the non linear models **(M1)**. We consider two other non-linear functions: exponential and quadratic models given by $T = 0.8 \exp(X) + \varepsilon$ and $T = 0.9X^2 + \frac{5}{2} + \varepsilon$ respectively. The results are similar to those of **(M1)**. Therefore, we show below only the results of **(M1)**. We evaluate the performance of the LLRER estimator according to the presence of censorship and outliers in data. Figures 2a show as the censoring rate increases, the LLRER curve deviates from the true one and similarly when we increase the M.C. (see Figure 2b). Thus, the quality of the fit is as good as in the linear case **(M0)**.

4.3. COMPARISON TO OTHER KERNEL ESTIMATORS

In this subsection, we compare our estimator to CR, LLR and RER estimators in terms of censoring rate and outliers effect. We note the following remarks. Figure 3 shows an improvement of the LLRER estimator over the CR, LLR and RER estimators near the right tail where the data points are sparse and mostly uncensored.

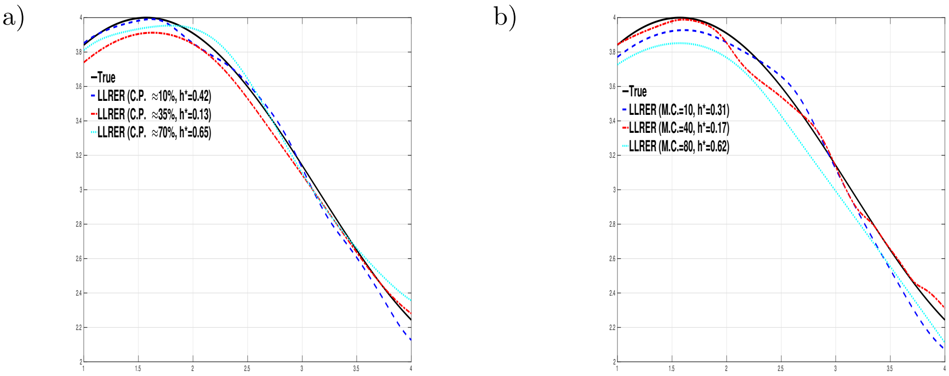

Fig. 2. Performance of the LLRER estimator for **(M1)**: a) censoring effect for $n = 400$, b) outliers effect for $n = 400$ and C.P.≈ 34

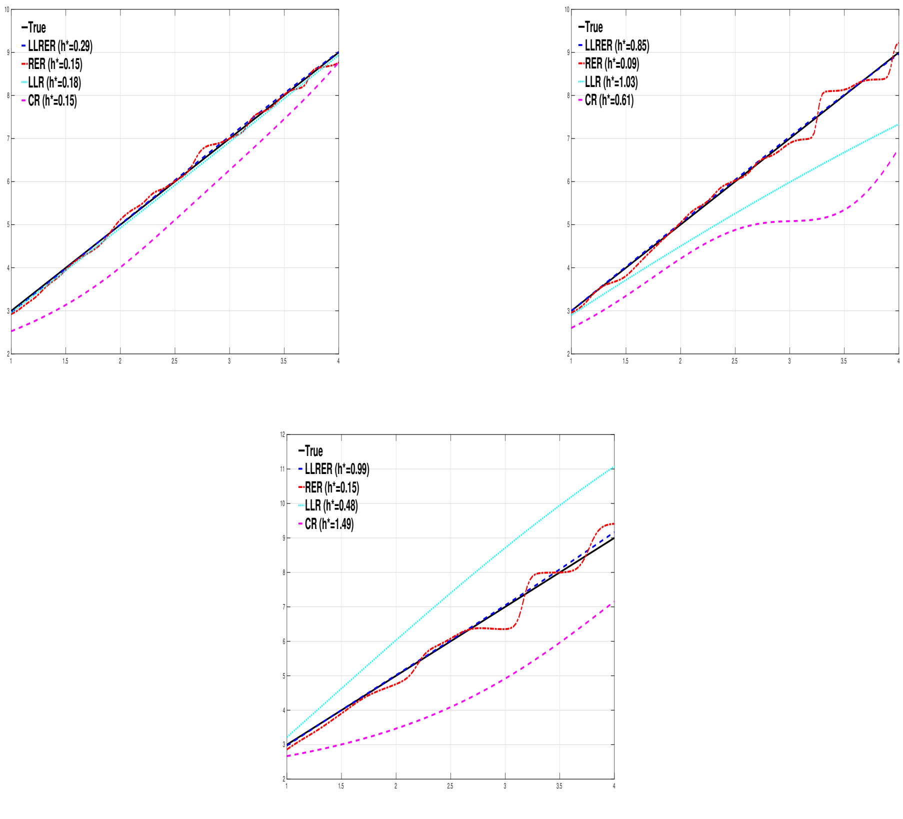

Fig. 3. Comparison of the CR, LLR and RER to LLRER in terms of censoring effect for **(M0)** with $n = 400$ and C.P.$\approx 13,\ 40$ and 69%, respectively

We note the instability of the estimators CR, LLR and RER when the censoring rate increases. We indicate the value of the optimal bandwidth for each estimator calculated from the equations (3.4). Then, the LLRER estimator is much more robust to censoring than the CR, LLR and RER estimators.

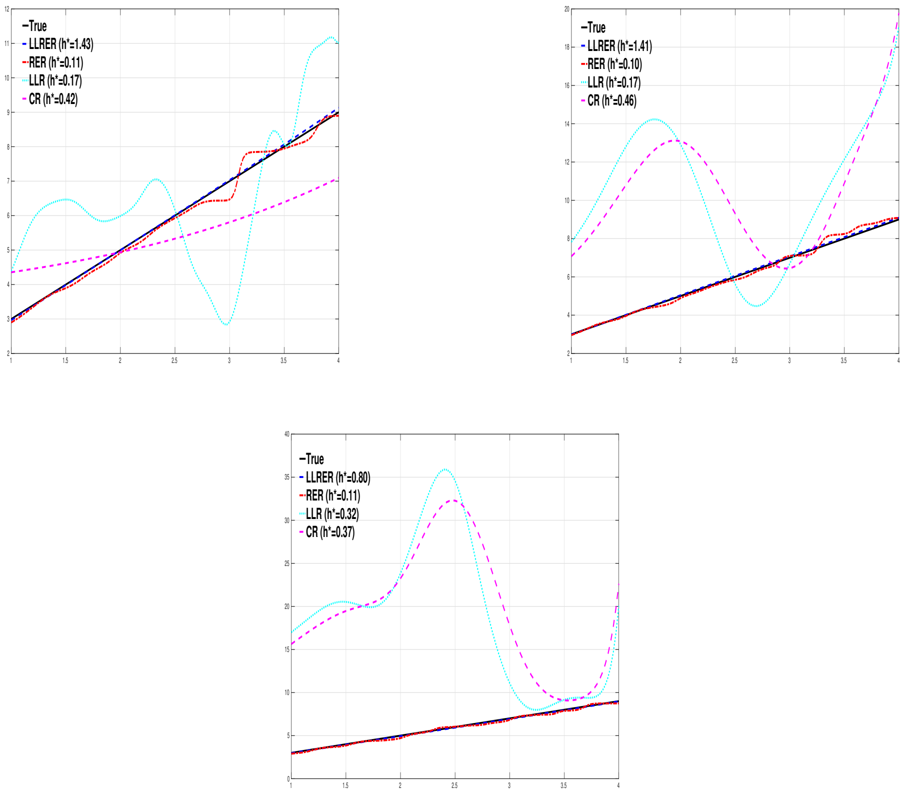

Fig. 4. Comparison of the CR, LLR and RER to LLRER in terms of outliers effect for **(M0)** with $n = 400$, C.P. $\approx 42.5\%$ and M.C. $= 10, 40$ and 80, respectively

