L∞/L1 Duality Results In Optimal Control Problems

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$\mathbb{L}^\infty/\mathbb{L}^1$ duality results in optimal control problems

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January 16, 2024

Abstract

We consider optimal control problems which consist in minimizing the L^∞ norm of an output function under an isoperimetric or L^1 inequality. These problems typically arise in control applications when one looks to minimizing the maximum trajectory deviation or "peak" under a budget constraint. We show a duality with more classical problems which amount to minimizing the L^1 cost under the state constraint given by an upper bound on the L^∞ norm of the output. More precisely, we provide a result linking the value functions of these two problems, as functions of the levels of the two kind of constraints. This is obtained for initial conditions at which lower semi-continuity of the value functions can be guaranteed, and is completed with optimality considerations. When the duality holds, we show that the two problems have the same optimal controls. Furthermore, we provide structural assumptions on the dynamics under which the semi-continuity of the value functions can be established. We illustrate theses results on non-pharmaceutically controlled epidemics models under peak or budget restrictions.

Key-words. Optimal control, L^∞ cost, iso-perimetric inequality, state constraint, value function, duality.

1 Introduction

Optimal control theory traditionally deals with integral or terminal costs, or both. For such criteria, necessary optimal conditions such as Maximum Principle of Pontryagin, or sufficient conditions based on dynamic programming and value functions are available under certain regularity hypotheses (see for instance [7]). However, these criteria do not reflect transient behaviors and, in particular, risky situations. For some applications, maximum trajectory deviations [16, 15, 19] or epidemic peaks [20] need to be minimized, leading to L^∞-type criterion [3, 10, 21] or "minimax" [23] control problems. However, minimizing excursions in the state space can be penalizing in terms of budget, usually represented by an integral or L^1 criterion. In this work, we consider that problem of minimizing a supremum cost under an isoperimetric inequality-type budget constraint. Although L^∞-type problems have been addressed in the literature, there are few theoretical tools, essentially based on the Bellman equation [4]. A practical way of dealing with these problems is to impose a constraint on the cost value, which amounts to imposing a state constraint, and then to search for the smallest value for which this state constraint can be satisfied under the budget constraint; or to minimize the budget necessary to satisfy the state constraint. Handling a running-cost problem, even under state constraints, is, perhaps, more accessible, albeit the need for structural conditions of the domain describing the constraints (see, for instance, [22], [11], [13], [12], [6]). Furthermore, such problems fall under the realm of Pontryagin's Maximum Principle and are, therefore, more likely to provide a candidate for optimality. There is thus a dual way of tackling the minimization of L^∞ cost under L^1 constraint (problem 1): minimizing the L^1 criterion under L^∞ constraint (problem 2). From this point of view, a result linking the value functions of the these two problems finds its importance, especially if this is accompanied by links between the optimal controls. The aim of the present paper is precisely to to investigate this L^∞/L^1 duality, which has not been yet addressed in the literature, to our best knowledge.

In the following, $\mathbb{R}$ and $\mathbb{R}_+$ stand for the sets of real and non-negative real numbers, respectively. For any function $\varphi : X \mapsto \mathbb{R} \cup \{\pm\infty\}$ defined on a non empty topological set X, its domain denoted $\mathrm{Dom}\, \phi$ refer to the the subset of points of X for which the image by φ is finite.

Let us illustrate this duality on a simple example in $\mathbb{R}^2$, which consists in controlling the double integrator

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & u \in [-1, 1] \end{array} \right.$$

for which one seeks to minimize the supremum of the coordinate x_1 under a L^1 budget on the control

$$\int_0^{+\infty} |u(t)| dt \leq \bar{g}$$

(problem 1). One can straightforwardly check that the optimal solution consists in taking the control $u = -1$ until x_1 reaches its maximum when $x_2(0) > 0$, which gives the following value function

$$V_g(x; \bar{g}) = \begin{cases} x_1 & \text{if } x_2 \leq 0, \\ x_1 + \frac{x_2^2}{2} & \text{if } 0 < x_2 \leq \bar{g}, \\ +\infty & \text{otherwise.} \end{cases}$$

For the dual problem, which consists in minimizing the L^1 norm of the control to keep x_1 below a threshold

$$\sup_{t \geq 0} x_1(t) \leq \bar{h}$$

(problem 2), one gets also straightforwardly that the optimal control is $u = -1$ when $x_2(0) > 0$ until x_1 reaches its maximum at the time equal to $x_2(0)$, which gives the following expression of the value function

$$V_h(x; \bar{h}) = \begin{cases} 0 & \text{if } x_1 \leq \bar{h} \text{ and } x_2 \leq 0, \\ x_2 & \text{if } x_1 + \frac{x_2^2}{2} \leq \bar{h} \text{ and } x_2 > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We note that for any x, $V_g(x, \cdot)$, $V_h(x; \cdot)$ are lower semi-continuous and a duality holds in the sense that one has

$$V_h(x, h) = \inf\{g;\ V_g(x, g) \leq h\},\ h \in \mathrm{Dom}\, V_h(x, \cdot),$$
$$V_g(x, g) = \inf\{h;\ V_h(x, h) \leq g\},\ g \in \mathrm{Dom}\, V_g(x, \cdot).$$

Regularity and characterization of value functions for optimal control problems with L^∞ cost have already been investigated in the literature (see for instance [2, 17]). However, let us underline that the consideration of L^1 constraints and the dependency of the value functions with respect to the level of the constraint have not been yet studied in the literature.

The paper is organized as follows. In Section 2, we give precise formulations of our control problems, and their assumptions. A particular emphasis is put on viability kernels as support domains of the value functions. Our main contributions are given in Section 3, where we provide a duality result under lower semi-continuity of the value functions (sub-section 3.1). Then, we give sufficient conditions for lower semi-continuity in sub-section 3.2. Section 4 is devoted to the illustration of our result on an epidemiological model with non-pharmaceutical control.

2 Preliminaries

2.1 Dynamics and Assumptions

In this work, we shall deal with a controlled dynamics

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \geq 0; \\ x(0) = x_0 \in \Omega \end{cases} \tag{1}$$

where Ω is a subset of $\mathbb{R}^n$ (where n is a positive integer) with non-empty interior, and $u(t)$ belongs to a subset U of a metric space. We require the following standard assumptions.

Assumptions 1.

1. *U is compact and the map $f : \Omega \times U \to \mathbb{R}^n$ is continuous and $[f]_1$-Lipschitz in x uniformly in u i.e.*

$$[f]_1 := \sup_{u \in U} \sup_{x,y \in \Omega,\ x \neq y} \frac{|f(x,u) - f(y,u)|}{|x - y|} < +\infty.$$

   Ω is forward invariant, i.e. any solution $x(\cdot)$ of (1) with $x_0 \in \Omega$, $u \in \mathcal{U}$ verifies $x(t) \in \Omega$ for any $t \geq 0$, where $\mathcal{U} := \mathbb{L}^0(\mathbb{R}_+; U)$, the set of Borel-measurable functions $u : \mathbb{R}_+ \to U$, will be referred to as admissible controls.

