Invexity of constraint maps in mathematical programs

Manh-Hung Nguyen

Nonlinear Funct. Anal. & Appl., Vol. 9, No. 2 (2004), pp. 289–304

INVEXITY OF CONSTRAINT MAPS IN MATHEMATICAL PROGRAMS

Do Van Luu and Nguyen Manh Hung

Abstract. In this paper, we study a multiobjective programming problem with constraints of equality and inequality types which are maps from a Banach space into other Banach spaces. Sufficient conditions for the invexity of constraint maps with respect to the same scale map are established together with a new constraint qualification involving a invexity condition and a generalized Slater condition.

1. Introduction

In the last two decades, theory of invex functions has been the subject of much development (see, e.g., [2], [4]-[7], [11], [14]). The invexity of functions occuring in mathematical programming problems plays an important role in the theory of optimality conditions and duality. A question arises as to when constraints in a mathematical programming are invex at a point with respect to the same scale. Recently, Ha-Luu [4] have shown that the constraint qualifications of Robinson [15], Nguyen-Strodiot-Mifflin [13] and Jourani [9] types are sufficient conditions ensuring constraints of Lipschitzian mathematical programs to be invex with respect to the same scale. It should be noted that the single-objective mathematical programs there involve finitely many constraints of equality and inequality types which are locally Lipschitzian real-valued functions defined on a Banach space.Motivated by the results due to Ha-Luu [4], in this paper we shall deal with a multiobjective programming problems with constraints maps from a Banach space into other Banach spaces which are directionally differentiable. Sufficient conditions for the invexity of constraint maps with respect to the same scale map are established

Received April 25, 2003.
2000 Mathematics Subject Classification: 52A01, 90C46.
Key words and phrases: Invexity, scale, constraint qualification, nearly S-convelike map.
This research was partially supported by the Natural Science Council of Vietnam.

together with showing that the invexity of constraint maps along with an another suitable condition gives a new constraint qualification. After Introduction, Section 2 is devoted to derive sufficient conditions for the invexity of constraint maps with respect to the same scale map. The results show that known constraint qualifications together with a condition on the existence of interior points will ensure constraint maps to be invex with respect to the same scale. Section 3 gives a new constraint qualification which comprises an invexity condition and a generalized Slater condition.

2. Sufficient conditions for invexity

Let X, Y, Z, V be real Banach spaces, and let f, g, h be maps from X into V, Y, Z, respectively. Let Q, S be closed convex cones in V, Y, respectively, with vertices at the origin, $\operatorname{int} Q \neq \emptyset$ and $\operatorname{int} S \neq \emptyset$. Let C be a nonempty convex subset of X. In this paper, we shall be concerned with the following mathematical programming problem:

$$
(P) \qquad \begin{aligned} & W-\min f(x), \\ & \text{subject to} \\ & \quad -g(x) \in S, \\ & \quad h(x) = 0, \\ & \quad x \in C, \end{aligned}
$$

where W-min denotes the weak minimum with respect to the cone Q.

Denote by M the feasible set of (P):

$$
M = \big\{x \in C : -g(x) \in S,\ h(x) = 0\big\}.
$$

For $\overline{x} \in C$, we define the following set

$$
C(\overline{x}) = \big\{\alpha(x - \overline{x}) : x \in C, \alpha \geq 0\big\}.
$$

Then $C(\overline{x})$ is a convex cone with vertex at the origin. Denote by S^* the dual cone of S

$$
S^* = \big\{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall\, y \in S\big\},
$$

where $\langle y^*, y\rangle$ is the value of the linear function $y^* \in Y^*$ at the point $y \in Y$. Y^* and Z^* will denote the topological duals of Y and Z, respectively.

The following notions are needed in the sequel.

Definition 2.1 [1]. A subset D of X is said to be nearly convex if there exists $\alpha \in (0,1)$ such that for each $x_1, x_2 \in D$,

$$\alpha x_1 + (1-\alpha)x_2 \in D.$$

Note that if D is nearly convex, then $\operatorname{int} D$ is a convex set (see, e.g., [8, Lemma 2.1]. $\operatorname{int} D$ here may be empty.

Definition 2.2 [8]. A map $F : D \to Y$ is called nearly S-convexlike on D if there exists $\alpha \in (0,1)$ such that for every $x_1, x_2 \in D$, there is $x_3 \in D$ such that

$$\alpha F(x_1) + (1-\alpha)F(x_2) - F(x_3) \in S.$$

Note that such a nearly S-convexlike map is simply called S-convexlike in [8]. A special case of nearly S-convexlike maps is nearly S-convex one.

Definition 2.3. Let D be a convex subset of X. A map $F : D \to Y$ is said to be nearly S-convex on D if there exists $\alpha \in (0,1)$ such that for every $x_1, x_2 \in D$,

$$\alpha F(x_1) + (1-\alpha)F(x_2) - F(\alpha x_1 + (1-\alpha)x_2) \in S.$$

Recall that the directional derivative of f at $\overline{x}$, with respect to a direction d, is the following limit

$$f'(\overline{x}; d) = \lim_{t \downarrow 0} \frac{f(\overline{x} + td) - f(\overline{x})}{t},$$

if it exists. Throughout this paper, we suppose that f, g, h are directionally differentiable at $\overline{x}$ in all directions.

