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INVEXITY OF CONSTRAINT MAPS IN MATHEMATICAL PROGRAMS

DO VAN LUU AND NGUYEN MANH HUNG

ABSTRACT. In this paper, we study a multiobjective programming problem with constraints of equality and inequality types which are maps from a Banach space into other Banach spaces. Sufficient conditions for the invexity of constraint maps with respect to the same scale map are established together with a new constraint qualification involving an invexity condition and a generalized Slater condition.

1. INTRODUCTION

In the last two decades, theory of invex functions has been the subject of much development (see, e.g., [2], [4]-[7], [11], [14]). The invexity of functions occurring in mathematical programming problems plays an important role in the theory of optimality conditions and duality. A question arises as to when constraints in a mathematical programming are invex at a point with respect to the same scale. Recently, Ha-Luu [4] have shown that the constraint qualifications of Robinson [15], Nguyen-Strodiot-Mifflin [13] and Jourani [9] types are sufficient conditions ensuring constraints of Lipschitzian mathematical programs to be invex with respect to the same scale. It should be noted that the single-objective mathematical programs there involve finitely many constraints of equality and inequality types which are locally Lipschitzian real-valued functions defined on a Banach space. Motivated by the results due to Ha-Luu [4], in this paper we shall deal with a multiobjective programming problems with constraints maps from a Banach space into other Banach spaces which are directionally differentiable. Sufficient conditions for the invexity of constraint maps with respect to the same scale map are established

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together with showing that the invexity of constraint maps along with an another suitable condition gives a new constraint qualification. After Introduction, Section 2 is devoted to derive sufficient conditions for the invexity of constraint maps with respect to the same scale map. The results show that known constraint qualifications together with a condition on the existence of interior points will ensure constraint maps to be invex with respect to the same scale. Section 3 gives a new constraint qualification which comprises an invexity condition and a generalized Slater condition.

2. SUFFICIENT CONDITIONS FOR INVEXITY

Let X, Y, Z, V be real Banach spaces, and let f, g, h be maps from X into V, Y, Z , respectively. Let Q, S be closed convex cones in V, Y , respectively, with vertices at the origin, $\text{int } Q \neq \emptyset$ and $\text{int } S \neq \emptyset$. Let C be a nonempty convex subset of X . In this paper, we shall be concerned with the following mathematical programming problem:

$$(P) \quad \begin{aligned} & W - \min f(x), \\ & \text{subject to} \\ & -g(x) \in S, \\ & h(x) = 0, \\ & x \in C, \end{aligned}$$

where W -min denotes the weak minimum with respect to the cone Q .

Denote by M the feasible set of (P):

$$M = \{x \in C : -g(x) \in S, h(x) = 0\}.$$

For $\bar{x} \in C$, we define the following set

$$C(\bar{x}) = \{\alpha(x - \bar{x}) : x \in C, \alpha \geq 0\}.$$

Then $C(\bar{x})$ is a convex cone with vertex at the origin. Denote by S^* the dual cone of S

$$S^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in S\},$$

where $\langle y^*, y \rangle$ is the value of the linear function $y^* \in Y^*$ at the point $y \in Y$. Y^* and Z^* will denote the topological duals of Y and Z , respectively.

The following notions are needed in the sequel.

Definition 2.1 [1]. A subset D of X is said to be nearly convex if there exists $\alpha \in (0, 1)$ such that for each $x_1, x_2 \in D$,

$$\alpha x_1 + (1 - \alpha)x_2 \in D.$$

Note that if D is nearly convex, then $\text{int } D$ is a convex set (see, e.g., [8, Lemma 2.1]. $\text{int } D$ here may be empty.

Definition 2.2 [8]. A map $F : D \rightarrow Y$ is called nearly S -convexlike on D if there exists $\alpha \in (0, 1)$ such that for every $x_1, x_2 \in D$, there is $x_3 \in D$ such that

$$\alpha F(x_1) + (1 - \alpha)F(x_2) - F(x_3) \in S.$$

Note that such a nearly S -convexlike map is simply called S -convexlike in [8]. A special case of nearly S -convexlike maps is nearly S -convex one.

Definition 2.3. Let D be a convex subset of X . A map $F : D \rightarrow Y$ is said to be nearly S -convex on D if there exists $\alpha \in (0, 1)$ such that for every $x_1, x_2 \in D$,

$$\alpha F(x_1) + (1 - \alpha)F(x_2) - F(\alpha x_1 + (1 - \alpha)x_2) \in S.$$

Recall that the directional derivative of f at \bar{x} , with respect to a direction d , is the following limit

$$f'(\bar{x}; d) = \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t},$$

if it exists. Throughout this paper, we suppose that f, g, h are directionally differentiable at \bar{x} in all directions.