Figure 4 is a comparison of estimators when the data contains outliers (generated as in Step $2'$). As expected, in presence of outliers, the LLRER estimator performs better than the CR and LLR estimators. Concerning the comparison between RER and LLRER estimators, we can see that both are resistant to outliers, but the LLRER performs better due to the high censoring rate. To conclude, in the presence of censoring and outliers separately or combined, the new estimator obtained by mixing the LL and RER methods is much more efficient compared to other kernel estimators existing in the literature.

Our purpose is to compare the mean square error (MSE) of the LLRER estimator with the CR, RER and LLR estimators respectively which are defined as:

$$\begin{aligned}
\mathrm{MSE}_{CR} &= \frac{1}{n}\sum_{i=1}^{n}\left(m_{CR}(x_i) - \widehat{m}_{CR}(x_i)\right)^2,\\
\mathrm{MSE}_{LLR} &= \frac{1}{n}\sum_{i=1}^{n}\left(m_{CR}(x_i) - \widehat{m}_{LLR}(x_i)\right)^2,\\
\mathrm{MSE}_{RER} &= \frac{1}{n}\sum_{i=1}^{n}\left(m_{RER}(x_i) - \widehat{m}_{RER}(x_i)\right)^2,\\
\mathrm{MSE}_{LLRER} &= \frac{1}{n}\sum_{i=1}^{n}\left(m_{RER}(x_i) - \widehat{m}_{LLRER}(x_i)\right)^2.
\end{aligned}$$

In Table 1, we can see that the MSEs decrease as the sample size increases compared to the CR, LLR and RER estimators. The proposed method performs globally well even when the C.P. and the M.C. are large. We note that the RER estimator remains stable in the presence of strong outliers and is sensitive in the presence of censoring. Comparing now between CR and LLR estimators. The CR estimator is more resistant to censoring and outliers in data.

Table 1
Comparative table of MSE for **(M0)**

n	C.P.$\approx$ %	M.C.	CR	LLR	RER	LLRER
100	10	10	0.1984	2.8510	0.0247	0.0021
		40	16.8018	23.8393	0.5098	0.0044
		80	339.2484	$2.0200 \cdot 10^4$	0.4770	0.0051
	40	10	0.1506	111.4210	0.0903	0.0041
		40	526.6807	$1.2169 \cdot 10^4$	0.3992	0.0123
		80	$1.3097 \cdot 10^4$	$1.0922 \cdot 10^4$	0.6686	0.0901
	80	10	4.5205	17.0810	1.3934	0.3134
		40	11.5675	36.8406	2.0637	0.5295
		80	$4.4740 \cdot 10^4$	$3.6106 \cdot 10^5$	7.0942	0.6426
800	10	10	0.0757	5.9148	0.0070	0.0052
		40	15.6419	106.4302	0.0140	$1.5495 \cdot 10^{-4}$
		80	128.8892	539.5578	0.0191	$2.4732 \cdot 10^{-4}$
	40	10	0.4938	0.9962	0.0226	$2.6819 \cdot 10^{-5}$
		40	26.2118	320.5188	0.0447	$1.1116 \cdot 10^{-4}$
		80	306.3905	379.6419	0.0678	$4.6621 \cdot 10^{-4}$
	80	10	5.4846	7.5194	0.9469	0.0013
		40	7.8674	11.5200	1.0250	0.0088
		80	$1.5028 \cdot 10^3$	$1.0530 \cdot 10^4$	2.2704	0.0293

4.4. EXPERIMENTAL PREDICTION

In this part, we evaluate the performance of the LLRER predictor and compared its performance to those of the CR, LLR and RER predictors for the same generated data set. For that, we consider $n = 400$ observations generated as in **Step 1** and we consider the model **(M0)**. The n-sample was then randomly split into two subsets: a training sample of size $n^\star = 350$ is used to calculate the estimators and a test sample of size $n - n^\star$ is used to verify the goodness of the predictions. The kernel and the optimal bandwidth are chosen as in **Step 4**. Note that, for the sake of brevity, we restrict ourselves to C.P. $\approx 41\%$.

In Figure 5, predicted values are plotted against true values. Please notice that, we eliminate the censored data from the predicted values.

The LLRER method seems to improve the quality of prediction compared to the CR, LLR and RER estimators. An interesting fact can be seen in Figure 5 which is the small difference between the predicted values obtained by the RER and LLRER estimators. We conclude that the LLRER estimator is more efficient and more precise compared to other kernel predictors.

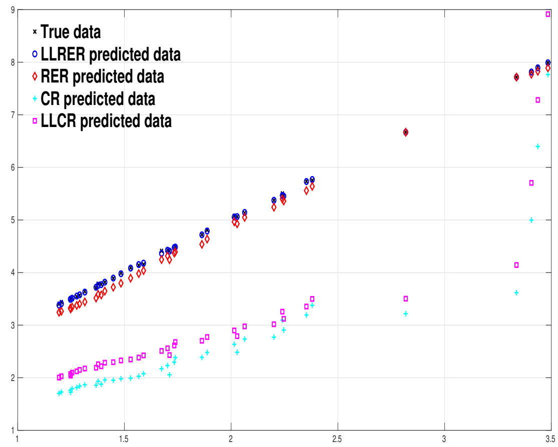

Fig. 5. Performance of the LLRER predictor in comparison to CR, LLR and RER predictors for **(M0)**

4.5. REAL DATA EXAMPLE: DEATH TIMES OF KIDNEY TRANSPLANT PATIENTS

In this part, we analyze a real data set to illustrate the efficiency of the LLRER in presence of censoring data. Then, we perform predictions using our approach which we compare with CR, LLR and RER approaches.