Following [2, 14], the map g is called S-invex at $\overline{x}$ if there exists a map ω from X into $C(\overline{x})$ such that for all $x \in X$,

$$g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) \in S.$$

Such a map ω is called a scale. When $S = \{0\}$ we get the notion of $\{0\}$-invexity.

A sufficient condition for invexity of constraints in Problem (P) without equality constraints can be formulated as follows.

Theorem 2.1. *Assume that $h = 0$ and $g'(\overline{x}; .)$ is nearly S-convexlike on $C(\overline{x})$. Suppose also that there exists $d_0 \in C(\overline{x})$ such that*

$$-g'(\overline{x}; d_0) \in int\, S \tag{1}$$

Then there exists a map $\omega : X \to C(\overline{x})$ such that g is S-invex at $\overline{x}$ with respect to ω.

Proof. Put $A := g'(\overline{x}; C(\overline{x})) + S$, where $g'(\overline{x}; C(\overline{x})) := \{g'(\overline{x}; d) : d \in C(\overline{x})\}$. We first begin with showing that A is nearly convex.

For $y_1, y_2 \in A$, there exist $d_i \in C(\overline{x})$ and $s_i \in S$ $(i = 1, 2)$ such that

$$y_i = g'(\overline{x}; d_i) + s_i \quad (i = 1, 2). \tag{2}$$

Since $g'(\overline{x}; .)$ is nearly S-convexlike on $C(\overline{x})$, there exist $\alpha \in (0, 1)$ and $d_3 \in C(\overline{x})$ such that

$$\alpha g'(\overline{x}; d_1) + (1 - \alpha) g'(\overline{x}; d_2) - g'(\overline{x}; d_3) \in S. \tag{3}$$

Combining (2) and (3) yields that

$$\begin{aligned} \alpha y_1 + (1 - \alpha) y_2 &= \alpha g'(\overline{x}; d_1) + (1 - \alpha) g'(\overline{x}; d_2) + \alpha s_1 + (1 - \alpha) s_2 \\ &\in g'(\overline{x}; d_3) + S + S \\ &\subset g'(\overline{x}; C(\overline{x})) + S = A, \end{aligned}$$

which means that the set A is nearly convex. We invoke Lemma 2.1 in [8] to deduce that $\operatorname{int} A$ is convex. Note that $\operatorname{int} A \neq \emptyset$, since $\operatorname{int} S \neq \emptyset$.

We now show that $A = Y$. Assume the contrary, that $A \underset{\neq}{\subset} Y$. Then there exists $y_0 \in Y \setminus A$, and so $y_0 \notin \operatorname{int} A$. Applying a separation theorem for the disjoint convex sets $\{y_0\}$ and $\operatorname{int} A$ in Y (see, e.g., [3, Theorem 3.3]) yields the existence of $0 \neq y^* \in Y^*$ such that

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall\, y \in \operatorname{int} A).$$

Since y^* is continuous on Y and $\operatorname{int} A \neq \emptyset$, we obtain

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall\, y \in \overline{\operatorname{int} A} = \overline{A}),$$

which implies that

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall\, y \in A). \tag{4}$$

Since $g'(\overline{x}; .)$ is positively homogeneous, $C(\overline{x})$ and S are cones, it follows that A is cone. Making use of Lemma 5.1 in [3], it follows from (4) that

$$\langle y^*, y_0 \rangle \leq 0 \leq \langle y^*, y \rangle \quad (\forall\, y \in A). \tag{5}$$

Observing that $0 \in S$, we have

$$\langle y^*, y\rangle \geq 0 \quad (\forall\, y \in g'(\overline{x}; C(\overline{x}))). \tag{6}$$

Moreover, since $g'(\overline{x}; .)$ is positively homogeneous, it follows from (5) that

$$\langle y^*, y\rangle \geq 0 \quad (\forall\, y \in S)$$

which means that $y^* \in S^*$.

On the other hand, it follows readily from (6) that

$$\langle y^*, g'(\overline{x}; d)\rangle \geq 0 \quad (\forall d \in C(\overline{x})),$$

which leads to the following

$$\langle y^*, g'(\overline{x}; d_0)\rangle \geq 0,$$

which contradicts (1). Consequently, $A = Y$, i.e.,

$$g'(\overline{x}; C(\overline{x})) + S = Y. \tag{7}$$

It follows from (7) that for all $x \in X$,

$$g(x) - g(\overline{x}) \in g'(\overline{x}; C(\overline{x})) + S,$$

which implies that there exists $d \in C(\overline{x})$ such that

$$g(x) - g(\overline{x}) \in g'(\overline{x}; d) + S.$$

Defining a map $\omega : x \mapsto \omega(x) = d$, we obtain

$$g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) \in S.$$

The proof is complete. □

Denote by $B(\overline{x}; \delta)$ the open ball of radius δ around $\overline{x}$.