2. *The functions $h : \Omega \to \mathbb{R}$, $g : \Omega \times U \to \mathbb{R}_+$ are bounded, continuous and Lipschitz in x uniformly in u, i.e.*

$$\sup_{u \in U} \sup_{x,y \in \Omega,\ x \neq y} \frac{|g(x,u) - g(y,u)| + |h(x) - h(y)|}{|x - y|} < +\infty.$$

   We will denote by $g_\infty := \sup_{(x,u) \in \Omega \times U} g(x,u)$.

Under Assumption 1.1, system (1) admits a unique absolutely continuous solution denoted $x^{x_0,u}(\cdot)$ for $x_0 \in \Omega$, $u \in \mathcal{U}$.

2.2 The Control Problems

Let us consider the extended dynamics with an additional scalar component z that integrates the running cost g, that is

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ \dot{z}(t) = g(x(t), u(t)), \\ x(0) = x_0 \in \Omega,\ z(0) = z_0 \in \mathbb{R}_+ \end{cases} \tag{2}$$

whose solution is denoted $(x^{x_0,u}(\cdot), z^{x_0,z_0,u}(\cdot))$. As z is the integral of a bounded output function, existence and uniqueness of solutions of (2) are preserved under Assumption 1.1.

We recall our aim to address problems in which the running maximum is minimized while obeying an area upper bound or, vice-versa, minimize the area while imposing a running constraint on the trajectories. Given $x_0 \in \Omega$, $h_0 \in \mathbb{R}$, $g_0 \in \mathbb{R}_+$, we define the sets of *viable* controls related to system (2)

$$\mathcal{U}_h(x_0, h_0) := \{u \in \mathcal{U};\ h(x^{x_0,u}(t)) \leq h_0,\ \forall t \geq 0\},$$
$$\mathcal{U}_g(x_0, g_0) := \{u \in \mathcal{U};\ z^{x_0,0,u}(t) \leq g_0,\ \forall t \geq 0\},$$

and the parameterized viability kernels as follows.

$$\mathrm{Viab}_h(h_0) := \{x_0 \in \Omega : \mathcal{U}_h(x_0, h_0) \neq \emptyset\},$$
$$\mathrm{Viab}_g(g_0) := \{x_0 \in \Omega : \mathcal{U}_g(x_0, g_0) \neq \emptyset\},$$
$$\mathrm{Viab}_{hg}(h_0, g_0) := \{x_0 \in \Omega : \mathcal{U}_h(x_0, h_0) \cap \mathcal{U}_g(x_0, g_0) \neq \emptyset\}.$$

Remark 1. *$Viab_h$, $Viab_g$ considered as set-valued maps enjoy monotonicity properties (with the partial order given by the inclusion of sets). Similar assertions can be given for the set-valued map $Viab_{hg}$ if one considers the order relation $(h_0, g_0) \preceq (h'_0, g'_0)$ defined by $h_0 \le h'_0$ and $g_0 \le g'_0$.*

We first consider the optimal problem with state constraint.

Problem 1. *Given $x_0 \in \Omega$ and $h_0 \in \mathbb{R}$,*

$$\mathcal{P}_h(x_0; h_0): \quad \text{minimize } G(x_0, u) \text{ over } u \in \mathcal{U}_h(x_0, h_0)$$
$$\text{where } G(x_0, u) := \int_0^{+\infty} g(x^{x_0,u}(t), u(t))dt.$$

The value function is denoted by $V_h(x_0; h_0)$, which is set to $+\infty$ when $x_0 \notin \mathrm{Viab}_h(h_0)$.

We consider the dual problem with integral constraint.

Problem 2. *Given $x_0 \in \Omega$ and $g_0 \in \mathbb{R}_+$,*

$$\mathcal{P}_g(x_0; g_0): \quad \text{minimize } H(x_0, u) \text{ over } u \in \mathcal{U}_g(x_0, g_0)$$
$$\text{where } H(x_0, u) := \sup_{t \ge 0} h(x^{x_0,u}(t))$$

The value function is denoted by $V_g(x_0; g_0)$, which is set to $+\infty$ when $x_0 \notin \mathrm{Viab}_g(g_0)$.

We shall denote in the following partial inverses of the viability kernel map V_{hg} as follows

$$\mathrm{Viab}_{hg}^{-g}(x_0; h_0) := \{g_0 \in \mathbb{R}_+ \,;\, x_0 \in \mathrm{Viab}_{hg}(h_0, g_0)\},$$
$$\mathrm{Viab}_{hg}^{-h}(x_0; g_0) := \{h_0 \in \mathbb{R} \,;\, x_0 \in \mathrm{Viab}_{hg}(h_0, g_0)\}$$

Then, one can formulate the following observations.

Remark 2. *Note that our problems amount to finding*

$$\begin{aligned} V_h(x_0; h_0) &= \inf \, Viab_{hg}^{-g}(x_0; h_0), \\ V_g(x_0; g_0) &= \inf \, Viab_{hg}^{-h}(x_0; g_0). \end{aligned} \tag{3}$$

This is consistent with setting $+\infty$ whenever the sets to which the inf *operator is to be applied are empty. The viability kernel $Viab_{hg}$ offers thus a complete description of the domains*

$$\begin{aligned} Dom\, V_h(x_0; \cdot) &= \bigcup_{h_0 \in \mathbb{R}} Dom\left(Viab_{hg}^{-g}(x_0; h_0)\right), \\ Dom\, V_g(x_0; \cdot) &= \bigcup_{g_0 \in \mathbb{R}_+} Dom\left(Viab_{hg}^{-h}(x_0; g_0)\right) \end{aligned} \tag{4}$$

where the domain $Dom(F)$ of a set-valued map $F : \mathbb{R} \rightsquigarrow \mathbb{R}^n$ is the set of points $\theta \in \mathbb{R}$ for which $F(\theta) \neq \emptyset$. W

The functions $V_h(x_0; \cdot)$, $V_g(x_0; \cdot)$ are bounded on their domains, as g and h are bounded functions, by Assumption 1.2.

We begin with some elementary properties.

Proposition 1. *Let $x_0 \in \Omega$.*

1. *$V_g(x_0; \cdot)$ and $V_h(x_0; \cdot)$ are non-increasing.*

2. *If $V_h(x_0;\cdot)$, resp. $V_g(x_0;\cdot)$, is lower-semi-continuous, then it is right-continuous on its domain.*

Proof.

