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Stefano Bosi, Carmen Camacho, Thai Ha-Huy

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Abstract

We consider a simple discrete-time version of Lucas (1988). When the speed of human capital accumulation is high (low), the Balanced Growth (Degrowth) Path is the unique optimal solution.

Keywords: human capital, balanced growth path.
JEL codes: C61, D50, O40.

1 Introduction

We present a simple discrete-time version of Lucas (1988) which allows for human capital depreciation and logarithmic (and hence unbounded) preferences. The supermodularity of the value function grants the optimality and uniqueness of the BGP. Moreover, the economy can enter a process of endogenous (optimal) degrowth if technology in human capital production is relatively low.

In continuous time, early works on the uniqueness of the optimal solution to the Lucas' model focused on numerical simulations like Mulligan and Sala-i-Martin (1993); on local stability like Benhabib and Perli (1994); on global dynamics like Xie (1994). Boucekkine and Ruiz-Tamarit (2004), Ruiz-Tamarit (2008) and Hiraguchi (2009) have addressed the uniqueness issue, by solving analytically the system of necessary optimal conditions.[1]

In discrete time, Mitra (1998) proves the existence of equilibria with physical and human capital. To our knowledge, only Gourdel et al. (2004) prove the uniqueness of the optimal solution. Their social planner's solutions internalizes the external effects of human capital in production. However, the uniqueness

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[1]See also See Ladrón de Guevara et al. (1997), Gómez (2003), La Torre and Marsiglio (2010), Bucci et al. (2011), and Gorostiaga et al. (2013).

proofs provided in Gourdel et al. (2004) and the sustained growth result in Ha-Huy and Tran (2020) can not encompass positive human capital depreciation. Furthermore, both papers require a bounded utility function, while our contribution covers logarithmic preferences. Moreover, we provide a straightforward proof of uniqueness based on the supermodularity of the value function, and compute the explicit trajectory of all economic variables along the BGP. Considering a convex or a linear technology with only one production factor, we also show that any other alternative trajectory is inefficient.

Interestingly, a strictly concave production function for final goods can be compatible with the existence, uniqueness and optimality of the BGP as long as the production function for human capital remains linear. In this respect, the model with human capital accumulation is different from the seminal AK model or other isomorphic models as Romer (1986) (with productive externalities) or Barro (1990) (with public spending externalities).

We prove that positive growth requires a high speed of capital accumulation. Conversely, under a low speed, the economy experiences degrowth at a constant rate.

Finally, note that human capital depreciation can be reinterpreted as human mortality under a stationary population age structure, or as an aging process, where individuals' health and intellectual capabilities decline with time.

Section 2 introduces the model fundamentals; Section 3 derives the BGP and proves its optimality and uniqueness. All proofs are gathered in the Appendix.

2 A simple version of Lucas' model

Let agents share the same preferences and endowments, and let the size of population be constant and equal to one. Hence, variable l_t denotes at the same time individual and aggregate labor supply at period t. Labor is the only factor required for the production of the unique final good: $y_t = Al_t^\alpha$ with $\alpha \in (0, 1]$ (notice that the linear case is considered with $\alpha = 1$). Let us assume that all production is entirely consumed, that is $c_t = y_t$.

At any period t, each worker is endowed with one unit of labor, which she can spend either working or investing in human capital. Accordingly, labor services are the product of the amount of the agent's human capital h_t and her working time u_t: $l_t \equiv h_t u_t$, with $u_t \in [0, 1]$. The remaining time $1 - u_t$ is devoted to human capital accumulation according to:

$$h_{t+1} - (1 - \delta) h_t \leq B (1 - u_t) h_t \tag{1}$$

for any $t \geq 0$. Note that human capital depreciates at a constant rate $\delta \in [0, 1]$.

The representative agent maximizes an intertemporal utility function where all utility comes from consumption, and where instantaneous utility is measured by a logarithmic function. Using that $c_t = y_t = Al_t^\alpha$, the agent maximizes

$$\sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \ln (Al_t^\alpha) = \frac{\ln A}{1 - \beta} + \alpha \sum_{t=0}^{\infty} \beta^t \ln l_t \tag{2}$$

which is equivalent to maximizing $\sum_{t=0}^{\infty}\beta^t \ln l_t$ by choosing the sequence of working times $(u_t)_{t=0}^{\infty}$, subject to (1).

