About the identifiability and observability of the SIR epidemic model with quarantine

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Abstract: We analyze the identifiability and observability of the well-known SIR epidemic model with an additional compartment Q of the sub-population of infected individuals that are placed in quarantine (SIQR model), considering that the flow of individuals placed in quarantine and the size of the quarantine population are known at any time. Then, we focus on the problem of identification of the model parameters and review different techniques.

Keywords: Parameters estimation, identifiability, observability, parameters identification.

1. INTRODUCTION

Many papers in epidemiology proposing a mathematical model using dynamical systems, face the problem of parameter estimation. In general, some parameters are given, extracted from the literature, while the remaining unknown parameters are estimated by fitting the model to some observed data, usually by means of an optimization algorithm based on least squares or maximum likelihood methods. Nevertheless, there is relatively few studies about the intrinsic property of a model to admit a unique set of parameters values for a given choice of measured variables. On the other hand, this is a question that is well known in automatic control. Investigation of identifiability in mathematical epidemiology is relatively recent Tuncer et al. (2016); Perasso et al. (2011); Xia and Moog (2003); Saccomani (2011); Miao et al. (2011); Eisenberg et al. (2013); Evans et al. (2005). Indeed, to the best of our knowledge, the first paper considering the identifiability problem in a epidemic model is the paper of Xia and Moog (2003). It is also surprising that the observability and identifiability of the original Kermack-Mckendrick model has not been more studied, since this system has been widely used to model an outbreak of an infection. The observability and identifiability of the classical model SIR, with demography and constant population, was first studied in 2005 Evans et al. (2005). For practical parameters identification with state reconstruction, different approaches can be considered, most of them using observers (Bichara et al. (2014); Diaby et al. (2015); Degue et al. (2021); Iggidr and Souza (2019); Hamelin et al. (2020); Fang et al. (2022) and references therein) but these approaches are rarely considered in the literature compared to least-square methods, differently to other applications domains, such as bio-processing or manufacturing industries.

The global health crisis of COVID-19 outbreak has led to a spectacular resurgence of interest in this type of models, but with specificities related to the detection and isolation of infected individuals. This is why we revisit here the issues of identification and observability for an extended 'SIQR' model (Hethcote et al. (2002)) for which such an analysis has not yet been performed (to the best of our knowledge). Once the question of identification has been settled (Sections 3,4,5), we tackle in Section 6 the task of proposing a practical strategy for reconstructing the values of the parameters.

2. THE MODELS

Inspired by Hethcote et al. (2002), we consider the classical SIR model, where S, I, R denote the size of the populations of respectively susceptible, infected and recovered individuals, with an additive compartment where Q denotes the size of the population of identified and isolated infectious individuals that have been removed from the infected population and placed in quarantine:

$$\begin{cases} \dot{S} = -\beta S \dfrac{I}{N-Q}, & \dot{I} = \beta S \dfrac{I}{N-Q} - (\rho+\alpha)I \\ \dot{Q} = \alpha I - \rho Q, & \dot{R} = \rho I + \rho Q. \end{cases} \tag{1}$$

When the size of the total population N is large and the size of the population placed in quarantine remains small compared to N during the considered interval of time, one can consider a simplified model:

$$\begin{cases} \dot{S} = -\beta S \dfrac{I}{N}, & \dot{I} = \beta S \dfrac{I}{N} - (\rho+\alpha)I, \\ \dot{Q} = \alpha I - \rho Q, & \dot{R} = \rho I + \rho Q. \end{cases} \tag{2}$$

For both models, one has $S(t) + I(t) + Q(t) + R(t) = N, \quad \forall t \geq 0$. Since $S(t) + I(t) + Q(t) + R(t)$ is constant

and R has no influence on the first three equations, it is sufficient to consider the following system

$$\begin{cases} \dot{S} = -\beta S\dfrac{I}{N}, \qquad \dot{I} = \beta S\dfrac{I}{N} - (\rho+\alpha)I, \\ \dot{Q} = \alpha I - \rho Q, \end{cases} \tag{3}$$

It can be proved that the open set $\Omega = \{S > 0,\ I > 0,\ Q > 0,\ S + I + Q < N\}$ is positively invariant.