The data consists on the time to death of $n = 863$ kidney transplant patients available on [23] web site. All patients had their transplant performed at *The Ohio State University Transplant Center* during the period 1982–1992. The maximum follow-up time for this study was 9.47 years. Patients were censored if they moved from Columbus (lost to follow-up) or if they were alive on June 30, 1992. The data set provides information on the gender (male/female), age (in years) at the time of the transplantation and it is also known whether a survival time was right censored or not. The general rate of censoring is approximately equal to 84%.

The time variable viewed as most important is time since the transplantation. We consider the link between the survival time (as a response variable) and the age at the moment of the transplantation (as a conditioning variable). A similar procedure as the experimental prediction is now applied to the kidney transplant patients data in order to compare the prediction performances of the LLRER estimator to CR, LLR, RER estimators. Hence, $n^\star = 763$ data points were randomly selected as training data, denoted as $(X_i, Y_i, \delta_i),\ i = 1, \dots, n^\star$ and the data points remaining were treated as testing data, denoted as $(X_i, Y_i, \delta_i),\ i = n^\star + 1, \dots, n$. The bandwidth and the kernel are taken as in **Step 4**. Note that, we eliminate the censoring data from the predicted values (i.e. if for any $n^\star + 1 \leq i \leq n$ the observed variable $Y_i = C_i$, we remove the observation Y_i from the predicted values).

We can observe from Figure 6 that the predictions resulting from our approach are superimposed on the true values even if the level of censoring is very high.

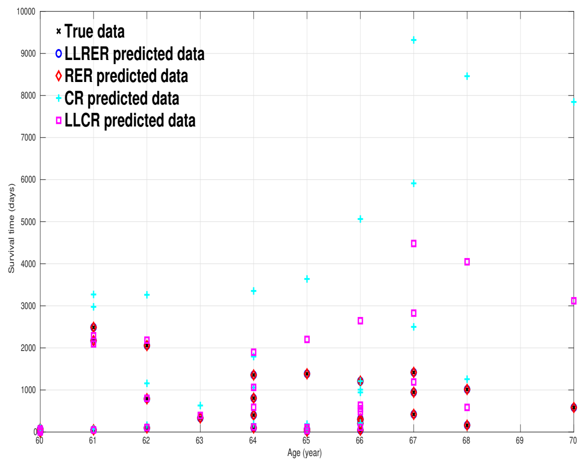

Fig. 6. Performance of the LLRER predictor in comparison to CR, LLR and RER predictors

In addition, we remark from that the CR, LLR predictors are less effective compared to the LLRER predictor. We can notice that there are a few predicted data points that match with the true values and that the majority of the predicted values are very far away. Hence, the proposed predictor shows an improvement over the CR and LLR estimates but we can not see a really difference between RER and LLRER performances in term of prediction.

5. PROOFS AND AUXILIARY RESULTS

Proof. To deal with Proposition 3.2, we use the following decomposition:

$$\begin{aligned}\widehat{r}_\ell(x) - \widetilde{r}_\ell(x) &= \left(\widehat{S}_{2,2}(x)\widehat{S}_{\ell,0}(x) - \widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x)\right) - \left(\widehat{S}_{2,1}(x)\widehat{S}_{\ell,1}(x) - \widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x)\right)\\ &=: \mathcal{B}_{\ell,1}(x) - \mathcal{B}_{\ell,2}(x).\end{aligned}$$

On the one hand, for $\ell = 1, 2$, we get

$$\begin{aligned}\mathcal{B}_{\ell,1}(x) &= \left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x)\right)\left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x)\right)\\ &\quad + \left(\widetilde{S}_{\ell,0}(x) - \mathbb{E}[\widetilde{S}_{\ell,0}(x)]\right)\left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x)\right)\\ &\quad + \mathbb{E}[\widetilde{S}_{\ell,0}(x)]\left(\widehat{S}_{2,2}(x) - \widetilde{S}_{2,2}(x)\right)\\ &\quad + \left(\widetilde{S}_{2,2}(x) - \mathbb{E}[\widetilde{S}_{2,2}(x)]\right)\left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x)\right)\\ &\quad + \mathbb{E}[\widetilde{S}_{2,2}(x)]\left(\widehat{S}_{\ell,0}(x) - \widetilde{S}_{\ell,0}(x)\right).\end{aligned} \tag{5.1}$$

On the other hand, for $\ell = 1, 2$, we get

$$\begin{aligned}\mathcal{B}_{\ell,2}(x) &= \left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x)\right)\left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x)\right)\\ &\quad + \left(\widetilde{S}_{2,1}(x) - \mathbb{E}[\widetilde{S}_{2,1}(x)]\right)\left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x)\right)\\ &\quad + \mathbb{E}[\widetilde{S}_{2,1}(x)]\left(\widehat{S}_{\ell,1}(x) - \widetilde{S}_{\ell,1}(x)\right)\\ &\quad + \left(\widetilde{S}_{\ell,1}(x) - \mathbb{E}[\widetilde{S}_{\ell,1}(x)]\right)\left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x)\right)\\ &\quad + \mathbb{E}[\widetilde{S}_{\ell,1}(x)]\left(\widehat{S}_{2,1}(x) - \widetilde{S}_{2,1}(x)\right).\end{aligned} \tag{5.2}$$

It remains to study each term of the decomposition (5.1) and (5.2). For this, we will state and proof the following three Lemmas 5.1–5.3.

Lemma 5.1. *Under assumptions* (H1), (H2) *and* (H4), *for* $\ell = 1, 2$, $\gamma = 0, \ell$ *and* n *large enough, we have*

$$\sup_{x\in\mathcal{C}}\left|\widetilde{S}_{\ell,\gamma}(x) - \mathbb{E}\left[\widetilde{S}_{\ell,\gamma}(x)\right]\right| = O_{a.s.}\left(\sqrt{\frac{\log n}{nh^{2-2\gamma}}}\right).$$

Proof. Let us consider the i.i.d sequence $(X_1, Y_1, \delta_1), \dots, (X_n, Y_n, \delta_n)$ and define for $\ell = 1, 2$, $\gamma = 0, 2$ and $n \geq 1$

$$\Phi_n^\gamma = \left\{ \theta_x^\gamma : \mathbb{R} \times \mathbb{R} \times \{0,1\} \to \mathbb{R} \,|\, \theta_x^\gamma(u,y,\delta) = \frac{\delta y^{-\ell}}{nh\, \overline{G}(y)} (u-x)^\gamma K_h(u-x),\, x \in \mathcal{C} \right\}.$$

By Lemma 3b in [15], Φ_n^γ is the Vapnik–Cervonenkis (V-C) class of measurable functions. Now, for $\gamma = 1$, we define

$$\Phi_n^1 = \left\{ \theta_x^1 : \mathbb{R} \times \mathbb{R} \times \{0,1\} \to \mathbb{R} \,|\, \theta_x^1(u,y,\delta) = \frac{\delta y^{-\ell}}{nh\, \overline{G}(y)} (u-x) K_h(u-x),\, x \in \mathcal{C} \right\}.$$

