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Abstract

We address the fundamental issue of the optimality of the Balanced Growth Path (BGP) in a discrete-time version of Lucas (1988). After proving that the value function is supermodular and that any optimal solution is monotone, we prove that the BGP is optimal and that it is the unique optimal solution. Because of human capital depreciation, we also show that the economy can experience optimal endogenous degrowth.

Keywords: human capital, balanced growth path.
JEL codes: C61, D50, O40.

1 Introduction

Lucas (1988) is the most popular continuous-time model of economic growth with human capital. Because of its simplicity and versatility, it has served as a basis to many extensions either deterministic or stochastic.[1] One of the challenging issues this model faced was to ensure the uniqueness of the optimal solution. Here, we present a simple discrete-time version of Lucas (1988) which keeps intact the main ingredients of the original model, allowing for human capital depreciation and logarithmic preferences (and, hence, unbounded). Relying on the supermodularity of the value function, we prove that the BGP is the unique optimal solution. Moreover, if technology in human capital production is relatively low, then the economy can enter a process of endogeneous (optimal) degrowth.

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[1]See Ladrón de Guevara et al. (1997), Gómez (2003), Boucekkine and Ruiz-Tamarit (2004), La Torre and Marsiglio (2010), Bucci et al. (2011), and Gorostiaga et al. (2013).

The problem of the uniqueness of the optimal solution to the Lucas' model has been approached from many different angles in continuous-time settings. Early works focused on numerical simulations like Mulligan and Sala-i-Martin (1993); on local stability like Benhabib and Perli (1994); on global dynamics like Xie (1994). Xie (1994) shows that a continuum of equilibria may exist in the Lucas model with physical and human capital. Multiple equilibria exist if the external effect of human capital in goods production is large enough. Boucekkine and Ruiz-Tamarit (2004), Ruiz-Tamarit (2008) and Hiraguchi (2009) have addressed the uniqueness issue, by solving analytically the system of necessary optimal conditions.

In a discrete-time version of the Lucas model, Mitra (1998) proves the existence of equilibria with physical and human capital. Bethmann (2007, 2013) addresses the optimality problem and raises the uniqueness issue. Although he does obtain an optimal solution in both papers, he cannot prove that this solution is unique given that the value function is not necessarily unique. To the best of our knowledge, only Gourdel et al. (2004) proves the uniqueness of the optimal solution in a discrete-time version of the model.

Differently from us, Gourdel et al. (2004) consider externalities of human capital in production and do not allow for human capital depreciation. Worth noting, their social planner's solution internalizes the external effects of human capital and, in this sense, it is more general than ours. However, our contribution remains more general because we do consider human capital depreciation, which may trigger a process of optimal and endogenous degrowth. In addition, the uniqueness proofs provided in Gourdel et al. (2004) and the sustained growth result in Ha-Huy and Tran (2020), cannot be extended to include positive human capital depreciation. Furthermore, both papers require a bounded utility function, while our contribution encompasses the case of logarithmic preferences. Finally, the simplicity of our framework allows us, first, to provide a straightforward proof of uniqueness based on the supermodularity of the value function, and second, to compute the explicit trajectory of all economic variables along the BGP.

More precisely, the supermodularity of the value function entails the monotonicity of any optimal solution. By considering a convex or a linear technology with only one production factor, we show not only that the BGP is optimal, but also that any other alternative trajectory is inefficient.

Interestingly, a strictly concave production function is not incompatible with the existence, uniqueness and optimality of the BGP. In this respect, the model with human capital accumulation is different from the seminal AK model or other isomorphic models such as Romer (1986) (with productive externalities) or Barro (1990) (with public spending externalities).

Finally, note that human capital depreciation can be reinterpreted as human mortality under a stationary population age structure. Under this interpretation, it is the health component of human capital that is emphasized instead of the education component. Otherwise, it can also capture the aging process, where not only individuals' health declines with time, but also their intellectual capabilities diminish in terms of memory and reasoning power.

The article is structured as follows. In the first section, we introduce the model fundamentals and we write an equivalent agent's program involving an indirect utility. In the second section, we derive the BGP and prove its optimality and uniqueness. The last section concludes. All proofs are gathered in the Appendix.