These models have three parameters: the infectivity parameter β, the recovery rate ρ, that we assume to be identical for the infected populations placed in quarantine or not, and the rate of placement in quarantine α. Theses parameters are unknown but we assume the following hypothesis.

Assumption 1. The reproduction number $\mathcal{R}_0$ verifies

$$\mathcal{R}_0 := \frac{\beta}{\rho+\alpha} > 1\,.$$

This assumption implies that the epidemic can spread in the population i.e. at initial time with $S(0) = N - I(0)$ close to N one has $\dot{I}(0) > 0$.

3. THE IDENTIFICATION PROBLEM

We assume that

- the flow $\alpha I(t)$ of infected people placed in quarantine is known at any time $t \geq 0$
- the size $Q(t)$ of the population placed in quarantine is perfectly known at any time $t \geq 0$
- the size N of the total population is known
- at initial time 0, one has $S(0) = N - \varepsilon$, $I(0) = \varepsilon$, $Q(0) = 0$, $R(0) = 0$ with $\varepsilon \in (0, N)$.

We consider then the observation function

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} := \begin{bmatrix} \alpha I(t) \\ Q(t) \end{bmatrix} \tag{4}$$

and follow the usual definitions of identifiability and observability of systems Walter and Pronzato (1997); Gauthier and Kupka (1996). However, note that when $Q = 0$ or $I = 0$ the system is not infinitesimally identifiable: the knowledge of the outputs and all its derivative do not allow to determine formally ρ. We adopt the following definition of identifiability for these models.

Definition 2. Given $N > 0$ and $\varepsilon \in (0, N)$, we shall say that system (1) resp. (2) is identifiable for the observation (4) if there exists $t > 0$ such that the map

$$(\,\alpha, \beta, \rho\,) \in \left(\mathbb{R}_+^\star\right)^3 \quad \longmapsto \quad y(\cdot) \in \mathcal{C}^\infty([0,t], \mathbb{R}_+^2)$$

is injective, where $(S(\cdot), I(\cdot), Q(\cdot), R(\cdot))$ is solution of the Cauchy problem for the differential system (1) resp. (2) with $S(0) = N - \varepsilon$, $I(0) = \varepsilon$, $Q(0) = 0$ and $R(0) = 0$.

If moreover the map

$$(\,\alpha, \beta, \rho, \varepsilon\,) \in \left(\mathbb{R}_+^\star\right)^3 \times (0, N) \longmapsto \quad y(\cdot) \in \mathcal{C}^\infty([0,t], \mathbb{R}_+^2)$$

is injective, then the system (1) resp. (2) is identifiable and observable for the observation (4).

4. ANALYSIS OF THE FIRST MODEL

Proposition 3. System (1) is identifiable and observable for the observation (4), in the sense of Definition 2.

Proof. It consists in showing that parameters and unmeasured state variables S, I and R can be expressed as functions of the successive derivatives of the output vector y. As the variable I cannot reach 0 in finite time, we shall assume $I \neq 0$ in the following.

Note first that with $Q(0) = 0$ one has $\dot{Q}(0) > 0$ and then $y_2(t) = Q(t) > 0$ for any $t > 0$. The dynamics of Q gives directly the expression of the parameter ρ as:

$$\rho = \frac{y_1(t) - \dot{y}_2(t)}{y_2(t)}\,, \quad t > 0. \tag{5}$$

Posit $h_1 := \dfrac{\dot{y}_1}{y_1}$. One has from the dynamics of I

$$h_1 = \frac{\beta\, S}{N - Q} - \alpha - \rho. \tag{6}$$

and then

$$(N - Q)\,\dot{h}_1 = \frac{\beta\, S}{N - Q}\left(-\beta\, I + \dot{Q}\right). \tag{7}$$

Using the equality $\dfrac{\beta\, S}{N - Q} = \alpha + h_1 + \rho$ from (6), one obtains from (7)

$$h_2 := (N - y_2)\,\dot{h}_1 = (h_1 + \alpha + \rho)\,(-\beta\, I + \dot{Q}). \tag{8}$$