The agent solves the following equivalent program:

$$\max \sum_{t=0}^{\infty} \beta^t \ln(h_t u_t) \tag{3}$$
$$h_{t+1} - (1-\delta) h_t \leq B(1-u_t) h_t$$

subject to $h_{t+1} \leq (1-\delta) h_t + B(1-u_t) h_t \in \Gamma(h_t)$ with

$$\Gamma(h_t) \equiv \{h_{t+1} \text{ such that } (1-\delta) h_t \leq h_{t+1} \leq (1-\delta+B) h_t\}$$

Since

$$l_t \equiv u_t h_t \leq \frac{1-\delta+B}{B} h_t - \frac{1}{B} h_{t+1} \tag{4}$$

we can introduce an indirect function V

$$V(h_t, h_{t+1}) \equiv \ln\left(\frac{1-\delta+B}{B} h_t - \frac{1}{B} h_{t+1}\right) = \ln l_t \tag{5}$$

Program (3) can be rewritten in terms of V as

$$\max \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) \tag{6}$$
$$h_{t+1} \in \Gamma(h_t)$$

for any t.

3 Balanced growth and degrowth

The cross-derivative of V, V_{12}, is positive

$$V_{12}(h_t, h_{t+1}) = \frac{1}{B} \frac{1-\delta+B}{B} l_t^{-2} > 0$$

for any $t \geq 0$, proving that function V is supermodular.[2] Then, by Lemma 2.1 in Ha-Huy and Tran (2020), any optimal path to our problem is either strictly monotonic or constant.

The proof in Gourdel et al. (2004) no longer works when the depreciation rate is strictly positive and, thus, economic degrowth is impossible. To ensure sustained growth, Ha-Huy and Tran (2020) consider the condition: $V_2(h,h) + \beta V_1(h,h) > 0$ for every $h > 0$, which is equivalent here to $g_1 > 1$, where

$$g_1 \equiv \beta(1-\delta+B)$$

[2] See Amir (1996) among others.

as we will see, is a balanced growth factor. This condition, combined with the bounded from below utility function, ensures that every optimal path is strictly increasing. Neither Gourdel et al. (2004) nor Ha-Huy and Tran (2020) cover the case of unbounded utility function.[3] The results with logarithmic preferences we obtain, are not a limit case of their model but a significant added value.

Additionally, according to (1), any sequence $(h_t)_{t=0}^{\infty}$ satisfies that $(1-\delta)\,h_t \leq h_{t+1}$ for any $t \geq 0$. Then, along any optimal path, either $h_t = h_0$ for every $t \geq 0$, or $h_t < h_{t+1}$ for any $t \geq 0$, or $(1-\delta)\,h_t \leq h_{t+1} < h_t$ for any $t \geq 0$.

Proposition 1 (dynamic system) *Any optimal path to program (6) satisfies the sequence of first-order necessary conditions:*

$$\lambda_t = \frac{\beta^{t+1}}{h_{t+1}} + \lambda_{t+1}\left[1-\delta+B\left(1-u_{t+1}\right)\right] \leq \frac{\beta^t}{Bh_t u_t} \tag{7}$$

where λ_t is the Lagrangian multiplier associated to (1). The inequality on the right binds when $u_t < 1$. In this case, the dynamic system becomes

$$\begin{aligned} h_{t+1}u_{t+1} &= \beta\left(1-\delta+B\right)h_t u_t & (8)\\ h_{t+1}/h_t &\leq 1-\delta+B\left(1-u_t\right) & (9)\end{aligned}$$

for any $t \geq 0$.

The "speed" of human capital accumulation B is key for economic growth. Its critical value is given by

$$B^* \equiv (1-\delta)\,\frac{1-\beta}{\beta}$$

In the next proposition, we will show that, under a high speed ($B > B^*$), the economy experiences a balanced growth rate $g_1 - 1 \gtreqless 0$ and, under a low speed ($B \leq B^*$), a balanced degrowth rate $g_2 - 1 = -\delta \leq 0$ where

$$g_2 \equiv 1-\delta$$

is the capital depreciation factor.