2 A simple version of Lucas' model

Let agents share the same preferences and endowments, and let the size of population be constant and equal to one. Hence, variable l_t will denote at the same time both the individual and the aggregate supply of labor services at period t. Labor is the only factor required for the production of the unique final good, and technology is linear: $y_t = Al_t^\alpha$ with $\alpha \in (0, 1]$ (notice that the linear case is considered with $\alpha = 1$). Let us assume that all production is entirely consumed, that is $c_t = y_t$.

At any period t, each worker is endowed with one unit of labor, which she can spend either working or investing in human capital through education and health care. Accordingly, labor services are the product of the amount of the agent's human capital h_t and her working time u_t: $l_t \equiv h_t u_t$, with $u_t \in [0, 1]$. The remaining time $1 - u_t$ is devoted to human capital accumulation according to:

$$h_{t+1} - (1 - \delta) h_t \leq B (1 - u_t) h_t \tag{1}$$

for any $t \geq 0$. Note that human capital depreciates at a constant rate $\delta \in [0, 1]$.

The representative agent maximizes an intertemporal utility function where all utility comes from consumption, and where instantaneous utility is measured by a logarithmic function. Using that $c_t = y_t = Al_t^\alpha$, the agent maximizes

$$\sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \ln (Al_t^\alpha) = \frac{\ln A}{1 - \beta} + \alpha \sum_{t=0}^{\infty} \beta^t \ln l_t \tag{2}$$

which is equivalent to maximizing $\sum_{t=0}^{\infty} \beta^t \ln l_t$ by choosing the sequence of working times $(u_t)_{t=0}^{\infty}$, under the law of motion for the accumulation of human capital in (1).

Thus, the agent solves the following equivalent program:

$$\max \sum_{t=0}^{\infty} \beta^t \ln (h_t u_t) \tag{3}$$

$$h_{t+1} - (1 - \delta) h_t \leq B (1 - u_t) h_t$$

subject to $h_{t+1} \leq (1 - \delta) h_t + B (1 - u_t) h_t \in \Gamma (h_t)$ with

$$\Gamma (h_t) \equiv \{h_{t+1} \text{ such that } (1 - \delta) h_t \leq h_{t+1} \leq (1 - \delta + B) h_t\}$$

Since

$$l_t \equiv u_t h_t \leq \frac{1 - \delta + B}{B} h_t - \frac{1}{B} h_{t+1} \tag{4}$$

we can introduce an indirect function V, defined as

$$V\left(h_t, h_{t+1}\right) \equiv \ln\left(\frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}\right) = \ln l_t \tag{5}$$

Eventually, program (3) can be rewritten in terms of function V as

$$\max \sum_{t=0}^{\infty} \beta^t V\left(h_t, h_{t+1}\right) \tag{6}$$

subject to $h_{t+1} \in \Gamma\left(h_t\right)$ for any t.

In order to ensure that u_t remains in the interval $[0, 1]$ along the optimal path, we require the following assumption.

Assumption 1 *The speed of human capital accumulation is bounded from below:*

$$B > (1-\delta)\frac{1-\beta}{\beta} \tag{7}$$

3 The Balanced Growth Path

Observe that the cross-derivative of V, V_{12}, is positive, that is

$$V_{12}\left(h_t, h_{t+1}\right) = \frac{1}{B}\frac{1-\delta+B}{B}l_t^{-2} > 0$$

for any $t \geq 0$, proving that function V is supermodular.[2] Then, by Lemma 2.1 in Ha-Huy and Tran (2020), any optimal path to our problem is either strictly monotonic or constant. Hence, the optimal path is strictly increasing, strictly decreasing, or constant.

In Gourdel et al. (2004), the possibility of economic degrowth is discarded because human capital does not depreciate. Most importantly, their proof no longer works when the depreciation rate is strictly positive. To ensure sustained growth, Ha-Huy and Tran (2020) consider the following condition: $V_2\left(h, h\right) + \beta V_1\left(h, h\right) > 0$ for every $h > 0$, which, in our case, is equivalent to $\beta\left(1-\delta+B\right) > 1$. This condition, combined with the property that the utility function is bounded from below, ensures that every optimal path is strictly increasing.