Let us write the derivative of h_2:

$$\begin{aligned} \dot{h}_2 \ &= \dot{h}_1\,(-\beta\, I + \dot{Q}) \\ &+(h_1 + \alpha + \rho)\,\left[-\beta\,\frac{\beta\, S}{N - Q}\, I + \beta\,(\alpha + \rho)\, I \ + \ddot{Q}\right] \end{aligned}$$

which can be also expressed as

$$\dot{h}_2 \ = \dot{h}_1\,(-\beta\, I + \dot{Q}) + (h_1 + \alpha + \rho)\,(-h_1\,\beta\, I + \ddot{Q}).$$

Then, using relation (8), one obtains the expression

$$\begin{aligned} \dot{h}_2 \ = \ & \dot{h}_1\,(-\beta\, I + \dot{Q}) \\ & + \frac{h_2}{(-\beta\, I + \dot{Q})}\,\left[h_1\,(-\beta\, I + \dot{Q}) - h_1\,\dot{Q} + \ddot{Q}\right] \end{aligned}$$

which implies

$$\begin{aligned} \dot{h}_1\,(-\beta\, I + \dot{Q})^2 + (h_2\, h_1 - \dot{h}_2)(-\beta\, I + \dot{Q}) \\ + h_2\,(-h_1\,\dot{Q} + \ddot{Q}) = 0 \end{aligned}$$

Observe that this last equation is a second order polynomial in the variable $X = -\beta\, I + \dot{Q}$. Since $I(0) = \epsilon, R(0) = Q(0) = 0$ and $S(0) = N - \epsilon$. From this and $\mathcal{R}_0 > 1$, one has (using expression (7))

$$\dot{h}_1(0) = \frac{\beta\,\varepsilon}{N}\,(-\beta + \alpha)\left(1 - \frac{\varepsilon}{N}\right) < 0 \tag{9}$$

and this allows us to show that one also has

$$\begin{aligned} & h_2(0)\,\Big(\, - h_1(0)\,\dot{Q}(0) + \ddot{Q}(0)\Big) \\ & = \alpha\ \ \beta\,\rho\,\varepsilon^2(\beta - \alpha)\left(1 - \frac{\varepsilon}{N}\right) > 0 \end{aligned} \tag{10}$$

Observe also that one has $X(0) = -\beta\, I(0) + \dot{Q}(0) = \varepsilon\,(-\beta + \alpha) < 0$. Therefore, by continuity w.r.t. t, we obtain

that for $t > 0$ small enough, X is the unique negative solution of

$$\dot{h}_1 X^2 + (h_2 h_1 - \dot{h}_2) X + h_2 (-h_1 \dot{Q} + \ddot{Q}) = 0.$$

that is

$$X = \frac{-(h_2 h_1 - \dot{h}_2) - \sqrt{(h_2 h_1 - \dot{h}_2)^2 + 4\dot{h}_1(h_1 \dot{y}_2 - \ddot{y}_2)}}{2\dot{h}_1}.$$

The parameter α can be then obtained from equations (6)-(7)

$$\alpha = \frac{(N - y_2)\dot{h}_1}{X} - h_1 - \rho$$

where ρ is given by (5). The initial condition ε is simply reconstructed by $\varepsilon = y_1(0)/\alpha$ and finally one obtains the parameter $\beta = \alpha - X(0)/\varepsilon$.

5. ANALYSIS OF THE SIMPLIFIED MODEL

Proposition 4. System (2) is identifiable and observable for the observation (4), in the sense of Definition 2.

Proof. The proof is somehow simpler than the precedent. Using similar arguments we show that the parameters can be expressed as follows:

$$\begin{cases} \rho = \dfrac{y_1(t) - \dot{y}_2(t)}{y_2(t)}, \quad t > 0, \\ \alpha = -\dfrac{N\dot{g}_1(t)}{\beta I(t)} - g_1(t) = -\dfrac{N\dot{g}_1(t)}{g_2(t)} - g_1(t) := g_3(t), \\ \beta = \alpha \dfrac{\beta I(t)}{y_1(t)} = g_3(t)\dfrac{g_2(t)}{y_1(t)} = \dfrac{-N\dot{g}_1(t) - g_1(t)g_2(t)}{y_1(t)}. \end{cases}$$

Where $\begin{cases} g_1(t) := \dfrac{\dot{y}_1(t)}{y_1(t)} + \rho, \\ g_2(t) := -N\left(\dfrac{\ddot{g}_1(t)}{\dot{g}_1(t)} - g_1(t) + \rho\right) = \beta I(t), \\ g_3(t) := -\dfrac{N\dot{g}_1(t)}{g_2(t)} - g_1(t). \end{cases}$

At last, the initial condition is recovered as $\varepsilon = y_1(0)/\alpha$.