Let for $\ell = 1, 2$

$$\theta_{x,1}^1(u,y,\delta) = \frac{\delta y^{-\ell}}{nh\, \overline{G}(y)} (u-x+h) K_h(u-x) \quad \text{and} \quad \theta_{x,2}^1(u,y,\delta) = \frac{\delta y^{-\ell}}{n\, \overline{G}(y)} K_h(u-x).$$

Obviously, we have $\theta_x^1(u,y,\delta) = \theta_{x,1}^1(u,y,\delta) - \theta_{x,2}^1(u,y,\delta)$ which is a difference between two measurable functions. Then, by Lemma 3c in [15] θ_x^1 is also V-C class of measurable functions. These are uniformly bounded with respective envelopes $\Theta = C\, \frac{h^{\gamma-1}\|K\|_\infty}{n\, \overline{G}(\tau)}$. On the one hand, under the assumption (H2) (i), for $\ell = 1, 2$ and $\gamma = 0, \ell$, we get

$$\sup_{x \in \mathcal{C}} |\theta_x^\gamma(X_1, Y_1, \delta_1)| \leq \frac{h^\gamma}{nh\overline{G}(\tau)} \|K\|_\infty = \frac{h^{\gamma-1}}{n} c_1 =: U_n$$

with $c_1 = \frac{\|K\|_\infty}{\overline{G}(\tau)}$. On the other hand, using the conditional expectation property, we have

$$\begin{aligned}
\mathbb{E}\left[\theta_x^{2\gamma}(X_1, Y_1, \delta_1)\right] &= \frac{1}{n^2h^2} \mathbb{E}\left[\frac{\delta_1 Y_1^{-2\ell}}{\overline{G}^2(Y_1)} (X_1 - x)^{2\gamma} K_h^2(X_1 - x) \right] \\
&= \frac{1}{n^2h^2} \mathbb{E}\left[(X_1 - x)^{2\gamma} K_h^2(X_1 - x) \mathbb{E}\left[\frac{T_1^{-2\ell}}{\overline{G}(T_1)} \big| X_1 \right] \right] \\
&= \frac{1}{n^2h^2} \int (u-x)^{2\gamma} K_h^2(u-x) \int \frac{t^{-2\ell}}{\overline{G}(t)} f_{T|X}(t|u) dt f(u) du \\
&= \frac{1}{n^2h^2} \int (u-x)^{2\gamma} K_h^2(u-x) \int \frac{t^{-2\ell}}{\overline{G}(t)} f_{X,T}(u,t) dt du \\
&= \frac{1}{n^2h^2} \int (u-x)^{2\gamma} K_h^2(u-x) \phi_\ell(u) du.
\end{aligned}$$

Hence, under (H2)(i) and (H4) for $\ell = 2, 4$ and $\gamma = 0, 1, 2$, we obtain

$$\begin{aligned}
\sup_{x \in \mathcal{C}} |Var\left[\theta_x^\gamma(X_1, Y_1, \delta_1)\right]| &\leq \sup_{x \in \mathcal{C}} \mathbb{E}\left| \left[\theta_x^{2\gamma}(X_1, Y_1, \delta_1)\right] \right| \\
&\leq \frac{h^{2\gamma} \|K\|_\infty^2 \|\phi_\ell\|_\infty}{n^2\, h^2} = \frac{h^{2\gamma-2}}{n^2} c_2 =: \sigma_n^2
\end{aligned}$$

with $c_2 = \|K\|_\infty^2 \|\phi_\ell\|_\infty$ and $\sigma_n \leq U_n$ for n large enough. Now applying Talagrand's inequality (see Proposition 2.2 in [16]) there exist three positive constants C_1, C_2 and C_3, with $t \geq C_1 \sqrt{\dfrac{\log n}{nh^{2-2\gamma}}}$ for $\gamma = 0, 1, 2$, we have

$$\mathbb{P}\left[\sup_{\theta_x^\gamma \in \Phi_n^\gamma} \left| \sum_{i=1}^n \left(\theta_x^\gamma(X_i, Y_i, \delta_i) - \mathbb{E}\left[\theta_x^\gamma(X_i, Y_i, \delta_i)\right]\right) \right| > C_1 \sqrt{\frac{\log n}{nh^{2-2\gamma}}}\right]$$

$$\leq C_2 \exp\left(- \frac{C_1 \sqrt{\frac{\log n}{nh^{2-2\gamma}}}}{C_2\ c_1 \frac{h^{\gamma-1}}{n}} \log\left[1 + \frac{C_1 \sqrt{\frac{\log n}{nh^{2-2\gamma}}}\ c_1\ \frac{h^{\gamma-1}}{n}}{C_2 \left(\sqrt{n}\ \frac{h^{\gamma-1}}{n} \sqrt{c_2} + c_1 \frac{h^{\gamma-1}}{n} \sqrt{\log C_3 \frac{c_1}{\sqrt{c_2}}}\right)^2}\right]\right),$$

and using $\log(1+x) \approx x$ (for $x \to 0$), the right-hand of the last equation becomes an order of

$$C_2 \exp\left(- \frac{C_1}{C_2} \frac{\sqrt{n\ \log n}}{c_1} \times \frac{C_1 \sqrt{\frac{\log n}{nh^{2-2\gamma}}}\ c_1\ \frac{h^{\gamma-1}}{n}}{C_2 n\ \frac{h^{2\gamma-2}}{n^2} c_2}\right) = C_2 n^{-\frac{C_1^2}{C_2^2 c_2}}$$

which by an appropriate choice of the constants C_1, C_2 and c_2, can be made $O(n^{-3/2})$. The latter is a general term of summable series and by Borel-Cantelli's Lemma we conclude the proof of Lemma 5.1. □

Lemma 5.2. *Under assumptions* (H2)(i) *and* (H5), *for* $\ell = 1, 2$, $\gamma = 0, \ell$ *and* n *large enough, we have*

$$\sup_{x \in \mathcal{C}} \left| \mathbb{E}\left[\widetilde{S}_{\ell,\gamma}(x)\right] \right| = O\left(h^\gamma\right).$$

Proof. Using the conditional expectation property, a change of variable for $\ell = 1, 2$ and $\gamma = 0, \ell$, we have

$$\begin{aligned}
\mathbb{E}\left[\widetilde{S}_{\ell,\gamma}(x)\right] &= \frac{1}{nh} \mathbb{E}\left[\sum_{i=1}^n \widetilde{T}_i^{-\ell} (X_i - x)^\gamma K_h(X_i - x)\right] \\
&= h^{-1} \mathbb{E}\left[(X_1 - x)^\gamma K_h(X_1 - x) \mathbb{E}\left[\widetilde{T}_1^{-\ell} | X_1\right]\right] \\
&= h^{-1} \int (u - x)^\gamma K_h(u - x) m_\ell(u) f(u) du \\
&= h^{-1} \int (u - x)^\gamma K_h(u - x) r_\ell(u) du \\
&= h^\gamma \int v^\gamma K(v) r_\ell(x + vh) dv.
\end{aligned}$$