Following [2, 14], the map g is called S -invex at \bar{x} if there exists a map ω from X into $C(\bar{x})$ such that for all $x \in X$,

$$g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) \in S.$$

Such a map ω is called a scale. When $S = \{0\}$ we get the notion of $\{0\}$ -invexity.

A sufficient condition for invexity of constraints in Problem (P) without equality constraints can be formulated as follows.

Theorem 2.1. *Assume that $h = 0$ and $g'(\bar{x}; \cdot)$ is nearly S -convexlike on $C(\bar{x})$. Suppose also that there exists $d_0 \in C(\bar{x})$ such that*

$$-g'(\bar{x}; d_0) \in \text{int } S \tag{1}$$

Then there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex at \bar{x} with respect to ω .

Proof. Put $A := g'(\bar{x}; C(\bar{x})) + S$, where $g'(\bar{x}; C(\bar{x})) := \{g'(\bar{x}; d) : d \in C(\bar{x})\}$. We first begin with showing that A is nearly convex.

For $y_1, y_2 \in A$, there exist $d_i \in C(\bar{x})$ and $s_i \in S$ ($i = 1, 2$) such that

$$y_i = g'(\bar{x}; d_i) + s_i \quad (i = 1, 2). \quad (2)$$

Since $g'(\bar{x}; \cdot)$ is nearly S -convexlike on $C(\bar{x})$, there exist $\alpha \in (0, 1)$ and $d_3 \in C(\bar{x})$ such that

$$\alpha g'(\bar{x}; d_1) + (1 - \alpha)g'(\bar{x}; d_2) - g'(\bar{x}; d_3) \in S. \quad (3)$$

Combining (2) and (3) yields that

$$\begin{aligned} \alpha y_1 + (1 - \alpha)y_2 &= \alpha g'(\bar{x}; d_1) + (1 - \alpha)g'(\bar{x}; d_2) + \alpha s_1 + (1 - \alpha)s_2 \\ &\in g'(\bar{x}; d_3) + S + S \\ &\subset g'(\bar{x}; C(\bar{x})) + S = A, \end{aligned}$$

which means that the set A is nearly convex. We invoke Lemma 2.1 in [8] to deduce that $\text{int } A$ is convex. Note that $\text{int } A \neq \emptyset$, since $\text{int } S \neq \emptyset$.

We now show that $A = Y$. Assume the contrary, that $A \subsetneq Y$. Then there exists $y_0 \in Y \setminus A$, and so $y_0 \notin \text{int } A$. Applying a separation theorem for the disjoint convex sets $\{y_0\}$ and $\text{int } A$ in Y (see, e.g., [3, Theorem 3.3]) yields the existence of $0 \neq y^* \in Y^*$ such that

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall y \in \text{int } A).$$

Since y^* is continuous on Y and $\text{int } A \neq \emptyset$, we obtain

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall y \in \overline{\text{int } A} = \overline{A}),$$

which implies that

$$\langle y^*, y_0 \rangle \leq \langle y^*, y \rangle \quad (\forall y \in A). \quad (4)$$

Since $g'(\bar{x}; \cdot)$ is positively homogeneous, $C(\bar{x})$ and S are cones, it follows that A is cone. Making use of Lemma 5.1 in [3], it follows from (4) that

$$\langle y^*, y_0 \rangle \leq 0 \leq \langle y^*, y \rangle \quad (\forall y \in A). \quad (5)$$

Observing that $0 \in S$, we have

$$\langle y^*, y \rangle \geq 0 \quad (\forall y \in g'(\bar{x}; C(\bar{x}))). \quad (6)$$

Moreover, since $g'(\bar{x}; \cdot)$ is positively homogeneous, it follows from (5) that

$$\langle y^*, y \rangle \geq 0 \quad (\forall y \in S)$$

which means that $y^* \in S^*$.