6. PRACTICAL ESTIMATION OF THE PARAMETERS

Till now we have studied structural observability / identifiability. While structural identifiability is a property of the model structure, given a set of outputs, practical identifiability is related to the experimental data. A model can be structurally identifiable, but still be practically unidentifiable due to poor data quality, e.g., bad signal-to-noise ratio, errors in measurement or sparse sampling Raue et al. (2009). This means parameters are identifiable with ideal data (in continuous time, noise-free data). However it does not guarantee that they will be practically identifiable with a finite number of noisy data points. Moreover estimation of parameters will use numerical algorithm and the distance, for the problem considered, to the nearest ill-posed problem (Demmel (1987)). Particularly this is the case when the system is near a non-observable equilibrium Banks et al. (2007). The main difficulty with epidemic models is that often parameters need to be reconstructed as early as possible at the beginning of the epidemic, in other words in the neighborhood of the Disease Free Equilibrium (DFE), which is a non-observable equilibrium.

Here after, we assume that the initial condition of model (1) or (2) is $(N - \varepsilon, \varepsilon, 0, 0)$ with unknown small $\varepsilon > 0$. Table 1 gives the parameters values used for data generation. We have considered daily measurements of variables

Table 1. Parameters used for the simulations

N	α	β	ρ	$I(0)$
10^6	0.1	0.4	0.1	20

y_1 and y_2 over a period of 30 days, first as "perfect" data (i.e. without noise) generated by the integration of model (3) (indeed data produced by model (1) or (2) are almost indistinguishable over this time period).

6.1 The classical least-square method

We have first tested the classical least-square method on the perfect data (without noise) over increasing time intervals $[0, T]$ with $T = 2, \cdots, 30$, using the Levenberg-Marquardt algorithm (`lsqrsolve` function in `Scilab` software). Fig. 1 shows that parameters values found by the algorithm are quite unstable depending on the time interval, making this method practically unreliable. Yet, one can see on Fig. 2 that the variables y_1 and y_2 are quite well reproduced (apart a few unrealistic points that are not represented).

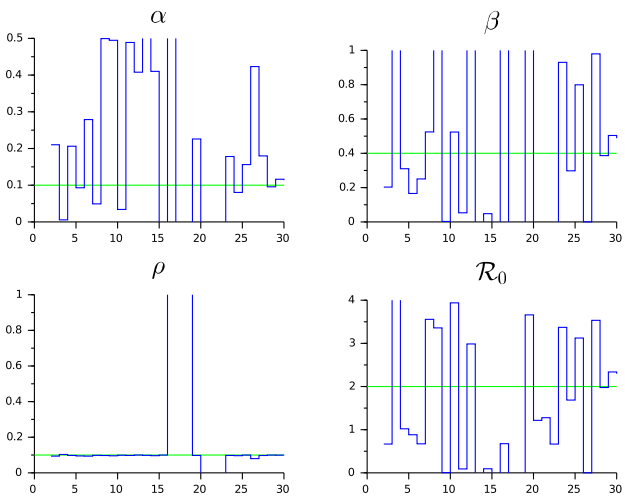

Fig. 1. Application of the least-square method with perfect data on increasing time intervals.

6.2 An alternative method

Here, we aim at exploiting the structure of the model (3) by decoupling the reconstruction of the parameter ρ from the estimations of α and β, and using an approximation of the dynamics of the variable I that is valid as long as S stays very large compared to I, for the reconstruction of α and β once ρ has been estimated.