The following result shows that a generalized constraint qualification of Mangasarian-Fromovitz [12] type for infinite dimensional cases is a sufficient condition ensuring g to be S-invex and h is $\{0\}$-invex at $\overline{x}$ with respect to the same scale.

Theorem 2.2. *Assume that h is Fréchet differentiable at $\overline{x}$ with Fréchet derivative $h'(\overline{x})$ and $g'(\overline{x}; .)$ is nearly S-convex on $C(\overline{x})$. Suppose, in addition, that there exists $d_0 \in C(\overline{x})$ such that*

(i) $-g'(\overline{x}; d_0) \in \operatorname{int} S,\ \ h'(\overline{x})d_0 = 0$*;*
(ii) $h'(\overline{x})$ *is a surjective map from* X *onto* Z*;*
(iii) *there exists* $\delta > 0$ *such that* $B(d_0; \delta) \subset C(\overline{x})$*, and for every* $z \in h'(\overline{x})(B(d_0; \delta))$*, there exists* $d \in B(d_0; \delta)$ *satisfying*

$$-g'(\overline{x}; d) \in S, \quad h'(\overline{x})d = z.$$

Then, there exists a map $\omega : X \to C(\overline{x})$ *such that* g *is* S*-invex and* h *is* $\{0\}$*-invex at* $\overline{x}$ *with respect to the same scale* ω*, which means that for all* $x \in X$*,*

$$\begin{aligned} &g(x) - g(\overline{x}) - g'(\overline{x}, \omega(x)) \in S, \\ &h(x) - h(\overline{x}) = h'(\overline{x})\omega(x). \end{aligned}$$

Note that the condition on existence of interior point like condition (iii) was introduced by Tamminen [17].

Proof of Theorem 2.2. We invoke assumption (i) to deduce that for all $\mu \in S^* \setminus \{0\}$, and $\nu \in Z^*$,

$$\langle \mu, g'(\overline{x}, d_0) \rangle + \langle \nu, h'(\overline{x})d_0 \rangle < 0. \tag{8}$$

In view of the differentiability of h at $\overline{x}$, putting $G = (g, h)$, one gets $G'(\overline{x}; .) = (g'(\overline{x}; .), h'(\overline{x})(\cdot))$.

We now show that

$$G'(\overline{x}; C(\overline{x})) + S \times \{O_z\} = Y \times Z. \tag{9}$$

Assume the contrary, that

$$G'(\overline{x}; C(\overline{x})) + S \times \{O_z\} \underset{\neq}{\subset} Y \times Z.$$

This leads to the existence of a point $u := (u_1, u_2) \in Y \times Z$, but $u \notin G'(\overline{x}; C(\overline{x})) + S \times \{O_z\}$. Setting $B := G'(\overline{x}; C(\overline{x})) + S \times \{O_z\}$, we shall prove that B is nearly convex.

It is easy to see that

$$\begin{aligned} B = \big\{(y, z) \in Y \times Z : \exists\, d \in C(\overline{x}), \\ y - g'(\overline{x}; d) \in S,\ h'(\overline{x})d = z\big\}. \end{aligned}$$

Hence, taking $(y_i, z_i) \in B$ $(i = 1, 2)$, there exist $d_i \in C(\overline{x})$ $(i = 1, 2)$ such that

$$y_i - g'(\overline{x}; d_i) \in S, \quad h'(\overline{x})d_i = z_i \quad (i = 1, 2). \tag{10}$$

Since $g'(\overline{x};.)$ is nearly S-convex, there exists $\alpha \in (0,1)$ such that

$$\alpha g'(\overline{x};d_1) + (1-\alpha)g'(\overline{x};d_2) - g'(\overline{x};\alpha d_1 + (1-\alpha)d_2) \in S. \tag{11}$$

Moreover, it follows from (10) that

$$\alpha y_1 + (1-\alpha)y_2 - \alpha g'(\overline{x};d_1) - (1-\alpha)g'(\overline{x};d_2) \in S. \tag{12}$$

Combining (11) and (12) yields that

$$\begin{aligned} \alpha y_1 + (1-\alpha)y_2 &\in g'(\overline{x};\alpha d_1 + (1-\alpha)d_2) + S + S \\ &\subset g'(\overline{x};\alpha d_1 + (1-\alpha)d_2 + S, \end{aligned}$$

which means that

$$\alpha y_1 + (1-\alpha)y_2 - g'(\overline{x};\alpha d_1 + (1-\alpha)d_2) \in S \tag{13}$$

On the other hand,

$$\alpha z_1 + (1-\alpha)z_2 = h'(\overline{x})(\alpha d_1 + (1-\alpha)d_2),$$

which along with (13) yields that

$$\alpha(y_1,z_1) + (1-\alpha)(y_2,z_2) \in B$$

Consequently, B is nearly convex. Due to Lemma 2.1 in [8], $\operatorname{int} B$ is convex.

Next we shall prove that $\operatorname{int} B \neq \emptyset$.

According to assumption (ii), $h'(\overline{x})$ is a surjective linear map from X onto Z, and hence $h'(\overline{x})$ is an open map. Therefore, $h'(\overline{x})(B(d_0;\delta))$ is an open nonempty subset of Z.