Proposition 2 (balanced growth and degrowth) *(1) If $B > B^*$, the set of optimal solutions described in (8)-(9) admits a BGP with*

$$u_t = u = \frac{1-\beta}{\beta}\frac{g_1}{B} \in (0,1) \tag{10}$$

for any $t \geq 0$, that is

$$\begin{aligned} h_t &= g_1^t h_0 & (11)\\ l_t &= g_1^t h_0 u & (12)\\ c_t &= g_1^{\alpha t} A h_0^{\alpha} u^{\alpha} & (13)\end{aligned}$$

[3] Ha-Huy and Tran (2020) also consider an unbounded utility function from below, but their condition (3.2) in Proposition 3.3 fails in the case of our article.

where g_1 is the balanced growth factor. The intertemporal utility along the BGP is

$$U = \frac{1}{1-\beta}\left(\ln c_0 + \frac{\alpha\beta}{1-\beta}\ln g_1\right) \tag{14}$$

with $c_0 = Ah_0^\alpha u^\alpha$.[4]

(2) If $B \leq B^*$, set of optimal solutions described in (8)-(9) admits a balanced degrowth with $u_t = 1$ for any $t \geq 0$, that is

$$\begin{aligned} h_t &= g_2^t h_0 && (15)\\ l_t &\equiv g_2^t h_0 && (16)\\ c_t &= Ag_2^{\alpha t} h_0^\alpha && (17) \end{aligned}$$

The intertemporal utility along the BGP is still given by (14) where the degrowth factor g_2 replaces g_1.

Growth is balanced in both the cases since human capital and labor services grow at the same constant factor $h_{t+1}/h_t = l_{t+1}/l_t = g$, while production and consumption grow at the common rate: $y_{t+1}/y_t = c_{t+1}/c_t = g^\alpha$, with $g = g_1$ in case (1) of Proposition 2 and $g = g_2$ in case (2) of Proposition 2.

Proposition 3 provides with the main results of this paper: a sustained growth requires a sufficiently high speed of capital accumulation. Conversely, when B is low, it is better to work than to accumulate human capital.

Proposition 3 (uniqueness) *(1) If $B > B^*$, the BGP (11)-(12) is the unique optimal path with*

$$u_0 = u = \frac{1-\beta}{\beta}\frac{g_1}{B}$$

(2) If $B \leq B^*$, the BGP (15)-(16) is the unique optimal path with $u_t = 1$ for any $t \geq 0$.

Interestingly, the economy can experience an optimal degrowth even in the case (1) of Proposition 3.

Corollary 4 (optimal degrowth) *Let $B > B^*$ and $\delta < 1$. The economic system experiences an optimal endogenous degrowth if and only if*

$$B < \frac{1}{\beta} - (1-\delta) \tag{18}$$

[4] If utility is isoelastic:

$$u(c_t) \equiv \frac{c_t^{1-1/\sigma}}{1-1/\sigma}$$

the results are the same: $h_t = g_1^t h_0$, $l_t = g_1^t h_0 u$, $c_t = y_t = Ag_1^{\alpha t} h_0^\alpha u^\alpha$ with $u = 1 - (g_1 - 1 + \delta)/B$. However, now, the generalized growth factor involves σ:

$$g_1 \equiv [\beta(1-\delta+B)]^{\frac{\sigma}{\sigma+\alpha-\sigma\alpha}}$$

4 Appendix

Proof of Proposition 1

We maximize the Lagrangian function

$$\sum_{t=0}^{\infty} \beta^t \ln (h_t u_t) + \sum_{t=0}^{\infty} \lambda_t \left[(1-\delta) h_t + B(1-u_t) h_t - h_{t+1}\right] + \sum_{t=0}^{\infty} \mu_t (1-u_t)$$

with respect to the sequence $(h_{t+1}, u_t, \lambda_t)_{t=0}^{\infty}$.