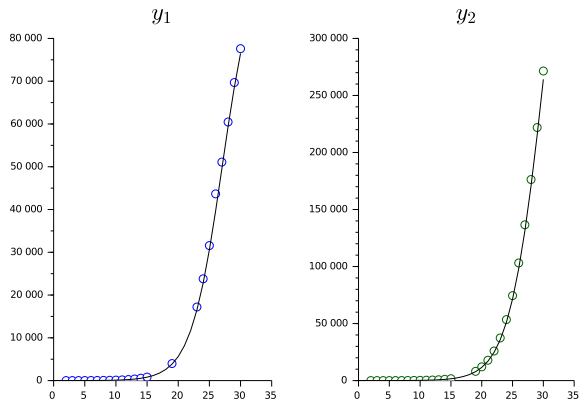

Fig. 2. Observations predicted by the least square estimation (output variables of the system in plain lines).

Reconstruction of parameter ρ. Following the identifiability analysis, parameter ρ can be easily identified integrating expression (5). Thus, the parameter ρ can be obtained by linear regression using the available values of the two outputs at the measurement times t_i by

$$\int_{t_i}^{t_{i+1}} y_1(\tau)d\tau + y_2(t_i) - y_2(t_{i+1}) = \rho \int_{t_i}^{t_{i+1}} y_2(\tau)d\tau$$

Assuming that measurements are done daily (the unit of time here is a day) the latter can be approximated by the expression

$$\frac{y_1(t+1)+y_1(t)}{2} + y_2(t) - y_2(t+1) = \rho \frac{y_2(t+1)+y_2(t)}{2}$$

for $t = 0, 1, ..., n$.

Reconstruction of parameters α and β. Here, we suppose that ρ is known, i.e. we use the approximation of ρ given before. Posit

$$\eta := \frac{\beta}{N\alpha}.$$

Observe that the solution $S(\cdot)$ of model (2) satisfies

$$S(t) = S(0)e^{-\eta \int_0^t y_1(\tau)d\tau}, \quad t > 0$$

and thus the solution $I(\cdot)$ fulfills

$$\dot{I}(t) = \left(\beta \frac{S(0)}{N} e^{-\eta \int_0^t y_1(\tau)d\tau} - (\alpha+\rho)\right) I(t), \quad t > 0.$$

Under the assumption that $I(0) = \varepsilon$ is small compared to N, the dynamics of $y_1(\cdot)$ can then be approximated by the expression

$$\dot{y}_1(t) = \left(\beta e^{-\eta \int_0^t y_1(\tau)d\tau} - (\alpha+\rho)\right) y_1(t) \qquad (11)$$

for small time $t > 0$, from which one can straightforwardly check that one has also

$$\log\left(\frac{d}{dt}\log y_1(t) + \alpha + \rho\right) = \log(\beta) - \eta \int_0^t y_1(\tau)d\tau.$$

Therefore, the parameter α possesses the property that the variables

$$Y(t,\alpha) := \log\left(\frac{d}{dt}\log y_1(t) + \alpha + \rho\right), \; X(t) := \int_0^t y_1(\tau)d\tau$$

fulfill a linear dependency: $Y(t,\alpha) = \log(\beta) - \eta X(t)$. Consequently, parameter α can be determined as the value that gives a linear regression between variables $Y(t)$ and $X(t)$ for a set of values t close to 0. Then parameter β can be determined from the Y-intercept of the regression line. Practically, one filters the data $\log y_1(\cdot)$ to estimate the derivative of $\log y_1(\cdot)$ (for instance with moving average or polynomial adjustment methods) and then α is sought to minimizing the residual sum of squares in the linear regression of the $\{Y(\cdot), X(\cdot)\}$ data (see for instance Walter and Pronzato (1997)). The alternative method has been applied first on the same "perfect" data set (without noise) as before, and the parameters estimation is given on Fig. 3. As expected, the estimation of ρ is quite faithful and

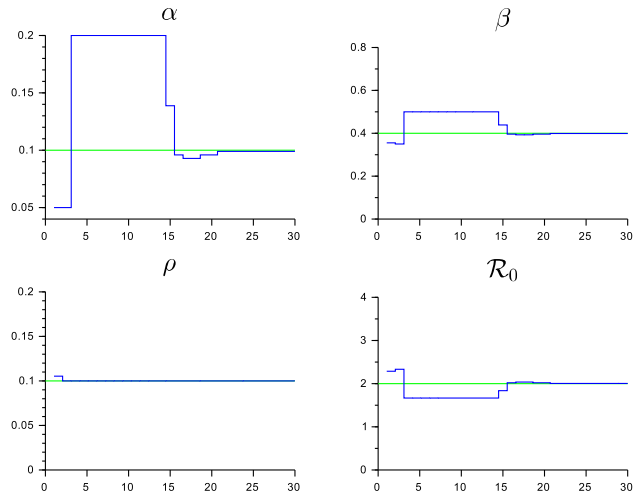

Fig. 3. Application of the alternative method on perfect data (without noise).