Using the first order Taylor expansion and under assumptions (H2)(i) and (H5), for $\ell = 1, 2$ and $\gamma = 0, \ell$, we get

$$\begin{aligned}
\sup_{x \in \mathcal{C}} \left| \mathbb{E}[\widetilde{S}_{\ell,\gamma}(x)] \right| &\leq h^\gamma \sup_{x \in \mathcal{C}} |r_\ell(x)| \int |v|^\gamma K(v) dv + h^{\gamma+1} \sup_{x \in \mathcal{C}} |r'_\ell(x)| \int |v|^{\gamma+1} K(v) dv \\
&= \mathrm{O}\left(h^\gamma\right).
\end{aligned}$$

□

Lemma 5.3. *Under assumptions* (H1), (H2), (H4) *and* (H5), *for* $\ell = 1, 2$, $\gamma = 0, \ell$, *and* n *large enough, we have*

$$\sup_{x \in \mathcal{C}} \left| \widehat{S}_{\ell,\gamma}(x) - \widetilde{S}_{\ell,\gamma}(x) \right| = O_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right).$$

Proof. For $\ell = 1, 2$, $\gamma = 0, \ell$, we have

$$\begin{aligned}
&\sup_{x \in \mathcal{C}} \left| \widehat{S}_{\ell,\gamma}(x) - \widetilde{S}_{\ell,\gamma}(x) \right| \\
&= \sup_{x \in \mathcal{C}} \left| \frac{1}{nh} \sum_{i=1}^{n} (X_i - x)^{\gamma} K_h(X_i - x) \left(\widehat{T}_i^{-\ell} - \widetilde{T}_i^{-\ell} \right) \right| \\
&= \sup_{x \in \mathcal{C}} \left| \frac{1}{nh} \sum_{i=1}^{n} \delta_i Y_i^{-\ell} (X_i - x)^{\gamma} K_h(X_i - x) \left(\frac{\overline{G}(Y_i) - \overline{G}_n(Y_i)}{\overline{G}_n(Y_i)\overline{G}(Y_i)} \right) \right| \\
&\leq \frac{1}{\overline{G}^2(\tau)} \sup_{t \leq \tau} \left| \overline{G}_n(t) - \overline{G}(t) \right| \times \sup_{x \in \mathcal{C}} \left| \frac{1}{nh} \sum_{i=1}^{n} T_i^{-\ell} (X_i - x)^{\gamma} K_h(X_i - x) \right| \\
&=: C \sup_{t \leq \tau} |\mathcal{D}_1(t)| \times \sup_{x \in \mathcal{C}} |\mathcal{D}_2(x)|.
\end{aligned}$$

From Lemma 4.2 in [9], the right-hand term is equal to

$$\sup_{t \leq \tau} |\mathcal{D}_1(t)| = \mathrm{O}_{a.s.} \left(\sqrt{\frac{\log \log n}{n}} \right) \quad \text{as} \quad n \to \infty. \tag{5.3}$$

Now, concerning the second term, for $\ell = 1, 2$ and $\gamma = 0, \ell$, we write

$$\begin{aligned}
\mathcal{D}_2(x) = \frac{1}{h} &\left\{ \frac{1}{n} \sum_{i=1}^{n} T_i^{-\ell} (X_i - x)^{\gamma} K_h(X_i - x) - \mathbb{E}\left[T_1^{-\ell} (X_1 - x)^{\gamma} K_h(X_1 - x) \right] \right\} \\
&+ \mathbb{E}\left[h^{-1} T_1^{-\ell} (X_1 - x)^{\gamma} K_h(X_1 - x) \right] \\
&=: \mathcal{D}_{2,1}(x) + \mathcal{D}_{2,2}(x).
\end{aligned}$$

On the one hand, $\mathcal{D}_{2,1}(\cdot)$ can be obtained analogously as Lemma 5.1 when the variable interest T is completely observed (i.e. $C = +\infty$). Then, for $\gamma = 0, 1, 2$, and n large enough, we obtain

$$\sup_{x \in \mathcal{C}} |\mathcal{D}_{2,1}(x)| = \mathrm{O}_{a.s.} \left(\sqrt{\frac{\log n}{nh^{2-2\gamma}}} \right). \tag{5.4}$$

On the other hand, we deal with $\mathcal{D}_{2,2}(\cdot)$ as in lemma 5.2. Hence, using the conditional expectation property, a change of variable and the Taylor expansion, for $\ell = 1, 2$ and $\gamma = 0, \ell$, we have

$$\begin{aligned}\mathcal{D}_{2,2}(x) &= h^{-1}\mathbb{E}\left[(X_1 - x)^{\gamma} K_h(X_1 - x)\mathbb{E}\left[T_1^{-\ell}|X_1\right]\right]\\ &= h^{-1}\int (u-x)^{\gamma} K_h(u-x) m_{\ell}(u) f(u)du\\ &= h^{-1}\int (vh)^{\gamma} K(v) r_{\ell}(x+vh)hdv\\ &= h^{\gamma} r_{\ell}(x)\int v^{\gamma}K(v)dv + h^{\gamma+1}\int v^{\gamma+1}K(v)r'_{\ell}(\xi)dv.\end{aligned}$$

Under (H2)(i) and (H5), we get

$$\sup_{x\in\mathcal{C}} |\mathcal{D}_{2,2}(x)| = \mathrm{O}\Big(h^{\gamma}\Big). \tag{5.5}$$

Combining the results in (5.3), (5.4) and (5.5) achieve the proof of the Lemma 5.3. □

Now, combining Lemma 5.1 with Lemma 5.3 according to the decomposition (5.1) and (5.2), we conclude the proof of Proposition 3.2. □

Proof. To deal with Proposition 3.3, we consider the following decomposition for $\ell = 1, 2$:

$$\begin{aligned}\widetilde{r}_{\ell}(x) - \mathbb{E}[\widetilde{r}_{\ell}(x)] &= \Big\{\widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x) - \mathbb{E}\Big[\widetilde{S}_{2,2}(x)\widetilde{S}_{\ell,0}(x)\Big]\Big\}\\ &\quad - \Big\{\widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x) - \mathbb{E}\Big[\widetilde{S}_{2,1}(x)\widetilde{S}_{\ell,1}(x)\Big]\Big\}\\ &=: \mathcal{E}_{\ell,1}(x) - \mathcal{E}_{\ell,2}(x).\end{aligned}$$