On the other hand, it follows readily from (6) that

$$\langle y^*, g'(\bar{x}; d) \rangle \geq 0 \quad (\forall d \in C(\bar{x})),$$

which leads to the following

$$\langle y^*, g'(\bar{x}; d_0) \rangle \geq 0,$$

which contradicts (1). Consequently, $A = Y$, i.e.,

$$g'(\bar{x}; C(\bar{x})) + S = Y. \quad (7)$$

It follows from (7) that for all $x \in X$,

$$g(x) - g(\bar{x}) \in g'(\bar{x}; C(\bar{x})) + S,$$

which implies that there exists $d \in C(\bar{x})$ such that

$$g(x) - g(\bar{x}) \in g'(\bar{x}; d) + S.$$

Defining a map $\omega : x \mapsto \omega(x) = d$, we obtain

$$g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) \in S.$$

The proof is complete. □

Denote by $B(\bar{x}; \delta)$ the open ball of radius δ around \bar{x} .

The following result shows that a generalized constraint qualification of Mangasarian-Fromovitz [12] type for infinite dimensional cases is a sufficient condition ensuring g to be S -invex and h is $\{0\}$ -invex at \bar{x} with respect to the same scale.

Theorem 2.2. *Assume that h is Fréchet differentiable at \bar{x} with Fréchet derivative $h'(\bar{x})$ and $g'(\bar{x}; \cdot)$ is nearly S -convex on $C(\bar{x})$. Suppose, in addition, that there exists $d_0 \in C(\bar{x})$ such that*

- (i) $-g'(\bar{x}; d_0) \in \text{int } S$, $h'(\bar{x})d_0 = 0$;
- (ii) $h'(\bar{x})$ is a surjective map from X onto Z ;
- (iii) there exists $\delta > 0$ such that $B(d_0; \delta) \subset C(\bar{x})$, and for every $z \in h'(\bar{x})(B(d_0; \delta))$, there exists $d \in B(d_0; \delta)$ satisfying

$$-g'(\bar{x}; d) \in S, \quad h'(\bar{x})d = z.$$

Then, there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex and h is $\{0\}$ -invex at \bar{x} with respect to the same scale ω , which means that for all $x \in X$,

$$\begin{aligned} g(x) - g(\bar{x}) - g'(\bar{x}, \omega(x)) &\in S, \\ h(x) - h(\bar{x}) &= h'(\bar{x})\omega(x). \end{aligned}$$

Note that the condition on existence of interior point like condition (iii) was introduced by Tamminen [17].

Proof of Theorem 2.2. We invoke assumption (i) to deduce that for all $\mu \in S^* \setminus \{0\}$, and $\nu \in Z^*$,

$$\langle \mu, g'(\bar{x}, d_0) \rangle + \langle \nu, h'(\bar{x})d_0 \rangle < 0. \quad (8)$$

In view of the differentiability of h at \bar{x} , putting $G = (g, h)$, one gets $G'(\bar{x}; \cdot) = (g'(\bar{x}; \cdot), h'(\bar{x})(\cdot))$.

We now show that

$$G'(\bar{x}; C(\bar{x})) + S \times \{O_z\} = Y \times Z. \quad (9)$$

Assume the contrary, that

$$G'(\bar{x}; C(\bar{x})) + S \times \{O_z\} \subsetneq Y \times Z.$$

This leads to the existence of a point $u := (u_1, u_2) \in Y \times Z$, but $u \notin G'(\bar{x}; C(\bar{x})) + S \times \{O_z\}$. Setting $B := G'(\bar{x}; C(\bar{x})) + S \times \{O_z\}$, we shall prove that B is nearly convex.

It is easy to see that

$$\begin{aligned} B = \{ &(y, z) \in Y \times Z : \exists d \in C(\bar{x}), \\ &y - g'(\bar{x}; d) \in S, \quad h'(\bar{x})d = z \}. \end{aligned}$$

Hence, taking $(y_i, z_i) \in B$ ($i = 1, 2$), there exist $d_i \in C(\bar{x})$ ($i = 1, 2$) such that

$$y_i - g'(\bar{x}; d_i) \in S, \quad h'(\bar{x})d_i = z_i \quad (i = 1, 2). \quad (10)$$

Since $g'(\bar{x}; \cdot)$ is nearly S -convex, there exists $\alpha \in (0, 1)$ such that

$$\alpha g'(\bar{x}; d_1) + (1 - \alpha)g'(\bar{x}; d_2) - g'(\bar{x}; \alpha d_1 + (1 - \alpha)d_2) \in S. \quad (11)$$

Moreover, it follows from (10) that

$$\alpha y_1 + (1 - \alpha)y_2 - \alpha g'(\bar{x}; d_1) - (1 - \alpha)g'(\bar{x}; d_2) \in S. \quad (12)$$