1. Let us consider $g_0 \le g_{0'}$ and $u(\cdot)$ a measurable control such that $\int_0^{+\infty} g(x^{x_0,u}(t),u(t)) \le g_0$, then one necessarily has $\int_0^{\infty} g(x^{x_0,u}(t),u(t)) \le g_{0'}$, which implies $V_g(x_0;g_{0'}) \le V_g(x_0;g_0)$. A similar argument implies that $V_h(x_0;\cdot)$ is non-increasing.

2. By monotonicity, if $h_0 \in \mathrm{Dom}\, V_h(x_0;\cdot)$, then $[h_0,\infty) \subset \mathrm{Dom}\, V_h(x_0;\cdot)$, and one has $\liminf_{h\to h_0+} V_h(x_0;h) \le V_h(x_0;h_0)$. Under the further assumption that $V_h(x_0;\cdot)$ is lower semi-continuous at h_0, one gets $\liminf_{h\to h_0+} V_h(x_0;h) = V_h(x_0;h_0)$, that is the right continuity of $V_h(x_0;\cdot)$ at h_0. The property for $V_g(x_0;\cdot)$ follows in the same way.

$\square$

Remark 3. *Here and in the following section, the semi-continuity of V_h and V_g are required with respect to the* constraint levels *h_0, g_0 and not with respect to the initial x_0. While this latter property is well studied (see, for instance, [8]), to our best knowledge, little is available for semi-continuity with respect to the level of constraints.*

3 The Main Results

We first show that a duality between problems $\mathcal{P}_g$ and $\underline{\mathcal{P}}$ can be established when the value functions V_h, V_g are lower semi-continuous. In a second step, we give sufficient conditions for these value functions to be semi-continuous.

3.1 The Duality Results

We first show how the value functions V_h, V_g can be linked.

Theorem 1. *Let $x_0 \in \Omega$.*

1. *If $V_g(x_0;\cdot)$ is right continuous at $g_0 \in \mathbb{R}_+$, then, for any $h_0 \in \mathbb{R}$ s.t. $V_h(x_0;h_0) \le g_0$, one has $V_g(x_0;g_0) \le h_0$.*
   If $V_h(x_0;\cdot)$ is right continuous at $h_0 \in \mathbb{R}$, then, for any $g_0 \in \mathbb{R}_+$ s.t. $V_g(x_0;g_0) \le h_0$ one has $V_h(x_0;h_0) \le g_0$.

2. *If the functions $V_h(x_0;\cdot)$, $V_g(x_0;\cdot)$ are lower semi-continuous on their domains, then V_h, V_g are generalized inverse i.e.*

$$\begin{array}{l} V_h(x_0;h_0) = \inf\{g_0 :\ V_g(x_0;g_0) \le h_0\},\ h_0 \in Dom\, V_h(x_0;\cdot), \\ V_g(x_0;g_0) = \inf\{h_0 :\ V_h(x_0;h_0) \le g_0\},\ g_0 \in Dom\, V_g(x_0;\cdot). \end{array} \tag{5}$$

Proof. 1) Assume $V_h(x_0;h_0) \le g_0 < +\infty$ for $h_0 < +\infty$. For any $\varepsilon > 0$, there exists an admissible control u^ε such that $\int_0^\infty g\big(x^{x_0,u^\varepsilon}(t),u^\varepsilon(t)\big)dt \le V_h(x_0;h_0)+\varepsilon \le g_0+\varepsilon$ with $h\big(x^{x_0,u^\varepsilon}(t)\big) \le h_0$, for all $t \ge 0$. Then, by definition, $V_g(x_0;g_0+\varepsilon) \le h_0$. The conclusion follows from the right-continuity of $V_g(x_0;\cdot)$ at g_0. The remaining assertion is shown in the same way.

2) By point 1. of Proposition 1, $g_0 = V_h(x_0;h_0)$ implies $V_g(x_0;g_0) \le h_0$. Then, to show

$$V_h(x_0;h_0) = \inf\{g_0' :\ V_g(x_0;g_0') \le h_0\}$$

we only need to prove the inequality $\ge$. We proceed by contradiction and assume that $V_h(x_0;h_0) = g_0 < g^0 := \inf\{g_0' :\ V_g(x_0;g_0') \le h_0\}$. By definition of the infimum one has $h^0 := V_g(x_0;\frac{g_0+g^0}{2}) > h_0$ and by monotonicity, $V_g(x_0;g_0') \ge h^0,\ \forall g_0' \in \left[g_0, \frac{g_0+g^0}{2}\right]$. This is in contradiction with $V_g(x_0;g_0) \le h_0$. The assertion concerning V_g is quite similar and its proof is omitted. $\square$

This result shows that a duality holds in the sense of equality (5) on the condition that the value functions V_h, V_g are lower semi-continuous with respect to h and g respectively.

Remark 4. *The second assertion in Theorem 1 can, alternatively, be written as in terms of viability kernels introduced in Section 2.1. Let $x_0 \in \Omega$ be such that $V_h(x_0, \cdot)$, respectively $V_g(x_0, \cdot)$, is lower semi-continuous on*

$$\bigcup_{\bar{h}\in\mathbb{R}} Dom\Big(Viab_{hg}(\bar{h},\cdot)^{-1}(x_0)\Big), \text{ respectively } \bigcup_{\bar{g}\in\mathbb{R}_+} Dom\Big(Viab_{hg}(\cdot,\bar{g})^{-1}(x_0)\Big)$$

Then, one has the equivalence

$$\inf Viab_{hg}^{-g}(x_0,\bar{h}) \le \bar{g} \iff \inf Viab_{hg}^{-h}(x_0,\bar{g}) \le \bar{h}.$$

Indeed, if $\bar{h}$ and $\bar{g}$ are such that $\inf Viab_{hg}^{-g}(x_0,\bar{h}) \le \bar{g}$*, then one has $V_h(x_0,\bar{h}) \le \bar{g}$ from the first equality in* (3) *and one gets* $\inf\{g \ : \ V_g(x_0,g) \le \bar{h}\} \le \bar{g}$ *with the first equality in* (5)*, which implies* $\inf Viab_{hg}^{-h}(x_0,\bar{g}) \le \bar{h}$*. The reverse implication is obtained similarly using* (3) *and* (5)*.*

It is our belief that the duality is more transparent in the initial formulation, while viability kernel formulations seem to hint to a hidden game-like behavior. In this direction, we refer the readers to [5].

We obtain as a consequence of Theorem 1 the following remarkable properties about functions $V_g(x_0;\cdot)$, $V_h(x_0;\cdot)$.