On the one hand, for $\ell = 1, 2$, we have

$$\begin{aligned}\mathcal{E}_{\ell,1}(x) &= \Big(\widetilde{S}_{2,2}(x) - \mathbb{E}\Big[\widetilde{S}_{2,2}(x)\Big]\Big)\Big(\widetilde{S}_{\ell,0}(x) - \mathbb{E}\Big[\widetilde{S}_{\ell,0}(x)\Big]\Big) + \mathbb{E}\Big[\widetilde{S}_{2,2}(x)\Big]\\ &\times\Big(\widetilde{S}_{\ell,0}(x) - \mathbb{E}\Big[\widetilde{S}_{\ell,0}(x)\Big]\Big) + \mathbb{E}\Big[\widetilde{S}_{\ell,0}(x)\Big]\Big(\widetilde{S}_{2,2}(x) - \mathbb{E}\Big[\widetilde{S}_{2,2}(x)\Big]\Big)\\ &- \mathrm{C}ov\Big(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x)\Big).\end{aligned} \tag{5.6}$$

On the other hand, for $\ell = 1, 2$, we have

$$\begin{aligned}\mathcal{E}_{\ell,2}(x) &= \Big(\widetilde{S}_{2,1}(x) - \mathbb{E}\Big[\widetilde{S}_{2,1}(x)\Big]\Big)\Big(\widetilde{S}_{\ell,1}(x) - \mathbb{E}\Big[\widetilde{S}_{\ell,1}(x)\Big]\Big) + \mathbb{E}\Big[\widetilde{S}_{2,1}(x)\Big]\\ &\times\Big(\widetilde{S}_{\ell,1}(x) - \mathbb{E}\Big[\widetilde{S}_{\ell,1}(x)\Big]\Big) + \mathbb{E}\Big[\widetilde{S}_{\ell,1}(x)\Big]\Big(\widetilde{S}_{2,1}(x) - \mathbb{E}\Big[\widetilde{S}_{2,1}(x)\Big]\Big)\\ &- \mathrm{C}ov\Big(\widetilde{S}_{2,1}(x), \widetilde{S}_{\ell,1}(x)\Big).\end{aligned} \tag{5.7}$$

It remains to study each term of the decomposition (5.6) and (5.7). The majority of the terms have already been dealt with in lemmas 5.1 and 5.2. The two terms that remain to be treated are the covariance terms that are the purpose of the two following lemmas.

Lemma 5.4. *Under assumptions* (H1), (H2), (H4) *and* (H5), *for $\ell = 1, 2$ and n large enough, we have*

$$\sup_{x\in\mathcal{C}} \left| \mathrm{Cov}\left(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x) \right) \right| = O\left(\sqrt{\frac{\log n}{nh^2}} \right).$$

Proof. By definition, for $\ell = 1, 2$, we have

$$\begin{aligned} \mathrm{Cov}\left(\widetilde{S}_{\ell,0}(x), \widetilde{S}_{2,2}(x) \right) &= \mathbb{E}\left[\widetilde{S}_{\ell,0}(x) \widetilde{S}_{2,2}(x) \right] - \mathbb{E}\left[\widetilde{S}_{\ell,0}(x) \right] \mathbb{E}\left[\widetilde{S}_{2,2}(x) \right] \\ &=: \Delta_1(x) - \Delta_2(x)\Delta_3(x). \end{aligned}$$

From this statement we proceed as follows. We first treat $\Delta_1(x)$, for $\ell = 1, 2$, we have

$$\begin{aligned} \Delta_1(x) &= \frac{1}{(nh)^2} \mathbb{E}\left[\sum_{i,j=1}^{n} \widetilde{T}_i^{-\ell} \widetilde{T}_j^{-2} (X_j - x)^2 K_h(X_i - x) K_h(X_j - x) \right] \\ &= \frac{1}{nh^2} \mathbb{E}\left[\frac{\delta_1 Y_1^{-\ell-2}}{\overline{G}^2(Y_1)} (X_1 - x)^2 K_h^2(X_1 - x) \right] \\ &\quad + \frac{(n-1)}{nh^2} \mathbb{E}\left[\widetilde{T}_1^{-\ell} \widetilde{T}_2^{-2} (X_2 - x)^2 K_h(X_1 - x) K_h(X_2 - x) \right] \\ &=: \Delta_{1,1}(x) + \Delta_{1,2}(x). \end{aligned}$$

On the one hand, we deal with $\Delta_{1,1}(\cdot)$. Using the conditional expectation property and a change of variable, for $\ell = 1, 2$, we have

$$\begin{aligned} \Delta_{1,1}(x) &= \frac{1}{nh^2} \mathbb{E}\left[(X_1 - x)^2 K_h^2(X_1 - x) \mathbb{E}\left[\frac{\delta_1 Y_1^{-\ell-2}}{\overline{G}^2(Y_1)} \Big| X_1 \right] \right] \\ &= \frac{1}{nh^2} \int (u - x)^2 K_h^2(u - x) \int \frac{t^{-\ell-2}}{\overline{G}(t)} f_{T_1|X_1}(t|u) dt f(u) du \\ &= \frac{1}{nh^2} \int (u - x)^2 K_h^2(u - x) \phi_\ell(u) du \\ &= \frac{h}{n} \int v^2 K^2(v) \phi_\ell(x + vh) dv. \end{aligned}$$

Using the Taylor expansion and under the assumptions (H2)(ii) and (H4), for $\ell = 3, 4$ and n large enough, we have

$$\sup_{x\in\mathcal{C}} |\Delta_{1,1}(x)| \leq \frac{h}{n} \sup_{x\in\mathcal{C}} |\phi_\ell(x)| \int v^2 K^2(v) dv + \frac{h^2}{n} \sup_{x\in\mathcal{C}} |\phi'_\ell(x)| \int |v|^3 K^2(v) dv = \mathrm{O}\left(\frac{h}{n} \right). \tag{5.8}$$

Now, we deal with $\Delta_{1,2}(\cdot)$. Using the same arguments as for $\Delta_{1,1}(\cdot)$, we get

$$\begin{aligned}\Delta_{1,2}(x) &= \frac{(n-1)}{nh^2}\mathbb{E}\left[(X_2-x)^2K_h(X_1-x)K_h(X_2-x)\mathbb{E}\left[\widetilde{T}_1^{-\ell}\widetilde{T}_2^{-2}\big|X_1,X_2\right]\right]\\ &= \frac{(n-1)}{nh^2}\int\int(v-x)^2K_h(u-x)K_h(v-x)m_\ell(u)m_2(v)f(u)f(v)dudv\\ &= \frac{(n-1)}{nh^2}\int\int(v-x)^2K_h(u-x)K_h(v-x)r_\ell(u)r_2(v)dudv\\ &= \frac{(n-1)h^2}{n}\int\int s^2K(t)K(s)r_\ell(x+th)r_2(x+sh)dtds.\end{aligned}$$