Combining (11) and (12) yields that

$$\begin{aligned} \alpha y_1 + (1 - \alpha)y_2 &\in g'(\bar{x}; \alpha d_1 + (1 - \alpha)d_2) + S + S \\ &\subset g'(\bar{x}; \alpha d_1 + (1 - \alpha)d_2) + S, \end{aligned}$$

which means that

$$\alpha y_1 + (1 - \alpha)y_2 - g'(\bar{x}; \alpha d_1 + (1 - \alpha)d_2) \in S \quad (13)$$

On the other hand,

$$\alpha z_1 + (1 - \alpha)z_2 = h'(\bar{x})(\alpha d_1 + (1 - \alpha)d_2),$$

which along with (13) yields that

$$\alpha(y_1, z_1) + (1 - \alpha)(y_2, z_2) \in B$$

Consequently, B is nearly convex. Due to Lemma 2.1 in [8], $\text{int } B$ is convex.

Next we shall prove that $\text{int } B \neq \emptyset$.

According to assumption (ii), $h'(\bar{x})$ is a surjective linear map from X onto Z , and hence $h'(\bar{x})$ is an open map. Therefore, $h'(\bar{x})(B(d_0; \delta))$ is an open nonempty subset of Z .

Taking $(\bar{y}, \bar{z}) \in (\text{int } S) \times h'(\bar{x})(B(d_0; \delta))$ yields that (\bar{y}, \bar{z}) is an interior point of B . Indeed, since $\bar{y} \in \text{int } S$ and $\bar{z} \in h'(\bar{x})(B(d_0; \delta))$, there exist neighborhoods U_1 of \bar{y} and U_2 of \bar{z} such that $U_1 \subset S$ and $U_2 \subset h'(\bar{x})(B(d_0; \delta))$, respectively. Taking any $(y, z) \in U_1 \times U_2$, due to assumption (iii), there exists $d \in B(d_0; \delta)$ such that

$$-g'(\bar{x}; d) \in S, \quad h'(\bar{x})d = z,$$

which implies that

$$y - g'(\bar{x}; d) \in S + S \subset S,$$

whence, $(y, z) \in B$. Consequently, $U_1 \times U_2 \subset B$ and (\bar{y}, \bar{z}) is an interior point of B , which means that $\text{int } B \neq \emptyset$.

Applying a separation theorem for the nonempty disjoint convex sets $\{u\}$ and $\text{int } B$ in $Y \times Z$ (see, e.g., [3, Theorem 3.3]) yields the existence of $(\mu^*, \nu^*) \in Y^* \times Z^* \setminus \{0\}$ satisfying

$$\langle \mu^*, u_1 \rangle + \langle \nu^*, u_2 \rangle \leq \langle \mu^*, y \rangle + \langle \nu^*, z \rangle \quad (\forall (y, z) \in \text{int } B)$$

Since B is a cone, making use of Lemma 5.1 in [3], we obtain

$$\langle \mu^*, u_1 \rangle + \langle \nu^*, u_2 \rangle \leq 0 \leq \langle \mu^*, y \rangle + \langle \nu^*, z \rangle \quad (\forall (y, z) \in \text{int } B).$$

Since $\text{int } B \neq \emptyset$, it follows that

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in \overline{\text{int } B} = \bar{B}),$$

where \bar{B} is the closure of B in normed topology. Hence,

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in B),$$

which leads to the following

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in G'(\bar{x}; C(\bar{x})), \quad (14)$$

$$\langle \mu^*, y \rangle \geq 0 \quad (\forall y \in S), \quad (15)$$

It follows from (14) that

$$\langle \mu^*, g'(\bar{x}; d) \rangle + \langle \nu^*, h'(\bar{x})d \rangle \geq 0 \quad (\forall d \in C(\bar{x})). \quad (16)$$

By (15) we get $\mu^* \in S^*$. We have to show that $\mu^* \neq 0$.

If it were not so, i.e. $\mu^* = 0$, then from (14) we should have

$$\langle \nu^*, h'(\bar{x})d \rangle \geq 0 \quad (\forall d \in C(\bar{x})).$$

Due to assumption (iii), $B(d_0; \delta) \subset C(\bar{x})$, and hence,

$$\langle \nu^*, h'(\bar{x})d \rangle \geq 0 \quad (\forall d \in B(d_0; \delta)) \quad (17)$$

For any $0 \neq d \in X$, since $B(d_0; \delta) - d_0$ is an open ball of radius δ centered at 0, it follows that $td \in B(d_0; \delta) - d_0$ ($\forall t \in (0, \frac{\delta}{\|d\|})$). Hence, $d_0 + td \in$