Taking $(\overline{y},\overline{z}) \in (\operatorname{int} S) \times h'(\overline{x})(B(d_0;\delta))$ yields that $(\overline{y},\overline{z})$ is an interior point of B. Indeed, since $\overline{y} \in \operatorname{int} S$ and $\overline{z} \in h'(\overline{x})(B(d_0;\delta))$, there exist neighborhoods U_1 of $\overline{y}$ and U_2 of $\overline{z}$ such that $U_1 \subset S$ and $U_2 \subset h'(\overline{x})(B(d_0;\delta))$, respectively. Taking any $(y,z) \in U_1 \times U_2$, due to assumption (iii), there exists $d \in B(d_0;\delta)$ such that

$$-g'(\overline{x};d) \in S, \quad h'(\overline{x})d = z,$$

which implies that

$$y - g'(\overline{x};d) \in S + S \subset S,$$

whence, $(y, z) \in B$. Consequently, $U_1 \times U_2 \subset B$ and $(\overline{y}, \overline{z})$ is an interior point of B, which means that $\operatorname{int} B \neq \emptyset$.

Applying a separation theorem for the nonempty disjoint convex sets $\{u\}$ and $\operatorname{int} B$ in $Y \times Z$ (see, e.g., [3, Theorem 3.3]) yields the existence of $(\mu^*, \nu^*) \in Y^* \times Z^* \setminus \{0\}$ satisfying

$$\langle \mu^*, u_1 \rangle + \langle \nu^*, u_2 \rangle \leq \langle \mu^*, y \rangle + \langle \nu^*, z \rangle \quad (\forall\, (y, z) \in \operatorname{int} B)$$

Since B is a cone, making use of Lemma 5.1 in [3], we obtain

$$\langle \mu^*, u_1 \rangle + \langle \nu^* u_2 \rangle \leq 0 \leq \langle \mu^*, y \rangle + \langle \nu^*, z \rangle \quad (\forall (y, z) \in \operatorname{int} B).$$

Since $\operatorname{int} B \neq \emptyset$, it follows that

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in \overline{\operatorname{int} B} = \overline{B}),$$

where $\overline{B}$ is the closure of B in normed topology. Hence,

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in B),$$

which leads to the following

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in G'(\overline{x}; C(\overline{x})), \tag{14}$$

$$\langle \mu^*, y \rangle \geq 0 \quad (\forall\, y \in S), \tag{15}$$

It follows from (14) that

$$\langle \mu^*, g'(\overline{x}; d) \rangle + \langle \nu^*, h'(\overline{x}) d \rangle \geq 0 \quad (\forall\, d \in C(\overline{x})). \tag{16}$$

By (15) we get $\mu^* \in S^*$. We have to show that $\mu^* \neq 0$.

If it were not so, i.e. $\mu^* = 0$, then from (14) we should have

$$\langle \nu^*, h'(\overline{x}) d \rangle \geq 0 \quad (\forall\, d \in C(\overline{x})).$$

Due to assumption (iii), $B(d_0; \delta) \subset C(\overline{x})$, and hence,

$$\langle \nu^*, h'(\overline{x}) d \rangle \geq 0 \quad (\forall\, d \in B(d_0; \delta)) \tag{17}$$

For any $0 \neq d \in X$, since $B(d_0; \delta) - d_0$ is an open ball of radius δ centered at 0, it follows that $td \in B(d_0; \delta) - d_0$ $\left(\forall\, t \in \left(0, \dfrac{\delta}{\|d\|}\right)\right)$. Hence, $d_0 + td \in$

$B(d_0;\delta)$ $\Big(\forall\, t \in \Big(0, \dfrac{\delta}{\|d\|}\Big)\Big)$. It follows from this and assumption (i) that for all $t \in \Big(0, \dfrac{\delta}{\|d\|}\Big)$,

$$\langle \nu^*, h'(\overline{x})(d_0 + td)\rangle = t\langle \nu^*, h'(\overline{x})d\rangle \geq 0.$$

Consequently,

$$\langle \nu^*, h'(\overline{x})d\rangle \geq 0 \quad \text{for all } d \in X,\ d \neq 0.$$

This inequality holds trivially if $d = 0$. Hence,

$$\langle \nu^*, h'(\overline{x})d\rangle = 0 \quad \text{for all } d \in X. \tag{18}$$

Since $h'(\overline{x})$ is surjective, it follows from (18) that $\nu^* = 0$, which conflicts with $(\mu^*, \nu^*) \neq 0$. Therefore $\mu^* \neq 0$. Thus we have proved that there exist $\mu^* \in S^* \setminus \{0\}$ and $\nu^* \in Z^*$ such that (16) holds. But this contradicts (8), and hence, (9) holds.