Lemma 1. *Whenever $V_g(x_0;\cdot)$ and $V_h(x_0;\cdot)$ are lower semi-continuous, one has*

$$V_g(x_0;\cdot) \text{ constant on } [V_h(x_0;V_g(x_0;g_0)), g_0], \ \forall g_0 \in \mathbb{R}_+;$$
$$V_h(x_0;\cdot) \text{ constant on } [V_g(x_0;V_h(x_0;h_0)), h_0], \ \forall h_0 \in \mathbb{R}.$$

Proof. From Proposition 1, $V_g(x_0;\cdot)$ and $V_h(x_0;\cdot)$ are everywhere right-continuous, and one gets $V_g(x_0;V_h(x_0;h_0)) \le h_0$ and $V_h(x_0;V_g(x_0;g_0)) \le g_0$ for any $h_0 \in \mathbb{R}$, $g_0 \in \mathbb{R}_+$.

Take $\tilde{h}_0 := V_g(x_0;V_h(x_0,h_0))$. One has then $h_0 \ge \tilde{h}_0$ and by monotonicity of $V_h(x_0;\cdot)$, one gets

$$V_h(x_0;h_0) \le V_h(x_0;\tilde{h}_0) = V_h(x_0;V_g(x_0;V_h(x_0;h_0))).$$

On another hand, take $\tilde{g}_0 := V_h(x_0;h_0)$. One has then $V_h(x_0;V_g(x_0;\tilde{g}_0)) \le \tilde{g}_0$ that is

$$V_h(x_0;V_g(x_0,V_h(x_0;h_0))) \le V_h(x_0;h_0).$$

One then concludes that

$$V_h(x_0;V_g(x_0;V_h(x_0;\cdot))) = V_h(x_0;\cdot),$$

and, in a similar way,

$$V_g(x_0;V_h(x_0;V_g(x_0;\cdot))) = V_g(x_0;\cdot).$$

As a consequence, $V_g(x_0;\cdot)$, respectively $V_h(x_0;\cdot)$ are constant on $[V_h(x_0;V_g(x_0;g_0)), g_0]$, respectively $[V_g(x_0;V_h(x_0;h_0)), h_0]$. □

We show now implications in terms of optimal controls, firstly for the problem $\mathcal{P}_g$.

Theorem 2. *Let $(x_0,h_0) \in \Omega\times\mathbb{R}$ be such that $V_h(x_0;h_0) < +\infty$ and $V_h(x_0;\cdot)$ is lower semi-continuous. Posit*

$$\underline{h}_0 := \inf\{h_0' : \ V_h(x_0;h_0') = V_h(x_0;h_0)\}, \ g_0 := V_h(x_0;h_0) = V_h(x_0;\underline{h}_0).$$

If u^* is optimal for $\mathcal{P}_h(x_0;\underline{h}_0)$, then u^* is optimal $\mathcal{P}_g(x_0;g_0)$. In particular, if u^* is unique, then one has

$$g_0 := V_h\Big(x_0;\sup_{t\ge 0} h\Big(x^{x_0,u^*}(t)\Big)\Big).$$

Proof. Let us fix u^* as in the statement. Clearly, u^* is admissible for Problem $\mathcal{P}_h(x_0; h_0)$. Indeed, by optimality of u^*, the area constraint is saturated i.e. $\int_0^\infty g\big(x^{x_0,u^*}(t), u^*(t)\big)dt = V_h(x_0; \underline{h}_0) = g_0$ and, as a consequence (point 1. of by Proposition 1), one gets

$$V_g(x_0; g_0) \leq \underline{h}_0.$$

Let us assume that there exists a control $\tilde{u}$ such that $\tilde{h}_0 := J_g(x_0, \tilde{u}, g_0) < \underline{h}_0$. Then $V_g(x_0; g_0) \leq \tilde{h}_0$ and, thus, $V_h(x_0; \tilde{h}_0) \leq g_0 = V_h(x_0; h_0)$. This inequality is established due to the first assertion combined with the right-continuity of $V_h(x_0; \cdot)$ (cf. Proposition 1). By monotonicity, this can only happen when $V_h(x_0; \tilde{h}_0) = V_h(x_0; h_0)$ which contradicts the choice of h_0.

When the optimal control $u^\star$ is unique, one has $V_h(x_0; h_0) = V_h(x_0; \inf_{t\geq 0} h\big(x^{x_0,u^*}(t)\big)) = g_0$ and

$$\underline{h}_0 = \inf_{t\geq 0} h\Big(x^{x_0,u^*}(t)\Big).$$

Then, $u^\star$ is optimal for $\mathcal{P}_h(x_0; \underline{h}_0)$, and therefore also optimal for $\mathcal{P}_g(x_0; g_0)$ with $g_0 = V_h(x_0; \underline{h}_0)$. □

One obtains analogously the following result for $\mathcal{P}_h$.

Theorem2bis. *Let $(x_0, g_0) \in \Omega \times \mathbb{R}_+$ be such that $V_g(x_0; g_0) < +\infty$ and $V_g(x_0; \cdot)$ is lower semi-continuous. Posit*

$$\underline{g}_0 = \inf\{g_0' \geq 0 \ : \ V_g(x_0; g_0') = V_g(x_0; g_0)\},\ h_0 := V_g(x_0; g_0) = V_g(x_0; \underline{g}_0).$$

If u^* is optimal for $\mathcal{P}_g(x_0; \underline{g}_0)$, then u^* is optimal for $\mathcal{P}_h(x_0; h_0)$. In particular, if u^* is unique, then one has

$$h_0 := V_g\left(x_0; \int_0^{+\infty} g\Big(x^{x_0,u^\star}(t), u^\star(t)\Big)dt\right).$$

These last two theorems show the practical interest of the duality (when it holds): the optimal solutions of both problems coincide. In applications, one can then choose the easiest problem to solve, analytically or numerically.

3.2 Criteria for lower semicontinuity

As we have seen in Proposition 1 and Theorem 1, the lower semi-continuity of the value functions is a crucial ingredient to obtain a duality. It is thus worthwhile to specify assumptions that ensure this property. For this purpose, we shall consider the family of optimal control problems with discounted cost.

Problem 1q. *Given $q > 0$, $x_0 \in \Omega$ and $h_0 \in \mathbb{R}$,*

$$\mathcal{P}_h^q(x_0; h_0): \text{ minimize } G^q(x_0, u) \text{ over } u \in \mathcal{U}_h(x_0, h_0)$$
$$\text{where } G^q(x_0, u) := \int_0^{+\infty} e^{-qt} g(x^{x_0,u}(t), u(t))dt$$

for which we denote by $V_h^q(x_0; h_0)$ the value function (set to $+\infty$ when $x_0 \notin Viab_h(h_0)$).

We also require the classical hypotheses in optimal control theory about the extended velocity set for problem $\mathcal{P}_h$.