$B(d_0; \delta) \left(\forall t \in \left(0, \frac{\delta}{\|d\|} \right) \right)$. It follows from this and assumption (i) that for all $t \in \left(0, \frac{\delta}{\|d\|} \right)$,

$$\langle \nu^*, h'(\bar{x})(d_0 + td) \rangle = t \langle \nu^*, h'(\bar{x})d \rangle \geq 0.$$

Consequently,

$$\langle \nu^*, h'(\bar{x})d \rangle \geq 0 \quad \text{for all } d \in X, d \neq 0.$$

This inequality holds trivially if $d = 0$. Hence,

$$\langle \nu^*, h'(\bar{x})d \rangle = 0 \quad \text{for all } d \in X. \tag{18}$$

Since $h'(\bar{x})$ is surjective, it follows from (18) that $\nu^* = 0$, which conflicts with $(\mu^*, \nu^*) \neq 0$. Therefore $\mu^* \neq 0$. Thus we have proved that there exist $\mu^* \in S^* \setminus \{0\}$ and $\nu^* \in Z^*$ such that (16) holds. But this contradicts (8), and hence, (9) holds.

Taking account of (9) yields that for any $x \in X$,

$$G(x) - G(\bar{x}) \in G'(\bar{x}, C(\bar{x})) + S \times \{O_z\},$$

which implies that there exists $d \in C(\bar{x})$ such that

$$G(x) - G(\bar{x}) \in G'(\bar{x}; d) + S \times \{O_z\}.$$

Setting $\omega(x) = d$, we obtain

$$G(x) - G(\bar{x}) - G'(\bar{x}; \omega(x)) \in S \times \{O_z\},$$

which means that

$$\begin{aligned} g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) &\in S, \\ h(x) - h(\bar{x}) &= h'(\bar{x})\omega(x) \end{aligned}$$

This concludes the proof. □

In case Y and Z are finite - dimensional, we have the following

Theorem 2.3. *Assume that $\dim Y < +\infty$ and $\dim Z < +\infty$. Suppose, furthermore, that h is Fréchet differentiable at \bar{x} , $g'(\bar{x}; \cdot)$ is nearly S -convex and there exists $d_0 \in C(\bar{x})$ such that*

- (i') $-g'(\bar{x}; d_0) \in \text{int } S, \quad h'(\bar{x})d_0 = 0;$
- (ii') $h'(\bar{x})$ is a surjective map from X onto Z .

Then, there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex and h is $\{0\}$ -invex at \bar{x} with respect to the same scale ω .

Proof. By an argument analogous to that used for the proof of Theorem 2.2, we get the conclusion. But it should be noted here that, in the case of the finite-dimensional spaces Y and Z , to separate nonempty disjoint convex sets $\{u\}$ and $B := G'(\bar{x}; C(\bar{x})) + S \times \{O_z\}$ in the finite - dimensional space $Y \times Z$ it is not necessarily to require that $\text{int } B$ is nonempty (see, for example, [16, Theorem 11.3]). Hence assumption (iii) in Theorem 2.2 can be omitted. \square

In case h is not Fréchet differentiable, we have the following sufficient condition for invexity.

Theorem 2.4. Assume that $G'(\bar{x}; \cdot)$ is nearly $S \times \{O_z\}$ -convexlike on $C(\bar{x})$, and the following conditions hold

(a) for all $(\mu, \nu) \in S^* \times Z^* \setminus \{0\}$, there exists $d \in C(\bar{x})$ such that

$$\langle \mu, g'(\bar{x}; d) \rangle + \langle \nu, h'(\bar{x}; d) \rangle < 0, \quad (19)$$

(b) $\text{int } h'(\bar{x}; C(\bar{x})) \neq \emptyset$, and there is an open set $U \subset \text{int } h'(\bar{x}; C(\bar{x}))$ such that for every $z \in U$, there exists $d \in C(\bar{x})$ satisfying

$$-g'(\bar{x}; d) \in S, \quad h'(\bar{x}; d) = z.$$

Then, there exists a map $\omega : X \rightarrow C(\bar{x})$ such that for every $x \in X$,

$$\begin{aligned} g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) &\in S, \\ h(x) - h(\bar{x}) &= h'(\bar{x}; \omega(x)). \end{aligned}$$