fast, while the estimation of α, β follows in sequence. We have also tested this method on the same data set but corrupted by an additive noise with a normal distribution of null mean value and standard deviation of 0.5 and 5 (see Fig. 4 and 5). In all cases, data have been filtered with a mobile average followed by a Lagrange interpolation. Time derivatives have been then computed analytically on the interpolated polynomial. One can see that the estimation of ρ remains always reliable, but requires a longer time to stabilize, which impacts the reconstruction of the other parameters, especially under large noise. However, the reproduction number $\mathcal{R}_0$ is estimated with a relatively good accuracy. Indeed, after 30 days, we move away from the validity of the approximation (11), which explains the bias obtained on α and β under large noise.

6.3 Numerical differentiation

For larger time windows, when the approximation (11) becomes less accurate, we investigate another approach based on numerical differentiation without any approximation of the system. Those techniques become today more

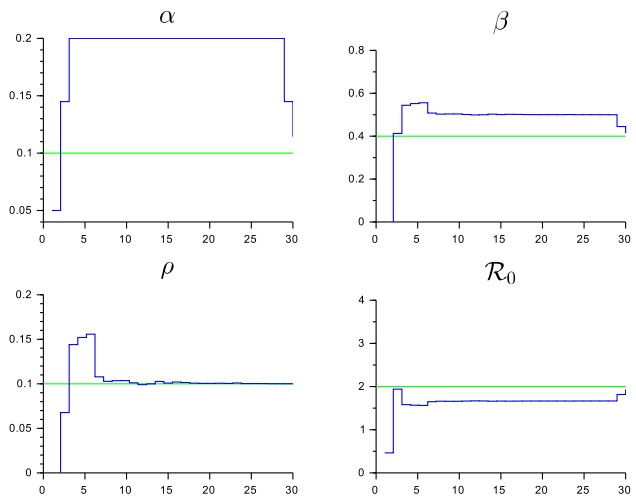

Fig. 4. Application of the alternative method on data with small noise.

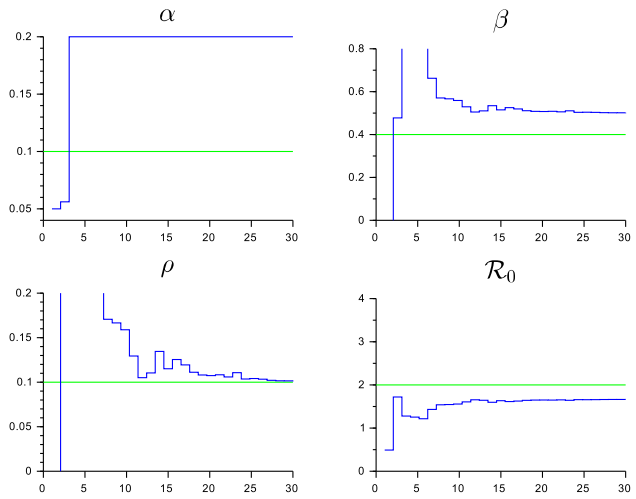

Fig. 5. Application of the alternative method on data with large noise.

prominent due to increasing interest in data driven models — cf. Kaiser et al. (2018) and references therein. Indeed, when the signals $y_1(\cdot)$ and $y_2(\cdot)$ become significantly larger (after 30 days in our simulations), we expect their time derivatives to be better estimated despite some noise.

Here we shall use a Total Variation Regularization (TVR) method for numerical differentiation firstly presented in Chartrand (2011) (see also Rudin et al. (1992)) for the original development of TVR applied to noise reduction in images. In what follows, we will refer to this approach to numerical differentiation as TVRD.