Taking account of (9) yields that for any $x \in X$,

$$G(x) - G(\overline{x}) \in G'(\overline{x}, C(\overline{x})) + S \times \{O_z\},$$

which implies that there exists $d \in C(\overline{x})$ such that

$$G(x) - G(\overline{x}) \in G'(\overline{x}; d) + S \times \{O_z\}.$$

Setting $\omega(x) = d$, we obtain

$$G(x) - G(\overline{x}) - G'(\overline{x}; \omega(x)) \in S \times \{O_z\},$$

which means that

$$g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) \in S,$$
$$h(x) - h(\overline{x}) = h'(\overline{x})\omega(x)$$

This concludes the proof. $\square$

In case Y and Z are finite - dimensional, we have the following

Theorem 2.3. *Assume that* $\dim Y < +\infty$ *and* $\dim Z < +\infty$. *Suppose, furthermore, that h is Fréchet differentiable at $\overline{x}$, $g'(\overline{x}; .)$ is nearly S-convex and there exists $d_0 \in C(\overline{x})$ such that*

(i') $-g'(\overline{x}; d_0) \in \operatorname{int} S,\ \ h'(\overline{x})d_0 = 0$;

(ii') $h'(\overline{x})$ *is a surjective map from X onto Z.*

Then, there exists a map $\omega : X \to C(\overline{x})$ such that g is S-invex and h is $\{0\}$-invex at $\overline{x}$ with respect to the same scale ω.

Proof. By an argument analogous to that used for the proof of Theorem 2.2, we get the conclusion. But it should be noted here that, in the case of the finite-dimensional spaces Y and Z, to separate nonempty disjoint convex sets $\{u\}$ and $B := G'(\overline{x}; C(\overline{x})) + S \times \{O_z\}$ in the finite - dimensional space $Y \times Z$ it is not necessarily to require that $\operatorname{int} B$ is nonempty (see, for example, [16, Theorem 11.3]). Hence assumption (iii) in Theorem 2.2 can be omitted. □

In case h is not Fréchet differentiable, we have the following sufficient condition for invexity.

Theorem 2.4. *Assume that $G'(\overline{x}; .)$ is nearly $S \times \{O_z\}$-convexlike on $C(\overline{x})$, and the following conditions hold*

(a) *for all $(\mu, \nu) \in S^* \times Z^* \setminus \{0\}$, there exists $d \in C(\overline{x})$ such that*

$$\langle \mu, g'(\overline{x}; d)\rangle + \langle \nu, h'(\overline{x}; d)\rangle < 0, \tag{19}$$

(b) $\operatorname{int} h'(\overline{x}; C(\overline{x})) \neq \emptyset$*, and there is an open set $U \subset \operatorname{int} h'(\overline{x}; C(\overline{x}))$ such that for every $z \in U$, there exists $d \in C(\overline{x})$ satisfying*

$$-g'(\overline{x}; d) \in S, \quad h'(\overline{x}; d) = z.$$

Then, there exists a map $\omega : X \to C(\overline{x})$ such that for every $x \in X$,

$$\begin{aligned}
&g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) \in S,\\
&h(x) - h(\overline{x}) = h'(\overline{x}; \omega(x)).
\end{aligned}$$

Proof. We shall begin with showing that

$$G'(\overline{x}; C(\overline{x})) + S \times \{O_z\} = Y \times Z. \tag{20}$$

Contrary to this, suppose that

$$G'(\overline{x}; C(\overline{x})) + S \times \{O_z\} \subsetneq Y \times Z.$$

Then, there exists $u := (u_1, u_2) \in Y \times Z \setminus [G'(\overline{x}; C(\overline{x})) + S \times \{O_z\}]$. Putting $B := G'(\overline{x}; C(\overline{x})) + S \times \{O_z\}$, we prove that B is nearly convex. Obviously,

$$\begin{aligned}
B = \big\{(y, z) \in & Y \times Z : \exists\, d \in C(\overline{x}),\\
& y - g'(\overline{x}; d) \in S,\ h'(\overline{x}; d) = z\big\}
\end{aligned}$$

So taking (y_1, z_1) and $(y_2, z_2) \in B$, there are d_1 and $d_2 \in C(\overline{x})$, respectively, such that for $i = 1, 2$,

$$y_i - g'(\overline{x}; d_i) \in S, \tag{21}$$

$$h'(\overline{x}; d_i) = z_i. \tag{22}$$

Since $G'(\overline{x}; .)$ is nearly $S \times \{O_z\}$-convexlike, there exist $\alpha \in (0, 1)$ and $d_3 \in C(\overline{x})$ such that

$$\alpha g'(\overline{x}; d_1) + (1 - \alpha) g'(\overline{x}; d_2) - g'(\overline{x}; d_3) \in S, \tag{23}$$

$$\alpha h'(\overline{x}; d_1) + (1 - \alpha) h'(\overline{x}; d_2) = h'(\overline{x}; d_3). \tag{24}$$

By virtue of (21) and (22), it follows that

$$\alpha y_1 + (1 - \alpha) y_2 - \alpha g'(\overline{x}; d_1) - (1 - \alpha) g'(\overline{x}; d_2) \in S, \tag{25}$$

$$\alpha z_1 + (1 - \alpha) z_2 = \alpha h'(\overline{x}; d_1) + (1 - \alpha) h'(\overline{x}, d_2). \tag{26}$$

Combining (23) - (26) yields that

$$\begin{aligned} \alpha y_1 + (1 - \alpha) y_2 &\in \alpha g'(\overline{x}; d_1) + (1 - \alpha) g'(\overline{x}; d_2) + S \\ &\subset g'(\overline{x}; d_3) + S + S \\ &\subset g'(\overline{x}; d_3) + S, \end{aligned} \tag{27}$$

$$\alpha z_1 + (1 - \alpha) z_2 = h'(\overline{x}; d_3). \tag{28}$$

It follows from (27) and (28) that $\alpha(y_1, z_1) + (1 - \alpha)(y_2, z_2) \in B$. Hence B is nearly convex.