Proof. We shall begin with showing that

$$G'(\bar{x}; C(\bar{x})) + S \times \{O_z\} = Y \times Z. \quad (20)$$

Contrary to this, suppose that

$$G'(\bar{x}; C(\bar{x})) + S \times \{O_z\} \subsetneq Y \times Z.$$

Then, there exists $u := (u_1, u_2) \in Y \times Z \setminus [G'(\bar{x}; C(\bar{x})) + S \times \{O_z\}]$. Putting $B := G'(\bar{x}; C(\bar{x})) + S \times \{O_z\}$, we prove that B is nearly convex. Obviously,

$$\begin{aligned} B = \{ &(y, z) \in Y \times Z : \exists d \in C(\bar{x}), \\ &y - g'(\bar{x}; d) \in S, h'(\bar{x}; d) = z \} \end{aligned}$$

So taking (y_1, z_1) and $(y_2, z_2) \in B$, there are d_1 and $d_2 \in C(\bar{x})$, respectively, such that for $i = 1, 2$,

$$y_i - g'(\bar{x}; d_i) \in S, \tag{21}$$

$$h'(\bar{x}; d_i) = z_i. \tag{22}$$

Since $G'(\bar{x}; \cdot)$ is nearly $S \times \{O_z\}$ -convexlike, there exist $\alpha \in (0, 1)$ and $d_3 \in C(\bar{x})$ such that

$$\alpha g'(\bar{x}; d_1) + (1 - \alpha)g'(\bar{x}; d_2) - g'(\bar{x}; d_3) \in S, \tag{23}$$

$$\alpha h'(\bar{x}; d_1) + (1 - \alpha)h'(\bar{x}; d_2) = h'(\bar{x}; d_3). \tag{24}$$

By virtue of (21) and (22), it follows that

$$\alpha y_1 + (1 - \alpha)y_2 - \alpha g'(\bar{x}; d_1) - (1 - \alpha)g'(\bar{x}; d_2) \in S, \tag{25}$$

$$\alpha z_1 + (1 - \alpha)z_2 = \alpha h'(\bar{x}; d_1) + (1 - \alpha)h'(\bar{x}; d_2). \tag{26}$$

Combining (23) - (26) yields that

$$\begin{aligned} \alpha y_1 + (1 - \alpha)y_2 &\in \alpha g'(\bar{x}; d_1) + (1 - \alpha)g'(\bar{x}; d_2) + S \\ &\subset g'(\bar{x}; d_3) + S + S \\ &\subset g'(\bar{x}; d_3) + S, \end{aligned} \tag{27}$$

$$\alpha z_1 + (1 - \alpha)z_2 = h'(\bar{x}; d_3). \tag{28}$$

It follows from (27) and (28) that $\alpha(y_1, z_1) + (1 - \alpha)(y_2, z_2) \in B$. Hence B is nearly convex.

We now show that $\text{int } B \neq \emptyset$. To do this, we take $(\bar{y}, \bar{z}) \in (\text{int } S) \times U$ and show that (\bar{y}, \bar{z}) is an interior point of B . Since $\bar{y} \in \text{int } S$ and $\bar{z} \in U$, there exists neighborhoods W_1 of \bar{y} and W_2 of \bar{z} such that $W_1 \subset S$ and $W_2 \subset U$. Taking any $(y, z) \in W_1 \times W_2$, in view of assumption (b), there exists $d \in C(\bar{x})$ such that

$$-g'(\bar{x}; d) \in S, \quad h'(\bar{x}; d) = z,$$

whence,

$$y - g'(\bar{x}; d) \in S + S \subset S.$$

So $(y, z) \in B$, and hence $W_1 \times W_2 \subset B$ and (\bar{y}, \bar{z}) is an interior point of B . Thus $\text{int } B \neq \emptyset$. Due to Lemma 2.1 in [8], it follows that $\text{int } B$ is convex.

According to the separation theorem 3.3 in [3], there exists $(\mu^*, \nu^*) \in Y^* \times Z^* \setminus \{0\}$ such that

$$\langle \mu^*, u_1 \rangle + \langle \nu^*, u_2 \rangle \leq \langle \mu^*, y \rangle + \langle \nu^*, z \rangle \quad (\forall (y, z) \in \text{int } B),$$

which implies that

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in \text{int } B),$$

since $\text{int } B$ is a cone. Hence,

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in \overline{\text{int } B} = \overline{B}),$$

which leads to the following

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in B).$$

Consequently,

$$\langle \mu^*, y \rangle + \langle \nu^*, z \rangle \geq 0 \quad (\forall (y, z) \in G'(\bar{x}; C(\bar{x}))), \quad (29)$$

$$\langle \mu^*, y \rangle \geq 0 \quad (\forall y \in S). \quad (30)$$

By (30) one gets $\mu^* \in S^*$. It follows from (29) that

$$\langle \mu^*, g'(\bar{x}; d) \rangle + \langle \nu^*, h'(\bar{x}; d) \rangle \geq 0 \quad (\forall d \in C(\bar{x})),$$

which contradicts (19), and hence (20) holds.