Assumption 2. *For any $x \in \Omega$, one has*

$$\bigcup_{u\in U, r\geq 0} \begin{bmatrix} f(x,u) \\ g(x,u)+r \end{bmatrix} \text{ is closed and convex.}$$

For convenience, we define, for any subset $L \subset \Omega$ and $(x_0, u) \in \Omega \times \mathcal{U}$ the hitting time function

$$\tau_L^{x_0,u} := \begin{cases} +\infty, & \text{if } x^{x_0,u}(t) \notin L, \ \forall t \geq 0, \\ \inf\{t;\ x^{x_0,u}(t) \in L\}, & \text{otherwise.} \end{cases}$$

Proposition 2. *Let $x_0 \in Viab_h(h_0)$ for $h_0 \in \mathbb{R}$.*

1. *For any $q > 0$, the map $V_h^q(x_0, \cdot)$ is bounded and lower semi-continuous on $[h_0, +\infty)$. Moreover, if $V_h(x_0; \cdot) = \sup_{q>0} V_h^q(x_0; \cdot)$, then it is also bounded and lower semi-continuous.*

2. *If there exists $\varepsilon > 0$ and a compact set $L \subset \Omega$ such that*

   i. *for any $\bar{h} \in [h_0, h_0 + \varepsilon)$, $L \cap Viab_h(\bar{h})$ is forward invariant by a viable and null-cost control, i.e.*

   $$\forall y_0 \in L \cap Viab_h(\bar{h}),\ \exists u(\cdot) \in \mathcal{U}_h(x_0, \bar{h}) \ \textit{s.t.}\ x^{y_0,u}(t) \in L \ \textit{and}\ g(x^{y_0,u}(t), u(t)) = 0, \forall t > 0$$

   ii. *L is finitely reached under viable controls i.e.*

   $$T^\star := \sup_{\bar{h} \in [h_0, h_0+\varepsilon)} \ \sup_{u \in \mathcal{U}_h(x_0, \bar{h})} \tau_L^{x_0,u} < +\infty \tag{6}$$

   then $V_h(x_0, \cdot)$ is bounded and lower semi-continuous on $[h_0, h_0 + \varepsilon)$.

Similar assertions hold true for $V_g(x_0; \cdot)$.

Proof. Let us fix $x_0 \in \Omega$ and, for the time being, $q > 0$. For any h_0 such that $x_0 \in \mathrm{Viab}_h(h_0)$, $V_h^q(x_0; \cdot)$ is well defined and bounded on $[h_0, \infty)$. Moreover, $Vh, q(x_0; \cdot)$ is non-increasing on $[h_0, \infty)$. As such, the lower semi-continuity of $V_h^q(x_0; \cdot)$ at h_0 only needs to be shown on decreasing sequences $h_n \to h_0$ $(n \geq 1)$. Posit

$$V_h := \liminf_{n\to\infty} V_h^q(x_0; h_n) < +\infty,$$

and consider, for any $n \geq 1$, a control $u_n \in \mathcal{U}_h(x_0, h_n)$ such that

$$\int_0^{+\infty} e^{-qt} g(x^{x_0,u_n}(t), u_n(t)) dt \leq V_h^q(x_0; h_n) + \frac{1}{n}.$$

We then define the sequence of functions

$$v_n(t) := \int_0^{+\infty} e^{-qs} g(x^{x_0,u_n}(s+t), u_n(s+t)) ds,\ t \geq 0.$$

Note that $v_n(\cdot)$ is the unique bounded solution of the equation

$$\dot{v}_n(t) = q v_n(t) - g(x^{x_0,u_n}(t), u_n(t)),\ t \geq 0.$$

Let us also define the set-valued map, for $(x, v) \in \Omega \times \mathbb{R}$

$$F(x, v) := \bigcup_{u \in U, \alpha \in [0,1]} \begin{bmatrix} f(x, u) \\ qv - \alpha g(x, u) - (1-\alpha) g_\infty \end{bmatrix}$$

which is Lipschitz continuous with compact convex values (from Assumptions 1.1 and 2). Clearly, $(x^{x_0,u_n}(\cdot), v_n(\cdot))$ is solution of the differential inclusion $(\dot{x}, \dot{v}) \in F(x, v)$. Passing to the limit (along some subsequence), for every compact time interval $[0, T]$, (x^{x_0,u_n}, v_n) converges uniformly to some solution (x, v) of $(\dot{x}, \dot{v}) \in F(x, v)$ with $x(0) = x_0$ and $v(0) = V_h$ (as a consequence of the Theorem of compactness of solutions of differential inclusions, see e.g. [9]). Furthermore, v is bounded since v_n are uniformly bounded by $\frac{1}{q}\|g\|_\infty$. The procedure can be repeated to obtain a solution (x,v) defined for any $t \in \mathbb{R}_+$. From Filippov selection Lemma, there exist admissible controls $(u(\cdot),\alpha(\cdot))$ such that

$$x(t) = x^{x_0,u}(t), \dot{v}(t) = qv(t) - \alpha(t)g(x(t),u(t)) - (1-\alpha(t))g_\infty, \text{ a.e. } t \geq 0.$$

Note that v is a bounded solution of

$$\dot{v}(t) = qv(t) - g(x^{x_0,u}(t),u(t)) - r(t), \quad t \geq 0, \tag{7}$$

where r is the bounded non-negative function

$$r(t) := (1-\alpha(t))(g_\infty - g(x^{x_0,u}(t),u(t)), \quad t \geq 0,$$

and that the unique bounded solution of (7) is given by

$$v(t) = \int_0^{+\infty} e^{-qs} g(x^{x_0,u}(s+t),u(s+t))ds + \int_0^{+\infty} e^{-qs} r(s+t)ds. \tag{8}$$

Moreover, for any $T \in (0,+\infty)$ and $p \geq 2$, the convergence of solutions x^{x_0,u_n} and the continuity and boundedness of h yields

$$\liminf_{n\to\infty} \int_0^T h^p(x^{x_0,u_n}(t))\ dt \geq \int_0^T h^p(x^{x_0,u}(t))\ dt,$$

from which one deduces

$$\begin{array}{rcl}
\sup\limits_{t\in[0,T]} \ h(x^{x_0,u}(t)) & = & \sup\limits_{p\geq 2} \|h(x^{x_0,u})\|_{\mathbb{L}^p([0,T];\mathbb{R})} \\
& \leq & \sup\limits_{p\geq 2} \liminf\limits_{n\to\infty} \|h(x^{x_0,u_n})\|_{\mathbb{L}^p([0,T];\mathbb{R})} \\
& = & \sup\limits_{p\geq 2} \sup\limits_{n\geq 1} \inf\limits_{m\geq n} \|h(x^{x_0,u_m})\|_{\mathbb{L}^p([0,T];\mathbb{R})} \\
& \leq & \sup\limits_{n\geq 1} \inf\limits_{m\geq n} \sup\limits_{p\geq 2} \|h(x^{x_0,u_m})\|_{\mathbb{L}^p([0,T];\mathbb{R})} \\
& = & \liminf\limits_{n\to\infty} \|h(x^{x_0,u_n})\|_{\mathbb{L}^\infty([0,T];\mathbb{R})} \\
& \leq & \liminf\limits_{n\to\infty} \sup\limits_{t\geq 0} \ h(x^{x_0,u_n}(t)) \leq h_0,
\end{array}$$