A brief review of TVR and numerical differentiation The use of Total Variation Regularization, as its name itself indicates, is a regularization approach of the following type:

$$F(u) = \kappa R(u) + M(Au - f),$$

where R is a functional that penalizes irregularity, A is the antiderivative operator and M is a metric functional that penalizes discrepancy between Au and f — κ is the regularization parameter that weighs both penalties. For TVR, as presented in Chartrand (2011), the metric functional is the L^2 norm, whereas the regularizing one is the total variation norm — namely, we have

$$F(u) = \kappa \int_0^L |u'| \, \mathrm{d}x + \frac{1}{2} \int_0^L |Au - f|^2 \, \mathrm{d}x. \tag{12}$$

For the numerical implementation of (12) we follow the original presentation in Chartrand (2011), for which we refer the reader for further details. In addition, we also use the selection framework developed in Van Breugel et al. (2020).

Estimation of ρ. We have

$$\rho y_2 = y_1 - \dot{y}_2 \tag{13}$$

The right hand side of (13) is computed by applying the TVRD to measurement y_2. An estimate $\hat{\rho}$ is then obtained by performing a linear regression.

Estimation of α and β. These parameters require a more involved estimation. We use the derivative of h_1 written in a convenient way as follows

$$-\dot{h}_1 = \eta y_1 \left(h_1 + \alpha\right), \quad \eta = \frac{\beta}{N\alpha}. \tag{14}$$

We then estimate η and α using a non linear least squares fitting. This estimation is usually accurate for η but not always for α. Thus we further estimate α by first computing

$$y_3 = \hat{\eta} y_1,$$

which is an estimation of $\frac{\beta I}{N}$.

In the sequel we then use a linear regression to estimate α using (14) in the following form

$$\left| -\left(\dot{h}_1 + y_3 h_1\right)\right| = \alpha y_3 \tag{15}$$

in order to obtain $\hat{\alpha}$. The absolute value is applied to ensure that the left hand side is non-negative and it works as a kind of filter for the errors introduced in the numerical differentiation.

Some numerical examples We simulated System (3) using the parameters $\rho = 0.1$, $\alpha = 0.07$, $\beta = 0.4$ and $N = 10^5$. The initial condition is $i_0 = 150N^{-1}$, $r_0 = 0$ and $s_0 = 1 - i_0$. This will be referred to as synthetic data. From this data we obtain the measurements y_1 and y_2. We also perturbed these measurements in two ways: a deterministic high frequency sinusoidal perturbation and a random one.

With synthetic data In this case, we plot the power spectra density of the data to be differentiated together with the cut off frequency used in each case. For data with slow decaying the power spectra we used half of the frequency of the grid, unless a sharper transition was identified. For data with fast decaying — typically $\ln y_1$ and h_1 we used either 10 or 20 times the grid frequency depending on how fast it is decaying. For these choices, we get the following estimates

$$\hat{\rho} = 0.107, \quad \hat{\alpha} = 0.0748, \quad \text{and} \quad \hat{\beta} = 0.413$$

With deterministic perturbation In this case we obtain, using a time sample of 40 days, the following estimates:

$$\hat{\rho} = 0.0972, \quad \hat{\alpha} = 0.0646, \quad \text{and} \quad \hat{\beta} = 0.375$$

In this case, the cutting frequency was first estimated as above, and if the result was still excessive wiggling (oscillations considerably greater than the average data magnitude) it was reduced until these oscillations were brought to this typical size.

With random perturbation. Selection of the cutting frequency was done in the same way as in the case of the deterministic perturbation.

Days	$\hat{\rho}$	$\hat{\alpha}$	$\hat{\beta}$
40	0.0987	0.162	0.659
50	0.098	0.0849	0.455
60	0.0996	0.0845	0.451
80	0.0987	0.0802	0.438

Further remarks

- The TVR method does not use any linear approximation and it is already in discrete form;
- TVR method can be improved by using TV of jerk (third derivative) which would give a smother differentiation;
- Integral version of the procedure can be also implemented — h_1 still need to be computed by differentiation however.

7. CONCLUSION

In this work, we have established the structural identifiability and observability of the SIR model with a quarantine compartment. We have also derived several methods to practically estimate the various parameters of the model. In particular, we have developed two complementary approaches adapted to the beginning of an epidemic or to its later development. In a forthcoming work, we shall investigate the practical estimation of these parameters using observers with application to some real data of COVID epidemics provided by various territories.

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