Deriving with respect to $(h_{t+1}, u_t, \lambda_t)$, we obtain the first-order conditions

$$\begin{aligned} \lambda_t &= \frac{\beta^{t+1}}{h_{t+1}} + \lambda_{t+1} \left[1 - \delta + B(1-u_{t+1})\right] \\ \lambda_t &= \frac{\beta^t}{B h_t u_t} - \frac{\mu_t}{B h_t} \leq \frac{\beta^t}{B h_t u_t} \end{aligned} \tag{19}$$

jointly with (9), now binding. When $\mu_t = 0$, after eliminating the multipliers λ_t, we get the first-order conditions (8) and (9). ■

Proof of Proposition 2

(1) Computing h_{t+1}/h_t from (8) and replacing it in (9),

$$\frac{1}{u_{t+1}} = \frac{1}{\beta}\left(\frac{1}{u_t} - \frac{B}{1-\delta+B}\right) \tag{20}$$

for any $t \geq 0$. Setting $u_{t+1} = u_t = u$ for any $t \geq 0$, we obtain the stationary state (10).

If $u_{t+1} = u_t = u$, then according to (8), we have that $h_{t+1} = \beta(1+B) h_t$, which, by induction yields (11). Since $l_t \equiv h_t u_t$, equation (8) also implies that $l_{t+1} = g_1 l_t$ and, by induction, that $l_t = g_1^t l_0$ with $l_0 = h_0 u$. Since $c_t = A l_t^{\alpha}$, we also get (13). Using (12), we can find the expression for overall welfare in (14). Indeed, one can write that

$$\sum_{t=0}^{\infty} \beta^t \ln l_t = \sum_{t=0}^{\infty} \beta^t \ln\left(g_1^t l_0\right) = \ln l_0 \sum_{t=0}^{\infty} \beta^t + \ln g_1 \sum_{t=0}^{\infty} t\beta^t = \frac{\ln(h_0 u)}{1-\beta} + \frac{\beta \ln g_1}{(1-\beta)^2} \tag{21}$$

Replacing (21) in (2), we obtain (14).

(2) Simply replace $u_t = 1$ in (1). ■

Proof of Proposition 3

(1) We show that, first, the BGP is optimal and, second, the optimal solution is unique.

(1.1) The BGP for human capital, $(h_t)_{t=0}^{\infty}$, satisfies equations (8) and (9) with $u_t = u$ for any t according to (10). Along this BGP, always according to (8) and (9), the optimal first-order condition of V must be verified, that is

$$V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2}) = 0 \tag{22}$$

for any $t \geq 0$, where V_1 and V_2 denote the partial derivatives of (5) with respect to h_t and h_{t+1}. Moreover,

$$V_1(h_t, h_{t+1}) = \frac{1-\delta+B}{Bl_t} \text{ and } V_2(h_t, h_{t+1}) = -\frac{1}{Bl_t} = -\frac{1}{Buh_t}$$

where

$$l_t = \frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}$$

This implies

$$\lim_{t\to\infty} \beta^t V_2(h_t, h_{t+1}) h_{t+1} = -\lim_{t\to\infty}\left(\beta^t \frac{1}{Bu}\frac{h_{t+1}}{h_t}\right) = -\frac{\beta}{1-\beta}\lim_{t\to\infty}\beta^t = 0 \quad (23)$$

since $h_{t+1}/h_t = g_1$.

Let us compare the BGP solution for human capital, $(h_t)_{t=0}^{\infty}$, with any other feasible path $(h_t')_{t=0}^{\infty}$ starting from h_0. Notice that $\ln l_t - \ln l_t' \geq (l_t - l_t')/l_t$ because of the concavity of ln, where $\ln l_t = V(h_t, h_{t+1})$. Then, we find the difference in the value function at time t associated to these two paths:

$$\begin{aligned} & V(h_t, h_{t+1}) - V\left(h_t', h_{t+1}'\right) = \ln l_t - \ln l_t' \\ \geq\ & \frac{l_t - l_t'}{l_t} = \frac{1-\delta+B}{Bl_t}(h_t - h_t') - \frac{1}{Bl_t}\left(h_{t+1} - h_{t+1}'\right) \\ =\ & V_1(h_t, h_{t+1})(h_t - h_t') + V_2(h_t, h_{t+1})\left(h_{t+1} - h_{t+1}'\right) \end{aligned}$$