and as this last inequality is valid for any $T > 0$, one deduces the inequality

$$\sup_{t\geq 0} \ h(x^{x_0,u}(t)) \leq h_0. \tag{9}$$

Finally, from (8) and (9) one obtains

$$V_h = v(0) = \int_0^{+\infty} e^{-qs} g(x^{x_0,u}(s),u(s))ds + \int_0^{+\infty} e^{-qs} r(s)ds \geq V_h^q(x_0;h_0),$$

that is

$$\liminf_{n\to\infty} V_h^q(x_0;h_n) \geq V_h^q(x_0;h_0),$$

which proves the lower semi-continuity and boundedness of $V_h^q(x_0;\cdot)$ at h_0. As the upper envelope $\sup\limits_{q>0} V_h^q(x_0;\cdot)$ is lower semi-continuous, we deduce that when the value function $V_h(x_0;\cdot)$ verifies $V_h(x_0;\cdot) = \sup_{q>0} V_h^q(x_0;\cdot)$, then it is also lower semi-continuous (and bounded as g is bounded).

Under assumption (6), one has clearly

$$V_h^q(x_0;\bar{h}) = \inf_{u\in\mathcal{U}_h(x_0,\bar{h})} \int_0^{T^\star} e^{-qt} g(x^{x_0,u}(t),u(t))dt,$$

for $\bar{h} \in [h_0, h_0 + \varepsilon)$ and, thus,

$$V_h(x_0; \bar{h}) = \inf_{u \in \mathcal{U}_h(x_0, \bar{h})} \int_0^{T^\star} g(x^{x_0,u}(t), u(t))dt = \sup_{q>0} V_h^q(x_0; \bar{h}), \quad \bar{h} \in [h_0, h_0 + \varepsilon).$$

□

Point 1. of Proposition 2 is a theiretical result while point 2. is more "practical" in the sense that it can be checked on data of the problem, but requires stronger conditions. Its application is illustrated in the next section on a concrete example, for which we show also how the duality results of Theorems 1 and 2 hold.

4 Illustration on an epidemiological model

The present work has been initially motivated by the study of the classical epidemiological SIR model with a non-pharmaceutical control

$$\begin{cases} \dot{s} = -\beta(1-u)si, \\ \dot{i} = \beta(1-u)si - \gamma i, \\ \dot{r} = \gamma i, \end{cases} \tag{10}$$

where s, i, r stand for the densities of the susceptible, infected and recovered populations, respectively. The control variable u takes values in $U = [0, \overline{u}]$ with $\overline{u} \leq 1$, and represent the effort to isolate the susceptible population from contacts with infected individuals, typically by reducing social distance. The objective is to investigate how to reduce the peak of the infected population under a budget constraint on the control variable, given by the L^1 norm of $u(\cdot)$. Two complementary contributions appear recently in the literature.

I. In [1], the authors consider the problem of minimizing the budget functional $||u||_{L^1}$, while maintaining constrained the infection peak to an upper ICU (intensive care unit) constraint $i \leq i_{\max}$ (see also [18]). It is shown in [1, Theorem 5.6] that the "greedy" control acting only as the trajectory reaches the boundary of viability kernel related to $i_{\max}$ is the unique optimal one. Further insights on the geometry and Hamilton-Jacobi approaches make the object of [14].

II. In [20], the aim is to minimize the peak of the infection $\sup_{t>0} i(t)$ given a budgetary constraint $||u||_{L^1} \leq Q$. Using Green's theorem, the main result in [20, Proposition 2] proves directly the optimality of the same type of greedy policy (under an assumption on $\bar{u}$ to be large enough for maintaining i constant when s reaches the immunity level $\frac{\gamma}{\beta}$). The reader is invited to look at Figure 1 for an illustration of an optimal solution in coordinates (s, i, z), where the variable $z(t)$ represents the available budget at time t, with the corresponding control $u(\cdot)$.

Our goal here is to explain why the same control is optimal for both problems, and how the duality results of Section 3 apply on this model. One can check on (10) that the property $s(t) + i(t) + r(t) = 1$ is satisfied for any control $u(\cdot)$ any $t \geq 0$. To keep it simple, we take here $n = 2$ with state variable $(s, i) \in \Omega$ where

$$\Omega := \left\{(s, i) \in \mathbb{R}^2;\ s > 0,\ i > 0,\ s + i \leq 1\right\},$$

and consider

$$h(s, i) := i, \quad g(s, i, u) = u.$$

Then the dual Problems 1 and 2 are exactly the ones described in I and II above. The reader can straightforwardly check that Assumptions 1 and 2 are satisfied. For convenience, we shall write $i^* \in [0, 1]$ instead of h_0. In [20], it is shown that the values the optimal control always takes values below $1 - \frac{\gamma}{\beta}$. We assume without loss of generality

$$\bar{u} < 1 - \frac{\gamma}{\beta}.$$

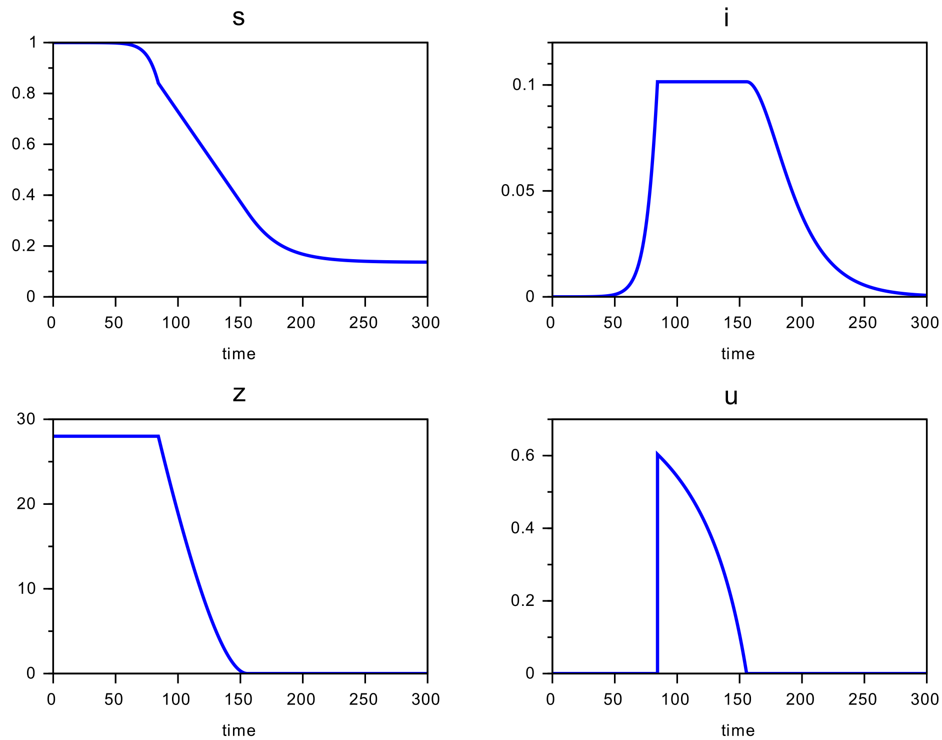

Figure 1: Example of an optimal solution for $\beta = 0.21$, $\gamma = 0.07$ with $||u||_{L^1} = 28$ and $i^* = 0.0115$ (from [20]).