As already mentioned, neither Gourdel et al. (2004) nor Ha-Huy and Tran (2020) can cover the case of an unbounded utility function.[3] Here, we do consider a logarithmic (and hence unbounded) utility function, which necessitates the development of a new approach to prove uniqueness. Therefore, our contribution is not simply a particular case or a limit case of Gourdel et al. (2004) or Ha-Huy and Tran (2020), but it does provide an added value to the literature and it complements their contributions.

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[2]See Amir (1996) among others.

[3]Ha-Huy and Tran (2020) also consider the case where utility function is unbounded from below, but their condition (3.2) in Proposition 3.3 is not satisfied in the case of our article.

Additionally, according to (1), any sequence $(h_t)_{t=0}^{\infty}$ satisfies that $(1-\delta)\,h_t \leq h_{t+1}$ for any $t \geq 0$. Then, along any optimal path, either $h_t = h_0$ for every $t \geq 0$, or $h_t < h_{t+1}$ for any $t \geq 0$, or $(1-\delta)\,h_t \leq h_{t+1} < h_t$ for any $t \geq 0$.

Proposition 1 (dynamic system) *Any optimal path to program (6) satisfies the sequence of first-order necessary conditions:*

$$h_{t+1}u_{t+1} = \beta\,(1-\delta+B)\,h_t u_t \tag{8}$$
$$h_{t+1}/h_t \leq 1-\delta+B\,(1-u_t) \tag{9}$$

for any $t \geq 0$.

The following proposition provides the explicit expression for the BGP.

Proposition 2 (Balanced Growth Path) *The set of optimal solutions described in (8)-(9) admits a BGP*

$$h_t = g^t h_0 \tag{10}$$
$$l_t = g^t h_0 u \tag{11}$$
$$c_t = g^{\alpha t} A h_0^{\alpha} u^{\alpha} \tag{12}$$

where $g \equiv \beta\,(1-\delta+B)$ *is the balanced growth factor, with*

$$u_t = u = \frac{1-\beta}{\beta}\frac{g}{B} \in (0,1) \tag{13}$$

for any $t \geq 0$*, because of Assumption 1. The intertemporal utility along the BGP is given by*

$$U = \frac{1}{1-\beta}\left(\ln c_0 + \frac{\alpha\beta}{1-\beta}\ln g\right) \tag{14}$$

with $c_0 = A h_0^{\alpha} u^{\alpha}$.[4]

Growth is balanced in the sense that human capital and labor services grow at the same constant rate: $h_{t+1}/h_t = l_{t+1}/l_t = g$, while production and consumption grow at the common rate: $y_{t+1}/y_t = c_{t+1}/c_t = g^{\alpha}$.

Next we provide with the main result of this paper. Proposition 3 below proves, first, that the BGP is optimal and, second, that the BGP is the unique optimal path.

[4] In the case of an isoelastic utility:

$$u\,(c_t) \equiv \frac{c_t^{1-1/\sigma}}{1-1/\sigma}$$

the results are the same: $h_t = g^t h_0$, $l_t = g^t h_0 u$, $c_t = y_t = A g^{\alpha t} h_0^{\alpha} u^{\alpha}$ with $u = 1 - (g-1+\delta)/B$, but, now, the mathematical expressions of the balanced growth factor involves the elasticity of intertemporal substitution σ:

$$g \equiv [\beta\,(1-\delta+B)]^{\frac{\sigma}{\sigma+\alpha-\sigma\alpha}}$$

Proposition 3 (uniqueness) *The balanced growth path* $(h_t, l_t)_{t=0}^{\infty}$ *starting from* (h_0, l_0) *and evolving in time according to (10) and (11) with*

$$u_0 = u = \frac{1-\beta}{\beta}\frac{g}{B}$$

is the unique optimal path.

Summarizing our results this far, we have proved that our version of Lucas' model in discrete time has an optimal solution, that this optimal solution is a BGP, and that this solution is unique. Furthermore, we have provided the explicit trajectory for capital and labor from $t = 0$.

Let us conclude our analysis by proving that, if technology in human capital production is relatively low, then the economy enters a process of endogeneous (optimal) degrowth.