Taking account of (20) we deduce that

$$G(x) - G(\bar{x}) \in G'(\bar{x}; C(\bar{x})) + S \times \{O_z\} \quad (\forall x \in X).$$

Hence, there is $d \in C(\bar{x})$ such that

$$G(x) - G(\bar{x}) \in G'(\bar{x}; d) + S \times \{O_z\} \quad (\forall x \in X).$$

Defining a map $\omega : x \mapsto \omega(x) = d$, we obtain

$$G(x) - G(\bar{x}) - G'(\bar{x}; \omega(x)) \in S \times \{O_z\} \quad (\forall x \in X),$$

which leads to the following

$$g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) \in S \quad (\forall x \in X),$$

$$h(x) - h(\bar{x}) = h'(\bar{x}; \omega(x)) \quad (\forall x \in X).$$

The proof is complete. □

In case Y and Z are finite-dimension, the following result shows that condition (b) in Theorem 2.4 can be omitted.

Theorem 2.5. *Assume that $\dim Y < +\infty$, $\dim Z < +\infty$ and $G'(\bar{x}; \cdot)$ is nearly $S \times \{O_z\}$ -convexlike on $C(\bar{x})$. Suppose, furthermore, that for all $(\mu, \nu) \in S^* \times Z^* \setminus \{0\}$, there exists $d \in C(\bar{x})$ such that*

$$\langle \mu, g'(\bar{x}; d) \rangle + \langle \nu, h'(\bar{x}; d) \rangle < 0.$$

Then, there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex and h is $\{0\}$ -invex at \bar{x} with respect to the same scale ω .

Proof. By using a separation theorem for nonempty disjoint convex sets in the finite-dimensional space $Y \times Z$ (see. e.g., [16, Theorem 11.3]) and by an argument similar to that used for the proof of Theorem 2.4, we obtain the assertion of Theorem 2.5. □

3. OPTIMALITY CONDITIONS

In this section, we show that invexity conditions to g and h with respect to the same scale can be used as a constraint qualification for Problem (P).

We now recall a Fritz-John necessary condition in [10].

Defining the map $F = (f, g, h)$, we obtain

$$F'(\bar{x}; \cdot) = (f'(\bar{x}; \cdot), g'(\bar{x}; \cdot), h'(\bar{x}; \cdot)).$$

Proposition 3.1 (Fritz-John necessary condition [10]). *Let \bar{x} be a local weak minimum of Problem (P). Assume that f and g are continuous and directionally differentiable at \bar{x} in any direction $d \in X$, h is continuously Fréchet differentiable at \bar{x} with Fréchet derivative $h'(\bar{x})$ is a surjective. Suppose, in addition, that $f'(\bar{x}; \cdot)$ is nearly Q -convex on $C(\bar{x})$, $g'(\bar{x}; \cdot)$ is nearly S -convex on $C(\bar{x})$, $\text{int } C(\bar{x}) \neq \emptyset$, and*

$$\text{int } [F'(\bar{x}; C(\bar{x})) + Q \times S \times \{O_z\}] \neq \emptyset.$$

Then, there exist $\bar{\lambda} \in Q^$, $\bar{\mu} \in S^*$ and $\bar{\nu} \in Z^*$ with $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \neq 0$ such that*

$$\begin{aligned} \langle \bar{\lambda}, f'(\bar{x}; d) \rangle + \langle \bar{\mu}, g'(\bar{x}; d) \rangle + \langle \bar{\nu}, h'(\bar{x})d \rangle &\geq 0 \quad (\forall d \in C(\bar{x})), \\ \langle \bar{\mu}, g(\bar{x}) \rangle &= 0. \end{aligned}$$

A Kuhn-Tucker necessary condition for (P) can be stated as follows

Theorem 3.1 (Kuhn-Tucker necessary condition). *Assume that all the hypotheses of Proposition 3.1 are fulfilled. Then, there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in S^*$ and $\bar{\nu} \in Z^*$ with $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \neq 0$ such that*

$$\langle \bar{\lambda}, f'(\bar{x}; d) \rangle + \langle \bar{\mu}, g'(\bar{x}; d) \rangle + \langle \bar{\nu}, h'(\bar{x})d \rangle \geq 0 \quad (\forall d \in C(\bar{x})), \quad (31)$$

$$\langle \bar{\mu}, g(\bar{x}) \rangle = 0. \quad (32)$$

Moreover, if the following regularity conditions hold

(i) there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex and h is $\{0\}$ -invex at \bar{x} with respect to the same scale ω ;

(ii) there exists $\hat{d} \in X$ such that

$$\langle \bar{\mu}, g(\hat{d}) \rangle + \langle \bar{\nu}, h(\hat{d}) \rangle < 0, \quad (33)$$

then $\bar{\lambda} \neq 0$.