Aggregating these differences in time, we can prove that the BGP dominates

$(h'_t)_{t=0}^{\infty}$:

$$
\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t V\left(h_t, h_{t+1}\right) - \sum_{t=0}^{\infty} \beta^t V\left(h'_t, h'_{t+1}\right) \\
= \; & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[V\left(h_t, h_{t+1}\right) - V\left(h'_t, h'_{t+1}\right)\right] \\
\geq \; & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[V_1\left(h_t, h_{t+1}\right)\left(h_t - h'_t\right) + V_2\left(h_t, h_{t+1}\right)\left(h_{t+1} - h'_{t+1}\right)\right] \\
= \; & V_1\left(h_0, h_1\right)\left(h_0 - h'_0\right) + \beta \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t V_1\left(h_{t+1}, h_{t+2}\right)\left(h_{t+1} - h'_{t+1}\right) \\
& + \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t V_2\left(h_t, h_{t+1}\right)\left(h_{t+1} - h'_{t+1}\right) \\
& + \lim_{T\to\infty} \beta^T V_2\left(h_T, h_{T+1}\right)\left(h_{T+1} - h'_{T+1}\right) \\
= \; & \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t \left[V_2\left(h_t, h_{t+1}\right) + \beta V_1\left(h_{t+1}, h_{t+2}\right)\right]\left(h_{t+1} - h'_{t+1}\right) \\
& + \lim_{T\to\infty} \beta^T V_2\left(h_T, h_{T+1}\right) h_{T+1} - \lim_{T\to\infty} \beta^T V_2\left(h_T, h_{T+1}\right) h'_{T+1} \\
= \; & -\lim_{T\to\infty} \beta^T V_2\left(h_T, h_{T+1}\right) h'_{T+1} \geq 0
\end{aligned}
$$

because $h_0 = h'_0$, $V_2\left(h_T, h_{T+1}\right) < 0$, (22) holds along the BGP and

$$\lim_{T\to\infty} \beta^T V_2\left(h_T, h_{T+1}\right) h_{T+1} = 0$$

according to (23).

Therefore, we have proven that $\sum_{t=0}^{\infty} \beta^t V\left(h_t, h_{t+1}\right) \geq \sum_{t=0}^{\infty} \beta^t V\left(h'_t, h'_{t+1}\right)$, so that the BGP dominates any other feasible path.

(1.2) In order to prove the uniqueness of the optimal path $(h_t)_{t=0}^{\infty}$, the BGP, consider an alternative optimal path $(h'_t)_{t=0}^{\infty}$. We want to prove that $h_t = h'_t$ for any $t \geq 0$.

Assume that, to the contrary, $(h_t)_{t=0}^{\infty} \neq (h'_t)_{t=0}^{\infty}$ with $h_0 = h'_0$. Let $(c_t)_{t=0}^{\infty}$ and $(c'_t)_{t=0}^{\infty}$ denote the consumption paths associated to the optimal paths $(h_t)_{t=0}^{\infty}$ and $(h'_t)_{t=0}^{\infty}$. Then, $(c_t)_{t=0}^{\infty} \neq (c'_t)_{t=0}^{\infty}$ because, otherwise,

$$c_t = A\left(\frac{1-\delta+B}{B} h_t - \frac{1}{B} h_{t+1}\right)^{\alpha} = c'_t = A\left(\frac{1-\delta+B}{B} h'_t - \frac{1}{B} h'_{t+1}\right)^{\alpha}$$

for any $t \geq 0$ and, since $h_0 = h'_0$, by induction, we would have that $h_t = h'_t$ for any $t \geq 0$, which would be a contradiction.