By the first order Taylor expansion for $r_\ell(\cdot)$ and under (H2)(i) and (H5), for $\ell=1,2$ and n large enough, we obtain

$$\begin{aligned}\sup_{x\in\mathcal{C}}|\Delta_{1,2}(x)| &\leq \frac{(n-1)h^2}{n}\sup_{x\in\mathcal{C}}|r_\ell(x)|\sup_{x\in\mathcal{C}}|r_2(x)|\int s^2K(s)ds\\ &= \mathrm{O}\left(h^2\right).\end{aligned}\tag{5.9}$$

Combining the results in (5.8) and (5.9) we get for n large enough

$$\sup_{x\in\mathcal{C}}|\Delta_1(x)| = \mathrm{O}\left(h^2\right).\tag{5.10}$$

On the other hand, from Lemma 5.2 for different values of ℓ and γ, we get

$$\sup_{x\in\mathcal{C}}|\Delta_2(x)| = \mathrm{O}\left(1\right)\quad\text{and}\quad\sup_{x\in\mathcal{C}}|\Delta_3(x)| = \mathrm{O}\left(h^2\right).\tag{5.11}$$

Finally, from (5.10) and (5.11), for $\ell=1,2$ and n large enough, we get

$$\sup_{x\in\mathcal{C}}\left|Cov\left(\widetilde{S}_{\ell,0}(x),\widetilde{S}_{2,2}(x)\right)\right| = \mathrm{O}\left(h^2\right).$$

By using the second part of the assumption (H1), the latter is negligible with respect to $\sqrt{\dfrac{\log n}{nh^2}}$. □

Lemma 5.5. *Under assumptions* (H1)*,* (H2)*,* (H4) *and* (H5)*, for $\ell=1,2$ and n large enough, we have*

$$\sup_{x\in\mathcal{C}}\left|Cov\left(\widetilde{S}_{\ell,1}(x),\widetilde{S}_{2,1}(x)\right)\right| = O\left(\sqrt{\frac{\log n}{nh^2}}\right).$$

Proof. In the same way as in Lemma 5.4, for $\ell=1,2$, write

$$\begin{aligned}Cov\left(\widetilde{S}_{\ell,1}(x),\widetilde{S}_{2,1}(x)\right) &= \mathbb{E}\left[\widetilde{S}_{\ell,1}(x)\widetilde{S}_{2,1}(x)\right]-\mathbb{E}\left[\widetilde{S}_{\ell,1}(x)\right]\mathbb{E}\left[\widetilde{S}_{2,1}(x)\right]\\ &=: \Gamma_1(x)-\Gamma_2(x)\Gamma_3(x).\end{aligned}$$

On the one hand, from Lemma 5.2 for different values of γ and ℓ, we have

$$\sup_{x\in\mathcal{C}}|\Gamma_2(x)| = \mathrm{O}\left(h\right) \qquad \text{and} \qquad \sup_{x\in\mathcal{C}}|\Gamma_3(x)| = \mathrm{O}\left(h\right). \tag{5.12}$$

On the other hand, we deal with $\Gamma_1(\cdot)$. For $\ell = 1, 2$, we have

$$\begin{aligned}\Gamma_1(x) &= \frac{1}{(nh)^2}\mathbb{E}\left[\sum_{i,j=1}^{n}\widetilde{T}_i^{-\ell}\widetilde{T}_j^{-2}(X_i-x)(X_j-x)K_h(X_i-x)K_h(X_j-x)\right]\\ &= \frac{1}{nh^2}\mathbb{E}\left[\frac{\delta_1 Y_1^{-\ell-2}}{\overline{G}^2(Y_1)}(X_1-x)^2K_h^2(X_1-x)\right]\\ &\quad+\frac{(n-1)}{nh^2}\mathbb{E}\left[\widetilde{T}_1^{-\ell}\widetilde{T}_2^{-2}(X_1-x)(X_2-x)K_h(X_1-x)K_h(X_2-x)\right]\\ &=: \Gamma_{1,1}(x)+\Gamma_{1,2}(x).\end{aligned}$$

Concerning $\Gamma_{1,1}(\cdot)$, from (5.8) we get

$$\sup_{x\in\mathcal{C}}|\Gamma_{1,1}(x)| = \sup_{x\in\mathcal{C}}|\Delta_{1,1}(x)| = \mathrm{O}\left(\frac{h}{n}\right). \tag{5.13}$$

Now, for $\Gamma_{1,2}(\cdot)$, by using the conditional expectation property and a change of variable, for $\ell = 1, 2$, we have

$$\begin{aligned}\Gamma_{1,2}(x) &= \frac{(n-1)}{nh^2}\mathbb{E}\left[(X_1-x)(X_2-x)K_h(X_1-x)K_h(X_2-x)\mathbb{E}\left[\widetilde{T}_1^{-\ell}\widetilde{T}_2^{-2}|X_1,X_2\right]\right]\\ &= \frac{(n-1)}{nh^2}\int\int(u-x)(v-x)K_h(u-x)K_h(v-x)m_\ell(u)m_2(v)f(u)f(v)dudv\\ &= \frac{(n-1)}{nh^2}\int\int(u-x)(v-x)K_h(u-x)K_h(v-x)r_\ell(u)r_2(v)dudv\\ &= \frac{(n-1)h^2}{n}\int\int tsK(t)K(s)r_\ell(x+th)r_2(x+sh)dtds.\end{aligned}$$

Using the first order Taylor expansion and under the assumptions (H2)(i) and (H5), for $\ell = 1, 2$ and n large enough, we obtain

$$\sup_{x\in\mathcal{C}}|\Gamma_{1,2}(x)| \leq \frac{(n-1)h^4}{n}\sup_{x\in\mathcal{C}}|r'_\ell(x)|\sup_{x\in\mathcal{C}}|r'_2(x)|\int\int t^2s^2K(t)K(s)dtds = \mathrm{O}\left(h^4\right). \tag{5.14}$$

Finally, combining (5.12), (5.13) and (5.14), for $\ell = 1, 2$ and n large enough, we obtain

$$\sup_{x\in\mathcal{C}}\left|\mathrm{C}ov\left(\widetilde{S}_{\ell,1}(x),\widetilde{S}_{2,1}(x)\right)\right| = \mathrm{O}\left(h^2\right).$$

By the last part of (H1), the later is negligible with respect to $\sqrt{\dfrac{\log n}{nh^2}}$. $\square$

Hence, Lemmas 5.1, 5.2, 5.4 and 5.5 according to the decomposition (5.6) and (5.7) conclude the proof of Proposition 3.3. □