Proof. We invoke Proposition 3.1 to deduce that there exist $\bar{\lambda} \in Q^*$, $\bar{\mu} \in S^*$ and $\bar{\nu} \in Z^*$ with $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \neq 0$ such that (31) and (32) hold.

Suppose now that assumptions (i) and (ii) hold. We have to prove that $\bar{\lambda} \neq 0$. If this were not so, that is $\bar{\lambda} = 0$, then from (31) we should have

$$\langle \bar{\mu}, g'(\bar{x}; d) \rangle + \langle \bar{\nu}, h'(\bar{x})d \rangle \geq 0 \quad (\forall d \in C(\bar{x})). \quad (34)$$

Observe that condition (i) means that for all $x \in X$,

$$\begin{aligned} g(x) - g(\bar{x}) - g'(\bar{x}; \omega(x)) &\in S, \\ h(x) - h(\bar{x}) - h'(\bar{x})\omega(x) &= 0, \end{aligned}$$

which leads to the following

$$G(x) - G(\bar{x}) - G'(\bar{x}; \omega(x)) \in S \times \{O_z\}.$$

Hence, there is $\hat{s} \in S$ such that

$$G(\hat{d}) - g(\bar{x}) - G'(\bar{x}; \omega(\hat{d})) = (\hat{s}, 0). \quad (35)$$

Combining (32), (33) and (35) yields that

$$\begin{aligned} \langle \bar{\mu}, g(\hat{d}) \rangle + \langle \bar{\nu}, h(\hat{d}) \rangle &= \langle \bar{\mu}, g(\bar{x}) + g'(\bar{x}; \omega(\hat{d})) \rangle \\ &+ \langle \bar{\nu}, h(\bar{x}) + h'(\bar{x})\omega(\hat{d}) \rangle + \langle \bar{\mu}, \hat{s} \rangle \\ &= \langle \bar{\mu}, g'(\bar{x}; \omega(\hat{d})) \rangle + \langle \bar{\nu}, h'(\bar{x})\omega(\hat{d}) \rangle + \langle \bar{\mu}, \hat{s} \rangle < 0. \end{aligned}$$

Since $\langle \bar{\mu}, \hat{s} \rangle \geq 0$, from this we obtain

$$\langle \bar{\mu}, g'(\bar{x}; \omega(\hat{d})) \rangle + \langle \bar{\nu}, h'(\bar{x})\omega(\hat{d}) \rangle < 0. \quad (36)$$

But $\omega(\hat{d}) \in C(\bar{x})$, so (36) conflicts with (34). Consequently, $\bar{\lambda} \neq 0$, as was to be shown. \square

Remark 3.1. The regularity condition (ii), which can be called the generalized Slater condition, together with the invexity of g and h with respect to the same scale gives a constraint qualification for Problem (P).

The following statement is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Assume that $h = 0$ and all the hypotheses of Proposition 3.1 are fulfilled. Then, there exist $\bar{\lambda} \in Q^*$ and $\bar{\mu} \in S^*$ with $(\bar{\lambda}, \bar{\mu}) \neq 0$ such that*

$$\begin{aligned} \langle \bar{\lambda}, f'(\bar{x}; d) \rangle + \langle \bar{\mu}, g'(\bar{x}; d) \rangle &\geq 0 \quad (\forall d \in C(\bar{x})), \\ \langle \bar{\mu}, g(\bar{x}) \rangle &= 0. \end{aligned}$$

Moreover, if the following conditions hold

- (i') there exists a map $\omega : X \rightarrow C(\bar{x})$ such that g is S -invex at \bar{x} ;
- (ii') there exists $\hat{d} \in X$ such that

$$-g(\hat{d}) \in \text{int } S,$$

then $\bar{\lambda} \neq 0$.

Remark 3.2. The Slater condition (ii') in Corollary 3.1 together with the invexity of g gives a constraint qualification for Problem (P) without equality constraints.

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