Define $h_t^{\lambda} \equiv \lambda h'_t + (1-\lambda) h_t$ for any t with $\lambda \in (0,1)$. Since $(1-\delta) h_t \leq h_{t+1} \leq (1-\delta+B) h_t$ and $(1-\delta) h'_t \leq h'_{t+1} \leq (1-\delta+B) h'_t$, then $(1-\delta) h_t^{\lambda} \leq$

$h_{t+1}^{\lambda} \leq (1-\delta+B)\, h_t^{\lambda}$ for any $t \geq 0$. Thus, the sequence $\left(h_t^{\lambda}\right)_{t=0}^{\infty}$ is feasible. Let u_t^{λ} be defined as

$$u_t^{\lambda} \equiv \frac{1-\delta+B}{B} - \frac{1}{B}\frac{h_{t+1}^{\lambda}}{h_t^{\lambda}}$$

The inequality $(1-\delta)\, h_t^{\lambda} \leq h_{t+1}^{\lambda} \leq (1-\delta+B)\, h_t^{\lambda}$ implies that $0 \leq u_t^{\lambda} \leq 1$. Therefore, according to (4), the consumption path $\left(c_t^{\lambda}\right)_{t=0}^{\infty}$ defined by $c_t^{\lambda} \equiv A\left(h_t^{\lambda} u_t^{\lambda}\right)^{\alpha}$ is also feasible. We observe that

$$\begin{aligned}
c_t^{\lambda} &= A\left(\frac{1-\delta+B}{B}h_t^{\lambda} - \frac{1}{B}h_{t+1}^{\lambda}\right)^{\alpha} \\
&= A\left[\lambda\left(\frac{1-\delta+B}{B}h_t' - \frac{1}{B}h_{t+1}'\right) + (1-\lambda)\left(\frac{1-\delta+B}{B}h_t - \frac{1}{B}h_{t+1}\right)\right]^{\alpha} \\
&= A\left[\lambda l_t' + (1-\lambda)\, l_t\right]^{\alpha} \geq \lambda A l_t'^{\alpha} + (1-\lambda)\, A l_t^{\alpha} = \lambda c_t' + (1-\lambda)\, c_t
\end{aligned}$$

Since $c_t \neq c_t'$ for some t, we have that $\ln c_t^{\lambda} \geq \ln\left[\lambda c_t' + (1-\lambda)\, c_t\right] > \lambda \ln c_t' + (1-\lambda) \ln c_t$ for some t, because of the strict concavity of ln. Overall welfare can be computed multiplying by β^t and computing the infinite sum of all the per-period utilities. We obtain that

$$\sum_{t=0}^{\infty} \beta^t \ln c_t^{\lambda} > \lambda \sum_{t=0}^{\infty} \beta^t \ln c_t' + (1-\lambda) \sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \ln c_t$$

Hence, $\sum_{t=0}^{\infty} \beta^t \ln c_t^{\lambda} > \sum_{t=0}^{\infty} \beta^t \ln c_t$. This would imply that $(c_t)_{t=0}^{\infty}$ is no longer optimal, which is a contradiction.

We conclude that $(h)_{t=0}^{\infty} = (h')_{t=0}^{\infty}$, demonstrating that the BGP is the unique optimal path.

(2) Focus now on the case $B \leq B^*$ (or, equivalently, $u \geq 1$).

We want to prove that $u_t = 1$ for every $t \geq 0$. Clearly, this entails also $h_t = g_2^t h_0$ for any t.

Suppose the contrary, that is $u_T < 1$ for some T. In this case, the inequality on the right in (7) binds and we obtain

$$\frac{1}{u_{T+1}} = \frac{1}{\beta}\left(\frac{1}{u_T} - \frac{B}{1-\delta+B}\right) \geq \frac{1}{u_T}$$

Hence $u_{T+1} \leq u_T < 1$. By induction, we get $u_t < 1$ for any $t \geq T$. The sequence $(u_t)_{t \geq T}$, being non-increasing, converges to some $u^* \geq 0$.

Let us prove that $u^* = 0$. For every $t \geq T$, we have $h_{t+1}u_{t+1} = g_1 h_t u_t$. $u^* > 0$ implies

$$\lim_{t\to\infty} (h_{t+1}/h_t) = g_1 \lim_{t\to\infty} (u_t/u_{t+1}) = g_1$$

Then, $h_{t+1}/h_t = 1-\delta+B\,(1-u_t)$ entails $u^* = 0$, a contradiction.