We now show that $\operatorname{int} B \neq \emptyset$. To do this, we take $(\overline{y}, \overline{z}) \in (\operatorname{int} S) \times U$ and show that $(\overline{y}, \overline{z})$ is an interior point of B. Since $\overline{y} \in \operatorname{int} S$ and $\overline{z} \in U$, there exists neighborhoods W_1 of $\overline{y}$ and W_2 of $\overline{z}$ such that $W_1 \subset S$ and $W_2 \subset U$. Taking any $(y, z) \in W_1 \times W_2$, in view of assumption (b), there exists $d \in C(\overline{x})$ such that

$$-g'(\overline{x}; d) \in S, \quad h'(\overline{x}; d) = z,$$

whence,

$$y - g'(\overline{x}; d) \in S + S \subset S.$$

So $(y, z) \in B$, and hence $W_1 \times W_2 \subset B$ and $(\overline{y}, \overline{z})$ is an interior point of B. Thus $\operatorname{int} B \neq \emptyset$. Due to Lemma 2.1 in [8], it follows that $\operatorname{int} B$ is convex.

According to the separation theorem 3.3 in [3], there exists $(\mu^*, \nu^*) \in Y^* \times Z^* \setminus \{0\}$ such that

$$\langle \mu^*, u_1\rangle + \langle \nu^*, u_2\rangle \leq \langle \mu^*, y\rangle + \langle \nu^*, z\rangle \quad (\forall\,(y,z) \in \operatorname{int} B),$$

which implies that

$$\langle \mu^*, y\rangle + \langle \nu^*, z\rangle \geq 0 \quad (\forall (y,z) \in \operatorname{int} B),$$

since $\operatorname{int} B$ is a cone. Hence,

$$\langle \mu^*, y\rangle + \langle \nu^*, z\rangle \geq 0 \quad (\forall\,(y,z) \in \overline{\operatorname{int} B} = \overline{B}),$$

which leads to the following

$$\langle \mu^*, y\rangle + \langle \nu^*, z\rangle \geq 0 \quad (\forall\,(y,z) \in B).$$

Consequently,

$$\langle \mu^*, y\rangle + \langle \nu^*, z\rangle \geq 0 \quad (\forall (y,z) \in G'(\overline{x}; C(\overline{x}))), \tag{29}$$

$$\langle \mu^*, y\rangle \geq 0 \quad (\forall\, y \in S). \tag{30}$$

By (30) one gets $\mu^* \in S^*$. It follows from (29) that

$$\langle \mu^*, g'(\overline{x}; d)\rangle + \langle \nu^*, h'(\overline{x}; d)\rangle \geq 0 \quad (\forall\, d \in C(\overline{x})),$$

which contradicts (19), and hence (20) holds.

Taking account of (20) we deduce that

$$G(x) - G(\overline{x}) \in G'(\overline{x}; C(\overline{x})) + S \times \{O_z\} \quad (\forall\, x \in X).$$

Hence, there is $d \in C(\overline{x})$ such that

$$G(x) - G(\overline{x}) \in G'(\overline{x}; d) + S \times \{O_z\} \quad (\forall\, x \in X).$$

Defining a map $\omega : x \mapsto \omega(x) = d$, we obtain

$$G(x) - G(\overline{x}) - G'(\overline{x}; \omega(x)) \in S \times \{O_z\} \quad (\forall\, x \in X),$$

which leads to the following

$$g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) \in S \quad (\forall\, x \in X),$$
$$h(x) - h(\overline{x}) = h'(\overline{x}; \omega(x)) \quad (\forall\, x \in X).$$

The proof is complete. □

In case Y and Z are finite-dimension, the following result shows that condition (b) in Theorem 2.4 can be omitted.

Theorem 2.5. *Assume that* $\dim Y < +\infty$, $\dim Z < +\infty$ *and* $G'(\overline{x};.)$ *is nearly* $S\times\{O_z\}$*-convexlike on* $C(\overline{x})$. *Suppose, furthermore, that for all* $(\mu,\nu) \in S^* \times Z^* \setminus \{0\}$, *there exists* $d \in C(\overline{x})$ *such that*

$$\langle \mu, g'(\overline{x};d)\rangle + \langle \nu, h'(\overline{x};d)\rangle < 0.$$

Then, there exists a map $\omega : X \to C(\overline{x})$ *such that* g *is* S*-invex and* h *is* $\{0\}$*-invex at* $\overline{x}$ *with respect to the same scale* ω.