Let us recall the expression of the greedy policy given in [1].

$$u^*_{i^\star}(s,i) := \begin{cases} \bar{u}, & \text{if } s > \frac{\gamma}{\beta(1-\bar{u})},\ i = \bar{\imath}(s,i^\star), \\ 1 - \frac{\gamma}{\beta s}, & \text{if } s \in \left[\frac{\gamma}{\beta}, \frac{\gamma}{\beta(1-\bar{u})}\right],\ i = i^*, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where $\bar{\imath}(s,i^\star) := i^* - s + \frac{\gamma}{\beta(1-\bar{u})}\left[1 + \log\left(\frac{\beta(1-\bar{u})s}{\gamma}\right)\right]$. When this control never reaches the value $\bar{u}$, it consists in $u = 0$ until i reaches $i^\star$, then to stay at $i = i^\star$ with a non-null (singular) control until s reaches the immunity threshold $s_h := \frac{\gamma}{\beta}$, and finally ends with $u = 0$ (a strategy called NSN for null-singular-null in [21]). If the expression of the singular control is above $\bar{u}$, then an anticipation with $u = \bar{u}$, given by $\bar{\imath}(s) < i^\star$, is necessary so that i reaches $i^\star$ with the singular control equal to $\bar{u}$. In terms of viability, this amounts to take $u = 0$ when the state is in the interior of $\mathrm{Viab}_h(i^\star)$ and a non non-null control when it reaches $\partial\mathrm{Viab}_h(i^\star)$.

We first study the domain of the value function V_h, and then we show its lower semi-continuity.

4.1 The structure of the domain of the value function V_h

Let $i^* \in [0,1]$ be fixed. Then, according to [1, Theorem 2.3], and provided that $i^* + \frac{\gamma}{\beta(1-\bar{u})} \leq 1$ if fulfilled, one has

$$\begin{aligned} (s_0, i_0) \in \mathrm{Viab}_h(i^*) \quad \Leftrightarrow \quad & \left\{s_0 \leq \tfrac{\gamma}{\beta(1-\bar{u})},\ i_0 \leq i^*\right\} \text{ or} \\ & \left\{s_0 > \tfrac{\gamma}{\beta(1-\bar{u})},\ i_0 \leq \tfrac{\gamma}{\beta(1-\bar{u})}\left(1 + \log\left(\tfrac{\beta(1-\bar{u})s_0}{\gamma}\right)\right) - s_0 + i^*\right\} \end{aligned} \tag{12}$$

The sets $\mathrm{Viab}_h(i^*) \subset \mathrm{Viab}_h(1)$ are compact. The last condition yields, in an equivalent form

$$\mathrm{Dom}\, V_h(s_0, i_0; \cdot) = \begin{cases} [i_0, \infty)\,, \text{ if } s_0 \le \frac{\gamma}{\beta(1-\overline{u})}, \\ \left[i_0 + s_0 - \frac{\gamma}{\beta(1-\overline{u})}\left[1 + \log\left(\frac{\beta(1-\overline{u})s_0}{\gamma}\right)\right], \infty\right)\,, \text{ otherwise.} \end{cases}$$

For further developments, we shall also introduce the invariance kernel associated to i^*

$$\mathrm{Inv}_h(i^\star) := \{(s_0, i_0) \in \Omega : \mathcal{U}_h((s_0, i_0), i^\star) = \mathcal{U}\}$$

Clearly, this set is determined keeping $u = 0$, and similar to $\mathrm{Viab}_h(i^*)$, one has

$$(s_0, i_0) \in \mathrm{Inv}_h(i^*) \quad \Leftrightarrow \quad \begin{array}{l} \left\{s_0 \le \frac{\gamma}{\beta},\, i_0 \le i^*\right\} \text{ or} \\ \left\{s_0 > \frac{\gamma}{\beta},\, i_0 \le \frac{\gamma}{\beta}\left[1 + \log\left(\frac{\beta s_0}{\gamma}\right)\right] - s_0 + i^*\right\} \end{array} \tag{13}$$

On Figure 2, the boundary of $\mathrm{Viab}_h(i^*)$ is represented by the graph of a function ψ depicted in yellow, while the set $\mathrm{Inv}_h(i^*)$ has a boundary represented in green as the graph of a function ϕ. The intermediate domain $\mathcal{B}$ which as a magenta boundary ($\partial\mathcal{B}$), in complement of the upper barrier $i = i^*$ in blue, is the set of initial conditions for which the greedy control is below $\bar{u}$ for almost any time (see [14]).

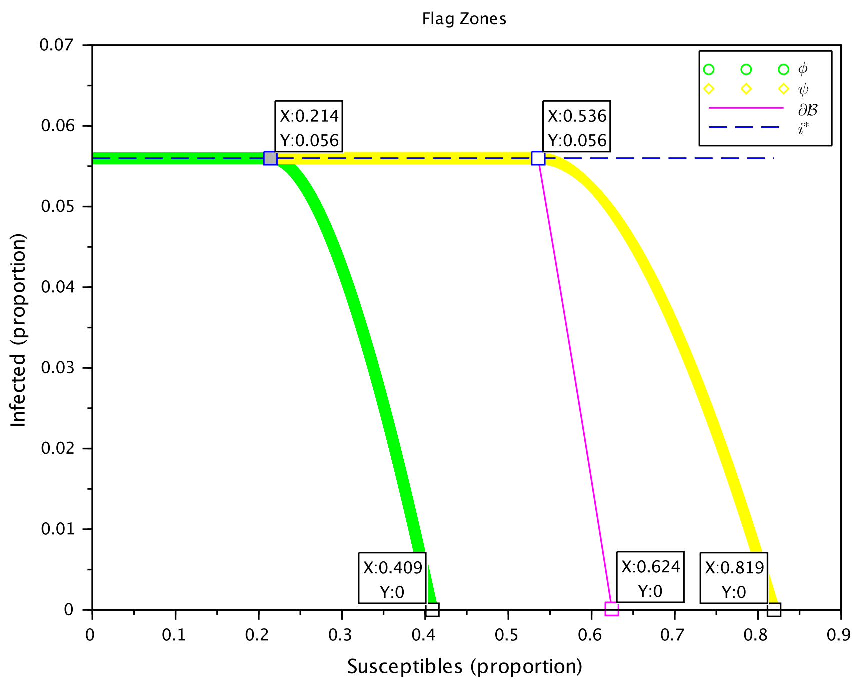

Figure 2: Geometric zones for $\beta = \frac{1}{3}$, $1 - \overline{u} = 0.4$, $\gamma = \frac{1}{14}$, and $i^* = 0.056$ (from [14]).