Corollary 4 (optimal degrowth) *Let* $\delta < 1$. *The economic system experiences an optimal endogenous degrowth if and only if*

$$\frac{(1-\beta)(1-\delta)}{\beta} < B < \frac{1-\beta(1-\delta)}{\beta} \tag{15}$$

4 Conclusion

The Lucas (1988) model has become a benchmark model to study the accumulation of human capital and perpetual growth. Despite its major role, relatively few papers have delved with the fundamental question of the uniqueness of the optimal solution. Indeed, most of the literature focuses on the BGP because of its practical properties. In this paper, we have presented a simple version of Lucas' model in discrete time, which keeps intact the main ingredients of the original model: the role of human capital in production and the competition for labor between the production sector and the human capital sector. Taking advantage of the supermodularity of the value function, we prove that all optimal solutions must be monotonically increasing and that the set of optimal necessary conditions admits a BGP. Finally, we show that the BGP is the unique optimal solution to our version of Lucas' model. Our proof of existence and uniqueness of the optimal path also holds in the case of human capital depreciation and unbounded preferences, in which the proofs by Gourdel et al. (2004) and Ha-Huy and Tran (2020) no longer work. Because of human capital depreciation, our model can also exhibit optimal endogenous degrowth trajectories.

5 Appendix

Proof of Proposition 1

We maximize the Lagrangian function

$$\sum_{t=0}^{\infty} \beta^t \ln(h_t u_t) + \sum_{t=0}^{\infty} \lambda_t \left[(1-\delta) h_t + B(1-u_t) h_t - h_{t+1}\right]$$

with respect to the sequence $(h_{t+1}, u_t, \lambda_t)_{t=0}^{\infty}$.

Deriving with respect to $(h_{t+1}, u_t, \lambda_t)$, we obtain the first-order conditions

$$\begin{aligned} \lambda_t &= \frac{\beta^{t+1}}{h_{t+1}} + \lambda_{t+1}\left[1-\delta+B\left(1-u_{t+1}\right)\right] \\ \lambda_t &= \frac{\beta^t}{Bh_t u_t} \end{aligned}$$

jointly with (9), now binding. After eliminating the multipliers λ_t, we get the first-order conditions (8) and (9). ■

Proof of Proposition 2

Computing h_{t+1}/h_t from (8) and replacing it in (9), we obtain

$$\frac{1}{u_{t+1}} = \frac{1}{\beta}\left(\frac{1}{u_t} - \frac{B}{1-\delta+B}\right) \tag{16}$$

for any $t \geq 0$. Setting $u_{t+1} = u_t = u$ for any $t \geq 0$, we obtain the stationary state (13).

If $u_{t+1} = u_t = u$, then according to (8), we have that $h_{t+1} = \beta\left(1+B\right)h_t$, which, by induction yields (10). Since $l_t \equiv h_t u_t$, equation (8) also implies that $l_{t+1} = gl_t$ and, by induction, that $l_t = g^t l_0$ with $l_0 = h_0 u$. Since $c_t = Al_t^{\alpha}$, we also get (12). Using (11), we can find the expression for overall welfare in (14). Indeed one can write that

$$\sum_{t=0}^{\infty}\beta^t \ln l_t = \sum_{t=0}^{\infty}\beta^t \ln\left(g^t l_0\right) = \ln l_0 \sum_{t=0}^{\infty}\beta^t + \ln g \sum_{t=0}^{\infty} t\beta^t = \frac{\ln\left(h_0 u\right)}{1-\beta} + \frac{\beta \ln g}{\left(1-\beta\right)^2} \tag{17}$$

Replacing (17) in (2), we obtain (14). ■

Proof of Proposition 3

We first show that the BGP is optimal and, then, that the optimal solution is unique.