Proof. By using a separation theorem for nonempty disjoint convex sets in the finite-dimensional space $Y \times Z$ (see. e.g., [16, Theorem 11.3]) and by an argument similar to that used for the proof of Theorem 2.4, we obtain the assertion of Theorem 2.5. □

3. Optimality conditions

In this section, we show that invexity conditions to g and h with respect to the same scale can be used as a constraint qualification for Problem (P).

We now recall a Fritz-John necessary condition in [10].

Defining the map $F = (f, g, h)$, we obtain
$F'(\overline{x};.) = (f'(\overline{x};.), g'(\overline{x};.), h'(\overline{x};.))$.

Proposition 3.1 (Fritz-John necessary condition [10]). *Let* $\overline{x}$ *be a local weak minimum of Problem (P). Assume that* f *and* g *are continuous and directionally differentiable at* $\overline{x}$ *in any direction* $d \in X$, h *is continuously Fréchet differentiable at* $\overline{x}$ *with Fréchet derivative* $h'(\overline{x})$ *is a surjective. Suppose, in addition, that* $f'(\overline{x};.)$ *is nearly* Q*-convex on* $C(\overline{x})$, $g'(\overline{x};.)$ *is nearly* S*-convex on* $C(\overline{x})$, $\operatorname{int} C(\overline{x}) \neq \emptyset$, *and*

$$\operatorname{int}\left[F'(\overline{x};C(\overline{x})) + Q \times S \times \{O_z\}\right] \neq \emptyset.$$

Then, there exist $\overline{\lambda} \in Q^*$, $\overline{\mu} \in S^*$ *and* $\overline{\nu} \in Z^*$ *with* $(\overline{\lambda},\overline{\mu},\overline{\nu}) \neq 0$ *such that*

$$\langle \overline{\lambda}, f'(\overline{x};d)\rangle + \langle \overline{\mu}, g'(\overline{x};d)\rangle + \langle \overline{\nu}, h'(\overline{x})d\rangle \geq 0 \quad (\forall\, d \in C(\overline{x})),$$
$$\langle \overline{\mu}, g(\overline{x})\rangle = 0.$$

A Kuhn-Tucker necessary condition for (P) can be stated as follows

Theorem 3.1 (Kuhn-Tucker necessary condition). *Assume that all the hypotheses of Proposition 3.1 are fulfilled. Then, there exist* $\overline{\lambda} \in Q^*$, $\overline{\mu} \in S^*$ *and* $\overline{\nu} \in Z^*$ *with* $(\overline{\lambda}, \overline{\mu}, \overline{\nu}) \neq 0$ *such that*

$$\langle \overline{\lambda}, f'(\overline{x}; d) \rangle + \langle \overline{\mu}, g'(\overline{x}; d) \rangle + \langle \overline{\nu}, h'(\overline{x}) d \rangle \geq 0 \quad (\forall\, d \in C(\overline{x})), \tag{31}$$

$$\langle \overline{\mu}, g(\overline{x}) \rangle = 0. \tag{32}$$

Moreover, if the following regularity conditions hold

(i) *there exists a map* $\omega : X \to C(\overline{x})$ *such that* g *is* S*-invex and* h *is* $\{0\}$*-invex at* $\overline{x}$ *with respect to the same scale* ω*;*

(ii) *there exists* $\hat{d} \in X$ *such that*

$$\langle \overline{\mu}, g(\hat{d}) \rangle + \langle \overline{\nu}, h(\hat{d}) \rangle < 0, \tag{33}$$

then $\overline{\lambda} \neq 0$.

Proof. We invoke Proposition 3.1 to deduce that there exist $\overline{\lambda} \in Q^*$, $\overline{\mu} \in S^*$ and $\overline{\nu} \in Z^*$ with $(\overline{\lambda}, \overline{\mu}, \overline{\nu}) \neq 0$ such that (31) and (32) hold.

Suppose now that assumptions (i) and (ii) hold. We have to prove that $\overline{\lambda} \neq 0$. If this were not so, that is $\overline{\lambda} = 0$, then from (31) we should have

$$\langle \overline{\mu}, g'(\overline{x}; d) \rangle + \langle \overline{\nu}, h'(\overline{x}) d \rangle \geq 0 \quad (\forall\, d \in C(\overline{x})). \tag{34}$$

Observe that condition (i) means that for all $x \in X$,

$$\begin{aligned} g(x) - g(\overline{x}) - g'(\overline{x}; \omega(x)) &\in S, \\ h(x) - h(\overline{x}) - h'(\overline{x}) \omega(x) &= 0, \end{aligned}$$

which leads to the following

$$G(x) - G(\overline{x}) - G'(\overline{x}; \omega(x)) \in S \times \{O_z\}.$$

Hence, there is $\hat{s} \in S$ such that

$$G(\hat{d}) - g(\overline{x}) - G'(\overline{x}; \omega(\hat{d})) = (\hat{s}, 0). \tag{35}$$