Proof. To deal with the bias term (see Proposition 3.4), we will use the below normalization. Notice that, for $\ell = 1, 2$, (2.4) can be written as

$$\widetilde{r}_\ell(x) = \frac{1}{n(n-1)h^2\mathbb{E}\left[v_{1,2}(x)\right]} \sum_{\substack{i,i=1\\ i\neq j}}^{n} \widetilde{w}^{\ell}_{i,j}(x)$$

where

$$v_{1,2}(x) = h^{-2}(X_1 - x)\left((X_1 - x) - (X_2 - x)\right) K_h(X_1 - x)K_h(X_2 - x). \tag{5.15}$$

First, we deal with the quantity (5.15). By a change of variable, we have

$$\begin{aligned}\mathbb{E}\left[v_{1,2}(x)\right] &= h^{-2} \int\int (u - x)\left((u - x) - (v - x)\right) K_h(u - x)K_h(v - x)f(u)f(v)dudv \\ &= h^2 \int\int \left(t^2 - ts\right) K(t)K(s)f(s + th)f(x + sh)dtds.\end{aligned}$$

By the first order Taylor expansion and under the assumptions (H2)(i) and (H3), we obtain

$$\sup_{x\in\mathcal{C}} \left|\mathbb{E}\left[v_{1,2}(x)\right]\right| = \mathrm{O}\left(h^2\right). \tag{5.16}$$

Now, we come back to our main calculation of the bias term, for $\ell = 1, 2$, given by

$$\begin{aligned}\mathbb{E}[\widetilde{r}_\ell(x)] - m_\ell(x)m_2(x) &= \frac{1}{n(n-1)h^2\mathbb{E}\left[v_{1,2}(x)\right]}\mathbb{E}\left[\sum_{\substack{i,j=1\\ i\neq j}}^{n} \widetilde{w}^{\ell}_{i,j}(x)\right] - m_\ell(x)m_2(x) \\ &= \frac{1}{h^2\mathbb{E}\left[v_{1,2}(x)\right]} \left\{\mathbb{E}[\widetilde{w}^{\ell}_{1,2}(x)] - m_\ell(x)m_2(x)h^2\mathbb{E}\left[v_{1,2}(x)\right]\right\}.\end{aligned} \tag{5.17}$$

We deal with term between braces. Then, using the conditional expectation property and a change of variable, for $\ell = 1, 2$, we have

$$\begin{aligned}&\mathbb{E}\left[\widetilde{w}^{\ell}_{1,2}(x)\right] - m_\ell(x)m_2(x)h^2\mathbb{E}\left[v_{1,2}(x)\right] \\ &= \mathbb{E}\left[(X_1 - x)\left((X_1 - x) - (X_2 - x)\right) K_h(X_1 - x)K_h(X_2 - x)\mathbb{E}\left[\widetilde{T}_1^{-2}\widetilde{T}_2^{-\ell}|X_1, X_2\right]\right] \\ &\quad - m_\ell(x)m_2(x)\mathbb{E}\left[(X_1 - x)\left((X_1 - x) - (X_2 - x)\right) K_h(X_1 - x)K_h(X_2 - x)\right] \\ &= \iint (u - x)\left((u - x) - (v - x)\right) \\ &\qquad \times K_h(u - x)K_h(v - x)f(u)f(v)\left[m_2(u)m_\ell(v) - m_\ell(x)m_2(x)\right] dudv \\ &= h^4 \iint \left(t^2 - ts\right) K(t)K(s)f(x + th) \\ &\qquad \times f(x + sh)\left[m_2(x + th)m_\ell(x + sh) - m_\ell(x)m_2(x)\right] dtds.\end{aligned} \tag{5.18}$$

The second order Taylor expansion of the function $m_\ell(\cdot)$ for $\ell = 1, 2$ gives:

$$\begin{aligned}
&m_2(x+th)m_\ell(x+sh) - m_2(x)m_\ell(x)\\
&= (m_2(x+th) - m_2(x))(m_\ell(x+sh) - m_\ell(x))\\
&\quad + m_\ell(x)\left(m_2(x+th) - m_2(x)\right) + m_2(x)\left(m_\ell(x+sh) - m_\ell(x)\right)\\
&= \left(thm_2'(x) + \frac{h^2t^2}{2}m_2''(\xi_1)\right)\left(shm_\ell'(x) + \frac{h^2s^2}{2}m_\ell''(\xi_2)\right)\\
&\quad + m_\ell(x)\left(thm_2'(x) + \frac{h^2t^2}{2}m_2''(\xi_1)\right) + m_2(x)\left(shm_\ell'(x) + \frac{h^2s^2}{2}m_\ell''(\xi_2)\right) \qquad (5.19)\\
&= h^2tsm_2'(x)m_\ell'(x) + \frac{h^3ts^2}{2}m_2'(x)m_\ell''(\xi_2) + \frac{h^3t^2s}{2}m_\ell'(x)m_2''(\xi_2)\\
&\quad + \frac{h^4t^2s^2}{4}m_2''(\xi_1)m_\ell''(\xi_2) + thm_\ell(x)m_2'(x) + \frac{h^2t^2}{2}m_\ell(x)m_2''(\xi_1)\\
&\quad + shm_2(x)m_\ell'(x) + \frac{h^2s^2}{2}m_2(x)m_\ell''(\xi_2),
\end{aligned}$$

and the first order Taylor expansion of the density function $f(\cdot)$ gives

$$\begin{aligned}
f(x+ht)f(x+sh) &= \left(f(x) + thf'(\xi_1)\right)\left(f(x) + shf'(\xi_2)\right)\\
&= f^2(x) + shf(x)f'(\xi_2) + thf(x)f'(\xi_1) + h^2stf'(\xi_1)f'(\xi_2).
\end{aligned} \qquad (5.20)$$

Then, replacing (5.19) and (5.20) in (5.18) and under (H2)(i), (H3) and (H6), for $\ell = 1, 2$, we get

$$\begin{aligned}
&\sup_{x\in\mathcal{C}}\left|\mathbb{E}\left[\widetilde{w}_{1,2}^{\ell}(x)\right] - h^2m_\ell(x)m_2(x)\mathbb{E}\left[v_{1,2}(x)\right]\right| \leq\\
&h^6\sup_{x\in\mathcal{C}}|f^2(x)|\sup_{x\in\mathcal{C}}|m_2'(x)|\sup_{x\in\mathcal{C}}|m_\ell'(x)|\int t^2s^2K(t)K(s)dtds \qquad (5.21)\\
&= \mathrm{O}\left(h^6\right).
\end{aligned}$$

Finally, combining the results in (5.16) and (5.21) according to (5.17), the bias term is of order

$$\sup_{x\in\mathcal{C}}|\mathbb{E}[\widetilde{r}_\ell(x)] - m_\ell(x)m_2(x)| = \mathrm{O}\left(h^2\right). \qquad \square$$

Remark 5.6. We point out that even if the LL method has the advantage of reducing the biais term, however, the combination of the two methods LL and RER has revealed several terms that do not allow us to get a better result than what we got.

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