(1) The BGP for human capital, $(h_t)_{t=0}^{\infty}$, satisfies equations (8) and (9) with $u_t = u$ for any t according to (13). Along this BGP, and always according to (8) and (9), the optimal first-order condition of V must be verified, that is

$$V_2\left(h_t, h_{t+1}\right) + \beta V_1\left(h_{t+1}, h_{t+2}\right) = 0 \tag{18}$$

for any $t \geq 0$, where V_1 and V_2 denote the partial derivatives of (5) with respect to h_t and h_{t+1}. Moreover,

$$V_1\left(h_t, h_{t+1}\right) = \frac{1-\delta+B}{Bl_t} \text{ and } V_2\left(h_t, h_{t+1}\right) = -\frac{1}{Bl_t} = -\frac{1}{Buh_t}$$

where

$$l_t = \frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}$$

This implies

$$\lim_{t\to\infty}\beta^t V_2\left(h_t, h_{t+1}\right)h_{t+1} = -\lim_{t\to\infty}\left(\beta^t \frac{1}{Bu}\frac{h_{t+1}}{h_t}\right) = -\frac{\beta}{1-\beta}\lim_{t\to\infty}\beta^t = 0 \tag{19}$$

since $h_{t+1}/h_t = g$.

Let us compare the BGP solution for human capital, $(h_t)_{t=0}^{\infty}$, with any other feasible path $(h'_t)_{t=0}^{\infty}$ starting from h_0. Notice that $\ln l_t - \ln l'_t \geq (l_t - l'_t)/l_t$ because of the concavity of ln, where $\ln l_t = V(h_t, h_{t+1})$. Then, we find the difference in the value function at time t associated to these two paths:

$$\begin{aligned} & V(h_t, h_{t+1}) - V\left(h'_t, h'_{t+1}\right) = \ln l_t - \ln l'_t \\ \geq\ & \frac{l_t - l'_t}{l_t} = \frac{1-\delta+B}{Bl_t}(h_t - h'_t) - \frac{1}{Bl_t}\left(h_{t+1} - h'_{t+1}\right) \\ =\ & V_1(h_t, h_{t+1})(h_t - h'_t) + V_2(h_t, h_{t+1})\left(h_{t+1} - h'_{t+1}\right) \end{aligned}$$

Aggregating these differences in time, we can prove that the BGP dominates $(h'_t)_{t=0}^{\infty}$:

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) - \sum_{t=0}^{\infty} \beta^t V\left(h'_t, h'_{t+1}\right) \\ =\ & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[V(h_t, h_{t+1}) - V\left(h'_t, h'_{t+1}\right)\right] \\ \geq\ & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[V_1(h_t, h_{t+1})(h_t - h'_t) + V_2(h_t, h_{t+1})\left(h_{t+1} - h'_{t+1}\right)\right] \\ =\ & V_1(h_0, h_1)(h_0 - h'_0) + \beta \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t V_1(h_{t+1}, h_{t+2})\left(h_{t+1} - h'_{t+1}\right) \\ & + \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t V_2(h_t, h_{t+1})\left(h_{t+1} - h'_{t+1}\right) \\ & + \lim_{T\to\infty} \beta^T V_2(h_T, h_{T+1})\left(h_{T+1} - h'_{T+1}\right) \\ =\ & \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t \left[V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2})\right]\left(h_{t+1} - h'_{t+1}\right) \\ & + \lim_{T\to\infty} \beta^T V_2(h_T, h_{T+1}) h_{T+1} - \lim_{T\to\infty} \beta^T V_2(h_T, h_{T+1}) h'_{T+1} \\ =\ & -\lim_{T\to\infty} \beta^T V_2(h_T, h_{T+1}) h'_{T+1} \geq 0 \end{aligned}$$

because $h_0 = h'_0$, $V_2(h_T, h_{T+1}) < 0$ and because the first-order condition in (18) holds along the BGP, that is

$$V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2}) = 0$$

and $\lim_{T\to\infty} \beta^T V_2(h_T, h_{T+1}) h_{T+1} = 0$ according to (19).

Therefore, we have proven that $\sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t V\left(h'_t, h'_{t+1}\right)$ so that the BGP dominates any other feasible path.

(2) In order to prove the uniqueness of the optimal path $(h_t)_{t=0}^{\infty}$, the BGP, consider an alternative optimal path $(h'_t)_{t=0}^{\infty}$. We want to prove that $h_t = h'_t$ for any $t \geq 0$.