Combining (32), (33) and (35) yields that

$$\begin{aligned} &\langle \overline{\mu}, g(\hat{d}) \rangle + \langle \overline{\nu}, h(\hat{d}) \rangle = \langle \overline{\mu}, g(\overline{x}) + g'(\overline{x}; \omega(\hat{d})) \rangle \\ &\quad + \langle \overline{\nu}, h(\overline{x}) + h'(\overline{x}) \omega(\hat{d}) \rangle + \langle \overline{\mu}, \hat{s} \rangle \\ &= \langle \overline{\mu}, g'(\overline{x}; \omega(\hat{d})) \rangle + \langle \overline{\nu}, h'(\overline{x}) \omega(\hat{d}) \rangle + \langle \overline{\mu}, \hat{s} \rangle < 0. \end{aligned}$$

Since $\langle \overline{\mu}, \hat{s} \rangle \geq 0$, from this we obtain

$$\langle \overline{\mu}, g'(\overline{x}; \omega(\hat{d})) \rangle + \langle \overline{\nu}, h'(\overline{x})\omega(\hat{d}) \rangle < 0. \tag{36}$$

But $\omega(\hat{d}) \in C(\overline{x})$, so (36) conflicts with (34). Consequently, $\overline{\lambda} \neq 0$, as was to be shown. □

Remark 3.1. The regularity condition (ii), which can be called the generalized Slater condition, together with the invexity of g and h with respect to the same scale gives a constraint qualification for Problem (P).

The following statement is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Assume that $h = 0$ and all the hypotheses of Proposition 3.1 are fulfilled. Then, there exist $\overline{\lambda} \in Q^*$ and $\overline{\mu} \in S^*$ with $(\overline{\lambda}, \overline{\mu}) \neq 0$ such that*

$$\begin{aligned} &\langle \overline{\lambda}, f'(\overline{x}; d) \rangle + \langle \overline{\mu}, g'(\overline{x}; d) \rangle \geq 0 \quad (\forall\, d \in C(\overline{x})), \\ &\langle \overline{\mu}, g(\overline{x}) \rangle = 0. \end{aligned}$$

Moreover, if the following conditions hold

(i') *there exists a map $\omega : X \to C(\overline{x})$ such that g is S-invex at $\overline{x}$;*

(ii') *there exists $\hat{d} \in X$ such that*

$$-g(\hat{d}) \in \operatorname{int} S,$$

then $\overline{\lambda} \neq 0$.

Remark 3.2. The Slater condition (ii') in Corollary 3.1 together with the invexity of g gives a constraint qualification for Problem (P) without equality constraints.

References

1. A. Aleman, *On some generalizations of convex sets and convex functions*, Mathematica: Revue d'Analyse Numérique et de Théorie de l'Approximation. **14** (1985), 1-6.
2. B. D. Craven, *Invex functions and constrained local minima*, Bull. Austral. Math. Soc. **24** (1981), 357-366.
3. I. V. Girsanov, *Lectures on Mathematical Theory of Extremum Problems*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
4. N. X. Ha and D. V. Luu, *Sufficient conditions for invexity*, Bull. Austral. Math. Soc. **68** (2003), 113-125.
5. N. X. Ha and D. V. Luu, *Invexity of supremum and infimum functions*, Bull. Austral. Math. Soc. **65** (2002), 289-306.
6. M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), 545-550.

7. M. A. Hanson and N. G. Rueda, *A sufficient condition for invexity*, J. Math. Anal. Appl. **138** (1989), 193-198.
8. T. Illés and G. Kassay, *Theorems of the alternative and optimality conditions for convexlike and general convexlike programming*, J. Optim. Theory Appl. **101** (1999), 243-257.
9. A. Jourani, *Constraint qualifications and Lagrange multipliers in nondifferentiable programming problems*, J. Optim. Theory Appl. **81** (1994), 533-548.
10. P. T. Kien and D. V. Luu, *Optimality conditions in terms of directional derivatives*, East-West J. of Mathematics. **4** (2002), 119-136.
11. D. V. Luu and N. X. Ha, *An invariant property of invex functions and application*, Acta Math. Vietnam. **25** (2000), 181-193.
12. O. L. Mangasarian and S. Fromovitz, *The Fritz-John necessary optimality conditions in the presence of equality and inequality constraints*, J. Math. Anal. Appl. **17** (1967), 37-47.
13. V. H. Nguyen, J. J. Strodiot and R. Mifflin, *On conditions to have bounded multipliers in locally Lipschitz programming*, Math. Program. **18** (1980), 100-106.
14. T. W. Reiland, *Nonsmooth invexity*, Bull. Austral.Math. Soc. **42** (1990), 437-446.
15. S. M. Robinson, *Stability theory for systems of inequalities*, Part II: Differentiable nonlinear systems, SIAM J. Number. Anal. **13** (1976), 497-513.
16. R. T. Rockafellar, *Convex Analysis. Vol* **20**, Princeton University Press, Princeton, New Jersey, 1970.
17. E. V. Tamminen, *Sufficient condition for the existence of multipliers and Lagrangian duality in abstract optimization problems*, J. Optim. Theory Appl. **82** (1994), 93-104.

D. V. Luu
Institute of Mathematics
18 Hoang Quoc Viet Road
10307 Hanoi
Vietnam
E-mail address: dvluu@math.ac.vn

N. M. Hung
Faculty of Information Technology
Hanoi Water Resources University, Hanoi
Vietnam
E-mail address: ngmhung76@yahoo.fr