We have $l_{t+1} = g_1 l_t$ for any $t \geq T$ and, then, $l_{T+t} = g_1^t l_T$. The solution to equation

$$\frac{1}{u_{t+1}} = \frac{1}{\beta}\left(\frac{1}{u_t} - \frac{B}{1-\delta+B}\right)$$

is given by

$$u_{T+t} = \frac{u}{1 + \frac{u-u_T}{u_T}\left(\frac{1}{\beta}\right)^t} \tag{24}$$

Consider now $T^* \geq T$ such that

$$\left(\frac{g_1}{1-\delta}\right)^\beta u_{T^*}^{1-\beta} < 1 \tag{25}$$

(this T^* exists because u_t converges to zero).

We want to prove that sequence (1) $(u_0, \ldots, u_{T-1}, u_T, u_{T+1}, \ldots)$, where u_{T+t} for any $t \geq 0$ is given by (24), can not be optimal.

Let us compare the sequence (1), with a sequence (2) $(u_0, \ldots, u_{T^*-1}, 1, 1, \ldots)$. Clearly, when $u_t = 1$ for any $t \geq T^*$, we have $h_{t+1}/h_t = g_2$, that is $h_t = g_2^{t-T^*} h_T$. In terms of utility, we obtain:

$$\begin{aligned}
U_1 &= \sum_{t=0}^{T^*-1} \beta^t \ln l_t + \sum_{t=T^*}^{\infty} \beta^t \ln l_t = \sum_{t=0}^{T^*-1} \beta^t \ln l_t + \sum_{t=T^*}^{\infty} \beta^t \ln \left(g_1^{t-T^*} l_{T^*}\right) \\
U_2 &= \sum_{t=0}^{T^*-1} \beta^t \ln l_t + \sum_{t=T^*}^{\infty} \beta^t \ln \left[(1-\delta)^{t-T^*} h_{T^*} * 1\right]
\end{aligned}$$

Hence, $U_1 < U_2$ if and only if

$$\begin{aligned}
\sum_{t=T^*}^{\infty} \beta^t \ln \left(g^{t-T^*} l_{T^*}\right) &< \sum_{t=T^*}^{\infty} \beta^t \ln \left[(1-\delta)^{t-T^*} h_{T^*} * 1\right] \\
\beta^{T^*} \sum_{t=T^*}^{\infty} \beta^{t-T^*} \ln g_1^{t-T^*} + \beta^{T^*} \ln l_{T^*} \sum_{t=0}^{\infty} \beta^t &< \beta^{T^*} \sum_{t=T^*}^{\infty} \beta^{t-T^*} \ln (1-\delta)^{t-T^*} \\
&\quad + \beta^{T^*} \ln h_{T^*} \sum_{t=0}^{\infty} \beta^t \\
\sum_{t=0}^{\infty} \beta^t \ln g_1^t + \frac{\ln l_{T^*}}{1-\beta} &< \sum_{t=0}^{\infty} \beta^t \ln (1-\delta)^t + \frac{\ln h_{T^*}}{1-\beta} \\
\ln g_1 \sum_{t=0}^{\infty} t\beta^t + \frac{\ln h_{T^*} u_{T^*}}{1-\beta} &< \ln (1-\delta) \sum_{t=0}^{\infty} t\beta^t + \frac{\ln h_{T^*}}{1-\beta} \\
\frac{\beta \ln g_1}{(1-\beta)^2} + \frac{\ln h_{T^*}}{1-\beta} + \frac{\ln u_{T^*}}{1-\beta} &< \frac{\beta \ln (1-\delta)}{(1-\beta)^2} + \frac{\ln h_{T^*}}{1-\beta} \\
\beta \ln \frac{g_1}{1-\delta} + (1-\beta) \ln u_{T^*} &< 0
\end{aligned}$$

Then, (25) implies $U_1 < U_2$ and the sequence $(u_0, \ldots, u_{T-1}, u_T, u_{T+1}, \ldots)$ with $u_T < 1$ (non-decreasing from T on) can not be optimal. We conclude that, when $B \leq B^*$, then $u_t = 1$ for any $t \geq 0$ along the optimal path. ■

Proof of Corollary 4

The LHS of (18) holds because of Assumption 1. The RHS is equivalent to $g_1 < 1$, entailing degrowth: $h_t = g_1^t h_0$. We observe that the interval $(B^*, 1/\beta - (1-\delta))$ is always nonempty if $\delta < 1$. ■

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