Assume that, to the contrary, $(h_t)_{t=0}^{\infty} \neq (h'_t)_{t=0}^{\infty}$ with $h_0 = h'_0$. Let $(c_t)_{t=0}^{\infty}$ and $(c'_t)_{t=0}^{\infty}$ denote the consumption paths associated to the optimal paths $(h_t)_{t=0}^{\infty}$ and $(h'_t)_{t=0}^{\infty}$. Then, $(c_t)_{t=0}^{\infty} \neq (c'_t)_{t=0}^{\infty}$ because, otherwise,

$$c_t = A\left(\frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}\right)^{\alpha} = c'_t = A\left(\frac{1-\delta+B}{B}h'_t - \frac{1}{B}h'_{t+1}\right)^{\alpha}$$

for any $t \geq 0$ and, since $h_0 = h'_0$, by induction, we would have that $h_t = h'_t$ for any $t \geq 0$, which would be a contradiction.

Define $h_t^{\lambda} \equiv \lambda h'_t + (1-\lambda) h_t$ for any t with $\lambda \in (0,1)$. We observe that since $(1-\delta) h_t \leq h_{t+1} \leq (1-\delta+B) h_t$ and $(1-\delta) h'_t \leq h'_{t+1} \leq (1-\delta+B) h'_t$, we have that $(1-\delta) h_t^{\lambda} \leq h_{t+1}^{\lambda} \leq (1-\delta+B) h_t^{\lambda}$ for any $t \geq 0$. Thus, the sequence $\left(h_t^{\lambda}\right)_{t=0}^{\infty}$ is feasible. Let u_t^{λ} be defined as

$$u_t^{\lambda} \equiv \frac{1-\delta+B}{B} - \frac{1}{B}\frac{h_{t+1}^{\lambda}}{h_t^{\lambda}}$$

The inequality $(1-\delta) h_t^{\lambda} \leq h_{t+1}^{\lambda} \leq (1-\delta+B) h_t^{\lambda}$ implies that $0 \leq u_t^{\lambda} \leq 1$. Therefore, according to (4), the consumption path $\left(c_t^{\lambda}\right)_{t=0}^{\infty}$ defined by $c_t^{\lambda} \equiv A\left(h_t^{\lambda} u_t^{\lambda}\right)^{\alpha}$ is also feasible. We observe that

$$\begin{aligned} c_t^{\lambda} &= A\left(\frac{1-\delta+B}{B}h_t^{\lambda} - \frac{1}{B}h_{t+1}^{\lambda}\right)^{\alpha} \\ &= A\left[\lambda\left(\frac{1-\delta+B}{B}h'_t - \frac{1}{B}h'_{t+1}\right) + (1-\lambda)\left(\frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}\right)\right]^{\alpha} \\ &= A\left[\lambda l'_t + (1-\lambda) l_t\right]^{\alpha} \geq \lambda A l_t'^{\alpha} + (1-\lambda) A l_t^{\alpha} = \lambda c'_t + (1-\lambda) c_t \end{aligned}$$

Since $c_t \neq c'_t$ for some t, we have that $\ln c_t^{\lambda} \geq \ln\left[\lambda c'_t + (1-\lambda) c_t\right] > \lambda \ln c'_t + (1-\lambda)\ln c_t$ for some t, because of the strict concavity of ln. We can then compute the associated overall welfare associated to both paths, multiplying by β^t and computing the infinite sum of all the per-period utilities. We obtain that

$$\sum_{t=0}^{\infty} \beta^t \ln c_t^{\lambda} > \lambda \sum_{t=0}^{\infty} \beta^t \ln c'_t + (1-\lambda) \sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \ln c_t$$

Hence, $\sum_{t=0}^{\infty} \beta^t \ln c_t^{\lambda} > \sum_{t=0}^{\infty} \beta^t \ln c_t$. This would imply that $(c_t)_{t=0}^{\infty}$ is no longer optimal, which is a contradiction.

We conclude that $(h)_{t=0}^{\infty} = (h')_{t=0}^{\infty}$, demonstrating that the BGP is the unique optimal path. ■

Proof of Corollary 4

The LHS of (15) holds because Assumption 1. The RHS is equivalent to $g < 1$ that is to $g < 1$, entailing a degrowth: $h_t = g^t h_0$. We observe that the interval

$$\left(\frac{(1-\beta)(1-\delta)}{\beta}, \frac{1-\beta(1-\delta)}{\beta}\right)$$

is always nonempty if $\delta < 1$. ■

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