4.2 Regularity of the value function V_h

In view of applying Proposition 2, we consider the set

$$L = \left[0, \frac{\gamma}{\beta}\right] \times [0, 1],$$

which has the following properties.

Proposition 3. *Let $i^* \in Dom\, V_h(s_0, i_0; \cdot)$ such that $i^* + \frac{\gamma}{\beta(1-\overline{u})} < 1$. Then, one has*

1. $L \cap Viab_h(i^*) = \left[0, \frac{\gamma}{\beta}\right] \times [0, i^*] \subset Inv_h(i^*)$.

2. $[i^*, 1 - \frac{\gamma}{\beta(1-\overline{u})}] \subset Dom\, V_h(s_0, i_0; \cdot)$. *If u is a 1-optimal*[1] *admissible control for $\mathcal{P}_h(s_0, i_0; i_2^*)$ with $i_2^* \in [i^*, 1 - \frac{\gamma}{\beta(1-\overline{u})}]$, then the time to reach L is bounded with*

$$\tau_L^{(s_0,i_0,u)} \leq \max \left\{ 0, \frac{\log \frac{s_0 \beta}{\gamma}}{\beta(1-\overline{u}) i_0} \exp\left[\gamma(V_h(s_0, i_0; i^*) + 1)\right] \right\}.$$

Proof. 1) The first assertion follows from the geometry of the domains given in (12) and (13).
2) Since $V_h(s_0, i_0; \cdot)$ is non-increasing, the inclusion of domains follows. Let $i_2^* \geq i^*$ and u be an admissible, 1-optimal control for problem $\mathcal{P}_h$. Then, one has

$$\int_0^\infty u(t)dt \leq V_h(s_0, i_0; i_2^*) + 1 \leq V_h(s_0, i_0; i^*) + 1.$$

As long as $s > \frac{\gamma}{\beta}$ (i.e., before reaching L), one gets $\dot{i} \geq -\gamma u i$ which implies

$$i(t) \geq i_0 e^{-\gamma \int_0^t u(\tau)d\tau} \geq i_0 e^{-\gamma \int_0^\infty u(\tau)d\tau}.$$

Then $\dot{s} < -\beta s(1-\overline{u}) i_0 e^{-\gamma \int_0^\infty u(\tau)d\tau}$ leading to

$$s(t) \leq s_0 \exp\left(-\beta(1-\overline{u}) i_0 e^{-\gamma \int_0^\infty u(\tau)d\tau} t\right)$$

and the conclusion follows. □

As a consequence, assertion 2. of Proposition 2 provides the lower semi-continuity of $V_h(s_0, i_0; \cdot)$ on its domain.

4.3 Optimality of the greedy policy

Now that we have proved the lower semi-continuity of $V_h(s_0, i_0; \cdot)$, Theorem 2 shows that if the (greedy) feedback policy given in (11) is the unique optimal control to the Problem 1, then it is also an optimal policy for Problem 2 for the budget

$$Q = ||u^*_{i^*}(\cdot)||_{L^1}.$$

Remark 5. *The right-continuity of $J_h(s_0, i_0; \cdot)$ can be easily checked, see [14, Lemma 1] for an idea of explicit computations of similar $\underline{J}(s_0, i_0; i^*)$. Whenever u^* is shown to be optimal for Problem 1 (resp. Problem 2), this implies the right-continuity of $V_h(s_0, i_0; \cdot)$ (resp. $V_g(s_0, i_0; \cdot)$). Since these functions are non-increasing, this is equivalent to their lower semi-continuity.*

Finally, to show the optimality of the feedback control $u^*_{i^*}$ for Problem 1, given in (11), we apply [14, Theorem 4]. We need to check the conditions coming from [14, Eq. (15)]. One writes

$$\tilde{l}_1(s, i, u) := \frac{u}{\gamma i u}.$$

1) The first condition in [14, Eq. (15)] requires, for $s_0 > \frac{\gamma}{\beta}$,

$$\frac{1}{\gamma i^*} \leq \frac{1}{\gamma i_0}, \tag{14}$$

[1] 1-optimal controls satisfy $J_h(s_0, i_0; u) \leq V_h(s_0, i_0; i_2^*) + 1$.

which is obviously due to the viability requirement $i \leq i^*$. In the case where $s_0 \leq \frac{\gamma}{\beta(1-\bar{u})}$, due to [14, Theorem 4], the NSN strategy mentioned above is optimal for problem $\mathcal{P}_h$. Therefore, it is also for problem $\mathcal{P}_h$. This is, of course, a vivid illustration of our main results in the present paper. Alternatively, we could have adapted the direct method in [20, Proposition 2] based on Green's theorem, to deal directly with problem $\mathcal{P}_h$.

2) The second condition, written as long as $s \geq \frac{\gamma}{\beta(1-\bar{u})}$, in [14, Eq. (15)] is implied if

$$\frac{1}{\gamma i^{s_0,i_0,0}} \leq \frac{1}{\gamma i_0}. \tag{15}$$

This is, of course, a simple consequence of the derivative of i being negative as long as null control is used and $s > \frac{\gamma}{\beta}$. It worth to note that under the condition (15), owing to the result on duality in this paper, we are able to extend the optimality result in [20] to any $(s_0, i_0) \in \Omega$ for which the greedy policy saturates at $\bar{u}$ i.e. for $s_0 > \frac{\gamma}{\beta(1-\bar{u})}$ (when $\bar{u} < 1 - \frac{\gamma}{\beta}$).

5 Conclusion

In this work, we have shown the role of the viability kernels associated to the L^∞ and L^1 constraints, not only for determining the domains of the associated value functions, but also related to the viable controls that allow to show the semi-continuity of the value functions (cf Proposition 2.2). The duality which is established between problems $\mathcal{P}_h$ and $\mathcal{P}_g$ under the lower semi-continuity of the value functions imply that both problems have the same optimal policies. These theoretical results are illustrated on an epidemiological model, for which one look to reduce epidemic peak under a budget constraints on non-pharmaceutical actions. These properties should lead to efficient numerical approaches for this type of problems, which could be the matter of a future work.

Acknowledgments. This research was funded in whole or in part by the French National Research Agency (ANR) under the NOCIME project (ANR-23-CE48-0004-03). D.G. acknowledges support from the NSF of Shandong Province (NO. ZR202306020015), the National Key R and D Program of China (NO. 2018YFA0703900), and the NSF of P.R. China (NO. 12031009).

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